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Weyl-Heisenberg frame in p-adic analysis

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Abstract

In this paper, we establish an one-to-one mapping between complex-valued functions defined on \( R^+ \cup \{0\} \) and complex-valued functions defined on \( p \)-adic number field \( \mathbb{Q}_p \), and introduce the definition and method of Weyl-Heisenberg frame on harmonic analysis to \( p \)-adic analysis.

1 Introduction

Wavelet transform was introduced to the field of \( p \)-adic numbers in [1]. In [4],[3] some theory of wavelet analysis and affine frame introduced to the field of \( p \)-adic numbers, respectively, on the basis of a mapping \( P : R^+ \cup \{0\} \rightarrow \mathbb{Q}_p \) (field of \( p \)-adic numbers). This paper considers on the basis of the mapping \( P \), gives the Weyl-Heisenberg frame in field of \( p \)-adic numbers.

The field \( \mathbb{Q}_p \) of the \( p \)-adic numbers is defined as the completion of field \( \mathbb{Q} \) of rational with respect to the \( p \)-adic metric induced by the \( p \)-adic norm \(|.|_p\), (see [5]). A \( p \)-adic number \( x \neq 0 \) is uniquely represented in the canonical form

\[
x = p^{-r} \sum_{k=0}^{\infty} x_k p^k, \quad |x|_p = p^r
\]

where \( p \) is prime and \( r \in Z \) (\( Z \) is integer set), \( 0 \leq x_k \leq p-1, x_0 \neq 0 \). For \( x, y \in \mathbb{Q}_p \), we define \( x < y \), either when \( |x|_p < |y|_p \) or when \( |x|_p = |y|_p \), but there exists an integer \( j \) such that \( x_0 = y_0, \ldots, x_{j-1} = y_{j-1}, x_j < y_j \) from viewpoint of (1.1). By interval \([a, b]\), we mean the set defined by \( \{x \in \mathbb{Q}_p | a \leq x \leq b\} \).

It is known that if \( x = p^r \sum_{k=0}^{n} x_k p^{-k} \in R^+ \cup \{0\} \) and \( x_0 \neq 0, \ 0 \leq x_k \leq p-1, k = 1, 2, \ldots \), then there is another expression;

\[
x = p^r \left( \sum_{k=0}^{n-1} x_k p^{-k} + (x_n - 1)p^{-n} + (p - 1) \sum_{k=n+1}^{\infty} p^{-k} \right)
\]
But we won’t use that expression (1.2) in this paper.

A mapping \( P : R^+ \cup \{0\} \rightarrow \mathbb{Q}_p \) was introduced in [4],[3],[2], as for
\[
x = p^r \sum_{k=0}^{\infty} x_k p^{-k} \in R, \hspace{1em} x_0 \neq 0, \hspace{1em} 0 \leq x_k \leq p - 1, \hspace{1em} k = 1, 2, \ldots
\]

\[
P(x) = p^{-r} \sum_{k=0}^{\infty} x_k p^k
\]

Let \( M_p = P(M_R) \),
\[
M_R = \{ x_R | x_R = p^r \sum_{k=0}^{n-1} x_k p^{-k} + (n-1)p^{-n} + (p-1) \sum_{k=0}^{\infty} p^{-k}, n \in \mathbb{Z}^+ \cup \{0\} \}
\]

In the following to distinguish between real number field \( R \) and \( p \)-adic number field \( \mathbb{Q}_p \), number in \( R \) denotes by the subscript \( R \), and number with the subscript \( p \) belongs to \( \mathbb{Q}_p \). For example \( x_R, a_R, b_R \) in \( R \); \( x_p, a_p \) in \( \mathbb{Q}_p \).

In [2] a measure is constructed using the mapping \( P \) from \( R^+ \cup \{0\} \) into \( \mathbb{Q}_p \setminus M_p \) and Lebesgue measure on \( R^+ \cup \{0\} \), the symbol \( \sum \) is the set of all compact subsets of \( \mathbb{Q}_p \), and \( S \) is the \( \sigma \)-ring generated by \( \sum \).

**Definition 1.1:** Let \( E \in S \), and put \( E_P = E \setminus M_p \), and \( E_R = P^{-1}(E_P) \). If \( E_R \) is a measurable set on \( R^+ \cup \{0\} \), the we call \( E \) is a measurable set on \( \mathbb{Q}_p \), and define a set function \( \mu_p(E) \) on \( S \)
\[
\mu_p(E) = \frac{1}{p} \mu(E_R)
\]
where \( \mu(E_R) \) is the Lebesgue measure on \( E_R \). This \( \mu_p(E) \) is called the measure on \( E \).

By the Definition 1.1, some examples can given immediately:
1. Let \( a_p, b_p \in \mathbb{Q}_p \), then \( \mu_p([a_p, b_p]) = (b_R - a_R)/p \)
2. Let \( B_r(0) = \{ x_p | x_p \leq p^r, x_p \in \mathbb{Q}_p \} \), then \( \mu_p\{B_r(0)\} = p^r \)
3. Let \( S_r(0) = \{ x_p | p^r, x_p \in \mathbb{Q}_p \} \), then \( \mu_p\{S_r(0)\} = p^r(1 - \frac{1}{p}) \)
4. \( \mu_p\{M_p\} = 0 \)

According to the above definition 1.1 of measure, we can define integration over measurable sets \( E \) in \( \mathbb{Q}_p \)
\[
\int_E f(x_p) d\mu_p(x_p) \hspace{1em} \text{or} \hspace{1em} \int_E f(x_p) dx_p
\]
By the definition 1.1 of measure we have following theorem.

**Theorem 1.2:** (see [2]) Suppose $f(x_p)$ is a Complex-Valued function on $Q_p$, the $f(x_p)$ is integrable over the interval $[a_p, b_p](a_p, b_p \in Q_p)$, if and if the real function $f_R(x_R)$ defined on $R^+ \cup \{0\}$ is Lebesgue integrable over the interval $[a_R, b_R]$, and

$$
\int_{[a_p, b_p]} f(x_p) dx_p = \frac{1}{p} \int_{a_R}^{b_R} f(x_R) dx_R
$$

(1.3)

where $f_R(x_R)$ is defined

$$
f(x_p) = f(P \circ P^{-1}(x_p)) = (f \circ P)(x_R) \overset{def}{=} f_R(x_R), x_p = P(x_R) \in Q_p \setminus M_p
$$

2 Weyl-Heisenberg frame on $p$-adic number field

In real analysis, if there exists constants $A$ and $B$, $A, B > 0$, such that

$$
A\|f\|^2_{L^2(R)} \leq \sum_{m,n} |(f, g_{m,n})_{L^2(R)}|^2 \leq B\|f\|^2_{L^2(R)}
$$

holds for $\forall f \in L^2(R)$, then $g_{mn}(x)$ is called the Weyl-Heisenberg frame, where $g \in L^2(R), p_0, q_0 \in R, g_{mn}(x) = g(x - nq_0)e^{2\pi i m x}, m, n \in Z$ and $(f, g_{mn})_{L^2(R)}$ is inner product in $L^2(R),

$$(f, g_{mn})_{L^2(R)} = \int_{R} f(x)\overline{g_{mn}(x)} dx.$$ 

In this section we give the definition of a Weyl-Heisenberg frame in $Q_p$ by

$$
g_{mn}(x_p) = g(\alpha_{mn}(x_p) - x_p) \exp(2\pi i m p_0 \rho(x_p))
$$

(2.1)

where

$$
\alpha_{mn}(x_p) = P(|x_R + nq_0|) + x_p
$$

(2.2)

and $m, n, p_0, q_0 \in Z, x_R = P^{-1}(x_p), x_p \in Q_p \setminus M_p$. If $g_{mn}(x_p)$ satisfies the frame condition:

$$
A\|f\|^2_{L^2} \leq \sum_{m,n} |(f, g_{m,n})_{L^2}|^2 \leq B\|f\|^2_{L^2}, A, B > 0, \forall f \in L^2(Q_p)
$$
then we have

\[ f(x_p) = \sum_{m,n} (f, g_{m,n}^*)_{L^2} g_{m,n}(x_p) = \sum_{m,n} (f, g_{m,n}) g_{m,n}^*(x_p), \quad x_p \in Q_p \]

where \( \{g_{m,n}^*\} \) is the dual frame of \( \{g_{m,n}\} \):

\[ g_{m,n}^* = S^{-1} g_{m,n} \]

and \( S \) is the frame operator:

\[ S f = \sum_{m,n} (f, g_{m,n})_{L^2} g_{m,n} \]

and

\[ (f, g_{m,n})_{L^2} = \int_{Q_p} f(x_p) g_{m,n}(x) dx \]

**Theorem 2.1:** Suppose \( f, g \in L^2(Q_p) \) are complex-valued functions defined on \( Q_p \), \( p_0 = p^r, q_0 = p^q, r_p, r_q \in \mathbb{Z} \), if support \( \hat{g}_R(|w|) \subset \left[ \frac{-1}{2q_0}, \frac{1}{2q_0} \right] \), and \( \exists A, B > 0 \), such that

\[ A \leq \sum_{m \in \mathbb{Z}} |\hat{g}_R(|w - mp_0|)|^2 \leq B, \quad w \in \mathbb{R}, \forall w \neq 0, \]

then functions \( g_{m,n}(x_p) \) defined by (2.1) construct a frame of \( L^2(Q_p) \), where \( g_R(|t|) = (g \circ P)(|t|) = g(x_p), (P(|t|) = x_p), g_R = g \circ P, t \in \mathbb{R}, \) and \( \hat{g}_R(|w|) \) is the Fourier transform of \( g_R(|t|) \) in real analysis

\[ \hat{g}_R(|w|) = \int_R g_R(|t|) \exp(-2\pi i wt) dt \]

**Proof.** From formula (2.2) or \( g(\alpha_{mn}(x_p) - x_p) = (g \circ P)(|x_p + nq_0|) = g_R(|x_p + nq_0|) \), for \( \forall f \in L^2(Q_p) \), we have

\[ \sum_{m,n \in \mathbb{Z}} |(f, g_{m,n})_{L^2}|^2 = \sum_{m,n \in \mathbb{Z}} |\int_{Q_p} f(x_p) \overline{g_{m,n}(x_p)} dx_p|^2 \]

\[ = \sum_{m,n \in \mathbb{Z}} |\int_{Q_p} f(x_p) \overline{g(\alpha_{m}^{(n)}(x_p) - x_p)} \exp(-2\pi i mp_0 \rho(x_p))|^2 \]
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\[ \frac{1}{p} \sum_{m,n \in \mathbb{Z}} \left| \int_{R^+} f_R(x_R) \overline{g_R(|x_R + nq_0|)} \exp(-2\pi imp_0 x_R) dx_R \right|^2 \] (2.3)

where we used (1.3) and \( f_R(x_R) = (f \circ P^{-1})(x_p) \) in section 1.

Let

\[ f_R^+(x) = \begin{cases} f_R(x), & x \geq 0 \\ 0, & x < 0 \end{cases} \]

From (2.3), we obtain

\[ \sum_{m,n \in \mathbb{Z}} |(f, g_{m,n})|^2 \]

\[ = \sum_{m,n \in \mathbb{Z}} \left| \int_R f_R^+(x) \overline{g_R(|x + nq_0|)} \exp(-2\pi imp_0 x) dx \right|^2 \]

\[ = \sum_{m,n \in \mathbb{Z}} \left| \int_R \hat{f}_R(w) \hat{g}_R(|\cdot + nq_0|) \exp(2\pi imp_0 \cdot) \hat{g}(w) dw \right|^2 \] (2.4)

where sign “:” is the argument on the function, for Fourier transform. But

\[ [g_R(|\cdot + nq_0|) \exp(2\pi imp_0 \cdot)](w) \]

\[ = \hat{g}_R(|w - mp_0|) \exp(2\pi inq_0 (w - mp_0)) \]

Hence from the support \( \hat{g} \subset [-\frac{1}{2q_0}, \frac{1}{2q_0}] \) in condition of the theorem and (2.4) we have

\[ \sum_{m,n \in \mathbb{Z}} |(f, g_{mn})|^2 \]

\[ = \sum_{m,n \in \mathbb{Z}} \left| \int_R \hat{f}_R^+(w + mp_0) \hat{g}_R(|w|) \exp(-2\pi inq_0 w) dw \right|^2 \]

\[ = \sum_{m,n \in \mathbb{Z}} \left| \int_{\frac{1}{2q_0}}^{\frac{1}{2q_0}} \hat{f}_R^+(w + mp_0) \hat{g}_R(|w|) \exp(-2\pi inq_0 w) \overline{dw} \right|^2 \] (2.5)
We know that
\[ c_n = q_0 \int_{-\frac{1}{2q_0}}^{\frac{1}{2q_0}} \hat{f}_R^+(w + mp_0) \hat{g}_R^+(|w|) \exp(-2\pi i q_0 w) dw, \quad n \in \mathbb{Z} \]
are Fourier coefficient of \( \hat{f}_R^+(w + mp_0) \hat{g}_R^+(|w|) \) on \( [\frac{-1}{2q_0}, \frac{1}{2q_0}] \). Hence by virtue of Parseval equality we have
\[ \sum_{n \in \mathbb{Z}} |c_n|^2 = q_0 \int_{-\frac{1}{2q_0}}^{\frac{1}{2q_0}} |\hat{f}_R^+(w + mp_0) \hat{g}_R^+(|w|)|^2 dw \]
\[ = q_0 \int_{-\frac{1}{2q_0}}^{\frac{1}{2q_0}} |\hat{f}_R^+(w + mp_0)|^2 dw 
\]
\[ = q_0 \int_{R} |\hat{f}_R^+(w + mp_0)|^2 dw 
\]
\[ = q_0 \int_{R} |\hat{f}_R^+(w)|^2 \hat{g}_R^+(|w - mp_0|)^2 dw \] (2.6)
Comparing (2.5) and (2.6), we have
\[ \sum_{m,n \in \mathbb{Z}} |(f, g_{m,n})_{L^2}|^2 = \frac{1}{q_0} \int_{R} |\hat{f}_R^+(w)|^2 G(w) dw \] (2.7)
where
\[ G(w) = \sum_{m \in \mathbb{Z}} |\hat{g}_R^+(|w - mp_0|)|^2 \]
Finally, by virtue of the conditions of the theorem, we have
\[ \sum_{m \in \mathbb{Z}, n \in \mathbb{Z}} |(f, g_{m,n})_{L^2}|^2 = \begin{cases} \geq \frac{A}{q_0} \int_{R} |\hat{f}_R^+(w)|^2 dw \\ \leq \frac{B}{q_0} \int_{R} |\hat{f}_R^+(w)|^2 dw \end{cases} \]
But
\[ \int_{R} |\hat{f}_R^+(w)|^2 dw = \int_{R} |f_R^+(x)|^2 dx = \int_{R} |f_R(x_R)|^2 dx_R = \int_{Q_p} |f(x_p)|^2 dx_p = \|f\|_{L^2}. \]
Hence we completed our proof.
3 Dual frame

In the section, we will give a formula to calculate the dual frame. By (2.7), we have

\[ (Sf, f)_{L^2} = \sum_{m,n \in \mathbb{Z}} |(f, g_{m,n})_{L^2}|^2 = \frac{1}{q_0} \int_{R} \hat{f_R}^+(w)G(w)\overline{\hat{f_R}^+(w)}dw \]

\[ = \frac{1}{q_0} \int_{R} (\hat{f_R}^+(\cdot)G(\cdot))\overline{\varphi(\cdot)}dx \]

\[ = \frac{1}{q_0} \int_{R^+} (\hat{f_R}^+(\cdot)G(\cdot))\overline{\varphi(\cdot)}\overline{\varphi(\cdot)}dx \]

where sign “ v ” is the inverse Fourier transform. Therefore

\[ (Sf, f)_{L^2} = \frac{1}{q_0} \int_{Q_p} (\hat{f_R}^+(\cdot)G(\cdot))\overline{(P^{-1}(x_p))}\overline{\varphi(\cdot)}dx_p \]

\[ = \frac{1}{q_0} ((\hat{f_R}^+(\cdot)G(\cdot))\overline{(P^{-1}(x_p))}, f(x_p))_{L^2} \]

Since \( f \) is an arbitrary function in \( L^2(Q_p) \), we conclude that

\[ (Sf)(x_p) = \frac{1}{q_0} (\hat{f_R}^+(\cdot)G(\cdot))\overline{(P^{-1}(x_p))} \]

or for \( x \in R^+ \cup \{0\} \) we conclude that

\[ (Sf)_R(x_R) = \frac{1}{q_0} (\hat{f_R}^+(\cdot)G(\cdot))\overline{x_R}, x_R \geq 0 \]

(3.1)

where \( (Sf)_R = (Sf)P^{-1} \)

Bases on (3.1), we will extend the domain of \( (Sf)_R(x_R) \) from \( R^+ \cup \{0\} \) onto \( R \) such that \( (Sf)_R(t), t \in R \) is an even function on \( R \). Taking Fourier transform on both sides of (3.1), we have

\[ ((Sf)_R)(w) = \frac{1}{q_0} \hat{f_R}^+(w)G(w) \]

(3.2)
After replace $f$ with $S^{-1}f$ in formula (3.2), we have

$$\hat{f}_R(w) = \frac{1}{q_0} \{(S^{-1}f)^+_R(w)\}^G(w)$$

Which leads to

$$\{(S^{-1}f)^+_R(w)\}^G(w) = \frac{q_0\hat{f}_R(w)}{G(w)} \quad (3.3)$$

Then we take Fourier inverse transformation on both sides of (3.3), we have

$$(S^{-1}f)^+_R(x) = \left\{\frac{q_0\hat{f}_R(\cdot)}{G(\cdot)}\right\}^\vee(x)$$

So, for $x \geq 0$,

$$(S^{-1}f)_R(x_R) = \left\{\frac{q_0\hat{f}_R(\cdot)}{G(\cdot)}\right\}^\vee(x_R)$$

is valied or

$$(S^{-1})f(x_p) = \left\{\frac{q_0\hat{f}_R(\cdot)}{G(\cdot)}\right\}^\vee(P^{-1}(x_p)) \quad (3.4)$$

Finally, let $f(x_p) = g_{mn}(x_p)$ in formula (3.4), we obtain

$$g^{*}_{m,n}(x_p) = \left\{\frac{q_0\hat{f}_R(\cdot)}{G(\cdot)}\right\}^\vee(P^{-1}(x_p))$$

References


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