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The Heisenberg uncertainty relation in harmonic analysis on p-adic numbers field


<http://ambp.cedram.org/item?id=AMBP_2005__12_1_181_0>
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Abstract

In this paper, two important geometric concepts—grapical center and width, are introduced in \( p \)-adic numbers field. Based on the concept of width, we give the Heisenberg uncertainty relation on harmonic analysis in \( p \)-adic numbers field, that is the relationship between the width of a complex-valued function and the width of its Fourier transform on \( p \)-adic numbers field.

1 Introduction

In reference \([1]\), wavelet transform is introduced to the field of \( p \)-adic numbers. In references \([2]\) and \([5]\), some theory of wavelet analysis and affine frame on harmonic analysis are introduced to the field of \( p \)-adic numbers respectively on the basis of a mapping \( P: \mathbb{R}^+ \cup \{0\} \to \mathbb{Q}_p \) (field of \( p \)-adic numbers).

In this paper, based on the same mapping \( P \) we will give the Heisenberg uncertainty relation in harmonic analysis on \( p \)-adic numbers field as

\[
\Delta f \Delta \widehat{f} \geq \frac{1}{4\pi^2}
\]

where \( \Delta f \), \( \Delta \widehat{f} \) are the widths of function \( f \) and its transform \( \widehat{f} \) respectively.

The field of the \( p \)-adic numbers is defined as the completion of field \( \mathbb{Q} \) of rationals with respect to the \( p \)-adic metric induced by the \( p \)-adic norm \(|\cdot|_p\) (see \([6]\)). A \( p \)-adic numbers \( x_p \neq 0 \) is uniquely represented in the canonical form

\[
x_p = p^{-r} \sum_{k=0}^{\infty} x_k p^k, |x_p|_p = p^r,
\]

where \( r \in \mathbb{Z} \) and \( x_k \in \mathbb{Z} \) such that \( 0 \leq x_k \leq p - 1, x_0 \neq 0 \). For \( x_p, y_p \in \mathbb{Q}_p \), we define \( x_p < y_p \) either when \( |x_p|_p < |y_p|_p \) or when \( |x_p|_p = |y_p|_p \), and there exist
an integer \( j \) such that \( x_0 = y_0, \ldots, x_{j-1} = y_{j-1}, x_j < y_j \) from the viewpoint of (1.1). By interval \([a_p, b_p]\), we mean the set defined by \( \{ x_p \in Q_p | a_p \leq x_p \leq b_p \} \).

The mapping \( P : \mathbb{R}^+ \cup \{ 0 \} \to Q_p \) are introduced in the references \([5]\) and \([2]\) as

\[
P(0) = 0; \ P \left( p^r \sum_{k=0}^{\infty} x_k p^{-k} \right) = p^{-r} \sum_{k=0}^{\infty} x_k p^k \in Q_p.
\]

(1.2)

It is known that if \( x_R = p^r \sum_{k=0}^{n} x_k p^{-k} \in \mathbb{R}^+ \cup \{ 0 \}, x_0 \neq 0 \) and \( 0 \leq x_k \leq p-1 \), then it has another expression

\[
x_R = p^r \sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1)p^{-n} + (p - 1) \sum_{k=n+1}^{\infty} p^{-k}.
\]

(1.3)

that we won’t use it in this paper. Let \( M_p \) be the set of numbers that is expressed by formula (1.3) and \( M_p = P(M_R) \). Let \( B_r(a_p) = \{ x_p \in Q_p \ | \ |x_p - a_p|_p \leq p^r, r \in \mathbb{Z} \}, S_r(a_p) = \{ x_p \in Q_p \ | \ |x_p - a_p|_p = p^r, r \in \mathbb{Z} \}. \) According to (1.2), we reach a conclusion that for an interval \([a_R, b_R]\) in \( \mathbb{R}^+ \cup \{ 0 \} \) and its corresponding interval \([a_p, b_p]\) in \( Q_p \)

\[
P\{ B_r(a_p) \} = [0, p^{r+1}],
\]

(1.4)

\[
P\{ S_r(a_p) \} = [p^r, p^{r+1}],
\]

(1.5)

\[
P\{ [a_p, b_p] \} = [a_R, b_R],
\]

(1.6)

\[
|a_R - b_R|_p \leq p|a_p - b_p|_p,
\]

(1.7)

where \( P(a_p) = a_p, P(b_p) = b_p (\text{see}[3]) \). Let \( f \) be a complex-valued function on \( Q_p \), for \( x_p \in Q_p \setminus M_p \), let

\[
f(x_p) = f(P \circ P^{-1}(x_p)) = (f \circ P)(x_R) \overset{\text{def}}{=} f_R(x_R), (f_R = f \circ P), x_R = P^{-1}x_p.
\]

(1.8)

From (1.7), we know that the inverse mapping \( P^{-1} \) is continuous on \( Q_p \setminus M_p \).

2 A Haar measure on \( Q \) and integration

In this section, a Haar measure is constructed by using the mapping \( P \) of \( \mathbb{R}^+ \cup \{ 0 \} \) into \( Q_p \setminus M_p \) and the Lebesque measure on \( \mathbb{R}^+ \cup \{ 0 \} \). The symbol
\[ \sum \] is the set of all compact subsets of \( \mathbb{Q}_p \), and \( S \) is the \( \sigma \)-ring generated by \( \sum \).

**Definition 2.1:** Let \( E \in S \), and put \( E_p = E \setminus M_p \), and \( E_r = \mathbb{P}^{-1}(E_p) \). If \( E_r \) is a measurable set on \( \mathbb{R}^+ \cup \{0\} \), then we call \( E \) a measurable set on \( \mathbb{Q}_p \), and define a set function \( \mu_p(E) \) on \( S \):

\[
\mu_p(E) = \frac{1}{p} \mu(E_r)
\]

(2.1)

where \( \mu(E_r) \) is the Lebesgue measure on \( E_r \). This \( \mu_p(E) \) is called the measure on \( E \).

By the Definition 2.1, some examples can be given immediately:

1. Let \( a_p, b_p \in \mathbb{Q}_p \), then \( \mu_p([a_p, b_p]) = (b_r - a_r)/p \) (see (1.7))

2. \( \mu_p(B_r(0)) = p^r \) (see (1.4))

3. \( \mu_p(S_r(0)) = p^r(1 - \frac{1}{p}) \) (see (1.5))

4. Let \( \{B_{r_i}(a_i)\}_i \) be disjoint discs covering \( E \), by the definition of measure \( \mu_p \) and definition of Lebesgue exterior measure on \( \mathbb{R}^+ \cup \{0\} \), it is evident that

\[
\mu_p(E) = \inf_{r_i \in Z} \mu_p(\bigcup_i B_{r_i}(a_i))
\]

(2.2)

(5) \( \mu_p(M_p) = 0 \).

It is obvious that \( \mu_p \), by Definition 2.1, is countably additive. In order to prove that \( \mu_p \) is a Haar measure, we will give the following lemma.

**Lemma 2.2:** If \( \alpha \in \mathbb{Q}_p \), then

\[
\mu_p(B_r(\alpha)) = \mu_p(B_r(0)).
\]

(2.3)

**Proof:** 1° Let \( \alpha = p^{-r_1} \), for \( r_1 > r, r_1, r \in Z \), and put \( x = p^{-r_1} + p^{-r} \sum_{0 \leq x_k < \infty} x_k p^k, x_0 \neq 0, 0 \leq x_k < p \). Then \( E \) is the set of all these \( p \)-adic numbers when \( x_k \) change for \( k = 0, 1, ..., p - 1 \). We write \( E_p = \{x_p| x_p \in E \setminus M_p\} \). For \( x_p \in E_p \), let

\[
\mathbb{P}^{-1}(x_p) = p^{r_1} + p^r \sum_{0 \leq k < \infty} x_k p^{-k}
\]

then \( M_r = [p^{r_1}, p^{r_1} + p^{r+1}] \) is the set of all real numbers as presented in (2.4) (see(1.4)). Hence

\[
\mu_p(\alpha + B_r(0)) = \mu_p(B_r(\alpha)) = \frac{1}{p} \mu(E_r) = p^r = \mu_p(B_r(0))
\]

(2.5)
Let $\alpha = p^{-r_2}$, for $r_2 \leq r, r_2, r \in \mathbb{Z}$, then $\alpha + B_r(0) = B_r(\alpha) = B_r(0)$, by $\alpha \in B_r(0)$. So that
\[
\mu_p(B_r(\alpha)) = \mu_p(B_r(0))
\] (2.6)

Let $\alpha = p^{-r_3} \sum_{0 \leq k < \infty} \alpha_k p^k$, and put $\alpha^n = p^{-r_3} \sum_{0 \leq k \leq n} \alpha_k p^k$, applying to the result of 1° and 2° repeatedly in this case, we have
\[
\mu_p(\alpha^n + B_r(0)) = \mu_p(B_r(\alpha^n)) = \mu_p(B_r(0))
\] (2.7)

However
\[
\lim_{n \to \infty} \mu_p(B_r(\alpha^n)) = \lim_{n \to \infty} \frac{1}{p} \mu_p(\mathbf{P}^{-1}(B_{r_p}(\alpha^n)))
\] (2.8)

where $B_{r_p}(\alpha^n) = B_r(\alpha^n) \setminus M_p$. By the continuity of the mapping $\mathbf{P}^{-1}$ (see(1.7)), we obtain
\[
\lim_{n \to \infty} \frac{1}{p} \mu_p(\mathbf{P}^{-1}(B_{r_p}(\alpha^n))) = \frac{1}{p} \mu_p(\mathbf{P}^{-1}(B_{r_p}(\alpha))) = \mu_p\{B_r(\alpha)\}
\] (2.9)

The part 3° follows from (2.7), (2.8) and (2.9).

**Theorem 2.3:** (The translation invariance of the measure $\mu_p$) Let $E \in \mathcal{S}$ and let $\alpha \in \mathbb{Q}_p$, then
\[
\mu_p(\alpha + E) = \mu_p(E)
\] (2.10)

**Proof:** Let $\{B_r(a_i)\}_{i=1}^\infty$ be disjoint discs covering $E$, then $\{B_r(a_i + \alpha)\}_{i=1}^\infty$ are disjoint discs covering $\alpha + E$. By the formula (2.2) in the example 4, we have
\[
\mu_p(\alpha + E) = \inf_{r_i \in \mathbb{Z}} \mu_p\{\cup \{\alpha + B_r(a_i)\}\}
\] (2.11)

Applying the lemma 2.2 to the right side of the above formula, then
\[
\inf_{r_i \in \mathbb{Z}} \mu_p\{\cup_i B_r(\alpha + a_i)\}
= \inf_{r_i \in \mathbb{Z}} \sum_i \mu_p\{B_r(\alpha + a_i)\}
= \inf_{r_i \in \mathbb{Z}} \sum_i \mu_p\{B_r(a_i)\}
= \inf_{r_i \in \mathbb{Z}} \mu_p\{\cup_i B_r(a_i)\}
= \mu_p(E)
\]
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Therefore, $\mu_p$ is a Haar measure. \hfill \Box

According to the above definition of Haar measure, we can define the integration over measurable sets $E$ in $Q_p$ (firstly define the integration of simple functions, then regard the limit of integration of simple functions as the definition of the integration of general functions (see [4]))

$$\int_E f(x_p) d\mu_p \quad (2.12)$$

By the theorem 2.3, the definition of measure and (1.8), we have the following theorem

**Theorem 2.4:** Suppose $f(x_p)$ is a complex-valued function on $Q_p$, then $f(x_p)$ is integrable over the interval $[a_p, b_p]$ ($a_p, b_p \in Q_p$), if and only if the real function $f_R(x_R)$ defined on $R^+ \cup \{0\}$ is integrable over the interval $[a_R, b_R]$, and

$$\int_{[a_p, b_p]} f(x_p) d\mu(x_p) = \frac{1}{p} \int_{a_R}^{b_R} f(x_R) d\mu(x_R) \quad (2.13)$$

where $f_R(x_R)$ is defined by (1.8), and $P(x_R) = x_p, P(a_R) = a_p, P(b_R) = b_p, a_p, b_p \in M_p$

**Corollary 2.5:** If $f(x_p)$ is a bounded continuous function on the interval $[a_p, b_p] \subset Q_p$, then $f(x_p)$ is integrable over $[a_p, b_p]$, where $[a_p, b_p]$ can be $Q_p$. Notice that under the condition of theorem $f_R(x_R)$ is a bounded piecewise continuous function on $R^+ \cup \{0\}$ by (1.4), By the theorem 2.4, $f(x_p)$ is integrable.

3 The indefinite integral and derivative of complex-valued function in $Q_p$

**Definition 3.1:** Let $f$ be a complex-valued function defined in $Q_p$ and for $\forall x_p \in Q_p$, $f$ is integrable on interval $[a_p, b_p]$, then

$$f(x_p) = \int_0^{x_p} g dx_p \quad (3.1)$$

is called on indefinite integral of $g$. 

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**Definition 3.2:** Let \( f \) be a complex-valued function defined in \( \mathbb{Q}_p \), if there exist an integrable complex-valued function \( g \) such that

\[
f(x_p) = \int_0^{x_p} g \, dx_p, \quad x_p \in \mathbb{Q}_p
\]

then \( g(x_p) \) is called the derivative of \( f \), which we will denote as \( f'(x_p) \).

In formula (3.2), let \( f = 1 \) then

\[
\mu([0, x_p]) = \int_0^{x_p} d\mu
\]

The equation (3.3) follows that

\[
\bar{\mu}'(x) \overset{\text{def}}{=} \mu'([0, x_p]) = 1
\]

**Theorem 3.3:** For complex-valued functions \( f, h \) on \( \mathbb{Q}_p \), if \( (f)_R(x_R) \) and \( (h)_R(x_R) \) are absolutely continuous, then

\[
f'_R(x_R) = (f')_R(x_R)
\]

\[
(f(x_p)h(x_p))' = f'(x_p)h(x_p) + f(x_p)h'(x_p)
\]

**Proof:** Let \( f' = g \) and \( g(x_p) = g(P(x_R)) = (g \circ P)(x_R) = g_R(x_R) \). By definition (3.2) and theorem 2.4, we have

\[
f(x_p) = \int_0^{x_p} g(x_p) \, dx_p
\]

\[
= \int_0^{x_R} g_R(x_R) \, dx_R
\]

\[
= f_R(x_R)
\]

and therefore

\[
(f_R)'(x_R) = g_R(x_R) = g(x_p) = f'(x_p) = (f')_R(x_R)
\]

and

\[
(f(x_p)h(x_p))' = (f_R(x_p)h_R(x_R))'
\]

\[
= (f'_R(x_R)h_R(x_R) + f_R(x_R)h'_R(x_R))
\]

\[
= (f'_R(x_R)h_R(x_R) + f_R(x_R)(h'_R(x_R))
\]

\[
= f'(x_p)h(x_p) + f(x_p)h'(x_p)
\]
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From (3.6) it follows that

Corollary 3.4: If a complex-valued function $h(x_R)$ is absolutely continuous on $R^+ \cup \{0\}$, then $f(x_p) \overset{\text{def}}{=} (hP^{-1})(x_p)$ is derivable on $Q_p \setminus M_p$.

Corollary 3.5: A locally constant function is derivable on $Q_p \setminus M_p$, and its derivative is equal to 0.

Similarly, we can prove

Theorem 3.6: If $f$ is derivable on $[a_p, b_p]$, then

$$\int_{a_p}^{b_p} f'(x_p) d\mu = f(b_p) - f(a_p) \quad (3.7)$$

4 Center and width of the graph of $f$

In this section, we will introduce the concepts of center and width of complex-valued function graph in the field of $p$-adic numbers $Q_p$.

Definition 4.1: Let $f$ be a complex-valued function of $p$-adic variable. We define the center $t_f$ of the graph $\{(x_p, f(x_p)) | x_p \in Q_p\}$ by

$$t_f = \frac{\int_{Q_p \setminus M_p} \left( P^{-1}(x_p) |f(x_p)|^2 dx_p / \int_{Q_p \setminus M_p} |f(x_p)|^2 dx_p \right)}{\int_{Q_p \setminus M_p} |f(x_p)|^2 dx_p} \quad (4.1)$$

if the integral (4.1) exists.

Definition 4.2: For a complex-valued function of $p$-adic variable, we define the width of $f$ by

$$\Delta_f = \left( \int_{Q_p} |x_p - t_f|^2 |f(x_p)|^2 dx_p / \int_{Q_p} |f(x_p)|^2 dx_p \right)^{1/2} \quad (4.2)$$

if the integral (4.2) exists.
**Theorem 4.3:** Let \( P(t_f^{(R)} - a_R) = P(t_f^{(R)}) - P(a_R) \), \( a_R = P^{-1}(a), a \in Q_p \setminus M_p \).

1. If \( f \) is increasing, then \( t_{a,f} = t_f - a \)
2. Suppose \( \text{supp} f \subseteq B_r(0) \). For \( a = p^{-\beta} \), if \( \beta > r \), then \( t_{a,f} = t_f - a \)
3. For \( a = p^{-\beta}, \beta \in \mathbb{Z} \), then
   \[ t_{a,f} = at_f, \]
where \( T_a f(x_p) = f(x_p + a), S_a f(x_p) = f\left(\frac{x_p}{a}\right) \).

**Proof:** (1) Under the condition of (1) in this theorem, using
\[ P(x_R + a_R) \geq P(x_R) + P(a_R) \]
we have
\[ (f \circ P)(x_R + a_R) \geq f(x + a) \]
where \( x_R = P^{-1}(x_p), x_p \in Q_p \setminus M_p \). Hence
\[ t_{a,f}^{(R)} = \int_{Q_p \setminus M_p} P^{-1}(x_p)|T_a f(x_p)|^2dx_p/ \int_{Q_p} |T_a f(x_p)|^2dx_p \]
\[ = \int_{Q_p \setminus M_p} P^{-1}(x_p)|f(x + a)|^2dx_p/ \int_{Q_p} |f(x_p)|^2dx_p \]
\[ \leq \int_{R^+} x_R[(f \circ P)(x_R + a_R)]^2dx_R/ \int_{Q_p} |f(x_p)|^2dx_p \]
\[ = \int_{R^+} (x_R - a_R)[(f \circ P)(x_R)]^2dx_R/ \int_{Q_p} |f(x_p)|^2dx_p \]
\[ = a_R + \int_{Q_p \setminus M_p} P^{-1}(x_p)|f(x_p)|^2dx_p/ \int_{Q_p} |f(x_p)|^2dx_p \]
\[ = -a_R + t_f^{(R)} \quad (4.3) \]
where we used \( \mu(M_p) = 0 \), and for \( x_p, a \in B_r(0) \cap (Q_p \setminus M_p), x_p + a \in B_r(0) \).
On the other hand, using inequation \( P^{-1}(x - a) \geq P^{-1}(x) - P^{-1}(a) \), we can easily obtain
\[ t_{a,f}^{(R)} \geq t_f^{(R)} - a_R \quad (4.4) \]
From (4.3) and (4.4), we have
\[ t_{a,f}^{(R)} = t_f^{(R)} - a_R. \]
Finally, from the condition of (1) in theorem, we have
\[ t_{R} = P(t^{(R)} - a_{R}) = P(t^{(R)} - P(a_{R}) = t - a \]

Conclusion of (1) in theorem is proved. (2) and (3) can be proved similarly.

**Theorem 4.4:** (1) If \( f(x) \) and \( a, t \) satisfy the condition of theorem 3.3, then
\[ \triangle_{T}a = \triangle_{f} \]

(2) \[ \triangle_{S}a = |a|_{p}\triangle_{f} \]

**Proof:** For (1), we have
\[
\triangle_{T}a = \left( \int_{Q_p} |x_{p} - t_{R}|^{2} |T_{f}|^{2}(x_{p})d{x}_{p}/ \int_{Q_p} |T_{f}|^{2}(x_{p})d{x}_{p} \right)^{1/2}
\]
\[
= \left( \int_{Q_p} |x_{p} - (t - a)|^{2} |f(x_{p} + a)|^{2}d{x}_{p}/ \int_{Q_p} |f(x_{p} + a)|^{2}d{x}_{p} \right)^{1/2}
\]
\[
= \left( \int_{Q_p} |t_{p} - t_{R}|^{2} |f(t_{p})|^{2}dt_{p}/ \int_{Q_p} |f(t_{p})|^{2}dt_{p} \right)^{1/2}
\]
\[
= \triangle_{f}
\]

(2) can be proved similarly.

After doing the preparation of section 1-4, we will give a theorem on harmonic analysis which is about the relation of the width of complex function in \( Q_{p} \) and the width of its Fourier transform. This theorem is similar to the Heisenberg uncertainty relation in quantum mechanics.

## 5 Main theorem

**Lemma 5.1:** Let \( x_{p} \in Q_{p} \), then \( \mu([0, x_{p}]) \leq |x_{p}|_{p} \)
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**Proof:** For \( x_p \in P_p \setminus M_p \)

\[
x_p = p^{-r} \sum_{k=0}^{\infty} x_k p^k \in Q_p, \quad x_0 \neq 0, 0 \leq x_k \leq p - 1
\]

and therefore, we have

\[
P^{-1}(x_p) = p^{r-1} \sum_{k=0}^{\infty} x_k p^{-k} \leq p^{r-1}(p - 1) \sum_{k=0}^{\infty} p^{-k} = |x_p|_p
\]

(5.1)

By definition of measure \( \mu_p \), we have

\[
\frac{1}{p} P^{-1}(x_p) = \mu([0, x_p])
\]

which leads to

\[
\overline{\mu}(x_p) \overset{\text{def}}{=} \mu([0, x_p]) \leq |x_p|_p/p
\]

(5.2)

**Theorem 5.2:** Let \( f \) be complex-valued function of \( p \)-adic variable. If \( f \in L^2(Q_p) \), \( f' \in L^2(Q_p) \) and

\[
\lim_{|b_p|_p \to \infty} |b_p|_p |f(b_p)|^2 = 0, \quad f(0) = 0
\]

(5.3)

then the following inequality is valid:

\[
\frac{1}{4\pi} \leq \triangle \hat{f}
\]

(5.4)

where \( \hat{f} \) is the transform of \( f \),

\[
\hat{f}(\xi_p) = \int_{Q_p} f(x_p) \exp(2\pi i \{\xi_p x_p\}) dx_p
\]

and by means of representation (1.1), \( \{x_p\} \) is defined as

\[
\{x_p\} = \begin{cases} 
0 & \text{if } r(x_p) \geq 0 \text{ or } x_p = 0 \\
p^r(x_0 + x_1 p + \cdots + x_{|r|-1} p^{|r|-1}) & \text{if } r(x_p) < 0
\end{cases}
\]

Inequality (5.4) is called the Heisenberg uncertainty relation in harmonic analysis on \( p \)-adic numbers field.
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PROOF: By using (3.4) and theorem 3.3, we have

\[
(\bar{\mu}(x_p - t_j)|f(x_p)|^2)' = (\bar{\mu}(x_p - t_j)f(x_p)\chi_p(t_jx_p)\overline{f(x_p)}\chi_p(t_jx_p))'
\]

\[
= |f(x_p)|^2 + \bar{\mu}(x_p - t_j)(f(x_p)\chi_p(t_jx_p))'\overline{f(x_p)}\chi_p(t_jx_p)
\]

\[
+\mu(x_p - t_j)f(x)\chi_p(t_jx_p)[f(x_p)\chi_p(t_jx_p)]'
\]

(5.5)

Therefore, from (3.7) we have

\[
\int_{Q_p}^{b_p} |f(x_p)|^2 dx = \bar{\mu}(x_p - t_j)|f(x_p)|^2|_{0}^{b_p}
\]

\[
-\int_{0}^{b_p} \mu(x_p - t_j)f(x_p)\chi_p(t_jx_p)(f(x_p)\chi_p(t_jx_p))' dx_p
\]

\[
-\int_{0}^{b_p} \bar{\mu}(x_p - t_j)f(x_p)\chi_p(t_jx_p)[f(x_p)\chi_p(t_jx_p)]' dx_p
\]

(5.6)

where the function \(\chi_p(t_jx_p) = \exp(2\pi i \{t_jx_p\})\)

By taking the limit of (5.5) as \(|b|_p \to \infty\) and using (5.2),(5.3), we obtain

\[
\int_{Q_p} |f(x_p)|^2 dx_p 
\]

\[
\leq 2 \left( \int_{Q_p} (\bar{\mu}(x_p - t_j))^2|f(x_p)|^2 dx_p \right)^{1/2} \left( \int_{Q_p} |[f(x_p)\chi_p(t_jx_p)]'|^2 dx_p \right)^{1/2}
\]

\[
= \left( \int_{Q_p} (\bar{\mu}(x_p - t_j))^2|f(x_p)|^2 dx_p \right)^{1/2} \left( \int_{Q_p} |f(\cdot)\chi_p(t_j\cdot)|^2 \xi^2 d\xi \right)^{1/2}
\]

\[
= 2 \left( \int_{Q_p} (\bar{\mu}(x_p - t_j))^2 |f(x_p)|^2 dx_p \right)^{1/2} \left( \int_{Q_p} 4\pi^2 |\xi|^2 |\widehat{f}(\xi + t_j)|^2 d\xi \right)^{1/2}
\]

(5.7)

where we used the Hölder inequality for the integral and \(f'(\cdot)^\wedge(\xi) = -2\pi i \xi \widehat{f}(\xi), \quad (f, f)_{L^2(Q_p)} = (\widehat{f}, \widehat{f})_{L^2(Q_p)}, \quad (f(\cdot)\chi_p(a\cdot))^\wedge(\xi) = \widehat{f}(\xi + a)\) From (5.2), we have

\[
\frac{1}{4\pi} \leq \left( \int_{Q_p} |x_p - t_j|^2|f(x_p)|^2 dx_p/ \int_{Q_p} |f(x_p)|^2 dx_p \right)^{1/2}
\]

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\[ \left( \int_{Q_p} |\xi_p - t_f|^2 |\hat{f}(\xi_p)|^2 d\xi_p / \int_{Q_p} |\hat{f}(\xi_p)|^2 d\xi_p \right)^{1/2} \]

\[ = \triangle_f \triangle \hat{f} \]  

(5.8)

Hence we have completed our proof. \(\square\)

References


THE HEISENBERG UNCERTAINTY RELATION

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