Michel Marias

$L^p$–boundedness of oscillating spectral multipliers on Riemannian manifolds


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\textit{L}^p-\textit{boundedness of oscillating spectral multipliers on Riemannian manifolds}

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Abstract

We prove endpoint estimates for operators given by oscillating spectral multipliers on Riemannian manifolds with $C^\infty$-bounded geometry and nonnegative Ricci curvature.

Keywords: spectral multipliers, wave equation, Riesz means

AMS Subject Classification: 58G03

1 Introduction and statement of the results

Let $M$ be an $n-$dimensional, complete, noncompact Riemannian manifold with nonnegative Ricci curvature and let us assume that it has $C^\infty$-bounded geometry, that is, the injectivity radius is positive and every covariant derivative of the curvature tensor is bounded (cf. [25]). Let $d(.,.)$ denote the Riemannian distance on $M$, $dx$ its volume element. Let us denote by $B(x,r)$ the ball of radius $r > 0$ centered at $x \in M$ and by $|B(x,r)|$ its volume. By the Bishop comparison theorem (cf. [5]), the assumption that $M$ has nonnegative Ricci curvature implies that

\[
\frac{|B(x,r)|}{|B(x,t)|} \leq \left( \frac{r}{t} \right)^n, \quad r \geq t > 0, \tag{1.1}
\]

and hence

\[
|B(x,2r)| \leq 2^n |B(x,r)|, \quad r > 0.
\]

This is the so called ‘doubling volume property’ and makes $M$ a ‘space of homogeneous type’ in the sense of Coifman and Weiss [8]. Thus we can define the atomic Hardy space $H^1(M)$ and $BMO(M)$, the space of functions of bounded mean oscillation, in the standard way (cf. [8]). Further, by Theorem B of [8], $BMO(M)$ is the dual of $H^1(M)$. 

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Let \( L \) be the Laplace-Beltrami operator. It admits a selfadjoint extension on \( L^2(M) \), also denoted by \( L \) and hence the spectral resolution

\[
L = \int_0^\infty \lambda dE_\lambda.
\]

Given a bounded measurable function \( m(\lambda) \), we can define, by the spectral theorem, the operator

\[
m(L) = \int_0^\infty m(\lambda)dE_\lambda.
\]

This operator is bounded on \( L^2(M) \). The function \( m(\lambda) \) is called multiplier. Oscillating multipliers are multipliers of the type

\[
m_{\alpha,\beta}(\lambda) = \psi(|\lambda|)|\lambda|^{-\beta/2}e^{i\lambda^{\alpha/2}}, \quad \alpha > 0, \beta \geq 0.
\]

with \( \psi \) a smooth function which is 0 for \(|\lambda| \leq 1\) and 1 for \(|\lambda| \geq 2\).

In this article we shall prove some endpoint results concerning the \( L^p \) boundedness of the family of operators

\[
m_{\alpha,\beta}(L) = \int_0^\infty m_{\alpha,\beta}(\lambda)dE_\lambda.
\]

We have the following:

**Theorem 1.1:** Let \( m_{\alpha,\beta} \) be as above and let \( \alpha \in (0,1) \). The following hold:

(i). If \( \beta = \frac{\alpha n}{2} \), then \( m_{\alpha,\beta}(L) \) is bounded from \( H^1(M) \) to \( L^1(M) \), on \( L^p(M) \), \( 1 \leq p < \infty \) and from \( L^\infty(M) \) to \( BMO(M) \).

(ii). If \( 0 \leq \beta < \frac{\alpha n}{2} \), then \( m_{\alpha,\beta}(L) \) is bounded on \( L^p(M) \), for \( \beta \geq \alpha n \left| \frac{1}{p} - \frac{1}{2} \right| \), \( 1 \leq p < \infty \).

(iii). If \( \beta > \frac{\alpha n}{2} \), then \( m_{\alpha,\beta}(L) \) is bounded on \( L^p(M) \) for \( 1 \leq p \leq \infty \).

Oscillating multipliers fall outside the scope of Calderón-Zygmund theory and they have been studied extensively. See for example [31, 14, 10, 11, 21, 22, 23, 28, 26] for \( \mathbb{R}^n \) and [9, 1, 20, 12] for more abstract settings.

The above result, in the context of \( \mathbb{R}^n \) and for \( 0 \leq \beta \leq \alpha n/2 \), has been proved by Fefferman and Stein in [11]. In the context of Riemannian manifolds of nonnegative Ricci curvature, Alexopoulos [1], has proved that
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for any $\alpha > 0$, $m_{\alpha,\beta}(L)$ is bounded on $L^p$ for $\beta > \alpha \left| \frac{1}{p} - \frac{1}{2} \right|$, $1 \leq p \leq \infty$. According to [11], the results above, for $0 \leq \beta \leq \alpha n/2$, are optimal.

For the proof of the $H^1 - L^1$ boundedness of $m_{\alpha,\beta}(L)$, we follow the strategy that Alexopoulos sketches at the end of the paper [1]. The idea, which is due to M. Taylor, is to express $m_{\alpha,\beta}(L)$ in terms of the wave operator $\cos t\sqrt{L}$ and then use the Hadamard parametrix method to get very precise estimates of its kernel near the diagonal. Away from the diagonal, we use the finite propagation speed property of $\cos t\sqrt{L}$ and the fast decay of the multiplier at infinity to obtain that $m_{\alpha,\beta}(L)$ is bounded on $L^p$, $p \geq 1$.

To prove that the operator $m_{\alpha,\beta}(L)$ is bounded on $L^p$ for $\beta = \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|$, $1 < p < \infty$, we compose $m_{\alpha,\beta}(L)$ with the imaginary powers of the Laplacian, which are bounded on $H^1$, (cf. [19]), and then use the $H^1 - L^1$ boundedness of $m_{\alpha,\beta}(L)$ and complex interpolation.

We shall apply Theorem 1.1 in order to obtain similar results for the Riesz means associated with the Schrödinger type group $e^{isL^{\alpha/2}}$ i.e. for the family of operators

$$I_{k,\alpha}(L) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isL^{\alpha/2}} ds, \quad 0 < \alpha < 1, \; k > 0.$$ 

We have the following

**Theorem 1.2:** For any $\alpha \in (0,1)$, the following hold:

(i). If $k = \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded from $H^1(M)$ to $L^1(M)$, on $L^p(M)$, $1 < p < \infty$, and from $L^\infty(M)$ to BMO $(M)$.

(ii). If $k < \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded on $L^p(M)$, for $k \geq n \left| \frac{1}{p} - \frac{1}{2} \right|$, $1 < p < \infty$.

(iii). If $k > \frac{n}{2}$, then $I_{k,\alpha}(L)$ is bounded on $L^p(M)$, $1 \leq p \leq \infty$.

In the context of $\mathbb{R}^n$, the operators $I_{k,\alpha}(L)$ are studied for example in [27] and [22]. According to [27], the results above, for $k \leq n/2$, are optimal. The operators $I_{k,\alpha}(L)$ have also been studied in more abstract contexts, see for example [1, 2, 17, 18, 4, 6].
It is worth mentioning that our approach is valid only for \( \alpha \in (0, 1) \). This is due to the fact that the estimates of the multiplier \( m_{\alpha, \beta}(\lambda) \) are available only for \( \alpha \in (0, 1) \), (cf. [31] and Section 5).

The paper is organized as follows. In Section 2 we recall some known facts about the Hardy space \( H^1 \) and \( BMO \) (Subsection 2.1), the wave operator and the construction of its parametrix (Subsection 2.2). In Section 3 the estimates of the Fourier transform of the derivatives of the multiplier \( m_{\alpha, \beta}(\lambda) \) are given. In Section 4 we give the estimates of the kernel of the operator \( m_{\alpha, \beta}(L) \) near the diagonal and in Section 5 we establish its \( L^p \)-boundedness when \( \beta > n/2 \). In Section 6 we prove the \( H^1 - L^1 \) boundedness of the operator \( m_{\alpha, \beta}(L) \) and in Section 7 we finish the proofs of Theorems 1.1 and 1.2.

Throughout this article the different constants will always be denoted by the same letter \( c \). When their dependence or independence is significant, it will be clearly stated.

## 2 Preliminaries

### 2.1 The Hardy space \( H^1 \) and \( BMO \)

Let us recall that a complex-valued function \( a \) on \( M \) is an atom if it is supported in a ball \( B(y_0, r) \) and satisfies
\[
\|a\|_\infty \leq |B(y_0, r)|^{-1} \quad \text{and} \quad \int_M a(x) \, dx = 0.
\]

A function \( f \) on \( M \) belongs to the Hardy space \( H^1(M) \) if there exist \( (\lambda_m)_{m \in \mathbb{N}} \in \ell^1 \) and a sequence of atoms \( (a_m)_{m \in \mathbb{N}} \) such that
\[
f = \sum_{m \in \mathbb{N}} \lambda_m a_m,
\]
where the series converges in \( L^1(M) \). The norm \( \|f\|_{H^1} \) is the infimum of \( \sum_{m \in \mathbb{N}} |\lambda_m| \) for all such decompositions of \( f \).

A function \( f \) belongs to \( BMO(M) \), if there exists a constant \( c > 0 \) such that for all balls \( B(x, r) \),
\[
\frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_B| \, dy < c,
\]
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where

\[ f_B = \frac{1}{|B(x,r)|} \int_{B(x,r)} f(y) dy. \]

The smallest of all such constants \( c \) is the BMO norm of \( f \).

Finally we note that the dual of \( H^1(M) \) is \( BMO(M) \), (cf. [8], Theorem B, p. 593).

2.2 The wave operator

Let \( G_t(x, y) \) be the kernel of the wave operator \( \cos t\sqrt{L} \). Note that \( G_t(x, y) \) is also the solution of the wave equation

\[
\begin{align*}
(\partial_t^2 + L_y) u(t, x, y) &= 0, \\
\partial_t u(0, x, y) &= \delta(x)(y), \\
\partial_t u(0, x, y) &= 0.
\end{align*}
\]

In this article we shall exploit the fact that \( G_t(x, y) \) propagates with finite propagation speed (cf. [7, 29]):

\[
\text{supp}(G_t) \subseteq \{(x, y) : d(x, y) \leq |t|\}.
\] (2.2)

Next we shall recall some facts about the Hadamard parametrix construction for the kernel \( G_t(x, y) \), (cf. [3, 4, 15]).

Let \( \delta \in (0, r_0) \), to be fixed later, and let us consider, for every ball \( B(x, \delta) \), \( x \in M \), the exponential normal coordinates centered at \( x \). Let \( g_{ij}(x, y) \), \( y \in B(x, \delta) \), be the metric tensor expressed in these coordinates and let us denote by \( (g^{ij}(x, y)) \) its inverse matrix. We have the following Taylor expansion of \( g_{ij} \):

\[
g_{ij}(x, y) = \delta_{ij} + 2 A_{ijkl}(y_k - x_k)(y_l - x_l) + \ldots
\]

where the \( k \) \( A_{ij...} \) are universal polynomials in the components of the curvature tensor and its first \( k - 2 \) covariant derivatives at the point \( x \), (cf. [24], p. 85). By the term “universal” we mean that the coefficients of the polynomials \( k A_{ij...} \) depend only on the dimension of the manifold.

It follows from (2.3) and the assumption of \( C^\infty \)-bounded geometry that for any multi-index \( \alpha \) there exists a positive constant \( c_\alpha \) such that

\[
|\partial_y^\alpha g_{ij}(x, y)| \leq c_\alpha, \ x \in M, \ y \in B(x, \delta).
\] (2.4)
Since \( g_{ij}(x, x) = \delta_{ij} \), there is \( c > 0 \) and \( \delta \in (0, r_0) \) such that
\[
c^{-1} \leq \det(g_{ij}(x, y)) \leq c.
\]

(2.5)

for all \( x \in M \) and \( y \in B(x, \delta) \).

In what follows, we shall fix a \( \delta \in (0, \min(1, r_0)) \) such that (2.5) is satisfied.

From (2.4) and (2.5) we also have that there is \( c'_{\alpha} > 0 \) such that
\[
\left| \partial_{y}^{\alpha} g^{ij}(x, y) \right| \leq c_{\alpha}.
\]

(2.6)

for all \( x \in M, y \in B(x, \delta) \).

Let \( \Theta(x, y) = \det(g_{ij}(x, y)) \). Then, the Laplace-Beltrami operator \( L \) can be written as follows:
\[
L = \frac{1}{(\Theta(x, y)))^{1/2}} \sum_{i,j} \frac{\partial}{\partial y_i} (\Theta(x, y)))^{1/2} g^{ij}(x, y) \frac{\partial}{\partial y_j}.
\]

Note that by (2.4), (2.5) and (2.6), the Laplacian can also be written as
\[
L = \sum_{|\alpha| \leq 2} c_{\alpha}(y) \partial_{y}^{\alpha}
\]

with the coefficients satisfying
\[
\left| \partial_{y}^{\beta} c_{\alpha}(y) \right| \leq c_{\alpha, \beta},
\]

(2.7)

for all \( x \in M, y \in B(x, \delta) \) and any multi-index \( \beta \).

Let us consider the following smooth functions:
\[
U_0(x, y) = \Theta^{-1/2}(x, y)
\]

and
\[
U_{k+1}(x, y) = \Theta^{-1/2}(x, y) \int_{0}^{1} s^k \Theta^{1/2}(x, y_s)L_2 U_k(x, y_s)ds,
\]

where \( y_s, s \in [0, 1] \), is the geodesic from \( x \) to \( y \) and \( L_2 \) denotes the Laplacian acting on the second variable. Note that \( U_0(x, x) = 1 \).

In what follows, we always assume that \( |t| \leq \delta \) and \( y \in B(x, \delta), x \in M \).

Let us consider the kernels
\[
E_N(t, x, y) = C_0 \sum_{k=0}^{N} (-1)^k U_k(x, y) |t| \frac{(t^2 - d(x, y)^2)^{k - \frac{n+1}{2}}}{4^k \Gamma \left( k - \frac{n-1}{2} \right)},
\]

(2.8)
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where $C_0$ is a normalizing constant.

They satisfy (cf. [3])

\[
(\partial_t^2 + L_y) E_N(t, x, y) = \frac{C_0(-1)^N}{4^N\Gamma(N - \frac{n+1}{2})} |t| (t^2 - d(x, y)^2)_+^{N - \frac{n+1}{2}} L_y U_N(x, y),
\]

\[E_N(0, x, y) = \delta_x(y), \]
\[\partial_t E_N(0, x, y) = 0.\]

Now, let us observe that by (2.4), (2.5) and (2.7) there exists a $c > 0$ such that

\[|U_0(x, y)| \leq c_0 \quad \text{and} \quad |L_y U_0(x, y)| \leq c_1. \tag{2.10}\]

These also imply that for any $k \in \mathbb{N}$ there is $c > 0$ such that

\[|U_k(x, y)| \leq \frac{c_1^k}{k!}, \quad |L_y U_k(x, y)| \leq \frac{c_1^{k+1}}{k!} \quad \text{and} \quad \|\nabla_y U_k(x, y)\| \leq \frac{c_1^k}{k!}, \tag{2.11}\]

for $x \in M$ and $y \in B(x, \delta)$.

If $k \geq \frac{n+1}{2}$, then (2.11) and the fact that

\[\Gamma\left(k - \frac{n+1}{2}\right) \sim k!, \quad \text{as} \quad k \to \infty,\]

imply that

\[\left|U_k(x, y)\right| t \left|t^2 - d(x, y)^2\right|_+^{\frac{k-n+1}{2}} \Gamma\left(k - \frac{n+1}{2}\right) \leq \frac{c_1^k \delta^{2k-(n+1)}_{\delta}}{4^k k!} \leq \frac{c_1^k \delta^{2k-n}_{\delta}}{4^k k!}. \tag{2.12}\]

From (2.8) and (2.12) we get that $E_N(t, x, y)$ converges uniformly as $N \to \infty$ and (2.9), (2.11) and (2.1) that the limit is $G_t(x, y)$. Thus we have the expansion

\[G_t(x, y) = C_0 \sum_{k=0}^{\infty} (-1)^k U_k(x, y) |t| \left|t^2 - d(x, y)^2\right|_+^{\frac{k-n+1}{2}} \Gamma\left(k - \frac{n+1}{2}\right) \frac{4^k k!}{4^k k!}, \tag{2.13}\]

the convergence being uniform for $|t| \leq \delta$ and $y \in B(y, \delta)$.

3 Estimates of the multiplier and of its derivatives

In this section we shall give some estimates for the derivatives of the Fourier transform of the multiplier $m_{\alpha, \beta}$. 

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Let us consider the function
\[ f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2) = \psi(t^2) |t|^{-\beta} e^{i|t|^\alpha}. \]

Let \( r_0 \) be the injectivity radius of \( M \) and fix \( \delta \in (0, r_0) \). Let \( \chi_\delta(t) \) be a smooth and nonnegative function such that \( \chi_\delta(t) = 1 \) for \( |t| \leq \delta/2 \) and 0 for \( |t| \geq \delta \). Set
\[ \hat{f}_{\alpha,\beta}^0(t) = \hat{f}_{\alpha,\beta}(t) \chi_\delta(t), \quad \hat{f}_{\alpha,\beta}^\infty(t) = \hat{f}_{\alpha,\beta}(t)(1 - \chi_\delta(t)). \] (3.1)

In this article we shall need the following:

**Lemma 3.1:** Let \( \alpha \in (0, 1) \) and \( \beta = \frac{\alpha n}{2} + \varepsilon, \varepsilon \geq 0 \). Then for all \( m, N \in \mathbb{N} \) and \( t \in \mathbb{R} \),
\[ \left| \partial_t^m \hat{f}_{\alpha,\beta}^0(t) \right| \leq c |t|^{-(1+m-\varepsilon - \frac{\alpha(n+1)}{4})/(1-\alpha)}, \] (3.2)
and
\[ \left| \partial_t^m \hat{f}_{\alpha,\beta}^\infty(t) \right| \leq c |t|^{-N}. \] (3.3)

Before proceeding to the proof of Lemma 3.1, let us recall the following estimates from Wainger [31], Theorem 9. For any \( \alpha \in (0, 1) \) and \( \varepsilon > 0 \), consider the function
\[ f_{\varepsilon,\alpha,b}(x) = e^{-\varepsilon \|x\|} \psi(\|x\|^2) \|x\|^{-b} e^{i\|x\|^\alpha}, \quad x \in \mathbb{R}^k. \]

We have that
\[ \hat{f}_{\varepsilon,\alpha,b}(\|x\|) = \|x\|^{rac{2k}{2-k}} \int_0^\infty e^{-\varepsilon u} \psi(u^2) u^{-b+k \varepsilon} e^{i u \alpha} J_{k-2}(u \|x\|) du \] (3.4)
where \( J_m(z) \) is the Bessel function.

Making use of this formula, Wainger proved that the limit
\[ \lim_{\varepsilon \to 0} \hat{f}_{\varepsilon,\alpha,b}(\|x\|) \]
exists and it is continuous for \( x \neq 0 \). Further, if \( b > k \left(1 - \frac{\alpha}{2}\right) \), then \( \hat{f}_{\alpha,b} \) is continuous also at \( x = 0 \), while if \( b \leq k \left(1 - \frac{\alpha}{2}\right) \) and \( M \in \mathbb{N} \), then
\[ \hat{f}_{\alpha,b}(\|x\|) = \|x\|^{-\left(k-b+k \varepsilon\right)/\alpha} e^{i\xi_\alpha \|x\|^{\alpha/(1-\alpha)}} \sum_{m=0}^M a_m \|x\|^{m\alpha/(1-\alpha)} + O(\|x\|^{(M+1)\alpha/(1-\alpha)}) + C(\|x\|), \] (3.5)
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where $a_0 \neq 0$, $\xi_\alpha$ is real and $\xi_\alpha \neq 0$; $C$ is a continuous function.

Furthermore

$$\left| \hat{f}_{a,b}(\|x\|) \right| = O(\|x\|^{-N}), \quad \text{as} \quad \|x\| \to \infty, \quad (3.6)$$

for any $N \in \mathbb{N}$.

PROOF OF LEMMA 3.1: If $m = 0$, then (3.2) and (3.3) are an immediate consequence of (3.5), with $k = 1$, and (3.6).

If $m = 2l$, $l \geq 1$, then $\partial^{2l} \hat{f}_{a,\beta}$ is the Fourier transform of the function

$$(-i\lambda)^{2l} f_{a,\beta}(\lambda) = (-i)^{2l} \psi(|\lambda|^2) |\lambda|^{-\beta+2l} e^{i|\lambda|^a} = (-i)^{2l} f_{a,\beta-2l}(\lambda).$$

Hence (3.2) and (3.3) follow again from (3.5) and (3.6) with $b = \beta - 2l$.

If $m = 2l + 1$, then $\partial^{2l+1} \hat{f}_{a,\beta}$ is the Fourier transform of the function

$$\varphi(\lambda) = (-i)^{2l+1} \psi(|\lambda|^2) \lambda |\lambda|^{-\beta+2l} e^{i|\lambda|^a}.$$

Since this function is odd, we have

$$\partial^{2l+1} \hat{f}_{a,\beta}(t) = -2i \int_0^{+\infty} \varphi(x) \sin(tx)dx$$

$$= -2i \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-\epsilon x} \varphi(x) \sin(tx)dx.$$

Since

$$\sin x = \sqrt{\frac{\pi x}{2}} J_\frac{1}{2}(x),$$

we have

$$\partial^{2l+1} \hat{f}_{a,\beta}(t) = c \sqrt{2\pi t} \lim_{\epsilon \to 0} \int_0^{+\infty} e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^a} J_\frac{1}{2}(tx)dx$$

$$= ct \lim_{\epsilon \to 0} \left\{ t^{-\frac{1}{2}} \int_0^{+\infty} e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^a} J_\frac{1}{2}(tx)dx \right\}.$$

The integral in brackets above is the same as the integral $\hat{f}_{\epsilon,a,b}(t)$ in formula (3.4), with $k = 3$ and $b = \beta - 2l$. This gives, as $\epsilon \to 0$, the Fourier transform of the multiplier $f_{a,b}(\lambda)$ in $\mathbb{R}^3$. Therefore, the estimates $\partial^{2l+1} \hat{f}_{a,\beta}(t)$ follow again from (3.5) and (3.6).
4 The estimates of the kernel near the diagonal

Let us express the operator \( m_{\alpha,\beta}(L) \) in terms of the wave operator \( \cos t\sqrt{L} \). If \( f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2) \), then \( m_{\alpha,\beta}(L) = f_{\alpha,\beta}(\sqrt{L}) \) and since \( f_{\alpha,\beta} \) is an even function, by the Fourier inversion formula we have that

\[
m_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{f}_{\alpha,\beta}(t) \cos t\sqrt{L} dt.
\]

Let \( m_{\alpha,\beta}(x, y) \) be the kernel of \( m_{\alpha,\beta}(L) \). Then by the finite propagation speed property (2.2)

\[
m_{\alpha,\beta}(x, y) = (2\pi)^{-1/2} \int_{|t| \geq d(x, y)} \hat{f}_{\alpha,\beta}(t) G(t, x, y) dt.
\]

This kernel is singular near the diagonal and integrable at infinity. We want to split \( m_{\alpha,\beta}(x, y) \) into these two parts and treat them separately. This can be done by considering the operators

\[
m^0_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{f}^0_{\alpha,\beta}(t) \cos t\sqrt{L} dt
\]

and

\[
m^\infty_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{f}^\infty_{\alpha,\beta}(t) \cos t\sqrt{L} dt,
\]

where \( f^0_{\alpha,\beta} \) and \( f^\infty_{\alpha,\beta} \) are defined in (3.1). We have

\[
m_{\alpha,\beta}(L) = m^0_{\alpha,\beta}(L) + m^\infty_{\alpha,\beta}(L).
\]

Let \( m^0_{\alpha,\beta}(x, y) \) and \( m^\infty_{\alpha,\beta}(x, y) \) denote the kernels of \( m^0_{\alpha,\beta}(L) \) and \( m^\infty_{\alpha,\beta}(L) \), respectively. Then

\[
m^0_{\alpha,\beta}(x, y) = (2\pi)^{-1/2} \int_{|t| \geq \delta(x, y)} \hat{f}^0_{\alpha,\beta}(t) G(t, x, y) dt \tag{4.1}
\]

and

\[
m^\infty_{\alpha,\beta}(x, y) = (2\pi)^{-1/2} \int_{|t| > \delta} \hat{f}^\infty_{\alpha,\beta}(t) G(t, x, y) dt.
\]
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In the present section we deal with the kernel $m_{\alpha,\beta}^0(x, y)$. This kernel contains the singular part of the kernel $m_{\alpha,\beta}(x, y)$ and from (4.1) it follows that

$$\text{supp}(m_{\alpha,\beta}^0) \subset \{(x, y) \in M \times M : d(x, y) \leq \delta\}. \quad (4.2)$$

We shall obtain very good $L^\infty$ estimates for $m_{\alpha,\beta}^0(x, y)$ by using the Hadamard parametrix construction for $G_t(x, y)$. These estimates allow us to prove in Section 6 that $m_{\alpha,\beta}(L)$ is bounded from $H^1$ to $L^1$ for $\beta = n\alpha/2$.

We have the following:

**Lemma 4.1:** Let $\alpha \in (0, 1)$. Then for all $\varepsilon \geq 0$, there exists a constant $c > 0$ such that for all $x, y \in M$

$$\left| m_{\alpha,\beta}^0(x, y) \right| \leq cd(x, y)^{-n+\frac{\varepsilon}{1-\alpha}} \quad (4.3)$$

and

$$\left| \nabla_y m_{\alpha,\beta}^0(x, y) \right| \leq cd(x, y)^{-(n+1)+\alpha'}, \quad (4.4)$$

where $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$.

For $\beta = \frac{n\alpha}{2} + \varepsilon$ and $k = -1, 0, 1, \ldots$, we set

$$I_k(x, y) = \int_{\mathbb{R}} f_{\alpha,\beta}^0(t) \left| t \right| \frac{(t^2 - d(x, y)^2)^{k-\frac{n+1}{2}}}{\Gamma(k-\frac{n-1}{2})} dt.$$

Lemma 4.1 is a consequence of the expansion (2.13) of $G_t(x, y)$ and of the following:

**Lemma 4.2:**  
(i). If $0 \leq k \leq \frac{n+1}{2}$, then there is a $c > 0$ such that

$$|I_k(x, y)| \leq cd(x, y)^{-n+\frac{\varepsilon}{1-\alpha}}, \quad \forall x, y \in M. \quad (4.5)$$

(ii). If $k > \frac{n+1}{2}$, then there is a $c > 0$ such that

$$|I_k(x, y)| \leq c\frac{\delta^{2k}}{\Gamma(k-\frac{n-1}{2})}, \quad \forall x, y \in M. \quad (4.6)$$

(iii). If $k = -1$ and $\varepsilon = 0$, then there is a $c > 0$ such that

$$|I_k(x, y)| \leq cd(x, y)^{-(n+2)+\alpha'}, \quad \forall x, y \in M. \quad (4.7)$$
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Proof: The proof is given in steps. Let us set, for simplicity, \( d = d(x, y) \).

Proof of (4.5) for \( n = 2p + 1 \). This is the simpler case. If we put \( t = ud \), then we have

\[
I_k(x, y) = d^{2k-n+1} \int_R |u| \hat{f}_{\alpha, \beta}^0(ud) (u^2-1)^{\frac{k-n+1}{2}} \frac{du}{\Gamma(k-\frac{n+1}{2})}
\]

\[
= d^{2k-n+1} \int_R |u| \hat{f}_{\alpha, \beta}^0(ud) (u+1)^{k-p-1} \frac{du}{\Gamma(k-p)}
\]

Since

\[
\frac{(u-1)^{k-p-1}}{\Gamma(k-p)} = \delta^{(p-k)}(u-1), \quad \text{for } k \leq p+1,
\]

(cf. [13], p. 56), we have

\[
I_k = d^{2k-n+1} \left( \partial_u^{p-k} |u| \hat{f}_{\alpha, \beta}^0(ud) (u+1)^{k-p-1} \right)_{u=1}
\]

\[
= d^{2k-n+1} \sum_{m=0}^{p-k} c_{m, p, k} \left( \partial_u^m \hat{f}_{\alpha, \beta}^0(ud) \partial_u^{p-k-m} \left( |u| (u+1)^{k-p-1} \right) \right)_{u=1}
\]

Making use of Lemma 3.1, we get that for all \( m = 0, ..., p-k \),

\[
\left| \partial_u^m \hat{f}_{\alpha, \beta}^0(ud)_{u=1} \right| \leq c d^m \frac{d^m}{d^{1+m-(\alpha+1)/\alpha}} \frac{d^{m-p}}{d^{1+m-(p+1)/\alpha}} \frac{d^{p-m}}{d^{1+m-(p+1)/\alpha}}
\]

\[
= c d^{1+m-\alpha} d^\alpha \frac{d^{m-p}}{d^{1+m-(p+1)/\alpha}} \frac{d^{p-m}}{d^{1+m-(p+1)/\alpha}}
\]

\[
= c d^{1+m-\alpha} d^\alpha \frac{d^{m-p}}{d^{1+m-(p+1)/\alpha}} \frac{d^{p-m}}{d^{1+m-(p+1)/\alpha}}
\]

This implies that for all \( k \geq 0 \),

\[
|I_k| \leq c d^{2k-n+1} d^{-1} d^\alpha \frac{d^{k(1-\alpha)}}{d^{1+m-(p+1)/\alpha}} \leq c d^{-n} d^\alpha
\]

which proves (4.5), when \( n = 2p + 1 \).
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Proof of (4.5), for \( n = 2p \). In this case we have

\[
I_k(x, y) = \int_{\mathbb{R}} |t| \hat{f}_{x, y}(t) \frac{(t^2 - d^2)^{k-p-rac{1}{2}}}{\Gamma(k - p + \frac{1}{2})} dt.
\]

The calculations now are more complicated because \( k - p - \frac{1}{2} \) is no more an integer. If we put \( t = du \) and \( v = u + 1 \), then

\[
I_k = cd^{2k-2p+1} \int_{|u|>1} |u| \hat{f}_{x, y}(du) (u^2 - 1)^{k-p-rac{1}{2}} du
\]

\[
= cd^{2k-2p+1} \int_{u>1} u \hat{f}_{x, y}(du) (u+1)^{k-p-rac{1}{2}} (u-1)^{k-p-rac{1}{2}} du
\]

\[
+ cd^{2k-2p+1} \int_{u<-1} (-u) \hat{f}_{x, y}(du) |u-1|^{k-p-rac{1}{2}} (-u+1)^{k-p-rac{1}{2}} du
\]

\[
= cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{x, y}(dv+1) (v+2)^{k-p-rac{1}{2}} (v-1)^{k-p-rac{1}{2}} dv
\]

\[
+ cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{x, y}(-d(v+1)) (v+2)^{k-p-rac{1}{2}} (v-1)^{k-p-rac{1}{2}} dv.
\]

Since \( \hat{f}_{x, y} \) is an even function

\[
I_k = 2 cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{x, y}(dv+1) (v+2)^{k-p-rac{1}{2}} (v-1)^{k-p-rac{1}{2}} dv.
\]

We shall only treat the term \( I_0 \) which is the most singular near \( v = 0 \). The integrals \( I_k, k > 0 \), can be treated similarly. We have

\[
I_0 = cd^{-2p+1} \int_{0}^{\infty} (v+1) \hat{f}_{x, y}(dv+1) (v+2)^{-p-rac{1}{2}} (v-1)^{-p-rac{1}{2}} dv.
\]

(4.9)

By replacing the term \( (v+2)^{-p-rac{1}{2}} \) by its Taylor’s expansion at \( v = 0 \), we can see that the most singular part of \( I_0 \) is the integral

\[
J_0 := d^{-2p+1} \int_{0}^{\infty} \hat{f}_{x, y}(dv+1) v_+^{-p-rac{1}{2}} dv.
\]

Let us observe that \( \hat{f}_{x, y}(dv+1) \) is the Fourier transform of the function

\[
\frac{1}{d} f_{x, y} \left( \frac{t}{d} \right) e^{it} = \frac{1}{d} \psi \left( \left| \frac{t}{d} \right| \right) \left| \frac{t}{d} \right|^{-\alpha/2} e^{it}.
\]

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Also, the Fourier transform of the distribution \( v_+^{-p+\frac{1}{2}} \) is equal to

\[
i\Gamma \left( -p + \frac{1}{2} \right) \left[ e^{-i\frac{\pi}{2}(p+\frac{1}{2})} t_+^{-\frac{1}{2}} - e^{+i\frac{\pi}{2}(p+\frac{1}{2})} t_-^{-\frac{1}{2}} \right],
\]

(cf. [13], p. 172). So,

\[
J_0 = d^{2p+1} \int_{-\infty}^{\infty} \frac{1}{2} \psi \left( \frac{1}{2} |u|^2 \right) |\frac{t}{d}|^{-\alpha/2} e^{it\frac{1}{2}} e^{iut} \left[ c_1 t_+^{p-\frac{1}{2}} - c_2 t_-^{p-\frac{1}{2}} \right] dt
\]

\[
= J_{0,1} + J_{0,2}.
\]

We shall only treat \( J_{0,1} \). The term \( J_{0,2} \) can be treated similarly. We have

\[
J_{0,1} = c_1 d^{2p+1} d^{p-\frac{1}{2}} \int_{0}^{\infty} \psi \left( u^2 \right) u^{-\frac{\alpha}{2} - \frac{\varepsilon}{2} + p - \frac{1}{2}} e^{iu^2} \cos(ud) du
\]

\[
+ ic_1 d^{2p+1} d^{p-\frac{1}{2}} \int_{0}^{\infty} \psi \left( u^2 \right) u^{-\frac{\alpha}{2} - \frac{\varepsilon}{2} + p - \frac{1}{2}} e^{iu^2} \sin(ud) du
\]

\[= d^{2p+1} d^{p-\frac{1}{2}} c_1 (L_1 + iL_2). \tag{4.10}\]

Now \( L_1 \) is the Fourier transform of the even function

\[
f_{\alpha,b}(u) = \psi \left( |u|^2 \right) |u|^{-\frac{\alpha}{2} - \frac{\varepsilon}{2} + p - \frac{1}{2}} e^{iu^2},
\]

with \( b = \frac{\alpha}{2} + \varepsilon - p + \frac{1}{2} \). So, by (3.5), with \( k = 1 \), we get that

\[
|L_1| \leq cd^{(1-\frac{\alpha}{2}-\varepsilon+p-\frac{1}{2})/(1-\alpha)}
\]

\[
= cd^{\left( 1-\frac{\alpha}{2}+p(1-\alpha) \right)/(1-\alpha)} d^{\frac{\varepsilon}{1-\alpha}} = d^{-\frac{1}{2}} d^{\frac{\varepsilon}{1-\alpha}}. \tag{4.12}\]

By the formula \( \sin x = \sqrt{\frac{\pi}{2}} J_{\frac{1}{2}}(x) \), we have

\[
L_2 = \int_{0}^{\infty} \psi \left( u^2 \right) u^{-\frac{\alpha}{2} - \frac{\varepsilon}{2} + p - \frac{1}{2}} e^{iu^2} \sin(ud) du
\]

\[
= c\sqrt{d} \int_{0}^{\infty} \psi \left( u^2 \right) u^{-\frac{\alpha}{2} - \frac{\varepsilon}{2} + p} e^{iu^2} J_{\frac{1}{2}}(ud) du
\]

\[
= cd \lim_{\rho \to 0} \left\{ \frac{1}{2} \int_{0}^{\infty} e^{-\rho u} \psi \left( u^2 \right) u^{-\left( \frac{\alpha}{2} + \varepsilon - p + \frac{3}{2} \right)} e^{iu^2} J_{\frac{1}{2}}(ud) du \right\}.
\]

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The integral in the brackets above is the same as the integral \( \hat{f}_{\epsilon, \alpha, b} \) in (3.4) with \( k = 3 \) and \( b = \frac{\alpha p}{2} + \epsilon - p + \frac{3}{2} \). Therefore, by (3.5), with \( k = 3 \), we get that, for

\[
|L_2| \leq cdd^{-\left(3 \frac{\alpha p}{2} - \epsilon + p - \frac{3}{2} - \frac{\alpha p}{2}\right)/(1 - \alpha)}
\]

\[
= cdd^{-\left(\frac{3}{2}(1 - \alpha) + p(1 - \alpha)\right)/(1 - \alpha)}d^{-\epsilon/(1 - \alpha)}
\]

\[
= cd^{-\frac{3}{2}d^{-\epsilon/(1 - \alpha)}} = cd^{-\frac{3}{2}d^{-p\epsilon/(1 - \alpha)}}.
\]

It follows from (4.11), (4.12) and (4.13) that

\[
|J_0,1| \leq cd^{d - 2}d^{-\frac{1}{2}d^{-p\epsilon^{-\frac{1}{2}}}} = cd^{-n}d^{-\frac{\epsilon}{(1 - \alpha)}}.
\]

Putting all together, from (4.9) to (4.14), we get

\[
|I_k(x, y)| \leq cd^{-n}d^{-\frac{\epsilon}{(1 - \alpha)}},
\]

which proves (4.5), for \( n = 2p \).

Proof of (4.6). If \( k > \frac{n+1}{2} \), then by (3.2) and (3.3) we get

\[
|I_k(x, y)| \leq c \int_{d \leq |t| \leq \delta} \left| \hat{f}_{\alpha, \beta}^0(t) \right| |t| \left( t^2 - \delta^2 \right)^{k - \frac{n + 1}{2}} \Gamma(k - \frac{n + 1}{2}) dt
\]

\[
\leq \frac{c}{\Gamma(k - \frac{n + 1}{2})} \int_{d \leq |t| \leq \delta} |t|^{-1 - \frac{\alpha(n + 1)}{2}} |t|^{2k - n} dt.
\]

But, if \( k > \frac{n+1}{2} \), then

\[
2k - n - \frac{1 - \epsilon - \alpha(n + 1)}{(1 - \alpha)} \geq \frac{2\epsilon + \alpha(n - 1)}{2(1 - \alpha)} > 0,
\]

so,

\[
|I_k(x, y)| \leq \delta^{2k - n + 1 - \frac{1 - \epsilon - \alpha(n + 1)}{2}} \Gamma(k - \frac{n - 1}{2}) \leq \frac{c \delta^{2k}}{\Gamma(k - \frac{n - 1}{2})}.
\]

Proof of (4.7). We shall only treat the case \( n = 2p + 1 \). The case \( n = 2p \) can be treated similarly. As in the proof of (4.5), we have to estimate the
integral

\[ I_{-1}(x, y) = \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(t) |t| \left( \frac{(t^2 - d(x,y)^2)^{\frac{n+1}{2}}}{\Gamma(-1 - \frac{n+1}{2})} \right) dt \]

\[ = d^{-n-1} \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(du) |u| \left( \frac{(u^2-1)^{-\frac{n+1}{2}}}{\Gamma(-\frac{n+1}{2})} \right) du \]

\[ = d^{-n-1} \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(du) |u| (u + 1)^{-p-2} \left( u^{-2\frac{n+1}{2}} \right) \Gamma(-\frac{n+1}{2}) dt \]

\[ = d^{-n-1} \partial_u^{p+1} \left( |u| \hat{f}_{\alpha, \beta}^0(ud) (u + 1)^{-p-2} \right) \bigg|_{u=1}. \]

So,

\[ |I_{-1}(x, y)| \leq cd^{-n-1} \sum_{m=0}^{p+1} c_{m,p} d^{m} t^{(1+m-(p+1)\alpha)/(1-\alpha)} \]

\[ = cd^{-n-1} \sum_{m=0}^{p+1} c_{m,p} d^{m} t^{(m\alpha-m+p\alpha)/(1-\alpha)} \]

\[ = cd^{-n-2} \sum_{m=0}^{p+1} c_{m,p} d^{m\alpha-m+p\alpha} \]

\[ = cd^{-n-2} \sum_{m=0}^{p+1} c_{m,p} d^{\frac{\alpha}{1-\alpha}(p-m)} \leq cd^{-n-2} d^{-\alpha/(1-\alpha)} = cd^{-n-2} d^\alpha'. \]

\[ \square \]

**Proof of Lemma 4.1:** (i). It is a consequence of (2.11) and Lemma 4.2. (ii) Making use of (2.13), we have

\[ \nabla_y G_t(x, y) = \sum_{k=0}^{\infty} (-1)^k \nabla_y U_k(x, y) |t| \left( \frac{(t^2 - d(x,y)^2)^{\frac{k}{2}}}{4^{k-1} k^{n+1} \Gamma(k-n/2)} \right) \]

\[ - \sum_{k=0}^{\infty} U_k(x, y) |t| \left( k - \frac{n+1}{2} \right) \left( \frac{(t^2 - d(x,y)^2)^{\frac{k}{2}}}{4^{k-1} k^{n+1} \Gamma(k-n/2)} \right) 2d\nabla_y (d) \]

\[ = I + II. \]

Now, it follows from (2.11) and the estimates (4.5), (4.6) for \( \varepsilon = 0 \), that

\[ |I| \leq cd(x, y)^{-n}. \]
To deal with $II$ we first note that $\|\nabla_y d(x, y)\| \leq 1$ for $d(x, y) \leq 1$. Then, by (4.6) and (4.7) we have

$$|II| \leq cd(x, y)^{-(n+1)+\alpha'}. $$

5 The $L^p$ boundedness of $m_{\alpha, \beta}(L)$ for $\beta > \frac{\alpha n}{2}$

In this Section we prove claim (iii) of Theorem 1.1 which states that for all $\alpha \in (0, 1)$ and $\beta \geq \frac{\alpha n}{2}$, $m_{\alpha, \beta}(L)$ is bounded on $L^p$, $p \geq 1$.

We note that the $L^p$ boundedness of $m_{\alpha, \beta}(L)$ for $\beta \geq \frac{\alpha n}{2}$, can be extracted from [1]. We shall give below a simple proof of this result by adapting an argument from [29].

**Proposition 5.1:** If $\alpha \in (0, 1)$ and $\beta \geq \frac{\alpha n}{2}$, then $m_{\alpha, \beta}(L)$ is bounded on $L^p$, $p \geq 1$.

**Proof:** We have that

$$m_{\alpha, \beta}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{\alpha, \beta}(t) \cos t \sqrt{\lambda} dt$$

and by the estimate (3.3) of $\hat{f}_{\alpha, \beta}(t)$ we get that $m_{\alpha, \beta}$ is bounded. Thus $m_{\alpha, \beta}(L)$ is bounded on $L^2$. Therefore, the Proposition will be a consequence of the following:

$$\sup_{x \in M} \int_M \left| m_{\alpha, \beta}(x, y) \right| dy < \infty. \quad (5.1)$$

Let us first notice that the Dirac mass $\delta_x$ at $x$ can be written as $\delta_x = L^k \varphi_x + \psi_x$, where $k = \lceil \frac{n}{4} \rceil + 1$ and where the functions $\varphi_x$ and $\psi_x$ are in $L^2(B(x, r_0))$, with $r_0$ the injectivity radius of $M$ (cf. [29], p. 776). Also by the assumption of $C^\infty$-bounded geometry, we can assume that there is $c > 0$ such that $\|\varphi_x\|_2 \leq c$ and $\|\psi_x\|_2 \leq c$ for all $x \in M$. We have

$$m_{\alpha, \beta}(x, y) = m_{\alpha, \beta}(L) \delta_x(y) = L^k m_{\alpha, \beta}(L) \varphi_x(y) + m_{\alpha, \beta}(L) \psi_x(y)$$

$$= (-i)^{-2k} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \partial^{2k} \hat{f}_{\alpha, \beta}(t) \cos t \sqrt{L} \varphi_x(y) dt$$

$$+ (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}_{\alpha, \beta}(t) \cos t \sqrt{L} \psi_x(y) dt$$

$$= I_1(x, y) + I_2(x, y). \quad (5.2)$$

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By the estimates (3.3) of $\partial_t^m \hat{f}_{\alpha,\beta}^\infty(t)$ and the finite propagation speed property we have that
\[
|I_1(x, y)| \leq c \int_{-\infty}^{\infty} \left| \partial^{2k} \hat{f}_{\alpha,\beta}^\infty(t) \cos t \sqrt{L} \phi_x(y) \right| \ dt
\]
\[
= c \sum_{j \geq 1} \int_{j \leq |t| \leq j + 1} \left| \partial^{2k} \hat{f}_{\alpha,\beta}^\infty(t) \right| \left| \cos t \sqrt{L} \phi_x(y) \right| \ dt
\]
\[
\leq c \sum_{j \geq 1} \frac{1}{j^N} \int_{j \leq |t| \leq j + 1} \left| 1_{B(x, r_0 + j + 1)}(y) \cos t \sqrt{L} \phi_x(y) \right| \ dt. 
\] (5.3)

By the Cauchy-Schwarz inequality
\[
\int_M \left| 1_{B(x, R)}(y) \cos t \sqrt{L} \phi_x(y) \right| \ dy \leq \left| B(x, R) \right|^\frac{1}{2} \left| \cos t \sqrt{L} \phi_x \right|_2 \leq c R^{n/2} \left| \cos t \sqrt{L} \right|_2 \left| \phi_x \right|_2 \leq c R^{n/2} 
\] (5.4)
since $\left| \cos t \sqrt{L} \right|_2 \leq 1$ and $\left| \phi_x \right|_2 \leq c$ for all $x \in M$.

Let $N > 2 + \frac{n}{2}$. Then, it follows from (5.2), (5.3) and (5.4) that
\[
\int_M |I_1(x, y)| \ dy \leq c \sum_{j \geq 1} (r_0 + j + 1) \frac{1}{j^{N-\frac{n}{2}}} \int_{j \leq |t| \leq j + 1} \ dt \leq c \sum_{j \geq 1} \frac{1}{j^{N-\frac{n}{2}}}
\]
and hence
\[
\sup_{x \in M} \int_M |I_1(x, y)| \ dy < \infty.
\]

The term $I_2(x, y)$ can be treated similarly. \( \square \)

**Proposition 5.2:** If $\alpha \in (0, 1)$ and $\beta > \frac{\alpha n}{2}$, then $m_{0,\beta}^\infty(L)$ is bounded on $L^p$, $p \geq 1$.

**Proof:** Since $m_{0,\beta}^\infty(L) = m_{\alpha,\beta}(L) - m_{0,\beta}^\infty(L)$, Proposition 5.1 implies that $m_{0,\beta}^\infty(L)$ is bounded on $L^2$. If $\beta = \frac{\alpha n}{2} + \varepsilon$, $\varepsilon > 0$, then from (4.2) and (4.3) we have that
\[
\sup_{x \in M} \int_M \left| m_{0,\beta}^\infty(x, y) \right| \ dy = \sup_{x \in M} \int_{B(x, \delta)} \left| m_{0,\beta}^\infty(x, y) \right| \ dy 
\]
\[
\leq c \sup_{x \in M} \int_{B(x, \delta)} d(x, y)^{-n + \frac{\varepsilon}{1+\varepsilon}} \ dy 
\]
\[
= c \sup_{x \in M} \int_{0}^{\delta} r^{-n + \frac{\varepsilon}{1+\varepsilon}} r^{n-1} \ dr = c \delta^{\frac{\varepsilon}{1+\varepsilon}}
\]
and the Proposition follows.

\section{H^1 - L^1 boundedness of the operator m_{\alpha, an}(L)}

In this section we prove claim (i) of Theorem 1.1. By the duality of $H^1$ with $BMO$, the $H^1 - L^1$ boundedness of $m_{\alpha, an}(L)$ is a consequence of the following

**Proposition 6.1:** If $\alpha \in (0, 1)$, then the operator $m_{\alpha, an}(L)$ is bounded from $L^\infty(M)$ to $BMO(M)$.

The $L^p$-boundedness of $m_{\alpha, an}(L)$ for $p \in (1, \infty)$, follows from the $L^2$ boundedness and Proposition 6.1 by interpolation and duality.

The strategy of the proof of Proposition 6.1 is inspired from [11]. It is based on the following Lemmata.

**Lemma 6.2:** There is a constant $A > 0$ such that

$$\int_{d(x,y) > 2d(y,y_1)^{1-a}} \left| m^0_{\alpha, an}(x, y) - m^0_{\alpha, an}(x, y_1) \right| dx < A, \quad (6.1)$$

for all $y_1 \in M$ and $y \in B(y_1, \delta)$.

**Proof:** Let us fix $y_1 \in M$ and $y \in B(y_1, \delta)$. Let $y(s)$, $s \in [0, d(y, y_1)]$, be the geodesic segment from $y$ to $y_1$. Then

$$m^0_{\alpha, an}(x, y) - m^0_{\alpha, an}(x, y_1) = \int_0^{d(y, y_1)} \nabla_y m^0_{\alpha, an}(x, y(s))ds.$$  

By (4.4) and the mean value theorem, we get that

$$\left| m^0_{\alpha, an}(x, y) - m^0_{\alpha, an}(x, y_1) \right| \leq c \frac{d(y, y_1)}{d(x, y^*)^{n+1-a}}, \quad (6.2)$$

for some $y^*$ on $y(s)$.

Let us set $d = d(y, y_1)$, $A_k = B(y_1, 2^{k+1}d^{1-a}) \setminus B(y_1, 2^{k}d^{1-a})$ and

$$I_k = \int_{A_k} \left| m^0_{\alpha, an}(x, y) - m^0_{\alpha, an}(x, y_1) \right| dx.$$
Then
\[
\int_{d(x,y_1)>2d(y,y_1)^{1-\alpha}} \left| m_{\alpha, a_n}^0(x, y) - m_{\alpha, a_n}^0(x, y_1) \right| \, dx \\
= \sum_{k \geq 1} \int_{A_k} \left| m_{\alpha, a_n}^0(x, y) - m_{\alpha, a_n}^0(x, y_1) \right| \, dx = \sum_{k \geq 1} I_k.
\]

Since \(d \leq \delta \leq 1\), we have
\[
d(x, y^*) \geq 2^k d^{1-\alpha} - d \geq 2^{k-1} d^{1-\alpha}, \quad \forall x \in A_k, \quad \forall k \geq 1.
\]
Now, by (6.2) and since \((1 - \alpha)(1 - \alpha') = 1\), we have
\[
I_k \leq c \int_{A_k} \frac{d(y,y_1)^{\alpha}}{d(x,y^*)^{\alpha + 1-\alpha'}} \leq c \int_{A_k} \frac{dx}{(2^{k-1} d^{1-\alpha})^{\alpha + 1-\alpha'}} \\
\leq \frac{cd|A_k|}{(2^k d^{1-\alpha})^{\alpha + 1-\alpha'}} \leq \frac{cd}{(2^k d^{1-\alpha})^{1-\alpha'(1-\alpha')}} \\
= \frac{cd}{(2^k d^{1-\alpha})^{1-\alpha'}}.
\]

It follows that
\[
\int_{d(x,y_1)>d(y,y_1)^{1-\alpha}} \left| m_{\alpha, a_n}^0(x, y) - m_{\alpha, a_n}^0(x, y_1) \right| \, dx \\
= \sum_{k=1}^{\infty} I_k \leq c \sum_{k=1}^{\infty} \frac{1}{(2^{k-1})^{1-\alpha'}} < \infty
\]
since \(1 - \alpha' > 0\) for \(\alpha \in (0, 1)\). \qed

The following Lemma is based on a local version of a generalization of Hardy-Littlewood-Sobolev theorem due to Varopoulos, (cf. [30], p. 12).

**Lemma 6.3:** For any \(\alpha \in (0, 1)\), \(m_{\alpha, a_n}(L)\) is bounded from \(L^2\) to \(L^{\frac{2}{\alpha}}\).

**Proof:** We write
\[
m_{\alpha, a_n}(L) = \psi(|L|) |L|^{-\alpha/4} e^{i[L]^{\alpha/2}} \\
= (1 + L)^{-\alpha/4} \psi(|L|) |L|^{-\alpha/4} (1 + L)^{\alpha/4} e^{i[L]^{\alpha/2}} \\
= (1 + L)^{-\alpha/4} \Phi(L),
\]
where \(\Phi(\lambda) = \psi(|\lambda|) |\lambda|^{-\alpha/4} (1 + \lambda)^{\alpha/4} e^{i[\lambda]^{\alpha/2}}\). Since \(\Phi(\lambda)\) is bounded, it suffices to show that the potential operator \((1 + L)^{-\alpha/4}\) is bounded from \(L^2\).
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to $L^{\frac{2}{1-\alpha}}$. To this end, let $q_t(x, y)$ be the kernel of the semigroup $e^{-t(1+L)}$ and $p_t(x, y)$ the heat kernel of $M$. Then

$$q_t(x, y) = e^{-t}p_t(x, y).$$

By the Li-Yau estimate of $p_t$:

$$p_t(x, y) \leq c e^{-d(x, y)^2/ct} \left| B \left( x, \sqrt{t} \right) \right|,$$

for all $t > 0$ and $x, y \in M$, (cf. [16]), it follows that

$$q_t(x, y) \leq \begin{cases} c t^{-n/2}, & \forall t \leq 1, \\ ce^{-t} \leq c t^{-n/2}, & \forall t \geq 1. \end{cases} \quad (6.3)$$

From (6.3) it follows that

$$\left\| e^{-t(1+L)} f \right\|_{\infty} \leq ct^{-n/2} \| f \|_1, \quad \forall f \in L^1, \quad \forall t > 0.$$ 

As it is shown by Varopoulos, (cf. [30], p. 12), this estimate implies that the operators $(1 + L)^{-\gamma/2}$, $\gamma > 0$, are bounded from $L^p$ to $L^q$ for $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$ and $1 < p < \infty$. The Lemma follows by taking $\gamma = \alpha n/2$ and $p = 2$. \hfill \Box

**Proof of Proposition 6.1:** In order to prove that $m_{\alpha, \frac{\alpha n}{2}}(L)$ is bounded from $L^\infty$ to $BMO$ it enough to show that there is a constant $c > 0$, such that for every ball $B(y_1, r) = B$ and every $f \in C_0^\infty(M)$

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L)f(x) - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B \right| \, dx \leq c \| f \|_{\infty} |B|, \quad (6.4)$$

where $(m_{\alpha, \frac{\alpha n}{2}}(L)f)_B$ is the mean value of $m_{\alpha, \frac{\alpha n}{2}}(L)f$ on $B$.

Let us then fix a ball $B(y_1, r) = B$ and let us set, in order to simplify the notation, $B_\alpha = B(y_1, 2r^{1-\alpha})$. If $f \in C_0^\infty(M)$, then we shall write $f = f\chi_{B_\alpha} + f\chi_{B_\alpha^c} := f_1 + f_2.$

To prove (6.4), we shall show that

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L)f_1(x) \right| \, dx \leq c \| f \|_{\infty} |B|, \quad (6.5)$$

and

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}(L)f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B \right| \, dx \leq c \| f \|_{\infty} |B|, \quad (6.6)$$

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Proof of (6.5). If \( r > 1 \), then \( r^{1-\alpha} \leq r \) and hence

\[
\int_B |m_{\alpha, \frac{n}{2}}(L)f_1(x)| \, dx \leq \|m_{\alpha, \frac{n}{2}}(L)f_1\|_2 |B|^{1/2} \leq c \|f_1\|_2 |B|^{1/2}
\]
\[
= c \|f\chi_B\|_2 |B|^{1/2} \leq c \|f\|_\infty |B_{\alpha}|^{1/2} |B|^{1/2}
\]
\[
= c \|f\|_\infty |B(y_1, 2r^{1-\alpha})|^{1/2} |B|^{1/2}
\]
\[
\leq c \|f\|_\infty |B(y_1, 2r)|^{1/2} |B|^{1/2} \leq c \|f\|_\infty |B|.
\]

In the case when \( r \leq 1 \), we proceed by arguing as in [11], Theorem 1, p. 143 (see also [9], Theorem 2.1). Let \( p = \frac{2}{(1 - \alpha)} \) and let \( p' \) be its conjugate exponent. Then by Lemma 6.3 and Hölder’s inequality

\[
\int_B |m_{\alpha, \frac{n}{2}} f_1(x)| \, dx \leq |B|^{1/p'} \|m_{\alpha, \frac{n}{2}} f_1\|_p \leq c |B|^{1/p'} \|f_1\|_2
\]
\[
\leq c |B|^{1/p'} \|f_1\|_2 = c |B|^{1/p'} \|f\chi_B\|_2
\]
\[
\leq c |B|^{1/p'} \|f\|_\infty |B(y_1, 2r^{1-\alpha})|^{1/2}
\]
\[
\leq c \|f\|_\infty r^{\frac{n}{p'} + (1-\alpha)\frac{n}{2}} = cr^n \|f\|_\infty \leq c |B| \|f\|_\infty,
\]

since \( \frac{n}{p'} + (1 - \alpha)\frac{n}{2} = \frac{n}{p'} + \frac{n}{p} = n \). This completes the proof of (6.5).

Proof of (6.6). We have

\[
|m_{\alpha, \frac{n}{2}}(L)f_2(x) - (m_{\alpha, \frac{n}{2}}(L)f)_B|
\]
\[
\leq |m_{\alpha, \frac{n}{2}}^0(L)f_2(x) - (m_{\alpha, \frac{n}{2}}^0(L)f)_B|
\]
\[
+ |(m_{\alpha, \frac{n}{2}}^0(L)f)_B - (m_{\alpha, \frac{n}{2}}^\infty(L)f)_B| + |m_{\alpha, \frac{n}{2}}^\infty(L)f_2(x)|. \tag{6.7}
\]

We write

\[
m_{\alpha, \frac{n}{2}}(L)f = m_{\alpha, \frac{n}{2}}^0(L)f_1 + m_{\alpha, \frac{n}{2}}^0(L)f_2 + m_{\alpha, \frac{n}{2}}^\infty(L)f,
\]

and we recall that the operator \( m_{\alpha, \frac{n}{2}}^0(L) \) is bounded on \( L^2 \) and that, by
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Proposition 5.1, the operator $m_{\alpha}^{\infty}(L)$ is bounded on $L^{\infty}$. Therefore,

$$
\left| (m_{\alpha}^{0}(L)f_{2})_{B} - (m_{\alpha}^{\infty}(L)f)_{B} \right|
\leq |B|^{-1} \left| \int_{B} m_{\alpha}^{0}(L)f_{2}(x)dx - \int_{B} m_{\alpha}^{\infty}(L)f(x)dx \right|
\leq |B|^{-1} \left| \int_{B} m_{\alpha}^{0}(L)f_{1}(x)dx + \int_{B} m_{\alpha}^{\infty}(L)f(x)dx \right|
\leq c |B|^{-1} \| f \|_{\infty} \ |B| + c \| f \|_{\infty} = c \| f \|_{\infty}.
$$

(6.8)

It follows from (6.7), (6.8) and the $L^{\infty}$ boundedness of $m_{\alpha}^{\infty}(L)$ that to prove (6.6), it is enough to show that

$$
\int_{B} \left| m_{\alpha}^{0}(L)f_{2}(x) - (m_{\alpha}^{0}(L)f_{2})_{B} \right| dx \leq c \| f \|_{\infty} |B|.
$$

(6.9)

Let us set

$$
c_{B} = \int_{B_{y}^{c}} m_{\alpha}^{0}(x, y_{1})f_{2}(x)dx.
$$

If $y \in B(y_{1}, r)$, then

$$
m_{\alpha}^{0}(L)f_{2}(y) - c_{B} = \int_{B_{y}^{c}} \left\{ m_{\alpha}^{0}(x, y) - m_{\alpha}^{0}(x, y_{1}) \right\} f_{2}(x)dx.
$$

Also, if $x \in B(y_{1}, 2r^{1-\alpha})$ and $y \in B(y_{1}, r)$, then

$$
d(x, y_{1}) > 2r^{1-\alpha} \geq 2d(y, y_{1})^{1-\alpha}.
$$

Therefore, by Lemma 6.2

$$
\left| m_{\alpha}^{0}(L)f_{2}(y) - c_{B} \right|
\leq \int_{B_{y}^{c}} \left| m_{\alpha}^{0}(x, y) - m_{\alpha}^{0}(x, y_{1}) \right| |f_{2}(x)| dx
\leq \| f \|_{\infty} \int_{d(x, y_{1}) > 2d(y, y_{1})^{1-\alpha}} \left| m_{\alpha}^{0}(x, y) - m_{\alpha}^{0}(x, y_{1}) \right| dx
\leq A \| f \|_{\infty}.
$$

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This implies that
\[
\int_B \left| m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2(y) - c_B \right| \, dy \leq A |B| \|f\|_{\infty}. \tag{6.10}
\]

By (6.10) we have
\[
\int_B \left| m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2(y) - m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2 \right| \, dy \\
\leq \int_B \left| m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2(y) - c_B \right| \, dy + \int_B \left| c_B - m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2 \right| \, dy \tag{6.11}
\]
\[
\leq A \|f\|_{\infty} |B| + |B| \left| c_B - m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2 \right|. 
\]

Finally, by using once more (6.10) we get
\[
\left| m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2 - c_B \right| = |B|^{-1} \left| \int_{B(y,r)} m_{\alpha,\frac{\alpha}{2}}^0 (L) f_2(y) \, dy - \int_B c_B \, dy \right| \\
\leq |B|^{-1} \left| \int_B m_{\alpha,\frac{\alpha}{2}}^0 (L) f(y) - c_B \right| \, dy \leq A \|f\|_{\infty}. \tag{6.12}
\]

7 Proof of the results

In this Section we shall finish the proofs of Theorems 1.1 and 1.2.

**Proof of Theorem 1.1:** The proof of claims (i) and (ii) of Theorem 1.1 are given in Sections 6 and 5 respectively. It remains to prove claim (ii). This will be done by complex interpolation as in Theorem 6 of [11]. Let us consider the analytic family of operators
\[
T_z(L) = e^{z^2} L^{\alpha z^2} m_{\alpha,\frac{\alpha}{2}}^0 (L), \quad Re z \in [0, 1].
\]

If \( t \in \mathbb{R} \), then
\[
T_{it}(L) = e^{-t^2} L^{i\alpha t} m_{\alpha,\frac{\alpha}{2}}^0 (L).
\]

But the imaginary powers of the Laplacian are bounded on \( H^1 \) and
\[
\|L^\gamma\|_{H^1 \rightarrow H^1} \leq c \left( 1 + \sqrt{|\gamma|} e^{\pi|\gamma|/2} \right), \quad \gamma \in \mathbb{R},
\]

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(cf. [19]). So, if we combine with Theorem 1.1(i), we get that \( T_{it}(L) \) is bounded from \( H^1(M) \) to \( L^1(M) \) and

\[
\|T_{it}(L)\|_{H^1 \to L^1} \leq ce^{-t^2} \left( c\sqrt{\pi} + \sqrt{\alpha n} |t| e^{\pi \alpha |t|/8} \right),
\]

for all \( t \in \mathbb{R} \).

Also, the operators \( T_{1+it}(L) \) are bounded on \( L^2(M) \) and

\[
\|T_{1+it}(L)\|_2 \leq ce^{-t^2}.
\]

By complex interpolation between \( \text{Re}z = 0 \) and \( \text{Re}z = 1 \), we obtain that for \( \theta \in (0, 1) \) and \( p \in (1, 2) \), the operator \( T_{\theta}(L) \) is bounded on \( L^p \) for \( \frac{1}{p} = 1 - \frac{\theta}{2} \).

If we choose \( \theta = 1 - \frac{2\beta}{\alpha n} \), then

\[
T_{\theta}(L) = e^{\theta^2 L^{\frac{\alpha}{2}}} L^{-\frac{\alpha}{4} - \frac{2\beta}{\alpha n}} m_{\alpha, \frac{\alpha}{2}}(L) = e^{\theta^2} m_{\alpha, \beta}(L)
\]

and \( \frac{1}{p} - \frac{1}{2} = \frac{\beta}{\alpha n} \). This is the desired result for \( p \in (1, 2) \). The case \( p \in (2, \infty) \) is just the dual result.

Proof of Theorem 1.2: As in [1], by replacing the operator \( L \) by \( L_1 = t^{2/\alpha}L \), the operators

\[
I_{k,\alpha}(L) = kt^{-k} \int_0^t (t - s)^{k-1} e^{isL^{\alpha/2}} ds, \quad 0 < \alpha < 1, \ k > 0,
\]

can be written in the form

\[
I_{k,\alpha}(L) = M_k(L_1^{\alpha/2}),
\]

with

\[
M_k(\lambda) = k \int_0^1 (1 - s)^{k-1} e^{is|\lambda|} ds.
\]

Further, the multiplier \( M_k(\lambda) \) can be written as

\[
M_k(\lambda) = C_k \psi(\lambda) \lambda^{-k} e^{i\lambda} + \Omega(\lambda),
\]

where \( \psi \) is as in (1.2) and \( \Omega(\lambda) \) satisfies

\[
\partial_\lambda^N \Omega(\lambda) = O(\lambda^{-N-1}), \quad \text{as} \ \lambda \to \infty,
\]

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for all \(N \in \mathbb{N}\), (cf. [1], [27], p. 336).

This implies that

\[
\left| \hat{\Omega}(t) \right| \leq \frac{c(N,R)}{|t|^{N+1}}, \quad \text{for} \quad |t| \geq R.
\]

Making use of this and by arguing in exactly the same way as in Proposition 5.1 we can prove that the operator \(\Omega(L)\) is bounded on \(L^p\), \(p \geq 1\). Furthermore, by Theorem 1.1(ii), \(C_k \psi(L_1)L_1^{-\alpha k/2}e^{it\nu_1/2}\) is bounded on \(L^p\) for \(\alpha k \geq \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|\) i.e. for \(k \geq n \left| \frac{1}{p} - \frac{1}{2} \right|\), \(1 < p < \infty\). This proves the claim (ii) of Theorem 1.2. The claims (i) and (iii) can be deduced in a similar way from Theorem 1.1(i) and (iii).

References


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**Michel Marias**

**Aristotle University of Thessaloniki**

**Department of Mathematics**

**Thessaloniki, 54.124**

**Greece**

marias@auth.gr