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Abstract

The main result of this note is that a finitely generated hyper–(Abelian–by–finite) group $G$ is finite–by-nilpotent if and only if every infinite subset contains two distinct elements $x, y$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$ for some positive integer $n = n(x, y)$ (respectively, $\langle x, x^y \rangle$ is an extension of a group satisfying the minimal condition on normal subgroups by an Engel group).

1. Introduction and results

Let $\mathcal{X}$ be a class of groups. Denote by $(\mathcal{X}, \infty)$ (respectively, $(\mathcal{X}, \infty)^*$) the class of groups $G$ such that for every infinite subset $X$ of $G$, there exist distinct elements $x, y \in X$ such that $\langle x, y \rangle \in \mathcal{X}$ (respectively, $\langle x, x^y \rangle \in \mathcal{X}$). Note that if $\mathcal{X}$ is a subgroup closed class, then $(\mathcal{X}, \infty) \subseteq (\mathcal{X}, \infty)^*$.

In answer to a question of Erdös, B.H. Neumann proved in [16] that a group $G$ is centre–by–finite if and only if $G$ is in the class $(\mathcal{A}, \infty)$, where $\mathcal{A}$ denotes the class of Abelian groups. Lennox and Wiegold showed in [13]...
that a finitely generated soluble group is in the class \((\mathcal{N}, \infty)\) (respectively, \((\mathcal{P}, \infty))\) if and only if it is finite-by-nilpotent (respectively, polycyclic), where \(\mathcal{N}\) (respectively, \(\mathcal{P}\)) denotes the class of nilpotent (respectively, polycyclic) groups. Other results of this type have been obtained, for example in \([1]—[3], [4]—[6], [7], [8], [13], [14]—[16], [21], [22]\) and \([23]\).

We say that a group \(G\) has finite depth if the lower central series of \(G\) stabilises after a finite number of steps. Thus if \(\gamma_n(G)\) denotes the \(n^\text{th}\) term of the lower central series of \(G\), then \(G\) has finite depth if and only if \(\gamma_n(G) = \gamma_{n+1}(G)\) for some positive integer \(n\). Denote by \(\Omega\) the class of groups which has finite depth. Moreover, if \(k\) is a fixed positive integer, let \(\Omega_k\) denotes the class of groups \(G\) such that \(\gamma_k(G) = \gamma_{k+1}(G)\).

Clearly, any group in the class \(\mathcal{FN}\) is of finite depth, where \(\mathcal{F}\) denotes the class of finite groups. From this and the fact that \(\mathcal{FN}\) is a subgroup closed class, we deduce that finite-by-nilpotent groups belong to \((\Omega, \infty)^*\). Here we shall be interested by the converse. In \([5]\), Boukaroura has proved that a finitely generated soluble group in the class \((\Omega, \infty)\) is finite-by-nilpotent. We obtain the same result when \((\Omega, \infty)^*\) is replaced by \((\Omega, \infty)^*\) and soluble by hyper-(Abelian-by-finite). More precisely we shall prove the following result.

**Theorem 1.1.** Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group. Then, \(G\) is in the class \((\Omega, \infty)^*\) if, and only if, \(G\) is finite-by-nilpotent.

Note that Theorem 1.1 improves the result of \([12]\) which asserts that a finitely generated soluble-by-finite group whose subgroups generated by two conjugates are of finite depth, is finite-by-nilpotent.

It is clear that an Abelian group \(G\) in the class \((\Omega_1, \infty)^*\) is finite. For if \(G\) is infinite, then it contains an infinite subset \(X = G \setminus \{1\}\). Therefore there exist two distinct elements \(x, y \neq 1\) in \(X\) such that \(\gamma_1(\langle x, x^y \rangle) = \gamma_2(\langle x, x^y \rangle) = 1\); so \(x = 1\), which is a contradiction. From this it follows that a hyper-(Abelian-by-finite) group \(G\) in the class \((\Omega_1, \infty)^*\) is hyper-(finite) as \((\Omega_1, \infty)^*\) is a subgroup and a quotient closed class. But it is not difficult to see that a hyper-(finite) group is locally finite \([17, \text{Part 1, page 36}]\). So \(G\) is locally finite. Now if \(G\) is infinite, then it contains an infinite Abelian subgroup \(A\) \([17, \text{Theorem 3.43}]\). Since \(A\) is in the class \((\Omega_1, \infty)^*\), it is finite; a contradiction and \(G\), therefore, is finite. As consequence of Theorem 1.1, we shall prove other results on the class \((\Omega_k, \infty)^*\).

**Corollary 1.2.** Let \(k\) be a positive integer and let \(G\) be a finitely generated hyper-(Abelian-by-finite) group. We have:
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(i) If $G$ is in the class $(\Omega_k, \infty)^*$, then there exists a positive integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\Omega_2, \infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If $G$ is in the class $(\Omega_3, \infty)^*$, then $G$ is in the class $\mathcal{FN}_3^{(2)}$, where $\mathcal{N}_3^{(2)}$ denotes the class of groups whose 2-generator subgroups are nilpotent of class at most 3.

Let $k$ be a fixed positive integer, denote by $\mathcal{M}$, $\mathcal{E}_k$ and $\mathcal{E}$ respectively the class of groups satisfying the minimal condition on normal subgroups, the class of $k$-Engel groups and the class of Engel groups. Using Theorem 1.1, we will prove the following results concerning the classes $(\mathcal{M}\mathcal{E}, \infty)^*$ and $(\mathcal{M}\mathcal{E}_k, \infty)^*$

**Theorem 1.3.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group. Then, $G$ is in the class $(\mathcal{M}\mathcal{E}, \infty)^*$ if, and only if, $G$ is finite-by-nilpotent.

Note that this theorem improves Theorem 3 of [23] (respectively, Corollary 3 of [5]) where it is proved that a finitely generated soluble group in the class $(\mathcal{CN}, \infty)^*$ (respectively, $(\mathcal{XN}, \infty)$) is finite-by-nilpotent, where $\mathcal{C}$ (respectively, $\mathcal{X}$) denotes the class of Chernikov groups (respectively, the class of groups satisfying the minimal condition on subgroups).

**Corollary 1.4.** Let $k$ be a positive integer and let $G$ be a finitely generated hyper-(Abelian-by-finite) group. We have:

(i) If $G$ is in the class $(\mathcal{M}\mathcal{E}_k, \infty)^*$, then there exists a positive integer $c = c(k)$, depending only on $k$, such that $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\mathcal{MA}, \infty)^*$, then $G/Z_2(G)$ is finite.

(iii) If $G$ is in the class $(\mathcal{M}\mathcal{E}_2, \infty)^*$, then $G$ is in the class $\mathcal{FN}_3^{(2)}$.

Note that these results are not true for arbitrary groups. Indeed, Golod [9] showed that for each integer $d > 1$ and each prime $p$, there are infinite $d$-generator groups all of whose $(d-1)$-generator subgroups are finite $p$-groups. Clearly, for $d = 3$, we obtain a group $G$ which belongs to the class $(\mathcal{F}, \infty)^*$. Therefore, $G$ belongs to the classes $(\Omega, \infty)^*$, $(\Omega_k, \infty)^*$, $(\mathcal{M}\mathcal{E}, \infty)^*$ and $(\mathcal{M}\mathcal{E}_k, \infty)^*$, but it is not finite-by-nilpotent.

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2. Proofs of Theorem 1.1 and Corollary 1.2

Let $\mathcal{E}(\infty)$ the class of groups in which every infinite subset contains two distinct elements $x, y$ such that $[x, n y] = 1$ for a positive integer $n = n(x, y)$. In [15], it is proved that a finitely generated soluble group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent. We will extend this result to finitely generated hyper-(Abelian-by-finite) groups (Proposition 2.5).

Our first lemma is a weaker version of Lemma 11 of [23], but we include a proof to keep our paper reasonably self contained.

**Lemma 2.1.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $(\mathcal{FN}, \infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be a finitely generated infinite Abelian-by-finite group in the class $(\mathcal{FN}, \infty)$. Hence there is a normal torsion-free Abelian subgroup $A$ of finite index. Let $x$ be a non trivial element in $A$ and let $g$ in $G$. Then the subset $\{x^i g : i \text{ a positive integer}\}$ is infinite, so there are two positive integers $m, n$ such that $\langle x^m g, x^n g \rangle$ is finite-by-nilpotent, hence $\langle x^r, x^n g \rangle$ is finite-by-nilpotent where $r = m - n$. Thus there are two positive integers $c$ and $d$ such that $[x^r c, x^n g]^d = 1$. The element $x$ being in $A$ which is Abelian and normal in $G$, we have $[x^r c, x^n g] = [x^r c, g] = [x, c g]^r$; so $[x, c g]^r d = 1$.

Now $[x, c g]$ belongs to the torsion-free group $A$, so $[x, c g] = 1$. It follows that $x$ is a right Engel element of $G$. Since $G$ is Abelian-by-finite and finitely generated, it satisfies the maximal condition on subgroups; so the set of right Engel elements of $G$ coincides with its hypercentre which is equal to $Z_i (G)$, the $(i + 1)$-th term of the upper central series of $G$, for some integer $i > 0$ [17, Theorem 7.21]. Hence, $A \leq Z_i (G)$; and since $A$ is of finite index in $G$, $G/Z_i (G)$ is finite. Thus, by a result of Baer [10, Theorem 1], $G$ is finite-by-nilpotent. \hfill \Box

**Lemma 2.2.** Let $G$ be a finitely generated Abelian-by-finite group. If $G$ is in the class $\mathcal{E}(\infty)$, then it is finite-by-nilpotent.

**Proof.** Let $G$ be an infinite finitely generated Abelian-by-finite group in $\mathcal{E}(\infty)$, and let $A$ be an Abelian normal subgroup of finite index in $G$. It is clear that all infinite subsets of $G$ contains two different elements $x, y$ such that $x A = y A$; so $y = x a$ for some $a$ in $A$ and $\langle x, y \rangle = \langle x, a \rangle$. Thus $\langle x, y \rangle$ is a finitely generated metabelian group in the class $\mathcal{E}(\infty)$. It follows by the result of Longobardi and Maj [15, Theorem 1], that $\langle x, y \rangle$
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is finite-by-nilpotent. Hence $G$ is in the class $(\mathcal{FN}, \infty)$. Now, by Lemma 2.1, $G$ is finite-by-nilpotent; as required.  

Lemma 2.3. A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is nilpotent-by-finite.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Since $\mathcal{E}(\infty)$ is a quotient closed class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that $G$ is not nilpotent-by-finite but every proper homomorphic image of $G$ is in the class $\mathcal{NF}$. Since $G$ is hyper-(Abelian-by-finite), $G$ contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so we have $G/H$ is in $\mathcal{NF}$. If $H$ is finite then $G$ is nilpotent-by-finite, a contradiction. Consequently $H$ is Abelian and so $G$ is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, $G$ is a finite extension of a soluble group; there is therefore a normal soluble subgroup $K$ of $G$ of finite index. Now, $K$ is a finitely generated soluble group in the class $\mathcal{E}(\infty)$; it follows, by the result of Longobardi and Maj [15, Theorem 1], that $K$ is finite-by-nilpotent. By a result of P. Hall [10, Theorem 2], $K$ is nilpotent-by-finite and so $G$ is nilpotent-by-finite, a contradiction. Now, the Lemma is shown.  

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 2.3 has the following consequence:

Corollary 2.4. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$. Then $G$ satisfies the maximal condition on subgroups.

Proposition 2.5. A finitely generated hyper-(Abelian-by-finite) group in the class $\mathcal{E}(\infty)$ is finite-by-nilpotent.

Proof. Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in $\mathcal{E}(\infty)$. According to Corollary 2.4, $G$ satisfies the maximal condition on subgroups. Now, since $\mathcal{E}(\infty)$ is a quotient closed class, we may assume that every proper homomorphic image of $G$ is in $\mathcal{FN}$, but $G$ itself is not in $\mathcal{FN}$. Our group $G$ being hyper-(Abelian-by-finite), contains a non-trivial normal subgroup $H$ such that $H$ is finite or Abelian; so by hypothesis $G/H$ is in the class $\mathcal{FN}$. If $H$ is finite, then $G$ is finite-by-nilpotent, a contradiction. Consequently $H$ is Abelian and so $G$ is in the class $A(\mathcal{FN})$, hence $G$ is in $(A\mathcal{F})\mathcal{N}$. Now, since $G$ satisfies the maximal condition on
subgroups, it follows from Lemma 2.2, that \( G \) is in \( \mathcal{F}(\mathcal{N}) \mathcal{N} \), so it is in \( \mathcal{F}(\mathcal{N}) \mathcal{N} \). Consequently, there is a finite normal subgroup \( K \) of \( G \) such that \( G/K \) is soluble. The group \( G/K \), being a finitely generated soluble group in the class \( \mathcal{E}(\infty) \), is in \( \mathcal{F} \mathcal{N} \mathcal{N} \), by the result of Longobardi and Maj [15, Theorem 1]. So \( G \) is in the class \( \mathcal{F} \mathcal{N} \), which is a contradiction and the Proposition is shown. \( \square \)

The remainder of the proof of Theorem 1.1 is adapted from that of Lennox’s Theorem [11, Theorem 3]

**Lemma 2.6.** Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). If \( G \) is residually nilpotent, then \( G \) is in the class \( \mathcal{F} \mathcal{N} \).

**Proof.** Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\) and assume that \( G \) is residually nilpotent. Let \( X \) be an infinite subset of \( G \), there are two distinct elements \( x \) and \( y \) of \( X \) such that \( \langle x, x^y \rangle \in \Omega \). It follows that there exists a positive integer \( k \) such that \( \gamma_k(\langle x, x^y \rangle) = \gamma_{k+1}(\langle x, x^y \rangle) \). The group \( \langle x, x^y \rangle \), being a subgroup of \( G \), is residually nilpotent, so \( \bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1 \). Hence \( \gamma_k(\langle x, x^y \rangle) = \bigcap_{i \in \mathbb{N}} \gamma_i(\langle x, x^y \rangle) = 1 \). Since \( \langle x, x^y \rangle = \langle [y, x], x \rangle; \gamma_k([y, x], x) = 1 \), thus \( [y, x] = 1 \). We deduce that \( G \) is a finitely generated hyper-(Abelian-by-finite) group in the class \( \mathcal{E}(\infty) \). It follows, by Proposition 2.5, that \( G \) is in the class \( \mathcal{F} \mathcal{N} \), as required. \( \square \)

**Lemma 2.7.** If \( G \) is a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\), then it is nilpotent-by-finite.

**Proof.** Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in \((\Omega, \infty)^*\). Since finitely generated nilpotent-by-finite groups are finitely presented and \((\Omega, \infty)^*\) is a quotient closed class of groups, by [17, Lemma 6.17], we may assume that every proper quotient of \( G \) is nilpotent-by-finite, but \( G \) itself is not nilpotent-by-finite. Since \( G \) is hyper-(Abelian-by-finite), it contains a non-trivial normal subgroup \( K \) such that \( K \) is finite or Abelian; so \( G/K \) is in \( N \mathcal{F} \). In this case, \( K \) is Abelian and so \( G \) is in the class \( \mathcal{A}(N \mathcal{F}) \) and therefore it is in the class \( (\mathcal{A}N) \mathcal{F} \). Consequently, \( G \) has a normal subgroup \( N \) of finite index such that \( N \) is Abelian-by-nilpotent. Moreover, \( N \) being a subgroup of finite index in a finitely generated group, is itself finitely generated, and so \( N \) is a finitely generated Abelian-by-nilpotent group. It follows, by a result of Segal [19,
Corollary 1], that \( N \) has a residually nilpotent normal subgroup of finite index. Thus, \( G \) has a residually nilpotent normal subgroup \( H \), of finite index. Therefore, \( H \) is residually nilpotent and it is a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). So, by Lemma 2.6, \( H \) is in the class \( \mathcal{FN} \), hence \( H \) is in the class \( \mathcal{NF} \). Thus \( G \) is in the class \( \mathcal{NF} \), a contradiction which completes the proof. \( \square \)

**Lemma 2.8.** Let \( G \) be a finitely generated group in the class \((\Omega, \infty)^*\) which has a normal nilpotent subgroup \( N \) such that \( G/N \) is a finite cyclic group. Then \( G \) is in the class \( \mathcal{FN} \).

**Proof.** We prove by induction on the order of \( G/N \) that \( G \) is in the class \( \mathcal{FN} \). Let \( n = |G/N| \); if \( n = 1 \), then \( G = N \) and \( G \) is nilpotent. Now suppose that \( n > 1 \) and let \( q \) be a prime dividing \( n \). Since \( G/N \) is cyclic, it has a normal subgroup of index \( q \). Thus \( G \) has a normal subgroup \( H \) of index \( q \) containing \( N \). Since \( |H/N| < |G/N| \), then by the inductive hypothesis, \( H \) is in the class \( \mathcal{FN} \). Let \( T \) be the torsion subgroup of \( H \). Since \( H \) is finitely generated, \( T \) is finite. So \( H/T \) is a finitely generated torsion-free nilpotent group. Therefore, by Gruenberg [18, 5.2.21], \( H/T \) is residually a finite \( p \)-group for all primes \( p \) and hence, in particular, \( H/T \) is residually a finite \( q \)-group. But \( H \) has index \( q \) in \( G \) from which we get that \( G/T \) is residually a finite \( q \)-group [20, Exercise 10, page 17]. This means that \( G/T \) is residually nilpotent. It follows, by Lemma 2.6, that \( G/T \) is in the class \( \mathcal{FN} \). So \( G \) itself is in \( \mathcal{FN} \). \( \square \)

**Proof of Theorem 1.1.** Let \( G \) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). Hence, by Lemma 2.7, \( G \) is in the class \( \mathcal{NF} \). Let \( K \) be a normal nilpotent subgroup of \( G \) such that \( G/K \) is finite. Since \( K \) is a finitely generated nilpotent group, it has a normal torsion-free subgroup of finite index [18, 5.4.15 (i)]. Thus, \( G \) has a normal torsion-free nilpotent subgroup \( N \) of finite index. Let \( x \) be a non-trivial element of \( G \). Since \( N \) is finitely generated, \( \langle N, x \rangle \) is a finitely generated hyper-(Abelian-by-finite) group in the class \((\Omega, \infty)^*\). Furthermore, \( \langle N, x \rangle /N \) is cyclic. Therefore, by Lemma 2.8, \( \langle N, x \rangle \) is in the class \( \mathcal{FN} \). Consequently, there is a finite normal subgroup \( H \) of \( \langle N, x \rangle \) such that \( \langle N, x \rangle /H \) is nilpotent. Therefore \( \gamma_{k+1}(\langle N, x \rangle) \leq H \) for some positive integer \( k \); so \( \gamma_{k+1}(\langle N, x \rangle) \) is finite. Hence, there is a positive integer \( m \) such that \( [g, k x]^m = 1 \), for all \( g \in N \). Since \( [g, k x] \) is an element of the torsion-free group \( N \), we get that \( [g, k x] = 1 \). Thus, \( g \) is a right Engel element of \( G \); so \( N \subseteq R(G) \),
where $R(G)$ denotes the set of right Engel elements of $G$. Moreover, since $G$ is a finitely generated nilpotent-by-finite group, it satisfies the maximal condition on subgroups. Therefore, from Baer [17, Theorem 7.21], $R(G)$ coincides with the hypercentre of $G$ which equal to $Z_n(G)$ for some positive integer $n$. Thus $N \leq Z_n(G)$, so $Z_n(G)$ is of finite index in $G$. It follows, by a result of Baer [10, Theorem 1], that $G$ is in the class $\mathcal{FN}$.

Proof of Corollary 1.2. (i) Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\Omega_k, \infty)^*$; from Theorem 1.1, $G$ is in the class $\mathcal{FN}$. Let $H$ be a normal finite subgroup of $G$ such that $G/H$ is nilpotent. It is clear that $G/H$ is in the class $(\Omega_k, \infty)^*$. Let $\bar{X}$ be an infinite subset of $G/H$; there are therefore two distinct elements $\bar{x} = xH$, $\bar{y} = yH$ $(x, y \in G)$ of $\bar{X}$ such that $\langle \bar{x}, \bar{y} \rangle \in \Omega_k$, so $\gamma_k(\langle \bar{x}, \bar{y} \rangle) = \gamma_{k+1}(\langle \bar{x}, \bar{y} \rangle)$. Now, since $\langle \bar{x}, \bar{y} \rangle$ is nilpotent, there is an integer $i$ such that $\gamma_i(\langle \bar{x}, \bar{y} \rangle) = 1$; so $\gamma_k(\langle \bar{x}, \bar{y} \rangle) = 1$. Since $\langle \bar{x}, \bar{y} \rangle = \langle [\bar{y}, \bar{x}], \bar{x} \rangle$, we have $\gamma_k(\langle [\bar{y}, \bar{x}], \bar{x} \rangle) = 1$ and thus $[\bar{y}, k \bar{x}] = 1$. Consequently, $G/H$ is in the class $\mathcal{E}_k(\infty)$ of groups in which every infinite subset contains two distinct elements $g$, $h$ such that $[g, kh] = 1$. The group $G/H$, being a finitely generated soluble group in the class $\mathcal{E}_k(\infty)$; it follows by a result of Abdollahi [2, Theorem 3], that there is an integer $c = c(k)$, depending only on $k$, such that $(G/H)/Z_c(G/H)$ is finite. By a result of Baer [10, Theorem 1], $\gamma_{c+1}(G/H) = \gamma_{c+1}(G)H/H$ is finite; and since $H$ is finite, $\gamma_{c+1}(G)$ is finite. According to a result of P. Hall [10, 1.5], $G/Z_c(G)$ is finite.

(ii) If $G$ is in the class $(\Omega_2, \infty)^*$, then by Theorem 1.1 $G$ is finite-by-nilpotent. Therefore, $G$ has a finite normal subgroup $H$ such that $G/H$ is nilpotent. Since $G/H$ is in the class $(\Omega_2, \infty)^*$, it is in the class $\mathcal{E}_2(\infty)$. Hence, by Abdollahi [1, Theorem], $(G/H)/Z_2(G/H)$ is finite, so $\gamma_3(G/H)$ is finite. Since $H$ is finite, $\gamma_3(G)$ is finite. It follows, by P. Hall [10, 1.5], that $G/Z_2(G)$ is finite.

(iii) Now if $G$ is in the class $(\Omega_3, \infty)^*$, then by Theorem 1.1 $G$ has a finite normal subgroup $H$ such that $G/H$ is nilpotent. Since $G/H$ is in the class $(\Omega_3, \infty)^*$, it is in the class $\mathcal{E}_3(\infty)$. Hence, by Abdollahi [2, Theorem 1] $G/H$ is in the class $\mathcal{FN}_3^{(2)}$; consequently $G$ is in the class $\mathcal{FN}_3^{(2)}$. □

3. Proofs of Theorem 1.3 and Corollary 1.4

We start by showing a weaker version of Theorem 1.3:
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**Lemma 3.1.** A finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{M}N, \infty)^*$ is finite-by-nilpotent.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{M}N, \infty)^*$, and let $X$ be an infinite subset of $G$. There are therefore two distinct elements $x, y$ of $X$ such that $\langle x, x^y \rangle$ is in the class $\mathcal{M}N$, so there exists a normal subgroup $N$ of $\langle x, x^y \rangle$ such that $N$ is in $\mathcal{M}$ and $\langle x, x^y \rangle / N$ is nilpotent. Now, $\gamma_{i+1}(\langle x, x^y \rangle) \leq N$ for some positive integer $i$, therefore $\gamma_{i+1}(\langle x, x^y \rangle) \geq \gamma_{i+2}(\langle x, x^y \rangle) \geq \ldots$ is an infinite descending sequence of normal subgroups of $N$; however $N$ is in $\mathcal{M}$, therefore there exists a positive integer $n \geq i + 1$ such that $\gamma_n(\langle x, x^y \rangle) = \gamma_{n+1}(\langle x, x^y \rangle)$. Hence, $G$ is in the class $(\Omega, \infty)^*$; it follows, by Theorem 1.1, that $G$ is finite-by-nilpotent.

**Lemma 3.2.** A finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{M}E, \infty)^*$ is nilpotent-by-finite.

**Proof.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{M}E, \infty)^*$. Since $(\mathcal{M}E, \infty)^*$ is a closed quotient class of groups and since finitely generated nilpotent-by-finite groups are finitely presented, we may assume that $G$ is not nilpotent-by-finite, but every proper homomorphic image of $G$ is nilpotent-by-finite. Since $G$ is hyper-(Abelian-by-finite), there exists a non-trivial normal subgroup $H$ of $G$ such that $H$ is finite or Abelian; so we have $G/H$ is nilpotent-by-finite. If $H$ is finite then $G$ is nilpotent-by-finite, a contradiction. Consequently $H$ is Abelian and so $G$ is Abelian-by-(nilpotent-by-finite) and therefore it is (Abelian-by-nilpotent)-by-finite. Hence, $G$ is a finite extension of a soluble group. Let $K$ be a normal soluble subgroup of $G$ of finite index. Clearly, $K$ is in $(\mathcal{M}E, \infty)^*$, and since all soluble Engel group coincides with its Hirsch-Plotkin radical which is locally nilpotent [17, Theorem 7.34], we deduce that $K$ is in the class $(\mathcal{M}N, \infty)^*$; it follows by Lemma 3.1 that $K$ is finite-by-nilpotent. According to a result of P. Hall [10, Theorem 2], $K$ is nilpotent-by-finite. Thus, $G$ is nilpotent-by-finite, a contradiction. The proof is now complete.

Since finitely generated nilpotent-by-finite groups satisfy the maximal condition on subgroups, Lemma 3.2 has the following consequence:

**Corollary 3.3.** Let $G$ be a finitely generated hyper-(Abelian-by-finite) group in the class $(\mathcal{M}E, \infty)^*$. Then $G$ satisfies the maximal condition on subgroups.
Proof of Theorem 1.3. It is clear that all finite-by-nilpotent groups are in the class \((\mathcal{ME}, \infty)^*\). Conversely, let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in \((\mathcal{ME}, \infty)^*\). According to Corollary 3.3, \(G\) satisfies the maximal condition on subgroups. Since Engel groups satisfying the maximal condition on subgroups are nilpotent \([18, 12.3.7]\), we deduce that \(G\) is in the class \((\mathcal{MN}, \infty)^*\). It follows, by Lemma 3.1, that \(G\) is in the class \(\mathcal{FN}\); as required. \(\square\)

Proof of Corollary 1.4. (i) Let \(G\) be a finitely generated hyper-(Abelian-by-finite) group in the class \((\mathcal{ME}_k, \infty)^*\); from Theorem 1.3, \(G\) is in the class \(\mathcal{FN}\). Let \(N\) be a normal finite subgroup of \(G\) such that \(G/N\) is nilpotent. Since \(G/N\) is nilpotent and finitely generated, its torsion subgroup \(T/N\) is finite, so \(T\) is finite and \(G/T\) is a torsion-free nilpotent group. Clearly, the property \((\mathcal{ME}_k, \infty)^*\) is inherited by \(G/T\), and since \(G/T\) is torsion-free and soluble, it belongs to \((\mathcal{E}_k, \infty)^*\) \([17, \text{Theorem 5.25}]\). Let \(\bar{X}\) be an infinite subset of \(G/T\); there are therefore two distinct elements \(\bar{x} = xT, \bar{y} = yT\) \((x, y \in G)\) of \(\bar{X}\) such that \(\langle \bar{x}, \bar{y}\rangle\) is a \(k\)-Engel group. Since \(\langle \bar{x}, \bar{y}\rangle = \langle [\bar{y}, \bar{x}], \bar{x}\rangle\), we have \([\bar{y}, k+1 \bar{x}] = [[\bar{y}, \bar{x}], k \bar{x}] = 1\). Hence, \(G/T\) is in the class \(\mathcal{E}_{k+1}(\infty)\). The group \(G/T\), being a finitely generated soluble group in the class \(\mathcal{E}_{k+1}(\infty)\); it follows by a result of Abdollahi \([2, \text{Theorem 3}]\), that there is an integer \(c = c(k)\), depending only on \(k\), such that \((G/T)/Z_c(G/T)\) is finite. By a result of Baer \([10, \text{Theorem 1}]\), \(\gamma_{c+1}(G/T) = \gamma_{c+1}(G)T/T\) is finite; and since \(T\) is finite, \(\gamma_{c+1}(G)\) is finite. According to a result of P. Hall \([10, 1.5]\), \(G/Z_c(G)\) is finite.

(ii) If \(G\) is in the class \((\mathcal{MA}, \infty)^* = (\mathcal{ME}_1, \infty)^*\), then by Theorem 1.3, \(G\) is finite-by-nilpotent. We proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_2(\infty)\). Hence, by Abdollahi \([1, \text{Theorem}]\), \((G/T)/Z_2(G/T)\) is finite, so \(\gamma_3(G/T)\) is finite. Since \(T\) is finite, \(\gamma_3(G)\) is finite. It follows, by P. Hall \([10, 1.5]\), that \(G/Z_2(G)\) is finite.

(iii) Now if \(G\) is in the class \((\mathcal{ME}_2, \infty)^*\), we proceed as in (i) until we obtain that \(G/T\) is in the class \(\mathcal{E}_3(\infty)\). Hence, by Abdollahi \([2, \text{Theorem 1}]\) \(G/T\) is in the class \(\mathcal{FN}_3(2)\); consequently \(G\) is in the class \(\mathcal{FN}_3(2)\). \(\square\)

References

A condition on infinite subsets


