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On an integral formula of Berndtsson related to the inversion of the Fourier-Laplace transform of $\bar{\partial}$-closed $(n, n-1)$-forms

Telemachos Hatziafratis

Abstract

We give a proof of an integral formula of Berndtsson which is related to the inversion of Fourier-Laplace transforms of $\bar{\partial}$-closed $(n, n-1)$-forms in the complement of a compact convex set in $\mathbb{C}^n$.

1 Introduction

Let $K$ be a compact and convex subset of $\mathbb{C}^n$ and $F(\zeta)$ an entire analytic function of the following exponential type: For every $\delta > 0$ there exists a constant $C_\delta > 0$ so that
\[ |F(\zeta)| \leq C_\delta \exp \left( H_K(\zeta) + \delta|\zeta| \right) \quad (\zeta \in \mathbb{C}^n), \quad (1.1) \]
where $H_K(\zeta) = \sup \{ \Re \langle z, \zeta \rangle : z \in K \}$ and $\langle z, \zeta \rangle = \sum_{j=1}^{n} z_j \zeta_j$. One way to produce functions $F(\zeta)$, which satisfy (1.1), is to take a $\bar{\partial}$-closed $(n, n-1)$-form $\theta(z)$ in $\mathbb{C}^n - K$ and consider its Fourier-Laplace transform $F_\theta(\zeta)$ defined by the integral
\[ F_\theta(\zeta) = \int_{z \in S} e^{\langle z, \zeta \rangle} \theta(z), \]
where $S$ is a smooth $(2n-1)$-dimensional surface surrounding $K$. Then it is easy to see that $F_\theta$ does not depend on the choice of $S$ and that it satisfies (1.1).

In [2], we showed that, conversely, any entire function $F(\zeta)$, which satisfies (1.1), is $F_\theta(\zeta)$ for some $\theta \in Z^{(n,n-1)}(\mathbb{C}^n - K)$. (Notation: $Z^{(n,n-1)}(\mathbb{C}^n - K)$ denotes sets of $\bar{\partial}$-closed $(n, n-1)$-forms.) The proof uses an integral of Berndtsson, which is defined as follows:
\[ \theta_F(\zeta) = a_n \left( \int_{t=0}^{\infty} t^{n-1} e^{-t(\zeta, \partial \rho/\partial z)} F(t \partial \rho/\partial z) dt \right) \partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}, \]
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for \( z \in \mathbb{C}^n - \{ \rho \leq 1 \} \), where \( \partial \rho / \partial z = (\partial \rho / \partial z_1, \ldots, \partial \rho / \partial z_n) \) and \( a_n = 1/[(n - 1)!(2\pi i)^n] \). Here we assume that \( 0 \in K \) and that the function \( \rho \) is chosen to be convex, positively homogeneous (i.e., \( \rho(sz) = s\rho(z) \) for \( s > 0 \)) and such that \( \{ \rho < 1 \} \) is a strictly convex neighborhood of \( K \). ((1.1) is needed for the convergence of the above integral.)

Then Berndtsson proved (in [1]) that if the entire function \( F(\zeta) \) satisfies (1.1) then

\[
\int_{\{ \rho(z) = 1 \}} e^{(z,\zeta)} \theta^\rho_F(z) = F(\zeta), \quad \text{for } \zeta \in \mathbb{C}^n. \tag{1.2}
\]

The proof given in [1] was based on an integral formula with weights and a change of variables, using some facts from convex analysis concerning the polar set of the convex set \( \{ \rho \leq 1 \} \).

In this note we will give a proof of (1.2) by a direct computation of the integral which is based on the following observations: First, the differential form \( \theta^\rho_F(z) \) is \( \bar{\partial} \)-closed in the set \( \mathbb{C}^n - \{ \rho \leq 1 \} \) (see [2, Lemma 1]) and therefore

\[
\int_{\{ \rho(z) = 1 \}} e^{(z,\zeta)} \theta^\rho_F(z) = \int_{\{|z|=R\}} e^{(z,\zeta)} \theta^\rho_F(z) \tag{1.3}
\]

when the sphere \( \{|z| = R\} \) surrounds the compact set \( \{ \rho \leq 1 \} \), and second, if we expand the entire function \( F(\zeta) \) in power series

\[
F(\zeta) = \sum_{k_1,\ldots,k_n \geq 0} c_{k_1\ldots k_n} \zeta_1^{k_1} \ldots \zeta_n^{k_n}
\]

and we if we substitute this expansion in the integral which defines \( \theta^\rho_F(z) \), then we may interchange the order of summation and integration, provided that \( R \) is sufficiently large.

After this interchange we see that \( \theta^\rho_F(z) \) is a combination of terms of the form

\[
a_n \left( \int_{t=0}^{\infty} t^{n+k_1+\cdots+k_n-1} e^{-t(z,\partial \rho/\partial z)} dt \right) \times \left( \frac{\partial \rho}{\partial z_1} \right)^{k_1} \ldots \left( \frac{\partial \rho}{\partial z_n} \right)^{k_n} \partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}. \tag{1.4}
\]

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Since \( \int_{t=0}^{\infty} t^N e^{-t\sigma} dt = \frac{N!}{\sigma^{N+1}} \) (for \( \text{Re} \sigma > 0 \)), we see that

\[ \int_{t=0}^{\infty} t^{n+k_1+\cdots+k_n-1} e^{-t(z,\partial \rho/\partial z)} dt = \frac{(n+k_1+\cdots+k_n-1)!}{\langle z, \partial \rho/\partial z \rangle^{n+k_1+\cdots+k_n}}. \]

It follows that (1.4) is the following derivative of the Cauchy-Fantappiè kernel:

\[ (n-1)!a_n \left. \frac{\partial^{k_1+\cdots+k_n}}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}} \right|_{w=0} \left( \frac{\partial \rho(z) \land [\partial \partial \rho(z)]^{n-1}}{\langle z-w, \partial \rho/\partial z \rangle^n} \right). \]  

(1.5)

Now recall the Cauchy-Fantappiè formula: For entire functions \( f \),

\[ \frac{1}{(2\pi i)^n} \int_{\{|z|=R\}} f(z) \frac{\partial \rho(z) \land [\partial \partial \rho(z)]^{n-1}}{\langle z-w, \partial \rho/\partial z \rangle^n} = f(w) \quad (|w| < R). \]

Differentiating both sides of this equation with respect to \( w \), we obtain that

\[ \int_{\{|z|=R\}} f(z) \mathcal{F}^\rho_{k_1,\ldots,k_n}(z) = \frac{\partial^{k_1+\cdots+k_n}}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}} f(0), \]  

(1.6)

where \( \mathcal{F}^\rho_{k_1,\ldots,k_n}(z) \) is the kernel (1.5) (which, as we pointed out, is equal to (1.4)).

These observations lead to a proof of the following theorem.

**Theorem 1.1:** If the entire function \( F(\zeta) = \sum c_k \zeta^k \) satisfies (1.1) then

\[ \int_{\{|\rho(z)|=1\}} f(z) \theta^\rho_F(z) = \sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} \frac{\partial^{k_1+\cdots+k_n}}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}} f(0), \quad \text{for every entire function } f. \]  

(1.7)

Notice that (1.2) is the formula (1.7) when \( f(z) = e^{\langle z,\zeta \rangle} \). Since the set of the functions \( e^{\langle z,\zeta \rangle}, \zeta \in \mathbb{C}^n \), is dense in the space of entire functions (with the topology of uniform convergence on compact sets), (1.2) is actually equivalent to (1.7).
2 The proof of the Theorem

First (1.1) guarantees the convergence of the integral which defines \( \theta^\rho_F(z) \), for \( \rho(z) \geq 1 \) (see [2, p.910]) and, as we pointed out before, \( \theta^\rho_F \in Z^{(n,n-1)}(\mathbb{C}^n - \{ \rho \leq 1 \}) \). Therefore, by Stokes’s theorem, the integral \( \int_{\rho=1} f \theta^\rho_F \) is equal to

\[
a_n \int_{\{ |z|=R \}} f(z) \int_{t=0}^\infty t^{n-1} e^{-t(z,\partial \rho/\partial z)} \sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} t^{k_1+\cdots+k_n} \times \left( \frac{\partial \rho}{\partial z_1} \right)^{k_1} \cdots \left( \frac{\partial \rho}{\partial z_n} \right)^{k_n} dt \partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}. \tag{2.1}
\]

We want to show that we may interchange the order of integration and summation in (2.1), provided that \( R \) is sufficiently large. By Lebesgue’s dominated convergence theorem, it suffices to choose \( R \) so that

\[
\sum_{k_1,\ldots,k_n} |c_{k_1,\ldots,k_n}| \int_{\{ |z|=R \}} \int_{t=0}^\infty |f(z)| t^{n-1} e^{-t(z,\partial \rho/\partial z)} t^{k_1+\cdots+k_n} \times \left( \frac{\partial \rho}{\partial z_1} \right)^{k_1} \cdots \left( \frac{\partial \rho}{\partial z_n} \right)^{k_n} \left| dt \partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1} \right| < \infty. \tag{2.2}
\]

For this purpose, we will need an estimate for the coefficients \( c_k \), which follows from (1.1). First (1.1) implies that \( |F(\zeta)| \leq Ae^{B|\zeta|} \) for \( \zeta \in \mathbb{C}^n \), where \( A \) and \( B \) are positive constants. Using this and Cauchy’s formula in the polydisc, we see that the coefficients

\[
c_{k_1,\ldots,k_n} = \frac{1}{k_1! \cdots k_n!} \frac{\partial^{k_1+\cdots+k_n} F}{\partial \zeta_1^{k_1} \cdots \partial \zeta_n^{k_n}}(0)
\]

satisfy the inequality

\[
|c_{k_1,\ldots,k_n}| \leq \frac{Ae^{B(r_1+\cdots+r_n)}}{r_1^{k_1} \cdots r_n^{k_n}}, \quad \text{for every } r_1, \ldots, r_n > 0.
\]

Applying this with \( r_1 = k_1/B, \ldots, r_n = k_n/B \), we obtain that

\[
|c_{k_1,\ldots,k_n}| \leq \frac{(eB)^{k_1+\cdots+k_n}}{k_1^{k_1} \cdots k_n^{k_n}}, \quad \text{for every } k_1, \ldots, k_n. \tag{2.3}
\]
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On the other hand \((\partial \rho / \partial z_j)(sz) = (\partial \rho / \partial z_j)(z)\) for \(s > 0\), and therefore

\[ \left| \frac{\partial \rho}{\partial z_j}(z) \right| \leq \beta \stackrel{def}{=} \max \left\{ \left| \frac{\partial \rho}{\partial z_j}(\xi) \right| : |\xi| = 1, j = 1, \ldots, n \right\} \quad (z \neq 0). \]

Also the function \(\gamma(z) \stackrel{def}{=} \text{Re}\langle z, \partial \rho / \partial z \rangle\) has the property \(\gamma(sz) = s\gamma(z)\) \((s > 0)\), and therefore

\[ \gamma(z) = |z|\gamma(z/|z|) \geq \epsilon |z| \quad \text{for } z \neq 0, \text{ where } \epsilon \stackrel{def}{=} \min \{\gamma(\xi) : |\xi| = 1\} > 0. \]

It follows that

\[ \int_{t=0}^{\infty} e^{-t(z,\partial \rho / \partial z)} \left| e^{-t(z,\partial \rho / \partial z)} \right|^k dt = \frac{(n + k_1 + \cdots + k_n - 1)!}{[\gamma(z)]^{n+k_1+\cdots+k_n}} \leq \frac{(n + k_1 + \cdots + k_n - 1)!}{(\epsilon |z|)^{n+k_1+\cdots+k_n}}. \]

Thus

\[ \int_{\{|z|=R\}} \int_{t=0}^{\infty} f(z) |t^{n-1} e^{-t(z,\partial \rho / \partial z)} \left| k_1 \cdots k_n \left( \partial \rho / \partial z_1 \right)^{k_1} \cdots \left( \partial \rho / \partial z_n \right)^{k_n} \right| dt \]

\[ \times |\partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}| \leq \frac{(n + k_1 + \cdots + k_n - 1)!}{(\epsilon R)^{n+k_1+\cdots+k_n}} \beta^{k_1+\cdots+k_n} \int_{\{|z|=R\}} |f(z)||\partial \rho(z) \wedge [\bar{\partial} \partial \rho(z)]^{n-1}|. \]

This inequality together with (2.3) imply that, in order to have (2.2), it suffices to choose \(R\) so that

\[ \sum_{k_1,\ldots,k_n} \frac{(n + k_1 + \cdots + k_n - 1)!}{(\epsilon R)^{k_1+\cdots+k_n}} \frac{(\beta eB)^{k_1+\cdots+k_n}}{k_1 \cdots k_n} < \infty. \quad (2.4) \]

But (2.4) holds, if \(R > n\beta eB/\epsilon\), since

\[ \sum_{k_1,\ldots,k_n} \frac{(k_1 + \cdots + k_n)!}{k_1! \cdots k_n!} \tau_1^{k_1} \cdots \tau_n^{k_n} = \frac{1}{1 - (\tau_1 + \cdots + \tau_n)} \]

for \(\tau_1 + \cdots + \tau_n < 1, \tau_j > 0\).
Thus, working with $R > n\beta eB/\epsilon$, we may interchange the order of integration and summation in (2.1). The result is that the integral $\int_{\{\rho=1\}} f^{\rho} \tilde{F}$ is equal to the sum

$$\sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} \int_{\{|z|=R\}} f(z) \tilde{F}_{k_1,\ldots,k_n}^{\rho}(z),$$

which, by (1.6), is equal to $\sum_{k_1,\ldots,k_n} c_{k_1,\ldots,k_n} \frac{\partial^{k_1+\cdots+k_n} f}{\partial w_1^{k_1} \cdots \partial w_n^{k_n}}(0)$.

This proves (1.7) and completes the proof of the Theorem.

References


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