Satoshi Ishiwata

Discrete version of Dungey’s proof for the gradient heat kernel estimate on coverings


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Abstract

We obtain another proof of a Gaussian upper estimate for a gradient of the heat kernel on cofinite covering graphs whose covering transformation group has a polynomial volume growth. It is proved by using the temporal regularity of the discrete heat kernel obtained by Blunck [2] and Christ [3] along with the arguments of Dungey [7] on covering manifolds.

1. Introduction

Let \( X = (V, E) \) be an oriented locally finite connected graph. We consider the reversible random walk on \( X \) defined by functions \( p : E \to \mathbb{R}_{>0} \) and \( m : V \to \mathbb{R}_{>0} \) satisfying

\[
p(e)m(o(e)) = p(\overline{e})m(t(e))
\]

and

\[
\sum_{e \in E_x} p(e) = 1 \quad x \in V,
\]

where \( o(e) \) is the origin of \( e \), \( t(e) \) is the end of \( e \), \( \overline{e} \) is the inverse edge of \( e \) and \( E_x = \{ e \in E \mid o(e) = x \} \). Here, \( p(e) \) is the probability that a particle at \( o(e) \) moves to \( t(e) \) in a unit time. The function \( m \) on \( V \) is a measure on \( V \). Then the transition probability \( p_n(x, y) \), \( x, y \in V \) is given by

\[
p_n(x, y) = \sum_{(e_1, e_2, \ldots, e_n) \in C_{x,n}, t(e_n) = y} p(e_1)p(e_2)\cdots p(e_n),
\]

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where $C_{x,n}$ is the set of paths from $x$ with length $n$. The transition operator $P$ associated with the random walk generates a discrete semigroup \( \{P^n\}_{n \in \mathbb{N}} \) acting on functions on $V$ defined by

\[
P^n f(x) = \sum_{y \in V} p_n(x, y) f(y).
\]

Then the kernel of $P^n$ on the weighted graph $(X, m)$ is given by

\[
k_n(x, y) = p_n(x, y) m(y)^{-1}.
\]

The purpose of this paper is to obtain a Gaussian upper estimate for the gradient of $k_n$ on $X$ assuming that the latter admits a cofinite group action with polynomial volume growth. By the results of Gromov [9], without loss of generality, we can always assume that the covering transformation group $\Gamma$ of $X$ is a nilpotent group of order $D$. Moreover, we also assume that the random walk on such $X$ is $\Gamma$-invariant, namely, $p : E \to \mathbb{R}_{>0}$ and $m : V \to \mathbb{R}_{>0}$ are $\Gamma$-invariant.

Under our assumptions, the Gaussian upper estimate for $k_n$

\[
k_n(x, y) \leq C n^{-D/2} e^{-C'n d(x,y)^2/n} \quad \forall x, y \in V \tag{1.1}
\]

is known (see [12] and also [10]). Here $d(x, y)$ is the graph distance, the length of the shortest path from $x$ to $y$ and $C, C'$ are some positive constants.

Moreover, the following theorem for the Gaussian upper estimate for the gradient of $k_n$ has been proved by [12] along the method of [10]. Let $\nabla_1$ be the gradient with respect to the first variable given by

\[
\nabla_1 k_n(x, y) = \left( \sum_{d(x, \omega) \leq 1} |k_n(\omega, y) - k_n(x, y)|^2 p_1(x, \omega) \right)^{1/2}
\]

\[
= \left( \sum_{e \in E_x} |dk_n(e, y)|^2 p(e) \right)^{1/2},
\]

where $d$ is the exterior derivative defined by

\[
df(e) = f(t(e)) - f(o(e)), \quad e \in E
\]
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for a function $f$ on $V$. Similarly we denote by $\nabla_2 k_n(x, y)$ the gradient with respect to the second variable, namely

$$
\nabla_2 k_n(x, y) = \left( \sum_{d(y, \omega) \leq 1} |k_n(x, \omega) - k_n(x, y)|^2 p_1(y, \omega) \right)^{1/2} = \left( \sum_{e \in E_y} |dk_n(x, e)|^2 p(e) \right)^{1/2}.
$$

Then we have

**Theorem 1.1.** Let $X = (V, E)$ be a non-bipartite covering graph whose covering transformation group has polynomial volume growth of order $D$. Then there exist $C, C' > 0$ such that

$$
\nabla_1 k_n(x, y) \leq Cn^{-(D+1)/2}e^{-C'd(x,y)^2/n} \quad (1.2)
$$

for $n \in \mathbb{N}^*$ and $x, y \in V$.

It should be noted that the estimate (1.2) is closely related to the boundedness of the Riesz transform (see [1], [13] and [11]). Let $\Delta$ be a discrete Laplacian on $X$ given by $\Delta = I - P$. Then we have

**Theorem 1.2.** Let $X$ be as above. For $1 < p < \infty$, there exists $C_p > 0$ such that

$$
\|\nabla\Delta^{-1/2} f\|_{L^p} \leq C_p\|f\|_{L^p}
$$

for all finitely supported functions $f$ on $V$. Here $\| \cdot \|_{L^p}$ is the $L^p$ norm with respect to the measure $m$ on $V$.

On the other hand, Dungey proved (1.2) for the heat kernel on covering manifolds with polynomial volume growth in [6]. Recently, in [7], he gave a new proof of (1.2) using the well-known temporal regularity of the heat kernel (see for instance [5]). In this paper, we give a shorter proof of Theorem 1.1 along with the arguments of the recent result by Dungey [7]. Let

$$
\partial_1 k_n(x, y) = k_{n+1}(x, y) - k_n(x, y).
$$

We use the following discrete version of the temporal regularity proved by Blunck [2], Christ [3]:

**Theorem 1.3.** Let $X$ be an oriented non-bipartite graph satisfying

$$
C^{-1}r^D \leq m(B(x, r)) \leq Cr^D \quad x \in V, r \geq 1
$$

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and (1.1), where $B(x,r)$ is the ball centered at $x \in V$ with radius $r$. Then there exist positive constants $C$ and $C'$ such that

$$\partial_1 k_n(x, y) \leq C n^{-(D+2)/2} e^{-C' d(x, y)^2 / n}$$  (1.3)

for $n \in \mathbb{N}^*$ and $x, y \in V$.

We remark that Dungey gave a short proof of (1.3) recently in [8].

2. Proof of Theorem 1.1

Let $\nu = m + n$ be a positive integer, where we choose $m = n$ or $m = n + 1$ depending on whether $\nu$ is even or odd. By the Cauchy-Schwarz inequality, it is easy to see that

$$e^{r d^2(u, z) / \nu} \nabla_1 k_\nu(u, z) \leq \| e^{2 r d^2(u, \cdot) / n} \nabla_1 k_n(u, \cdot) \|_{L^2} \| e^{2 r d^2(\cdot, z) / m} k_m(\cdot, z) \|_{L^2}$$

for $r > 0$.

By the Gaussian upper bound (1.1) for $k_n$, for small $r > 0$, there exists $C_r > 0$ such that

$$\| e^{2 r d^2(\cdot, z) / m} k_m(\cdot, z) \|_{L^2} \leq C_r m^{-D/4}.$$  (2.1)

Let $F \subset V$ be a fundamental domain for the action of the transformation group $\Gamma$ on $V$. Namely, $F$ is a subset in $V$ such that for all $x \in V$, there exists a unique pair $\gamma_x \in \Gamma$ and $x_0 \in F$ satisfying $x = \gamma_x x_0$. Then we denote $F_x = \gamma_x F$. The following lemma gives a comparison of the weighted integral for $\nabla_1 k_n$ and $\nabla_2 k_n$. Similar arguments can be found in [12].

**Lemma 2.1.** There exists a positive constant $C > 0$ such that

$$\| e^{2 r d^2(u, \cdot) / n} \nabla_1 k_n(u, \cdot) \|_{L^2}^2 \leq C \sum_{y_0 \in F_u} \| e^{2 r d^2(y_0, \cdot) / n} \nabla_2 k_n(y_0, \cdot) \|_{L^2}^2 m(y_0)$$

for all $u \in X$.

**Proof.** Since $m$ is $\Gamma$-invariant, there exists $C > 0$ such that $C^{-1} < \min \{ m(x) \mid x \in V \}$. Then we have

$$\| e^{2 r d^2(u, \cdot) / n} \nabla_1 k_n(u, \cdot) \|_{L^2}^2 \leq C \sum_{v_0 \in F_u} \sum_{y \in V} \left| e^{2 r d^2(v_0, y) / n} \nabla_1 k_n(v_0, y) \right|^2 m(y) m(v_0)$$

$$= C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2 r d^2(v_0, \gamma y_0) / n} \nabla_1 k_n(v_0, \gamma y_0) \right|^2 m(\gamma y_0) m(v_0).$$
From the $\Gamma$-invariance of the distance function $d$ and $k_n$, the latter is
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd} (\gamma^{-1} v_0, y_0) / n \nabla_1 k_n (\gamma^{-1} v_0, y_0) \right|^2 m(y_0) m(\gamma^{-1} v_0).
\]
By replacing $\gamma^{-1}$ with $\gamma$ in the sum of $\Gamma$, we get
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd} (\gamma v_0, y_0) / n \nabla_1 k_n (\gamma v_0, y_0) \right|^2 m(y_0) m(\gamma v_0).
\]
Since $\nabla_1 k_n (\gamma v_0, y_0) = \nabla_2 k_n (y_0, \gamma v_0)$, this is also
\[
C \sum_{v_0, y_0 \in F_u} \sum_{\gamma \in \Gamma} \left| e^{2rd} (y_0, \gamma v_0) / n \nabla_2 k_n (y_0, \gamma v_0) \right|^2 m(\gamma v_0) m(y_0)
= C \sum_{y_0 \in F_u} \sum_{v \in V} \left| e^{2rd} (y_0, v) / n \nabla_2 k_n (y_0, v) \right|^2 m(v) m(y_0)
= C \sum_{y_0 \in F_u} \left\| e^{2rd} (y_0, \cdot) / n \nabla_2 k_n (y_0, \cdot) \right\|^2_{L^2} m(y_0).
\]
\[\square\]

Remark 2.2. In this proof, we use only the $\Gamma$-invariance of $d$, $\nabla_1 k_n$ and $\nabla_1 k_n (x, y) = \nabla_2 k_n (y, x)$. Therefore, there are other definitions of $\nabla$ so that the previous lemma holds. For example, we can obtain the same results for
\[
\nabla_1^p k_n (x, y) := \left( \sum_{d(\omega, x) \leq 1} |k_n (\omega, y) - k_n (x, y)|^p p_1 (x, \omega) \right)^{1/p}, \quad 1 < p < \infty,
\]
\[
\nabla_1^\infty k_n (x, y) := \sup_{d(\omega, x) \leq 1} |k_n (\omega, y) - k_n (x, y)|,
\]
which are comparable with each other.

By the same arguments as the continuous case ([4]), we obtain the following by “discrete integration by parts”. Similar arguments can be found in [10] and [12].

Lemma 2.3. For $e \in E$, let $m(e)$ be a weight of $e$ defined by $m(e) = p(e) m(o(e))$. Then we have
\[
\left\| e^{2rd} (y_0, \cdot) / n \nabla_2 k_n (y_0, \cdot) \right\|^2_{L^2} = - \sum_{e \in E} e^{4rd^2 (y_0, e) / n} k_n (y_0, o(e)) dk_n (y_0, e) m(e)
- 2 \sum_{v \in V} e^{4rd^2 (y_0, v) / n} k_n (y_0, v) \partial_1 k_n (y_0, v) m(v).
\]
Proof.

\[
\sum_{v \in V} \left| e^{2rd(y_0,v)/n} \nabla_2 k_n(y_0, v) \right|^2 m(v)
= \sum_{v \in X} e^{4rd^2(y_0,v)/n} \sum_{e \in E_v} |dk_n(y_0, e)|^2 p(e)m(v)
\]
\[
= \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} |k_n(y_0, t(e)) - k_n(y_0, o(e))|^2 p(e)m(o(e))
\]
\[
= \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} \left( k_n(y_0, t(e))^2 - 2k_n(y_0, t(e))k_n(y_0, o(e)) + k_n(y_0, o(e))^2 \right) m(e)
\]
\[
= \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} k_n(y_0, t(e)) (k_n(y_0, t(e)) - k_n(y_0, o(e))) m(e)
+ \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e).
\]

Since \( m(e) = m(\bar{e}) \), by replacing \( e \) with \( \bar{e} \) in the sum of \( E \) in the first term,

\[
= \sum_{e \in E} e^{4rd^2(y_0,t(e))/n} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
\]
\[
+ \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
\]
\[
= \sum_{e \in E} \left( e^{4rd^2(y_0,t(e))/n} - e^{4rd^2(y_0,o(e))/n} \right) k_n(y_0, o(e))
\cdot (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
\]
\[
+ 2 \sum_{e \in E} e^{4rd^2(y_0,o(e))/n} k_n(y_0, o(e)) (k_n(y_0, o(e)) - k_n(y_0, t(e))) m(e)
\]
\[
= - \sum_{e \in E} de^{4rd^2(y_0,e)/n} k_n(y_0, o(e)) dk_n(y_0, e) m(e)
\]
\[
- 2 \sum_{v \in V} e^{4rd^2(y_0,v)/n} k_n(y_0, v) \sum_{e \in E_v} (k_n(y_0, t(e)) - k_n(y_0, o(e))) p(e)m(v).
\]
Since
\[
\sum_{e \in E_v} (k_n(y_0, t(e)) - k_n(y_0, o(e)))p(e)
= \sum_{e \in E_v} k_n(y_0, t(e))p(e) - k_n(y_0, v)
= k_{n+1}(y_0, v) - k_n(y_0, v)
= \partial_1 k_n(y_0, v),
\]
the lemma is proved. \(\square\)

Finally, we apply the temporal regularity for \(k_n\) along with the argument of Lemma 2.3 in [4]. Let

\[
I(n, y_0) = \sum_{v \in V} \left| e^{2r^2d^2(y_0, v)/n} \nabla_2 k_n(y_0, v) \right|^2 m(v),
I_1(n, y_0) = -2 \sum_{v \in V} e^{Ar^2d^2(y_0, v)/n} k_n(y_0, v) \partial_1 k_n(y_0, v)m(v),
I_2(n, y_0) = -\sum_{e \in E} d e^{Ar^2d^2(y_0, e)/n} k_n(y_0, o(e)) d k_n(y_0, e) m(e).
\]

Lemma 2.3 says that \(I(n, y_0) = I_1(n, y_0) + I_2(n, y_0)\). Using (1.1) and (1.3), for sufficiently small \(r > 0\), there exists \(C_r > 0\) such that
\[
|I_1(n, y_0)| \leq C_r n^{-D/2-1}.
\]

By the Cauchy-Schwarz inequality, we have that \(|I_2(n, y_0)|\) is less than

\[
\sum_{v \in V} \left( \sum_{e \in E_v} |d e^{Ar^2d^2(y_0, e)/n} p(e)|^{2} \right)^{1/2} \left( \sum_{e \in E_v} |d k_n(y_0, e)|^2 p(e) \right)^{1/2} k_n(y_0, v)m(v)
= \sum_{v \in V} \left( \sum_{e \in E_v} |d e^{Ar^2d^2(y_0, e)/n} p(e)|^{2} \right)^{1/2} \nabla_2 k_n(y_0, v) k_n(y_0, v) m(v).
\]

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Using the mean value theorem for $f(x) = e^{4rx^2/n}$, there exists $d(y_0, o(e)) < s < d(y_0, t(e))$ such that

$$
e^{4rd^2(y_0, t(e))/n} - e^{4rd^2(y_0, o(e))/n} = \frac{8rs}{n}e^{4rs^2/n}(d(y_0, t(e)) - d(y_0, o(e))) \\
\leq \frac{8r(d(y_0, o(e)) + 1)}{n}e^{4r(d(y_0, o(e)) + 1)^2/n} \\
\leq \frac{C_r}{\sqrt{n}}e^{16rd^2(y_0, o(e))/n}
$$

for some $C_r > 0$. Then we have that $|I_2(n, y_0)|$ is less than

$$
\frac{C_r}{\sqrt{n}} \sum_{v \in V} e^{16rd^2(y_0, v)/n} \nabla_2 k_n(y_0, v) k_n(y_0, v) m(v) \\
\leq \frac{C_r}{\sqrt{n}} \left( \sum_{v \in V} e^{28rd^2(y_0, v)/n} |k_n(y_0, v)|^2 m(v) \right)^{1/2} \\
\cdot \left( \sum_{v \in V} e^{2rd^2(y_0, v)/n} \nabla_2 k_n(y_0, v) |^2 m(v) \right)^{1/2}.
$$

For sufficiently small $r > 0$, we obtain

$$|I_2(n, y_0)| \leq C'_r n^{-D/4-1/2} I(n, y_0)^{1/2}.$$

Then we conclude that

$$I(n, y_0) \leq C_r n^{-D/2-1} + C'_r n^{-D/4-1/2} I(n, y_0)^{1/2},$$

namely,

$$I(n, y_0) \leq C n^{-D/4-1/2}$$

for some $C > 0$. Together with (2.1), this completes the proof of Theorem 1.1.

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References


S. ISHIWATA

Satoshi Ishiwata
Institute of Mathematics
University of Tsukuba
1-1-1 Tennoudai, 305-8571
Ibaraki
JAPAN
ishiwata@math.tsukuba.ac.jp