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Abstract

We are interested in the approximation and simulation of solutions for the backward stochastic differential equations. We suggest two approximation schemes, and we study the $\mathbb{L}^2$ induced error.

1. **Introduction**

Backward Stochastic Differential Equations (BSDEs in short) have been introduced by Pardoux and Peng [16]. One of the main motivations for studying the BSDE's has been to use it to solve diverse problems in mathematical finance [8], in optimal stochastic control [11] and stochastic games [6].

In this paper, we propose to simulate the solution (X, Y, Z) of the coupled Forward-Backward SDE defined for all $0 \leq t \leq T$ by

$$(FBSDE) \begin{cases} X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s & Forward \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s & Backward \end{cases}$$

where $x \in \mathbb{R}$ and W is a Brownian motion defined on some complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t\leq T})$. The forward component X will be approximated by the Euler or the Milshtein Scheme. The simulation of the couple (Y, Z) is more delicate. Douglas et al presented in [7] a numerical method for a class of Forward-Backward SDEs based on the finite difference approximation of the associated PDE and the four steps scheme developed in [13]. In [5], Chevance proposed a numerical method for BSDE's where the generator f is not dependent on the control gradient Z. The difficulty is in the approximation of the process Z that comes only from the martingale representation Theorem. In the case where f depends only on Z, Bally has developed in [1] a discretization scheme considers a random time partition. His method is more difficult to implement. The work of Zhang [17] is more interesting but doesn't answer the question of approximation of Z. Bouchard and Touzi [3] invested this work to give an implicit approximation scheme. Recently, Gobet et al [10] propose a new numerical scheme based on iterative regression function basis which coefficients are evaluated using Monte Carlo simulation.

Our approach is to give an approximation scheme via the representation for the solution (Y, Z) of the Backward SDE. We suggest a pseudo approximation scheme then an explicit discretization scheme, and we calculate the $\mathbb{L}^2$ error induced to these approximations.

Throughout the paper, we use the following notation:

Notation 1.1. Let $\pi : 0 = t_0 < t_1 < ... < t_n = T$ be a regular partition of the time interval $[0, T]$ such that $|\pi| = T/n$ and $t_i = i|\pi|$ for $i = 0, ..., n$. We denote by $\Delta W_{t_{i+1}} = W_{t_{i+1}} - W_{t_i}$ and $\mathbb{E}_i = \mathbb{E}[./\mathcal{F}_{t_i}]$ for all i. Finally, C will denote a generic constant independent of π that may take different values from line to line.

We shall make use of the following assumptions:

Condition 1. *$\sigma \in \mathcal{C}_b^{0,2}([0,T] \times \mathbb{R}, \mathbb{R}^*)$ and $b \in \mathcal{C}_b^{0,1}([0,T] \times \mathbb{R}, \mathbb{R})$, and all derivatives (respect to x) are uniformly bounded by a common constant $K > 0$. Further, there exists a constant $\kappa > 0$ such that for any $x \in \mathbb{R}$ and $t \in [0, T]$ $\quad \kappa \leq |\sigma(t, x)| \leq K$ and $|b(t, 0)| \leq K$.*

Condition 2. *The functions $f \in \mathcal{C}([0, T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and $g \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ are uniformly K-Lipschitz i.e $\forall\ x, x', y, y', z, z' \in \mathbb{R}$ and $t, t' \in [0, T]$*

$$|f(t, x, y, z) - f(t', x', y', z')| \leq K \left(|t - t'| + |x - x'| + |y - y'| + |z - z'|\right)$$

and

$$|g(x) - g(x')| \leq K|x - x'|.$$

Under these assumptions, there exists a unique adapted process (X, Y, Z) solution of the $(FBSDE)$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X_t|^2 + \sup_{0\leq t\leq T} |Y_t|^2 + \int_0^T |Z_s|^2 ds\right) \leq C(1 + |x|^2), \qquad (1.1)$$

The following estimates are standard [17]

$$\max_{1\leq i\leq n} \sup_{t\in(t_{i-1},t_i]} \mathbb{E}\left(|X_t - X_{t_{i-1}}|^2 + |Y_t - Y_{t_{i-1}}|^2\right) \leq C(1 + |x|^2)|\pi|, \quad (1.2)$$

and

$$\sum_{i=1}^{n}\mathbb{E}\int_{t_{i-1}}^{t_i}\left(|Z_s-Z_{t_{i-1}}|^2+|Z_s-Z_{t_i}|^2\right)ds\le C(1+|x|^2)|\pi|. \tag{1.3}$$

We shall cite main representation of the solution (X,Y,Z).
To begin with, let us define for $0<t<r<T$ the process

$$N_r^t=\frac{1}{r-t}\int_t^r\sigma^{-1}(s,X_s)\nabla X_s dW_s(\nabla X_t)^{-1},$$

where ∇X is the unique solution of the following Linear SDE

$$\nabla X_t=1+\int_0^t\partial_x b(s,X_s)\nabla X_s ds+\int_0^t\partial_x\sigma(s,X_s)\nabla X_s dW_s.$$

It is known that ∇X is invertible and

$$\begin{aligned}(\nabla X_t)^{-1} &= 1-\int_0^t\left(\partial_x b(s,X_s)-|\partial_x\sigma(s,X_s)|^2\right)(\nabla X_s)^{-1}ds\\ &- \int_0^t\partial_x\sigma(s,X_s)(\nabla X_s)^{-1}dW_s.\end{aligned}$$

As a consequence of these notations, one has the following estimates [15]:

$$\left(\mathbb{E}|N_r^t|^2\right)^{\frac{1}{2}}\le\frac{C}{\sqrt{r-t}},\qquad \mathbb{E}|N_{r_1}^t-N_{r_2}^t|^2\le C\frac{|r_1-r_2|}{(r_1-t)(r_2-t)}. \tag{1.4}$$

The representation formula can be written $\mathbb{P}$- almost surely [14]:

$$\begin{cases} Y_t=\mathbb{E}\left(g(X_T)+\int_t^T f(s,X_s,Y_s,Z_s)ds/\mathcal{F}_t\right),\\ Z_t=\mathbb{E}\left(g(X_T)N_T^t+\int_t^T f(s,X_s,Y_s,Z_s)N_s^t ds/\mathcal{F}_t\right)\sigma(t,X_t).\end{cases}$$

Remark 1.2. A direct consequence of the representation formula of the matringale integrand Z that: If condition 1 and condition 2 hold; for all $p\ge 2$, there exists a constant $C_p>0$ depending only on $T,\ K$ and p such that

$$\mathbb{E}\sup_{0\le t\le T}|Z_t|^p\le C_p(1+|x|^p).$$

(see [15] Lemma 2.5).

Now, we are ready to give the approximation schemes.

2. Backward approximation schemes

2.1. Pseudo-discretization

In this part, we consider that the forward component X will be approximated by the classical Euler scheme

$$\begin{cases} \bar{X}_0 = x, \\ \bar{X}_{t_{i+1}} = \bar{X}_{t_i} + b(t_i, \bar{X}_{t_i})|\pi| + \sigma(t_i, \bar{X}_{t_i})\Delta W_{t_{i+1}} \text{ for any } i = 0, ..., n-1. \end{cases}$$

We set for all $t_i \leq t \leq t_{i+1}$

$$\bar{X}_t = \bar{X}_{t_i} + b(t_i, \bar{X}_{t_i})(t - t_i) + \sigma(t_i, \bar{X}_{t_i})(W_t - W_{t_i}),$$

and we have the following estimates for all $p \geq 2$

$$\mathbb{E} \sup_{0 \leq t \leq T} |\bar{X}_t|^p < +\infty \quad \text{and} \quad \mathbb{E} \sup_{0 \leq t \leq T} |X_t - \bar{X}_t|^p \leq C_p |\pi|^{p/2}. \tag{2.1}$$

To approximate the solution of the Backward SDE, we consider the natural discrete time scheme

$$\begin{cases} \bar{Y}_{t_n} = g(\bar{X}_{t_n}), \text{ and for } \ i = n-1, ..., 0 \\ \bar{Y}_{t_i} = \mathbb{E}_i \left[\bar{Y}_{t_{i+1}} + f(t_{i+1}, \bar{X}_{t_{i+1}}, \bar{Y}_{t_{i+1}}, \bar{Z}_{t_{i+1}})|\pi| \right] \\ \bar{Z}_{t_i} = \ \mathbb{E}_i \left[\bar{Y}_{t_{i+1}} N^{t_i}_{t_{i+1}} + f(t_{i+1}, \bar{X}_{t_{i+1}}, \bar{Y}_{t_{i+1}}, \bar{Z}_{t_{i+1}}) N^{t_i}_{t_{i+1}} |\pi| \right] \sigma(t_i, \bar{X}_{t_i}), \end{cases}$$

By a backward induction argument, we can see that $(\bar{Y}_{t_i}, \bar{Z}_{t_i}) \in \mathbb{L}^2$ for any i, and they are deterministic functions of $\bar{X}_{t_i}$. For later use, we need to introduce a continuous time approximation for (Y, Z). Since, from the classical martingale representation Theorem, there exists a predictable process $\tilde{Z} \in \mathbb{L}^2([0, T] \times \Omega)$ such that

$$\bar{Y}_{t_{i+1}} + f(t_{i+1}, \bar{X}_{t_{i+1}}, \bar{Y}_{t_{i+1}}, \bar{Z}_{t_{i+1}})|\pi| = \bar{Y}_{t_i} + \int_{t_i}^{t_{i+1}} \tilde{Z}_s dW_s.$$

We then define for all $t \in (t_i, t_{i+1})$

$$\begin{cases} \bar{Y}_t = \bar{Y}_{t_{i+1}} + f(t_{i+1}, \bar{X}_{t_{i+1}}, \bar{Y}_{t_{i+1}}, \bar{Z}_{t_{i+1}})(t_{i+1} - t) - \int_t^{t_{i+1}} \tilde{Z}_s dW_s \\ \bar{Z}_t = \bar{Z}_{t_i}. \end{cases} \tag{2.2}$$

The following Theorem provides the error estimates due to this approximation

Theorem 2.1. *There exists a constant $C > 0$ depending only on T and K such that*

$$\sup_{0\le t\le T} \mathbb{E}|Y_t - \bar{Y}_t|^2 + \mathbb{E}\int_0^T |Z_t - \bar{Z}_t|^2 dt \le C|\pi|.$$

Proof. We denote $\Theta_s = (X_s, Y_s, Z_s)$ and $\bar{\Theta}_s = (\bar{X}_s, \bar{Y}_s, \bar{Z}_s)$. From the Itô's formula and the Lipschitz property of f, we get

$$\begin{aligned}
\mathbb{E}|Y_t - \bar{Y}_t|^2 + \mathbb{E}\int_t^{t_{i+1}} |Z_s - \tilde{Z}_s|^2 ds \;&=\; \mathbb{E}|Y_{t_{i+1}} - \bar{Y}_{t_{i+1}}|^2 \\
&\quad + 2\mathbb{E}\int_t^{t_{i+1}} (Y_s - \bar{Y}_s)(f(s,\Theta_s) - f(t_{i+1}, \bar{\Theta}_{t_{i+1}}))ds \\
&\le\; \mathbb{E}|Y_{t_{i+1}} - \bar{Y}_{t_{i+1}}|^2 \\
&\quad + \frac{C}{\alpha}\mathbb{E}\int_t^{t_{i+1}} \left(|s - t_{i+1}|^2 + |X_s - X_{t_{i+1}}|^2 + |X_{t_{i+1}} - \bar{X}_{t_{i+1}}|^2\right) ds \\
&\quad + \alpha\mathbb{E}\int_t^{t_{i+1}} |Y_s - \bar{Y}_s|^2 ds + \frac{C}{\alpha}\mathbb{E}\int_t^{t_{i+1}} \left(|Y_s - Y_{t_{i+1}}|^2 + |Y_{t_{i+1}} - \bar{Y}_{t_{i+1}}|^2\right) ds \\
&\quad + \frac{C}{\alpha}\mathbb{E}\int_t^{t_{i+1}} \left(|Z_s - Z_{t_{i+1}}|^2 + |Z_{t_{i+1}} - \bar{Z}_{t_{i+1}}|^2\right) ds
\end{aligned}$$

for every $t \in (t_i, t_{i+1})$ and $\alpha > 0$. Applying the Gronwall Lemma, we obtain by (1.2)

$$\mathbb{E}|Y_t - \bar{Y}_t|^2 \le \left((1 + C|\pi|)\mathbb{E}|Y_{t_{i+1}} - \bar{Y}_{t_{i+1}}|^2 + \frac{C}{\alpha}\Lambda_i\right) e^{\alpha|\pi|} \tag{2.3}$$

where $\Lambda_i = |\pi|^2 + \mathbb{E}\int_{t_i}^{t_{i+1}} |Z_s - Z_{t_{i+1}}|^2 ds + |\pi|\mathbb{E}|Z_{t_{i+1}} - \bar{Z}_{t_{i+1}}|^2$.
In particular for $t = t_i$, we have

$$\begin{aligned}
&\mathbb{E}|Y_{t_i} - \bar{Y}_{t_i}|^2 + \mathbb{E}\int_{t_i}^{t_{i+1}} |Z_s - \tilde{Z}_s|^2 ds \\
&\qquad \le (1 + C\alpha|\pi|)\left[(1 + C|\pi|)\mathbb{E}|Y_{t_{i+1}} - \bar{Y}_{t_{i+1}}|^2 + \frac{C}{\alpha}\Lambda_i\right].
\end{aligned}$$

Iterating the last inequality to get

$$\begin{aligned}
\mathbb{E}|Y_{t_i}-\bar{Y}_{t_i}|^2 &+ \mathbb{E}\int_{t_i}^{t_n}|Z_s-\tilde{Z}_s|^2ds\\
&\le (1+C|\pi|)^{n-i}(1+C\alpha|\pi|)^{n-i}\mathbb{E}|g(X_{t_n})-g(\bar{X}_{t_n})|^2\\
&+ (1+C\alpha|\pi|)\frac{C}{\alpha}\sum_{j=0}^{n-i-1}(1+C|\pi|)^j(1+C\alpha|\pi|)^j\Lambda_{i+j},\\
&\le C\left(\mathbb{E}|X_{t_n}-\bar{X}_{t_n}|^2+|\pi|+\sum_{j=0}^{n-1}\mathbb{E}\int_{t_j}^{t_{j+1}}|Z_s-Z_{t_{j+1}}|^2ds\right)\\
&+ (1+C\alpha|\pi|)\frac{C}{\alpha}|\pi|\sum_{j=0}^{n-1}\mathbb{E}|Z_{t_j}-\bar{Z}_{t_j}|^2,\\
&\le C|\pi|+(1+C\alpha|\pi|)\frac{C}{\alpha}|\pi|\sum_{j=0}^{n-1}\mathbb{E}|Z_{t_j}-\bar{Z}_{t_j}|^2 \qquad (2.4)
\end{aligned}$$

by the condition 2, (1.3) and (2.1). On the other hand, from the expression of Z and $\bar{Z}$, if we denote by

$$\Sigma_j=|\sigma^{-1}(t_j,X_{t_j})Z_{t_j}-\sigma^{-1}(t_j,\bar{X}_{t_j})\bar{Z}_{t_j}|$$

we can write

$$\begin{aligned}
\Sigma_j=\Big|\mathbb{E}_j\Big[&(Y_{t_{j+1}}-\bar{Y}_{t_{j+1}})N_{t_{j+1}}^{t_j}\\
&+\int_{t_j}^{t_{j+1}}\left(f(s,\Theta_s)N_s^{t_j}-f(t_{j+1},\bar{\Theta}_{t_{j+1}})N_{t_{j+1}}^{t_j}\right)ds\Big]\Big|\\
&=\left|\mathbb{E}_j\left[Y_{t_j}-\bar{Y}_{t_j}+\int_{t_j}^{t_{j+1}}(Z_s-\tilde{Z}_s)dW_s\right]N_{t_{j+1}}^{t_j}\right.\\
&\left.+\mathbb{E}_j\int_{t_j}^{t_{j+1}}f(s,\Theta_s)(N_s^{t_j}-N_{t_{j+1}}^{t_j})ds\right|.
\end{aligned}$$

Since (2.2) and (1.4) we have

$$\begin{aligned}
\Sigma_j &\leq \left(\mathbb{E}\int_{t_j}^{t_{j+1}} |Z_s - \tilde{Z}_s|^2 ds\right)^{\frac{1}{2}} \left(\mathbb{E}|N_{t_{j+1}}^{t_j}|^2\right)^{\frac{1}{2}} \\
&+ \int_{t_j}^{t_{j+1}} \left(\mathbb{E}_j|f(s,\Theta_s)|^2\right)^{\frac{1}{2}} \left(\mathbb{E}|N_s^{t_j} - N_{t_{j+1}}^{t_j}|^2\right)^{\frac{1}{2}} ds, \\
&\leq \frac{C}{\sqrt{|\pi|}} \left(\mathbb{E}\int_{t_j}^{t_{j+1}} |Z_s - \tilde{Z}_s|^2 ds\right)^{\frac{1}{2}} \\
&+ C\left(\mathbb{E}_j(1 + \sup_{0\leq s\leq T} |\Theta_s|^2)\right)^{\frac{1}{2}} \int_{t_j}^{t_{j+1}} \sqrt{\frac{t_{j+1}-s}{|\pi|(s-t_j)}} ds, \qquad (2.5) \\
&\leq \frac{C}{\sqrt{|\pi|}} \left(\mathbb{E}\int_{t_j}^{t_{j+1}} |Z_s - \tilde{Z}_s|^2 ds\right)^{\frac{1}{2}} + C\left(\mathbb{E}_j(1 + \sup_{0\leq s\leq T} |\Theta_s|^2)\right)^{\frac{1}{2}} \sqrt{|\pi|}.
\end{aligned}$$

The Lipschitz property of σ and the condition 1 insure that

$$|Z_{t_j} - \bar{Z}_{t_j}| \leq C\left(|Z_{t_j}||X_{t_j} - \bar{X}_{t_j}| + |\Sigma_j|\right).$$

Taking the $\mathbb{L}^2$ estimates and replacing by (2.5), then we have

$$\begin{aligned}
|\pi|\sum_{j=0}^{n-1} \mathbb{E}|Z_{t_j} - \bar{Z}_{t_j}|^2 \leq C\sum_{j=0}^{n-1} &\Big((\mathbb{E}|X_{t_j} - \bar{X}_{t_j}|^4)^{\frac{1}{2}} \\
&+ \mathbb{E}\int_{t_j}^{t_{j+1}} |Z_s - \tilde{Z}_s|^2 ds + |\pi|^2\Big) \\
&\qquad \leq C\mathbb{E}\int_0^T |Z_s - \tilde{Z}_s|^2 ds + C|\pi|. \qquad (2.6)
\end{aligned}$$

Now, we plug (2.6) in (2.4) to obtain for $i = 0$

$$|Y_0 - \bar{Y}_0|^2 + \mathbb{E}\int_0^T |Z_s - \tilde{Z}_s|^2 ds \leq C|\pi| + (1 + C\alpha|\pi|)\frac{C}{|\pi|}\mathbb{E}\int_0^T |Z_s - \tilde{Z}_s|^2 ds.$$

For α sufficiently larger to C we have $\sum_{j=0}^{n-1} \mathbb{E}|Z_{t_j} - \bar{Z}_{t_j}|^2 \leq C$, and we conclude by (2.4) and (2.3) that

$$\sup_{0\leq t\leq T} \mathbb{E}|Y_t - \bar{Y}_t|^2 \leq C|\pi|.$$

Together, with (1.3) we conclude that

$$\begin{aligned}\mathbb{E}\int_0^T |Z_s - \bar{Z}_s|^2 ds &\leq 2\sum_{j=0}^{n-1}\mathbb{E}\int_{t_j}^{t_{j+1}} |Z_s - Z_{t_j}|^2 ds + 2|\pi|\sum_{j=0}^{n-1}\mathbb{E}|Z_{t_j} - \bar{Z}_{t_j}|^2,\\ &\leq C|\pi|.\end{aligned}$$

This completes the proof of the Theorem. □

Remark 2.2. We can obtain a similar result if we replace in the approximation of the forward component X the Euler scheme by the Milshtein scheme.

In this scheme, the process N is only dependent on the forward component X, we henceforth call it a pseudo-approximation. The following backward scheme raises this handicap.

2.2. **Explicit Approximation scheme**

In this section we consider that the processus X will be approximated by the Milshtein scheme described by

$$\begin{cases} X_0^\pi &= x,\\ X_{t_{i+1}}^\pi &= X_{t_i}^\pi + (b - \frac{1}{2}\partial_x\sigma\sigma)(t_i, X_{t_i}^\pi)|\pi| + \sigma(t_i, X_{t_i}^\pi)\Delta W_{t_{i+1}}\\ &+ \dfrac{1}{2}(\partial_x\sigma\sigma)(t_i, X_{t_i}^\pi)(\Delta W_{t_{i+1}})^2.\end{cases}$$

For $t \in (t_i, t_{i+1})$, let us put

$$\begin{aligned}X_t^\pi &= X_{t_i}^\pi + (b - \frac{1}{2}\partial_x\sigma\sigma)(t_i, X_{t_i}^\pi)(t - t_i) + \sigma(t_i, X_{t_i}^\pi)(W_t - W_{t_i})\\ &+ \frac{1}{2}(\partial_x\sigma\sigma)(t_i, X_{t_i}^\pi)(W_t - W_{t_i})^2.\end{aligned}$$

It is known that [9]

$$\forall\, p \geq 2, \qquad \sup_{0\leq t\leq T}\mathbb{E}|X_t - X_t^\pi|^p \leq \mathbf{K}(T)|\pi|^p \tag{2.7}$$

where $\mathbf{K}$ is an increasing function. Let us define the processes ∇X^π and $(\nabla X^\pi)^{-1}$ by $\nabla X_{t_0}^\pi = (\nabla X_{t_0}^\pi)^{-1} = 1$, and for every $t \in (t_i, t_{i+1}]$

$$\begin{aligned}\nabla X_t^\pi &= \nabla X_{t_i}^\pi + (\partial_x b - \frac{1}{2}\partial_x\sigma^2)(t_i, X_{t_i}^\pi)\nabla X_{t_i}^\pi(t - t_i)\\ &+ \partial_x\sigma(t_i, X_{t_i}^\pi)\nabla X_{t_i}^\pi(W_t - W_{t_i}) + |\partial_x\sigma(t_i, X_{t_i}^\pi)|^2\nabla X_{t_i}^\pi(W_t - W_{t_i})^2\end{aligned}$$

and

$$\begin{aligned}(\nabla X_t^\pi)^{-1} &= (\nabla X_{t_i}^\pi)^{-1} - (\partial_x b - \frac{1}{2}\partial_x \sigma^2)(t_i, X_{t_i}^\pi)(\nabla X_{t_i}^\pi)^{-1}(t-t_i)\\ &- \partial_x\sigma(t_i, X_{t_i}^\pi)(\nabla X_{t_i}^\pi)^{-1}(W_t - W_{t_i})\\ &+ \frac{1}{2}|\partial_x\sigma(t_i, X_{t_i}^\pi)|^2(\nabla X_{t_i}^\pi)^{-1}(W_t - W_{t_i})^2.\end{aligned}$$

Now, we are ready to approximate the process N. We put for $i = 0, ..., n-1$

$$N_{t_{i+1}}^{\pi,t_i} = \frac{1}{|\pi|}\int_{t_i}^{t_{i+1}} \sigma^{-1}(s, X_s^\pi)\nabla X_s^\pi dW_s(\nabla X_{t_i}^\pi)^{-1},$$

and we have the following estimates

Proposition 2.3.

- $\sup_{0\le t\le T}\mathbb{E}\left(|\nabla X_t - \nabla X_t^\pi|^2 + |(\nabla X_t)^{-1} - (\nabla X_t^\pi)^{-1}|^2\right) \le C|\pi|^2,$
- $\max_{0\le i\le n-1}\mathbb{E}|N_{t_{i+1}}^{t_i} - N_{t_{i+1}}^{\pi,t_i}|^2 \le C|\pi|.$

Proof. We denote $\nabla\bar{X}$ the Milshtein approximation of ∇X defined by $\nabla\bar{X}_0 = 1$, and

$$\begin{aligned}\nabla\bar{X}_t &= \nabla\bar{X}_{t_i} + (\partial_x b - \frac{1}{2}\partial_x\sigma)(t_i, X_{t_i})\nabla\bar{X}_{t_i}(t-t_i)\\ &+ \partial_x\sigma(t_i, X_{t_i})\nabla\bar{X}_{t_i}(W_t - W_{t_i}) + |\partial_x\sigma(t_i, X_{t_i})|^2\nabla\bar{X}_{t_i}(W_t - W_{t_i})^2\end{aligned}$$

for $t \in (t_i, t_{i+1})$. Then we have $\sup_{0\le t\le T}\mathbb{E}|\nabla X_t - \nabla\bar{X}_t|^2 \le C|\pi|^2$.
On the other hand

$$\begin{aligned}\mathbb{E}|\nabla\bar{X}_t - \nabla X_t^\pi|^2 &\le C(1+|\pi|+|\pi|^2)\mathbb{E}|\nabla\bar{X}_{t_i} - \nabla X_{t_i}^\pi|^2\\ &+ C|\pi|(1+C|\pi|)(\mathbb{E}|\nabla\bar{X}_{t_i}|^4)^{\frac{1}{2}}\left(\mathbb{E}|X_{t_i} - X_{t_i}^\pi|^4\right)^{\frac{1}{2}}\\ &+ C|\pi|^2(\mathbb{E}|\nabla\bar{X}_{t_i}|^4)\frac{1}{2}|\left(\mathbb{E}X_{t_i} - X_{t_i}^\pi|^8\right)^{\frac{1}{2}}.\end{aligned}$$

In particular for $t = t_{i+1}$

$$\begin{aligned}\mathbb{E}|\nabla\bar{X}_{t_{i+1}} - \nabla X_{t_{i+1}}^\pi|^2 &\le (1+C|\pi|)\mathbb{E}|\nabla\bar{X}_{t_i} - \nabla X_{t_i}^\pi|^2 + C|\pi|^3,\\ &\le C|\pi|^2.\end{aligned}$$

The result is followed by the classical iteration.
By the same argument, we prove that $\mathbb{E}|(\nabla X_t)^{-1} - (\nabla X_t^\pi)^{-1}|^2 \le C|\pi|^2$.
For the second point, we have

$$\begin{aligned}|\pi|^2\mathbb{E}|N^{t_i}_{t_{i+1}} - N^{\pi,t_i}_{t_{i+1}}|^2 & \le C\Big\{\mathbb{E}|(\nabla X_{t_i})^{-1} - (\nabla X^\pi_{t_i})^{-1}|^2 \int_{t_i}^{t_{i+1}} \mathbb{E}|\nabla X_s|^2 ds \\ & +\mathbb{E}|(\nabla X^\pi_{t_i})^{-1}|^2 \int_{t_i}^{t_{i+1}} \mathbb{E}|\nabla X_s - \nabla X^\pi_s|^2 ds \\ & + \mathbb{E}|(\nabla X^\pi_{t_i})^{-1}|^2 \int_{t_i}^{t_{i+1}} \Big(\mathbb{E}|X_s - X^\pi_s|^4\Big)^{\frac{1}{2}} \Big(\mathbb{E}|\nabla X_s|^2\Big)^{\frac{1}{2}} ds\Big\}, \\ & \le C|\pi|^3\end{aligned}$$

by the condition 1 and the first point. This completes the proof of the proposition. □

Now, we present our explicit backward approximation scheme. We consider the process (Y^π, Z^π) defined by

$$\begin{cases} Y^\pi_{t_n} = g(X^\pi_{t_n}), \quad \text{and for } \ i = n-1, ..., 0 \\ Y^\pi_{t_i} = \mathbb{E}_i\Big[Y^\pi_{t_{i+1}} + f(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}}, Z^\pi_{t_{i+1}})|\pi|\Big] \\ Z^\pi_{t_i} = \ \mathbb{E}_i\Big[Y^\pi_{t_{i+1}} N^{\pi,t_i}_{t_{i+1}} + f(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}}, Z^\pi_{t_{i+1}}) N^{\pi,t_i}_{t_{i+1}}|\pi|\Big]\sigma(t_i, X^\pi_{t_i}). \end{cases}$$

The continuous version is defined in the same way that the pseudo-approximation:

$$Y^\pi_t = Y^\pi_{t_{i+1}} + f(t_{i+1}, X^\pi_{t_{i+1}}, Y^\pi_{t_{i+1}}, Z^\pi_{t_{i+1}})(t_{i+1} - t) - \int_t^{t_{i+1}} \tilde{Z}^\pi_s dW_s, \quad Z^\pi_t = Z^\pi_{t_i}$$

where $\tilde{Z}^\pi$ is the unique square integrable predictable process verifying

$$Y^\pi_{t_{i+1}} = \mathbb{E}_i[Y^\pi_{t_{i+1}}] + \int_{t_i}^{t_{i+1}} \tilde{Z}^\pi_s dW_s.$$

With these suggestions, we provide the induce error of this approximation.

Theorem 2.4.

$$\sup_{0\le t\le T} \mathbb{E}|Y_t - Y^\pi_t|^2 + \mathbb{E}\int_0^T |Z_t - Z^\pi_t|^2 dt \le C|\pi|$$

where $C > 0$ is only dependent on K, κ and T.

Proof. To prove the Theorem, it suffices to show that

$$\sup_{0\le t\le T} \mathbb{E}|\bar{Y}_t - Y^\pi_t|^2 + \mathbb{E}\int_0^T |\bar{Z}_s - Z^\pi_s|^2 ds \le C|\pi|.$$

We denote $\Theta_s^\pi = (X_s^\pi, Y_s^\pi, Z_s^\pi)$, and let $\beta > 0$ be a constant to be chosen later on. From the Lipschitz property of f we get

$$\mathbb{E}|\bar{Y}_t - Y_t^\pi|^2 + \mathbb{E}\int_t^{t_{i+1}} |\tilde{Z}_s - \tilde{Z}_s^\pi|^2 ds$$
$$= \mathbb{E}|\bar{Y}_{t_{i+1}} - Y_{t_{i+1}}^\pi|^2 + 2\mathbb{E}\int_t^{t_{i+1}} (\bar{Y}_s - Y_s^\pi) f(t_{i+1}, \bar{\Theta}_{t_{i+1}}) - f(t_{i+1}, \Theta_{t_{i+1}}^\pi) ds,$$
$$\leq \beta\mathbb{E}\int_t^{t_{i+1}} |\bar{Y}_s - Y_s^\pi|^2 ds + \frac{C}{\beta}|\pi|\mathbb{E}|\bar{Z}_{t_{i+1}} - Z_{t_{i+1}}^\pi|^2 + (1 + \frac{C}{\beta}|\pi|)\mathbb{E}|\bar{Y}_{t_{i+1}} - Y_{t_{i+1}}^\pi|^2.$$

Using Gronwall Lemma and the classical iteration to obtain

$$\mathbb{E}|\bar{Y}_{t_i} - Y_{t_i}^\pi|^2 + \mathbb{E}\int_{t_i}^{t_n} |\tilde{Z}_s - \tilde{Z}_s^\pi|^2 ds \leq (1 + C\beta|\pi|)\frac{C}{\beta}|\pi| \sum_{j=0}^{n-1} \mathbb{E}|\bar{Z}_{t_j} - Z_{t_j}^\pi|^2$$

On the other hand, σ is bounded by K, then we can write

$$\begin{aligned}
|\bar{Z}_{t_j} - Z_{t_j}^\pi| &\leq K\Big|\mathbb{E}_j\Big[\bar{Y}_{t_{j+1}} N_{t_{j+1}}^{t_j} - Y_{t_{j+1}}^\pi N_{t_{j+1}}^{\pi,t_j} + |\pi| f(t_{j+1}, \bar{\Theta}_{t_{j+1}}) N_{t_{j+1}}^{t_j} \\
&\qquad - f(t_{j+1}, \Theta_{t_{j+1}}^\pi) N_{t_{j+1}}^{\pi,t_j}\Big]\Big| \\
&\leq K\Big|\mathbb{E}_j\Big[(\bar{Y}_{t_{j+1}} - Y_{t_{j+1}}^\pi + |\pi| f(t_{j+1}, \bar{\Theta}_{t_{j+1}}) \\
&\qquad - |\pi| f(t_{j+1}, \Theta_{t_{j+1}}^\pi)) N_{t_{j+1}}^{t_j}\Big]\Big| \\
&\qquad + K\left|\mathbb{E}_j\left[(Y_{t_{j+1}}^\pi + |\pi| f(t_{j+1}, \Theta_{t_{j+1}}^\pi))(N_{t_{j+1}}^{t_j} - N_{t_{j+1}}^{\pi,t_j})\right]\right| \\
&\leq K\mathbb{E}\left|\int_{t_j}^{t_{j+1}} (\tilde{Z}_s - \tilde{Z}_s^\pi) dW_s . N_{t_{j+1}}^{t_j}\right| \\
&\quad + K\left(\mathbb{E}_j[Y_{t_{j+1}}^\pi + |\pi| f(t_{j+1}, \Theta_{t_{j+1}}^\pi)]^2\right)^{\frac{1}{2}} (\mathbb{E}|N_{t_{j+1}}^{t_j} - N_{t_{j+1}}^{\pi,t_j}|^2)^{\frac{1}{2}} \\
&\leq \frac{C}{\sqrt{|\pi|}}\left(\mathbb{E}\int_{t_j}^{t_{j+1}} |\tilde{Z}_s - \tilde{Z}_s^\pi|^2 ds\right)^{\frac{1}{2}} \\
&\qquad + C\sqrt{|\pi|}\left(\mathbb{E}_j(Y_{t_{j+1}}^\pi + |\pi| f(t_{j+1}, \Theta_{t_{j+1}}^\pi))^2\right)^{\frac{1}{2}}.
\end{aligned}$$

By the same way that the pseudo-approximation, we prove for β larger to C that $\sum_{j=0}^{n-1} \mathbb{E}|\bar{Z}_{t_j} - Z_{t_j}^\pi|^2 \leq C$, and we conclude that

$$\begin{aligned}
\mathbb{E}|\bar{Y}_t - Y_t^\pi|^2 + \mathbb{E}\int_0^T |\bar{Z}_s - Z_s^\pi|^2 ds &\leq |\pi| \sum_{j=0}^{n-1} \mathbb{E}|\bar{Z}_{t_j} - Z_{t_j}^\pi|^2 \\
&\leq C|\pi|.
\end{aligned}$$

This completes the proof of the Theorem. □

Remark 2.5. The conditional expectation in the above discretization scheme, reduced to the regression on the random variable $\bar{X}_{t_i}$ in the pseudo discretization and $X^{\pi}_{t_i}$ in the explicit scheme. Different methods of regression are developed, see for example [2], [4] and [12].

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