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Abstract

If \( X \) is a class of groups, then a group \( G \) is said to be minimal non-\( X \)-group if all its proper subgroups are in the class \( X \), but \( G \) itself is not an \( X \)-group. The main result of this note is that if \( c > 0 \) is an integer and if \( G \) is a minimal non \((LF)N\) (respectively, \((LF)N_c\))-group, then \( G \) is a finitely generated perfect group which has no non-trivial finite factor and such that \( G/Frat(G) \) is an infinite simple group; where \( N \) (respectively, \( N_c, LF \)) denotes the class of nilpotent (respectively, nilpotent of class at most \( c \), locally finite) groups and \( Frat(G) \) stands for the Frattini subgroup of \( G \).

Résumé

Si \( X \) est une classe de groupes, alors un groupe \( G \) est dit minimal non-\( X \)-groupe si tous ses sous-groupes propres sont dans la classe \( X \), alors que \( G \) lui-même n’est pas un \( X \)-groupe. Le principal résultat de cette note affirme que si \( c > 0 \) est un entier et si \( G \) est un groupe minimal non \((LF)N\) (respectivement, \((LF)N_c\))-groupe, alors \( G \) est un groupe parfait, de type fini, n’ayant pas de facteur fini non trivial et tel que \( G/Frat(G) \) est un groupe simple infini ; où \( N \) (respectivement, \( N_c, LF \)) désigne la classe des groupes nilpotents (respectivement, nilpotents de classe égale au plus \( c \), localement finis) et \( Frat(G) \) est le sous-groupe de Frattini de \( G \).

1. Introduction

If \( X \) is a class of groups, then a group \( G \) is said to be minimal non-\( X \) if all its proper subgroups are in the class \( X \), but \( G \) itself is not an \( X \)-group. Many results have been obtained by many authors on minimal non \( X \)-groups for various classes of groups \( X \), for example see [1], [2],

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In particular, in [13] it is proved that if $G$ is a finitely generated minimal non $\mathcal{F}\mathcal{N}$-group, then $G$ is a perfect group which has no non-trivial finite factor and such that $G/\text{Frat}(G)$ is an infinite simple group; where $\mathcal{N}$ (respectively, $\mathcal{F}$) denotes the class of nilpotent (respectively, finite) groups and $\text{Frat}(G)$ stands for the Frattini subgroup of $G$. The aim of the present note is to extend the above results to minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$ (respectively, $(\mathcal{L}\mathcal{F})\mathcal{N}_c$)-groups, and to prove that there are no minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$ (respectively, $(\mathcal{L}\mathcal{F})\mathcal{N}_c$)-groups which are not finitely generated; where $c > 0$ is an integer and $\mathcal{N}_c$ (respectively, $\mathcal{L}\mathcal{F}$) denotes the class of nilpotent groups of class at most $c$ (respectively, locally finite groups). More precisely we shall prove the following results.

**Theorem 1.1.** If $G$ is a minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$-group, then $G$ is a finitely generated perfect group which has no non-trivial finite factor and such that $G/\text{Frat}(G)$ is an infinite simple group.

Using Theorem 1.1, we shall prove the following result on minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}_c$-groups.

**Theorem 1.2.** Let $c > 0$ be an integer and let $G$ be a minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}_c$-group. Then $G$ is a finitely generated perfect group which has no non-trivial finite factor and such that $G/\text{Frat}(G)$ is an infinite simple group.

Note that if $\mathcal{X}_1$ and $\mathcal{X}_2$ are two classes of groups such that $\mathcal{X}_1 \subseteq \mathcal{X}_2$, then a minimal non $\mathcal{X}_1$-group is either a minimal non $\mathcal{X}_2$-group or an $\mathcal{X}_2$-group. From Xu’s results [13, Theorem 3.5], an infinitely generated minimal non $\mathcal{F}\mathcal{N}$-group is a locally finite-by-nilpotent group. So one might expect, as we shall prove in Proposition 2.1, that there are no infinitely generated minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$-group.

Note that minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$ (respectively, non $(\mathcal{L}\mathcal{F})\mathcal{N}_c$)-groups exist. Indeed, the group constructed by Ol’shanskii [7] is a simple torsion-free finitely generated group whose proper subgroups are cyclic.

**2. Minimal non $(\mathcal{L}\mathcal{F})\mathcal{N}$-groups**

A part of Theorem 1.1 is an immediate consequence of the following Proposition:
Proposition 2.1. Let \( G \) be a group whose proper subgroups are in the class \((\mathcal{LF})\mathcal{N}\). Then \( G \) belongs to \((\mathcal{LF})\mathcal{N}\) if it satisfies one of the following two conditions:

(i) \( G \) is finitely generated and has a proper subgroup of finite index,
(ii) \( G \) is not finitely generated.

Proof. (i) Suppose that \( G \) is finitely generated and let \( N \) be a proper subgroup of finite index in \( G \). By [10, Theorem 1.6.9] we may assume that \( N \) is normal in \( G \). So \( N \) is in \((\mathcal{LF})\mathcal{N}\) and it is also finitely generated. Hence \( \gamma_k+1(N) \) is locally finite for some integer \( k \geq 0 \). Since \( N \) is of finite index in \( G \), \( G/\gamma_k+1(N) \) is a finitely generated group in the class \( \mathcal{NF} \), so that it satisfies the maximal condition on subgroups. Therefore every proper subgroups of \( G/\gamma_k+1(N) \) is in \( \mathcal{FN} \). Now Lemma 4 of [3] states that a finitely generated locally graded group whose proper subgroups are finite-by-nilpotent is itself finite-by-nilpotent. Since groups in the class \( \mathcal{NF} \) are clearly locally graded, we deduce that \( G/\gamma_k+1(N) \) is in \( \mathcal{FN} \), so \( G \) is in \((\mathcal{LF})\mathcal{N}\).

(ii) Suppose that \( G \) is not finitely generated and let \( x_1, ..., x_n \) be \( n \) elements of finite order in \( G \). Since the subgroup \( \langle x_1, ..., x_n \rangle \) is proper in \( G \), it is in \((\mathcal{LF})\mathcal{N}\), hence it is finite. This means that the elements of finite order in \( G \) form a locally finite subgroup \( T \). If \( G/T \) is not finitely generated, then it is locally nilpotent and its proper subgroups are nilpotent as \( G/T \) is torsion-free. Now Theorem 2.1 of [11] states that a torsion-free locally nilpotent group with all proper subgroups nilpotent is itself nilpotent. Therefore \( G/T \) is nilpotent, so \( G \) is in \((\mathcal{LF})\mathcal{N}\). Now if \( G/T \) is finitely generated, then there exists a finitely generated subgroup \( H \) such that \( G = HT \). Since \( G \) is not finitely generated, \( H \) is proper in \( G \), so \( H \) is in \((\mathcal{LF})\mathcal{N}\). Since \( G/T \simeq H/H \cap T \), we deduce that \( G/T \) is in \((\mathcal{LF})\mathcal{N}\), hence \( G/T \) is nilpotent since it is torsion-free. Therefore \( G \) is in \((\mathcal{LF})\mathcal{N}\). \( \square \)

Since finitely generated locally graded groups have proper subgroups of finite index, the previous Proposition admits the following consequence:

Corollary 2.2. Let \( G \) be a locally graded group whose proper subgroups are in the class \((\mathcal{LF})\mathcal{N}\). Then \( G \) is in the class \((\mathcal{LF})\mathcal{N}\).

Corollary 2.3. Let \( G \) be a non perfect group whose proper subgroups are in the class \((\mathcal{LF})\mathcal{N}\). Then \( G \) is in the class \((\mathcal{LF})\mathcal{N}\).

Proof. If \( G \) is not finitely generated, then \( G \) is in \((\mathcal{LF})\mathcal{N} \) from (ii) of Proposition 2.1. Now suppose that \( G \) is finitely generated. Therefore \( G/G' \),
being a non trivial finitely generated locally graded group, has a non trivial finite image. So $G$ has a proper subgroup of finite index. Thus we deduce from (i) of Proposition 2.1 that $G$ is in $(\mathcal{LF})\mathcal{N}$.

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\textbf{\textit{Proof of Theorem 1.1.}} Let $G$ be a minimal non $(\mathcal{LF})\mathcal{N}$-group. It follows from Proposition 2.1 and Corollary 2.3 that $G$ is a finitely generated perfect group which has no non trivial finite factor. Now we prove that $G/F\text{rat}(G)$ is an infinite simple group. Since finitely generated groups have maximal subgroups, $G/F\text{rat}(G)$ is non trivial and therefore infinite. Let $N$ be a proper normal subgroup of $G$ properly containing $F\text{rat}(G)$. Then $N$ is in $(\mathcal{LF})\mathcal{N}$ and there is an $x \in N$ such that $x \notin F\text{rat}(G)$. Hence there is a maximal subgroup $M$ of $G$ such that $x \notin M$, so $N$ is not contained in $M$. Then $G = NM$ and we have $\gamma_{k+1}(M)$ is locally finite for some integer $k \geq 0$. Since $G$ is perfect, then

$$G = \gamma_{k+1}(G) = \gamma_{k+1}(NM).$$

We show by induction on $k$ that $\gamma_{k+1}(NM) \subseteq N\gamma_{k+1}(M)$. If $k = 0$, then the result follows immediately. Now let $k > 0$, suppose inductively that $\gamma_{k}(NM) \subseteq N\gamma_{k}(M)$ and let $g$ be an element of $\gamma_{k+1}(NM)$. Hence $g$ can be written as a finite product of elements of the form $[x_1y_1, ..., x_{k+1}y_{k+1}]$ with $x_i \in N$ and $y_i \in M$ for every $1 \leq i \leq k + 1$. It follows by the inductive hypothesis that the commutator $v = [x_1y_1, ..., x_ky_k]$ of weight $k$ is in $N\gamma_{k}(M)$. So we get $[v, x_{k+1}y_{k+1}] = [xy, x_{k+1}y_{k+1}]$ with $x \in N$ and $y \in \gamma_{k}(M)$. Therefore

$$[xy, x_{k+1}y_{k+1}] = [x, y_{k+1}]^y[y, y_{k+1}][x, x_{k+1}]^y[y, x_{k+1}]^{y_{k+1}}.$$ 

We have that $[y, y_{k+1}]$ is in $\gamma_{k+1}(M)$ and since $N$ is normal in $G$, we have that $[x, y_{k+1}]$ and $([x, x_{k+1}]^y[y, x_{k+1}])^{y_{k+1}}$ belong to $N$. Thus $[x_1y_1, ..., x_{k+1}y_{k+1}]$ is in $N\gamma_{k+1}(M)$ and consequently $g$ belongs to $N\gamma_{k+1}(M)$. Hence the inclusion $\gamma_{k+1}(NM) \subseteq N\gamma_{k+1}(M)$ hold. We deduce $G = N\gamma_{k+1}(M)$. Thus $G/N' = (N/N')(\gamma_{k+1}(M)N'/N')$. Since $\gamma_{k+1}(M)N'/N'$ is locally finite, $G/N'$ is in $A(\mathcal{LF})$, where $A$ denotes the class of abelian groups; so $G/N'$ is a locally graded group. We deduce from Corollary 2.2 that $G/N'$ is in $(\mathcal{LF})\mathcal{N}$. Now Theorem 1.2 of [4] states that if $N \lhd G$ such that $N$ and $G/N'$ are in $(\mathcal{LF})\mathcal{N}$, then $G$ is in $(\mathcal{LF})\mathcal{N}$. So $G$ is in $(\mathcal{LF})\mathcal{N}$, a contradiction. This means that $G/F\text{rat}(G)$ is a simple group. \qed
3. Minimal non \((\mathcal{L}\mathcal{F})\mathcal{N}_c\)-group

**Lemma 3.1.** Let \(G\) be a group and let \(F\) be the locally finite radical of \(G\). If \(G/F\) is nilpotent, then \(G/F\) is torsion-free.

*Proof.* Put \(\overline{G} = G/F\) and suppose that \(\overline{G}\) is nilpotent and let \(\overline{x}\) be an element of finite order in \(\overline{G}\). First of all, we show that the normal closure \(\overline{x}\overline{G}\) is locally finite; to this end, let \(\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \rangle\) be a finitely generated subgroup of \(\overline{x}\overline{G}\). Since every element \(\overline{h}_i\), where \(1 \leq i \leq n\), can be written as a finite product of elements \(\overline{x}\overline{g}_j\), there is an integer \(s > 0\) such that \(\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \rangle\) is a subgroup of \(\langle \overline{x}\overline{g}_1, \overline{x}\overline{g}_2, ..., \overline{x}\overline{g}_s \rangle\) for some \(\overline{g}_j \in \overline{G}\), with \(1 \leq j \leq s\). Moreover \(\langle \overline{x}\overline{g}_1, \overline{x}\overline{g}_2, ..., \overline{x}\overline{g}_s \rangle\) being nilpotent and generated by finitely many elements of finite order, is finite. So \(\langle \overline{h}_1, \overline{h}_2, ..., \overline{h}_n \rangle\) is finite and consequently \(\overline{x}\overline{G}\) is locally finite. Now since \(\overline{G}\) has no non trivial locally finite normal subgroup, then \(\overline{x}\overline{G}\) is trivial. Thus \(\overline{x} = 1\), hence \(\overline{G}\) is torsion-free. \(\Box\)

**Proposition 3.2.** Let \(c > 0\) be an integer and let \(G\) be a group whose proper subgroups are in the class \((\mathcal{L}\mathcal{F})\mathcal{N}_c\). Then \(G\) is in the class \((\mathcal{L}\mathcal{F})\mathcal{N}_c\) if it satisfies one of the following two conditions:

(i) \(G\) is finitely generated and has a proper subgroup of finite index,

(ii) \(G\) is not finitely generated.

*Proof.* Suppose that \(G\) satisfies one of the conditions (i) or (ii). Since \((\mathcal{L}\mathcal{F})\mathcal{N}_c\) is included in \((\mathcal{L}\mathcal{F})\mathcal{N}\), it follows from Proposition 2.1 that \(G\) is in \((\mathcal{L}\mathcal{F})\mathcal{N}\). So there is a normal subgroup \(N\) such that \(N\) is locally finite and \(G/N\) is nilpotent. Thus \(N\) is contained in \(F\), the locally finite radical of \(G\), and therefore \(G/F\) is nilpotent. Clearly, we may assume that \(G\) is not locally finite, so \(G/F\) is non trivial and by Lemma 3.1, it is torsion-free. If \(G/F\) is finitely generated, then as \(G/F\) is clearly locally graded, \(G/F\) has a proper normal subgroup of finite index. So \(G/F\) belongs to \(((\mathcal{L}\mathcal{F})\mathcal{N}_c)\mathcal{F}\), hence \(G/F\) is in \(\mathcal{N}_c\mathcal{F}\) as it is torsion-free. Therefore \(G/F\), being a nilpotent torsion-free group in the class \(\mathcal{N}_c\mathcal{F}\), is in \(\mathcal{N}_c\) by Lemma 6.33 of [9]. Consequently \(G\) is in \((\mathcal{L}\mathcal{F})\mathcal{N}_c\). Now suppose that \(G/F\) is not finitely generated and let \(a_1, ..., a_{c+1}\) be elements of \(G/F\). Since \(\langle a_1, ..., a_{c+1} \rangle\) is proper in \(G/F\), it is in \((\mathcal{L}\mathcal{F})\mathcal{N}_c\) and consequently it is in \(\mathcal{N}_c\) because \(G/F\) is torsion-free. So \([a_1, ..., a_{c+1}] = 1\), hence \(G/F\) is in \(\mathcal{N}_c\). Therefore \(G\) is in \((\mathcal{L}\mathcal{F})\mathcal{N}_c\). \(\Box\)
Since finitely generated locally graded groups have proper subgroups of finite index, Proposition 3.2 admits the following consequence:

**Corollary 3.3.** Let \( c > 0 \) be an integer and let \( G \) be a locally graded group whose proper subgroups are in the class \((\mathcal{LF})N_c\). Then \( G \) is in the class \((\mathcal{LF})N_c\).

**Proof of Theorem 1.2.** Let \( G \) be a minimal non \((\mathcal{LF})N_c\)-group. It follows that every proper subgroup of \( G \) is in \((\mathcal{LF})N\). Now suppose that \( G \) is in \((\mathcal{LF})N\), so there exists a normal subgroup \( N \) of \( G \) such that \( N \) is locally finite and \( G/N \) is nilpotent. By Corollary 3.3, \( G/N \) is in \((\mathcal{LF})N_c\), consequently we deduce that \( G \) is in \((\mathcal{LF})N_c\) because \( N \) is locally finite; a contradiction. Hence \( G \) is a minimal non \((\mathcal{LF})N\)-group, and by Theorem 1.1, \( G \) is a finitely generated perfect group which has no non-trivial finite factor and such that \( G/Frat(G) \) is an infinite simple group. \( \square \)

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**References**


Locally finite-by-nilpotent proper subgroups


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