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P-adic Spaces of Continuous Functions I


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Abstract

Properties of the so called $\theta_0$-complete topological spaces are investigated. Also, necessary and sufficient conditions are given so that the space $C(X, E)$ of all continuous functions, from a zero-dimensional topological space $X$ to a non-Archimedean locally convex space $E$, equipped with the topology of uniform convergence on the compact subsets of $X$ to be polarly barrelled or polarly quasi-barrelled.

Introduction

Let $\mathbb{K}$ be a complete non-Archimedean valued field and let $C(X, E)$ be the space of all continuous functions from a zero-dimensional Hausdorff topological space $X$ to a non-Archimedean Hausdorff locally convex space $E$. We will denote by $C_b(X, E)$ (resp. by $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of $E$. The dual space of $C_{rc}(X, E)$, under the topology $t_u$ of uniform convergence, is a space $M(X, E')$ of finitely-additive $E'$-valued measures on the algebra $K(X)$ of all clopen, i.e. both closed and open, subsets of $X$. Some subspaces of $M(X, E')$ turn out to be the duals of $C(X, E)$ or of $C_b(X, E)$ under certain locally convex topologies.

In section 2 of this paper, we give some results about the space $M(X, E')$. The notion of a $\theta_0$-complete topological space was given in [2]. In section 3 we study some of the properties of $\theta_0$-complete spaces. Among other results, we prove that a Hausdorff zero-dimensional space is $\theta_0$-complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces. In section 4, we give necessary and sufficient conditions for the space $C(X, E)$, equipped with the topology of uniform convergence on the compact subsets of $X$, to be polarly barrelled or polarly quasi-barrelled.

Keywords: Non-Archimedean fields, zero-dimensional spaces, locally convex spaces.
1. Preliminaries

Throughout this paper, $\mathbb{K}$ will be a complete non-Archimedean valued field, whose valuation is non-trivial. By a seminorm, on a vector space over $\mathbb{K}$, we will mean a non-Archimedean seminorm. Similarly, by a locally convex space we will mean a non-Archimedean locally convex space over $\mathbb{K}$ (see [9]). Unless it is stated explicitly otherwise, $X$ will be a Hausdorff zero-dimensional topological space, $E$ a Hausdorff locally convex space and $cs(E)$ the set of all continuous seminorms on $E$. The space of all $\mathbb{K}$-valued linear maps on $E$ is denoted by $E^\star$, while $E'$ denotes the topological dual of $E$. A seminorm $p$, on a vector space $G$ over $\mathbb{K}$, is called polar if $p = \sup\{|f| : f \in G^*, |f| \leq p\}$. A locally convex space $G$ is called polar if its topology is generated by a family of polar seminorms. A subset $A$ of $G$ is called absolutely convex if $\lambda x + \mu y \in A$ whenever $x, y \in A$ and $\lambda, \mu \in \mathbb{K}$, with $|\lambda|, |\mu| \leq 1$. We will denote by $\beta_oX$ the Banaschewski compactification of $X$ (see [3]) and by $v_oX$ the $\mathbb{N}$-repletion of $X$, where $\mathbb{N}$ is the set of natural numbers. We will let $C(X, E)$ denote the space of all continuous $E$-valued functions on $X$ and $C_b(X, E)$ (resp. $C_{rc}(X, E)$) the space of all $f \in C(X, E)$ for which $f(X)$ is a bounded (resp. relatively compact) subset of $E$. In case $E = \mathbb{K}$, we will simply write $C(X, \mathbb{K})$, $C_b(X, \mathbb{K})$ and $C_{rc}(X, \mathbb{K})$ respectively. For $A \subset X$, we denote by $\chi_A$ the $\mathbb{K}$-valued characteristic function of $A$. Also, for $X \subset Y \subset \beta_oX$, we denote by $\bar{B}^Y$ the closure of $B$ in $Y$. If $f \in E^X, p$ a seminorm on $E$ and $A \subset X$, we define

$$\|f\|_p = \sup_{x \in X} p(f(x)),$$

$$\|f\|_{A,p} = \sup_{x \in A} p(f(x)).$$

The strict topology $\beta_o$ on $C_b(X, E)$ (see [4]) is the locally convex topology generated by the seminorms $f \mapsto \|hf\|_p$, where $p \in cs(E)$ and $h$ is in the space $B_o(X)$ of all bounded $\mathbb{K}$-valued functions on $X$ which vanish at infinity, i.e. for every $\epsilon > 0$ there exists a compact subset $Y$ of $X$ such that $|h(x)| < \epsilon$ if $x \not\in Y$.

Let $\Omega = \Omega(X)$ be the family of all compact subsets of $\beta_oX \setminus X$. For $H \in \Omega$, let $C_H$ be the space of all $h \in C_{rc}(X)$ for which the continuous extension $h^{\beta_o}$ to all of $\beta_oX$ vanishes on $H$. For $p \in cs(E)$, let $\beta_{H,p}$ be the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p$, $h \in C_H$. For $H \in \Omega, \beta_H$ is the locally convex topology on $C_b(X, E)$ generated by the seminorms $f \mapsto \|hf\|_p$, $h \in C_H, p \in cs(E)$.
The inductive limit of the topologies $\beta_H, H \in \Omega$, is the topology $\beta$. Replacing $\Omega$ by the family $\Omega_1$ of all $\mathbb{K}$-zero subsets of $\beta_oX$, which are disjoint from $X$, we get the topology $\beta_1$. Recall that a $\mathbb{K}$-zero subset of $\beta_oX$ is a set of the form $\{x \in \beta_oX : g(x) = 0\}$, for some $g \in C(\beta_oX)$. We get the topologies $\beta_u$ and $\beta'_u$ replacing $\Omega$ by the family $\Omega_u$ of all $Q \in \Omega$ with the following property: There exists a clopen partition $(A_i)_{i \in I}$ of $X$ such that $Q$ is disjoint from each $\overline{A_i}^{\beta_oX}$. Now $\beta_u$ is the inductive limit of the topologies $\beta_Q$, $Q \in \Omega_u$. The inductive limit of the topologies $\beta_H, p$, as $H$ ranges over $\Omega_u$, is denoted by $\beta_{u,p}$, while $\beta'_u$ is the projective limit of the topologies $\beta_{u,p}$, $p \in cs(E)$. For the definition of the topology $\beta_e$ on $C_b(X)$ we refer to [7].

Let now $K(X)$ be the algebra of all clopen subsets of $X$. We denote by $M(X,E')$ the space of all finitely-additive $E'$-additive measures $m$ on $K(X)$ for which the set $m(K(X))$ is an equicontinuous subset of $E'$. For each such $m$, there exists a $p \in cs(E)$ such that $\|m\|_p = m_p(X) < \infty$, where, for $A \in K(X)$,

$$m_p(A) = \sup\{|m(B)s/p(s) : p(s) \neq 0, A \supset B \in K(X)|$$

The space of all $m \in M(X,E')$ for which $m_p(X) < \infty$ is denoted by $M_p(X,E')$. If $m \in M_p(X,E')$, then for $x \in X$ we define

$$N_{m,p}(x) = \inf\{m_p(V) : x \in V \in K(X)\}.$$ 

In case $E = \mathbb{K}$, we denote by $M(X)$ the space of all finitely-additive bounded $\mathbb{K}$-valued measures on $K(X)$. An element $m$ of $M(X)$ is called $\tau$-additive if $m(V_\delta) \to 0$ for each decreasing net $(V_\delta)$ of clopen subsets of $X$ with $\bigcap V_\delta = \emptyset$. In this case we write $V_\delta \downarrow \emptyset$. We denote by $M_\tau(X)$ the space of all $\tau$-additive members of $M(X)$. Analogously, we denote by $M_\sigma(X)$ the space of all $\sigma$-additive $m$, i.e. those $m$ with $m(V_n) \to 0$ when $V_n \downarrow \emptyset$. For an $m \in M(X,E')$ and $s \in E$, we denote by $ms$ the element of $M(X)$ defined by $(ms)(V) = m(V)s$.

Next we recall the definition of the integral of an $f \in E^X$ with respect to an $m \in M(X,E')$. For a non-empty clopen subset $A$ of $X$, let $D_A$ be the family of all $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\}$, where $\{A_1, \ldots, A_n\}$ is a clopen partition of $A$ and $x_k \in A_k$. We make $D_A$ into a directed set by defining $\alpha_1 \geq \alpha_2$ iff the partition of $A$ in $\alpha_1$ is a refinement of the one in $\alpha_2$. For an $\alpha = \{A_1, A_2, \ldots, A_n; x_1, x_2, \ldots, x_n\} \in D_A$ and $m \in M(X,E')$,
we define

\[ \omega_\alpha(f, m) = \sum_{k=1}^{n} m(A_k) f(x_k). \]

If the limit \( \lim \omega_\alpha(f, m) \) exists in \( K \), we will say that \( f \) is \( m \)-integrable over \( A \) and denote this limit by \( \int_A f \, dm \). We define the integral over the empty set to be 0. For \( A = X \), we write simply \( \int_X f \, dm \). It is easy to see that if \( f \) is \( m \)-integrable over \( X \), then it is \( m \)-integrable over every clopen subset \( A \) of \( X \) and \( \int_A f \, dm = \int \chi_A f \, dm \). If \( \tau_u \) is the topology of uniform convergence, then every \( m \in M(X, E') \) defines a \( \tau_u \)-continuous linear functional \( \phi_m \) on \( C_{rc}(X, E) \), \( \phi_m(f) = \int f \, dm \). Also every \( \phi \in (C_{rc}(X, E), \tau_u)' \) is given in this way by some \( m \in M(X, E') \).

2. Some results on \( M(X, E') \)

**Theorem 2.1.** Let \( m \in M(X, E') \) be such that \( m_s \in M_{\tau}(X) \), for all \( s \in E \), and let \( p \in cs(E) \) with \( \|m\|_p < \infty \). Then :

1. \( m_p(V) = \sup_{x \in V} N_{m,p}(x) \) for every \( V \in K(X) \).
2. The set
   \[ \text{supp}(m) = \bigcap \{ V \in K(X) : m_p(V^c) = 0 \} \]
   is the smallest of all closed support sets for \( m \).
\( (3) \) \( \text{supp}(m) = \{ x : N_{m,p}(x) \neq 0 \} \).

\( (4) \) If \( V \) is a clopen set contained in the union of a family \((V_i)_{i \in I}\) of clopen sets, then
\[
m_p(V) \leq \sup \{ m_p(V_i) : i \in I \}.
\]

**Proof:** (1). If \( x \in V \), then \( N_{m,p}(x) \leq m_p(V) \) and so
\[
m_p(V) \geq \alpha = \sup_{x \in V} N_{m,p}(x).
\]

On the other hand, let \( m_p(V) > d \). There exists a clopen set \( W \), contained in \( V \), and \( s \in E \) with \( |m(W)s|/p(s) > d \). Let \( \mu = ms \in M_\tau(X) \). Then
\[
|\mu|(W) = \sup_{x \in W} N_\mu(x).
\]

Let \( x \in W \) be such that \( N_\mu(x) > d \cdot p(s) \). Now \( N_{m,p}(x) \geq d \). In fact, assume the contrary and let \( Z \) be a clopen neighborhood of \( x \) contained in \( W \) and such that \( m_p(Z) < d \). Now
\[
N_\mu(x) \leq |\mu|(Z) = \sup \{ |m(Y)s| : Z \supset Y \in K(X) \} \leq p(s) \cdot m_p(Z) \leq d \cdot p(s).
\]
This contradiction proves (1).

(2).
\[
X \setminus \text{supp}(m) = \bigcup \{ W \in K(X) : m_p(W) = 0 \}.
\]

Let \( V \in K(X) \) be disjoint from \( \text{supp}(m) \). For each \( x \in V \), there exists \( W \in K(X) \), with \( x \in W \) and \( m_p(W) = 0 \) and so \( N_{m,p}(x) = 0 \). It follows that
\[
m_p(V) = \sup_{x \in V} N_{m,p}(x) = 0,
\]
which proves that \( \text{supp}(m) \) is a support set for \( m \). On the other hand, let \( Y \) be a closed support set for \( m \). There exists a decreasing net \((V_\delta)\) of clopen sets with \( Y = \bigcap V_\delta \). Let \( W \in K(X) \) be disjoint from \( Y \). For each clopen set \( V \) contained in \( W \) and each \( s \in E \), we have \( V \cap V_\delta \downarrow \emptyset \) and so \( \lim_\delta (ms)(V \cap V_\delta) = 0 \). Since \( V^c_\delta \) is disjoint from \( Y \), we have \( m(V^c_\delta) = 0 \) and so \( m(V) = m(V_\delta \cap V) \), which implies that \( m(V)s = 0 \), for all \( s \in E \), i.e. \( m(V) = 0 \), and hence \( m_p(W) = 0 \). Therefore \( \text{supp}(m) \subset W^c \). Taking \( V^c_\delta \) in place of \( W \), we get that \( \text{supp}(m) \subset \bigcap V_\delta = Y \), which proves (2).

(3) Let \( G = x : N_{m,p}(x) \neq 0 \). If \( V \in K(X) \) is disjoint from \( G \), then
\[
m_p(V) = \sup_{x \in V} N_{m,p}(x) = 0,
\]

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and so \( \text{supp}(m) \subset V^c \), which implies that \( \text{supp}(m) \subset G \). On the other hand, let \( x \notin \text{supp}(m) \). There exists a clopen neighborhood \( W \) of \( x \) disjoint from \( \text{supp}(m) \). Since \( \text{supp}(m) \) is a support set for \( m \), we have that \( m_p(W) = 0 \) and thus \( N_{m,p} = 0 \) on \( W \), which proves that \( x \notin G \). Thus \( G \) is contained in \( \text{supp}(m) \) and (3) follows.

(4). Let \( m_p(V) > \alpha > 0 \). There exists a clopen set \( A \) contained in \( V \) and \( s \in E \) such that \( |m(A)s|/p(s) > \alpha \). If \( \mu = ms \in M_\tau(X) \), then \( |\mu|(V) \geq |m(A)s| > \alpha \cdot p(s) \). In view of [9, p. 250] there exists an \( i \) such that \( m_p(V_i) \geq |\mu|(V_i)/p(s) > \alpha \), which clearly completes the proof.

**Theorem 2.2.** Let \( m \in M(X, E') \) be such that \( ms \in M_\tau(X) \) for all \( s \in E \) (this in particular holds if \( m \in M_\tau(X, E') \)). Let \( p \in \text{cs}(E) \) be such that \( m_p(X) < \infty \). If a clopen set \( V \) is contained in the union of a sequence \( (V_n) \) of clopen sets, then \( m_p(V) \leq \sup_n m_p(V_n) \).

**Proof:** We show first that, for \( \mu \in M_\tau(X) \), then there exists an \( n \) with \( |\mu|(V) \leq |\mu|(V_n) \). In fact, this is clearly true if \( |\mu|(V) = 0 \). Assume that \( |\mu|(V) > 0 \) and let \( W_n = \bigcup_{1 \leq k \leq n} V_k \). Since \( W_n^c \cap V \neq \emptyset \), there exists an \( n \) such that \( |\mu|(V \cap W_n^c) < |\mu|(V) \). Since \( V \subset (V \cap W_n^c) \cup W_n \), it follows that

\[
|\mu|(V) \leq |\mu|(W_n) = \max_{1 \leq k \leq n} |\mu|(V_k),
\]

and the claim follows for \( \mu \). Suppose now that \( m_p(V) > r > 0 \). There exists a clopen subset \( W \) of \( V \) and \( s \in E \) such that \( |m(W)s| > r \cdot p(s) \). Let \( \mu = ms \). Then \( \mu \in M_\tau(X) \) and \( |\mu|(V) \geq |m(W)s| > r \cdot p(s) \). By the first part of the proof, there exists an \( n \) such that \( |\mu|(V_n) > r \cdot p(s) \). Hence, there exists a clopen subset \( D \) of \( V_n \) such that \( |\mu(D)| > r \cdot p(s) \). Now \( |m|_p(V_n) \geq |m(D)s|/p(s) > r \), which completes the proof.

For \( X \subset Y \subset \beta_oX \), and \( m \in M(X) \), we denote by \( m^Y \) the element of \( M(Y) \) defined by \( m^Y(V) = m(V \cap X) \). We denote by \( m^{v_o} \) and \( m^{\beta_o} \) the \( m^Y \) for \( Y = v_oX \) and \( Y = \beta_oX \), respectively.

We have the following easily established

**Theorem 2.3.** Let \( m \in M(X, E') \) be such that \( ms \in M_\tau(X) \) for all \( s \in E \). Then:

1. \( \text{supp}(m^{\beta_o}) = \text{supp}(m)^{\beta_o}X \).
2. \( \text{supp}(m) = \text{supp}(m^{\beta_o}) \cap X \).
If \( m \) has compact support, then \( \text{supp}(m) = \text{supp}(m^{\beta_0}) \).

**Theorem 2.4.** Let \( m \in M_p(X, E') \) and \( \mu = m^{\beta_0} \). The following are equivalent:

1. \( \text{supp}(\mu) \subset \upsilon_0 X \).
2. If \( V_n \downarrow \emptyset \), then there exists an \( n_o \) such that \( m(V_n) = 0 \) for every \( n \geq n_o \).
3. If \( V_n \downarrow \emptyset \), then there exists an \( n \) such that \( m(V) = 0 \) for every clopen set \( V \) contained in \( V_n \).
4. For every \( Z \in \Omega_1 \) there exists a clopen subset \( A \) on \( \beta_0 X \) disjoint from \( Z \) and such that \( \text{supp}(\mu) \subset A \).
5. If \( V_n \downarrow \emptyset \), then there exists an \( n \) such that \( m_p(V_n) = 0 \).

**Proof:**

(1) \( \Rightarrow \) (2). If \( V_n \downarrow \emptyset \), then the set \( \bigcap V_n^{-\beta_0} \) is disjoint from \( \upsilon_0 X \) and so \( \text{supp}(\mu) \subset \bigcup V_n^{-\beta_0} \). In view of the compactness of \( \text{supp}(\mu) \), there exists an \( n_o \) with \( \text{supp}(\mu) \subset V_n^{-\beta_0} \). If now \( n \geq n_o \), then \( m(V_n) = 0 \).

(2) \( \Rightarrow \) (3). Let \( V_n \downarrow \emptyset \) and suppose that, for each \( n \), there exists a clopen subset \( A \) of \( V_n \) such that \( m(A) \neq 0 \).

**Claim.** For each \( n \), there exists \( k > n \) and a clopen set \( B \) with \( V_k \subset B \subset V_n \) and \( m(B) \neq 0 \). Indeed there exists a clopen subset \( A \) of \( V_n \) such that \( m(A) \neq 0 \). For each \( k \), let \( B_k = V_k \cap A, D_k = V_k \setminus B_k \). Then \( D_k \downarrow \emptyset \). By our hypothesis, there exists \( k > n \) such that \( m(D_k) = 0 \). Let \( B = A \cup D_k \). Then \( V_k \subset B \subset V_n \). Since \( A \) and \( D_k \) are disjoint, we have that \( m(B) = m(A) \neq 0 \) and the claim follows. By induction, we choose \( n_1 = 1 < n_2 < \ldots \) and clopen sets \( B_k \) such that \( V_{n_k+1} \subset B_k \subset V_{n_k} \) and \( m(B_k) \neq 0 \). Since \( B_k \downarrow \emptyset \) and \( m(B_k) \neq 0 \) for every \( k \), we arrived at a contradiction.

(3) \( \Rightarrow \) (4). Let \( Z \in \Omega_1 \). There exists a decreasing sequence \( (V_n) \) of clopen sets with \( Z = \bigcap V_n^{-\beta_0} \). By our hypothesis, there exists an \( n \) such that \( m(V) = 0 \) for each clopen subset \( V \) of \( V_n \). Now it suffices to take \( A = V_n^{-\beta_0} \).

(4) \( \Rightarrow \) (1). Let \( z \in \beta_0 X \setminus \upsilon_0 X \). There exists a decreasing sequence \( (V_n) \) of clopen sets with \( z \in Z = \bigcap V_n^{-\beta_0} \). Clearly \( Z \in \Omega_1 \). Thus, there exists a
clopen subset $A$ of $\beta_0X$ disjoint from $Z$ and containing $\text{supp}(\mu)$. Hence $z$ is not in $\text{supp}(\mu)$.

(3) $\Rightarrow$ (5). It is trivial.

(5) $\Rightarrow$ (1). Let $z \in \beta_0X \setminus \nu_0X$. There exists a decreasing sequence $(V_n)$ of clopen sets with $z \in Z = \bigcap V_n^{\beta_0X}$. Let $n$ be such that $m_p(V_n) = 0$. If $G = \overline{V_n^{\beta_0X}}$, then $\mu_p(G) = 0$ and so $\text{supp}(\mu) \subset \beta_0X \setminus G$, which implies that $z \notin \text{supp}(\mu)$. This completes the proof.

**Theorem 2.5.** For an $m \in M_p(X, E')$, the following are equivalent:

(1) $m$ has a compact support, i.e. $m \in M_c(X, E')$.

(2) $\text{supp}(m^{\beta_0}) \subset X$.

(3) If $V_\delta \downarrow \emptyset$, then there exists a $\delta_o$ such that $m(V_\delta) = 0$ for all $\delta \geq \delta_o$.

(4) If $V_\delta \downarrow \emptyset$, then there exists a $\delta$ such $m(V) = 0$ for each clopen subset $V$ of $V_\delta$.

(5) If $H \in \Omega$, then there exists a clopen subset $A$ of $\beta_0X$, disjoint from $H$ and containing $\text{supp}(m^{\beta_0})$.

(6) If $V_\delta \downarrow \emptyset$, then there exists a $\delta$ such that $m_p(V_\delta) = 0$.

**Proof:** In view of Theorem 2.3, (1) implies (2).

(2) $\Rightarrow$ (3). Let $V_\delta \downarrow \emptyset$. By the compactness of $\text{supp}(m^{\beta_0})$, there exists $\delta_o$ such that $\text{supp}(m^{\beta_0}) \subset V^{\beta_0}_o$ and so $m(V_\delta) = 0$ for $\delta \geq \delta_o$.

(3) $\Rightarrow$ (4). Let $V_\delta \downarrow \emptyset$ and suppose that, for each $\delta$, there exists a clopen subset $V$ of $V_\delta$ with $m(V) \neq 0$.

**Claim:** For each $\delta$ there exist $\gamma \geq \delta$ and a clopen set $A$ such that $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. In fact, there exists a clopen subset $G$ of $V_\delta$ with $m(G) \neq 0$. For each $\gamma$, let $Z_\gamma = V_\gamma \cap G$, $W_\gamma = V_\gamma \setminus Z_\gamma$. Then $W_\gamma \downarrow \emptyset$. By our hypothesis, there exists $\gamma \geq \delta$ with $m(V_\gamma) = 0$. Let $A = G \cup W_\gamma$. Since the sets $G$ and $W_\gamma$ are disjoint, we have that $m(A) = m(G) \neq 0$. Since $V_\gamma \subset A \subset V_\delta$, the claim follows.

Let now $F$ be the family of all clopen subsets $A$ of $X$ with the following property: There are $\gamma, \delta$, with $\gamma \geq \delta$, $V_\gamma \subset A \subset V_\delta$ and $m(A) \neq 0$. Since $F \downarrow \emptyset$, we got a contradiction.

(4) $\Rightarrow$ (5). If $H \in \Omega$, then there exists a decreasing net $(V_\delta)$ of clopen
subsets of $X$ with $\bigcap V_\delta^{\beta_oX} = H$. Since $V_\delta \downarrow \emptyset$, there exists $\delta$ such that $m(V) = 0$ for each clopen subset $V$ of $V_\delta$. Now it suffices to take $A = V_\delta^{\beta_oX}$.

(5) $\Rightarrow$ (1). Let $z \in \beta_oX \setminus X$. By (5), there exists a clopen subset $A$ of $\beta_oX$ containing $\text{supp}(m_{\beta_o})$ and not containing $z$.

(4) $\Rightarrow$ (6). It is trivial.

(6) $\Rightarrow$ (2). Let $z \in \beta_oX \setminus X$. There exists a decreasing net $(V_\delta)$ of clopen sets with $\{z\} = \bigcap V_\delta^{\beta_oX}$. Let $\delta$ be such that $m_p(V_\delta) = 0$. If $\mu = m_{\beta_o}$, then $\mu_p(V_\delta^{\beta_oX}) = m_p(V_\delta) = 0$ and so $\text{supp}(\mu)$ is disjoint from the closure of $V_\delta$ in $\beta_oX$, which implies that $z \notin \text{supp}(\mu)$.

This completes the proof.

3. $\theta_o$-Complete Spaces

Recall that $\theta_oX$ is the set of all $z \in \beta_oX$ with the following property: For each clopen partition $(V_i)$ of $X$ there exists $i$ such that $z \in V_i^{\beta_oX}$ (see [2]). By [2, Lemma 4.1] we have $X \subset \theta_oX \subset \nu_oX$. For each clopen partition $\alpha = (V_i)_{i \in I}$ of $X$, let

$$W_\alpha = \bigcup_{i \in I} V_i \times V_i.$$ 

Then the family of all $W_\alpha$, $\alpha$ a clopen partition of $X$, is a base for a uniformity $U_c = U_c^X$, compatible with the topology of $X$, and $(\theta_oX, U_c^{\theta_oX})$ coincides with the completion of $(X, U_c)$. We will say that $X$ is $\theta_o$-complete iff $X = \theta_oX$. As it is shown in [2], if $Y$ is a $\theta_o$-complete and $f : X \to Y$ is a continuous function, then $f$ has a continuous extension $f^{\theta_o} : \theta_oX \to Y$.

A subset $A$ of $X$ is called bounding if every $f \in C(X)$ is bounded on $A$. Note that several authors use the term bounded set instead of bounding. But in this paper we will use the term bounding to distinguish from the notion of a bounded set in a topological vector space. A set $A \subset X$ is bounding iff $A^{\nu_oX}$ is compact. In this case (as it is shown in [2, Theorem 4.6]) we have that $A^{\theta_oX} = A^{\nu_oX} = A^{\beta_oX}$. Clearly a continuous image of a bounding set is bounding. Let us say that a family $F$ of subsets of $X$ is finite on a subset $A$ of $X$ if the family $\{f \in F : F \cap A \neq \emptyset\}$ is finite. We have the following easily established

**Lemma 3.1.** For a subset $A$ of $X$, the following are equivalent:

1. $A$ is bounding.
2. $A$ is $\nu_o$-closed.
3. $A$ is $\theta_o$-complete.

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(1) $A$ is bounding.

(2) Every continuous real-valued function on $X$ is bounded on $A$.

(3) Every locally finite family of open subsets of $X$ is finite on $A$.

(4) Every locally finite family of clopen subsets of $X$ is finite on $A$.

By [1, Theorem 4.6] every ultraparacompact space (and hence every ultrametrizable space) is $\theta_o$-complete.

**Theorem 3.2.** Every complete Hausdorff locally convex space $E$ is $\theta_o$-complete.

**Proof:** Let $\mathcal{U}$ be the usual uniformity on $E$, i.e. the uniformity having as a base the family of all sets of the form

$$W_{p,\epsilon} = \{ (x, y) : p(x - y) \leq \epsilon \}, \quad p \in cs(E), \quad \epsilon > 0.$$  

Given $W_{p,\epsilon}$, we consider the clopen partition $\alpha = (V_i)_{i \in I}$ of $E$ generated by the equivalence relation $x \sim y$ iff $p(x - y) \leq \epsilon$. Then $W_{p,\epsilon} = W_\alpha$ and so $\mathcal{U}$ is coarser than $\mathcal{U}_c$. Since $(E, \mathcal{U})$ is complete and $\mathcal{U}_c$ is compatible with the topology of $E$, it follows that $(E, \mathcal{U}_c)$ is complete and the result follows.

**Corollary 3.3.** A subset $B$, of a complete Hausdorff locally convex space $E$, is bounding iff it is totally bounded.

**Proof:** If $B$ is bounding, then $\overline{B} = \overline{B}^{\theta_o E}$ is compact and hence totally bounded, which implies that $B$ is totally bounded. Conversely, if $B$ is totally bounded, then $\overline{B}$ is totally bounded. Thus $\overline{B}$ is compact and hence $B$ is bounding.

**Theorem 3.4.** If $G$ is a locally convex space (not necessarily Hausdorff), then every bounding subset $A$ of $G$ is totally bounded.

**Proof:** Assume first that $G$ is Hausdorff. Let $\hat{G}$ be the completion of $G$. The closure $\overline{B}$ of $A$ in $\hat{G}$ is bounding and hence $B$ is totally bounded, which implies that $A$ is totally bounded. If $G$ is not Hausdorff, we consider the quotient space $F = G/\{0\}$ and let $u : G \to F$ be the quotient map. Since $u$ is continuous, the set $u(A)$ is bounding, and hence totally bounded, in $F$. Let now $V$ be a convex neighborhood of zero in $G$. Then, $u(V)$
is a neighborhood of zero in \(F\). Let \(S\) be finite subset of \(A\) such that \(u(A) \subset u(S) + u(V)\). But then
\[
A \subset S + V + \{0\} \subset S + V + V = S + V,
\]
which proves that \(A\) is totally bounded.

**Theorem 3.5.** We have:

1. Closed subspaces of \(\theta_o\)-complete spaces are \(\theta_o\)-complete.
2. If \(X = \prod X_i\), with \(X_i \neq \emptyset\) for all \(i\), then \(X\) is \(\theta_o\)-complete iff each \(X_i\) is \(\theta_o\)-complete.
3. If \((Y_i)_{i \in I}\) is a family of \(\theta_o\)-complete subspaces of \(X\), then \(Y = \bigcap Y_i\) is \(\theta_o\)-complete.
4. \(\theta_oX\) is the smallest of all \(\theta_o\)-complete subspaces of \(\beta_oX\) which contain \(X\).

**Proof:** (1). Let \(Z\) be a closed subspace of a \(\theta_o\)-complete space \(X\) and let \((x_\delta)\) be a \(\mathcal{U}_c^Z\)-Cauchy net in \(Z\). Then \((x_\delta)\) is \(\mathcal{U}_c^X\)-Cauchy and hence \(x_\delta \to x \in X\). Moreover, \(x \in Z\) since \(Z\) is closed.

(2). Each \(X_i\) is homeomorphic to a closed subspace of \(X\). Thus \(X_i\) is \(\theta_o\)-complete if \(X\) is \(\theta_o\)-complete. Conversely, suppose that each \(X_i\) is \(\theta_o\)-complete. If \((x_i^\delta)\) is a \(\mathcal{U}_c^X\)-Cauchy net, then \((x_i^\delta)\) is a \(\mathcal{U}_c^{X_i}\)-Cauchy net in \(X_i\) and hence \(x_i^\delta \to x_i \in X_i\). If \(x = (x_i)\), then \(x^\delta \to x\), which proves that \((X, \mathcal{U}_c)\) is complete.

(3). Let \(X = \prod Y_i\) and consider the map \(f : Y \to X\), \(f(x)_i = x\) for all \(i\). Then \(f : Y \to \beta_o f(Y) = D\) is a homeomorphism. Also \(D\) is a closed subspace of \(X\). Since \(X\) is \(\theta_o\)-complete, it follows that \(D\) is \(\theta_o\)-complete and hence \(Y\) is \(\theta_o\)-complete.

(4). Since \(\theta_oX\) is \(\theta_o\)-complete (by [2, Theorem 4.9]) and \(X \subset \theta_oX \subset \beta_oX\), the result follows from (3).

**Theorem 3.6.** For a point \(z \in \beta_oX\), the following are equivalent:

1. \(z \in \theta_oX\).
2. If \(Y\) is a Hausdorff ultraparacompact space and \(f : X \to Y\) continuous, then \(f^{\beta_o}(z) \in Y\), where \(f^{\beta_o} : \beta_oX \to \beta_oY\) is the continuous extension of \(f\).
For every ultrametric space $Y$ and every $f : X \to Y$ continuous, we have that $f^{\beta_0}(z) \in Y$.

Proof: (1) $\Rightarrow$ (2). Since $\theta_0 Y = Y$, the result follows from [2, Theorem 4.4].

(2) $\Rightarrow$ (3). It is trivial.

(3) $\Rightarrow$ (1). Assume that $z \notin \theta_0 X$. Then, there exists a clopen partition $(A_i)$ of $X$ such that $z \notin \bigcup_i A_i^{\beta_0 X}$. Let $f_i = \chi_{A_i}$ and define

$$d : X \times X \to \mathbb{R}, \quad d(x, y) = \sup_i |f_i(x) - f_i(y)|.$$  

Then $d$ is a continuous ultrapseudometric on $X$. Let $Y = X_d$ be the corresponding ultrametric space and let $\pi : X \to Y_d$ be the quotient map, $x \mapsto \tilde{x}_d = \tilde{x}$. Since $\pi$ is continuous, there exists (by (3)) an $x \in X$ such that $\pi^{\beta_0}(z) = \tilde{x}_d$. Let $(x_{\delta})$ be a net in $X$ converging to $z$. Then $\tilde{x}_\delta = \pi^{\beta_0}(x_{\delta}) \to \pi^{\beta_0}(z) = \tilde{x}$, and so $d(x_{\delta}, x) \to 0$. If $x \in A_i$, then $|f_i(x_{\delta}) - 1| \to 0$, and so there exists $\delta_0$ such that $x_{\delta} \in A_i$ when $\delta \geq \delta_0$. But then $z \in A_i^{\beta_0 X}$, a contradiction. This completes the proof.

**Theorem 3.7.** Let $X$ be a dense subspace of a Hausdorff zero-dimensional space $Y$. The following are equivalent:

1. $Y \subset \theta_0 X$ (more precisely, $Y$ is homeomorphic to a subspace of $\theta_0 X$).
2. Each continuous function, from $X$ to any ultrametric space $Z$, has a continuous extension to all of $Y$.

Proof: (1) implies (2) by the preceding Theorem.

(2) $\Rightarrow$ (1). We will prove first that, for each clopen subset $V$ of $X$, we have that $\overline{V}^Y \cap V^{cY} = \emptyset$, and so $\overline{V}^Y$ is clopen in $Y$. Indeed, define

$$d : X \times X \to \mathbb{R}, \quad d(x, y) = \max\{|f_1(x) - f_1(y)|, |f_2(x) - f_2(y)|\},$$

where $f_1 = \chi_V$, $f_2 = \chi_{V^c}$. Then $d$ is a continuous ultrapseudometric on $X$. Let $\pi : X \to X_d$ be the quotient map. By our hypothesis, there exists a continuous extension $h : Y \to X_d$ of $\pi$. Suppose that $z \in \overline{V}^Y \cap V^{cY}$. There are nets $(x_{\delta}), (y_{\gamma})$, in $V$, $V^c$ respectively, such that $x_{\delta} \to z$, and $y_{\gamma} \to z$. Let $\tilde{d}$ be the ultrametric of $X_d$ and let $\delta_0, \gamma_0$ be such that

$$\tilde{d}(\pi(x_{\delta}), h(z)) < 1 \quad \text{and} \quad \tilde{d}(\pi(x_{\gamma}), h(z)) < 1.$$  

when $\delta \geq \delta_0, \gamma \geq \gamma_0$. Now
\[ d(x_{\delta_0}, y_{\gamma_0}) = \tilde{d}(\pi(x_{\delta_0}), \pi(y_{\delta_0})) < 1, \]
a contradiction. Thus $V^\gamma$ is clopen in $Y$. If $A = V^\gamma, B = V^{\gamma_0}$, then
\[ A^{\beta_0} \bigcap B^{\beta_0} = V^{\beta_0} \bigcap V^{\gamma_0} = \emptyset. \]
This, being true for each clopen subset $V$ of $X$, implies that $\beta_0X = \beta_0Y$ and so $X \subset Y \subset \beta_0Y = \beta_0X$. Now our hypothesis (2) and the preceding Theorem imply that $Y \subset \theta_0X$, and the result follows.

**Theorem 3.8.** For each continuous ultrapseudometric $d$ on $X$, there exists a continuous ultrapseudometric $d^{\theta_0}$ on $\theta_0X$ which is an extension of $d$. Moreover, $d^{\theta_0}$ is the unique continuous extension of $d$.

**Proof:** Consider the ultrametric space $X_d$ and let $\tilde{d}$ be its ultrametric. Let $h$ be the continuous extension of the quotient map $\pi: X \rightarrow X_d$ to all of $\theta_0X$. Define
\[ d^{\theta_0}: \theta_0X \times \theta_0X \rightarrow \mathbb{R}, d^{\theta_0}(y, z) = \tilde{d}(h(y), h(z)). \]
It is easy to see that $d^{\theta_0}$ is a continuous ultrapseudometric which is an extension of $d$. Finally, let $\varrho$ be any continuous ultrapseudometric on $\theta_0X$, which is an extension of $d$, and let $y, z \in \theta_0X$. There are nets $(y_\delta)_{\delta \in \Delta}, (z_\gamma)_{\gamma \in \Gamma}$ in $X$ which convergence to $y, z$, respectively. Let $\Phi = \Delta \times \Gamma$ and consider on $\Phi$ the order $(\delta_1, \gamma_1) \geq (\delta_2, \gamma_2)$ iff $\delta_1 \geq \delta_2$ and $\gamma_1 \geq \gamma_2$. For $\phi = (\delta, \gamma) \in \Phi$, we let $a_\phi = y_\delta, b_\phi = z_\gamma$. Then $a_\phi \rightarrow y, b_\phi \rightarrow z$. Thus
\[ \varrho(y, z) = \lim \varrho(a_\phi, b_\phi) = \lim \tilde{d}(h(a_\phi), h(b_\phi)) \]
\[ = \lim d^{\theta_0}(a_\phi, b_\phi) = d^{\theta_0}(y, z) \]
and hence $\varrho = d^{\theta_0}$, which completes the proof.

**Theorem 3.9.** Let $(H_n)$ be a sequence of equicontinuous subsets of $C(X)$. If $z \in \theta_0X$, then there exists $x \in X$ such that $f^{\theta_0}(z) = f(x)$ for all $f \in \bigcup H_n = H$.

**Proof:** Define
\[ d: X^2 \rightarrow \mathbb{R}, d(x, y) = \max \min \{1/n, \sup_{f \in H_n} |f(x) - f(y)| \}. \]
Then $d$ is a continuous ultrapseudometric on $X$. Take $Y = X_d$ and let $\pi: X \rightarrow Y$ be the quotient map. Then $\pi^{\beta_0}(z) = u \in Y$. Choose $x \in \theta_0X$. Now
X with \( \pi(x) = u \), and let \((x_\delta)\) be a net in \( X \) converging to \( z \) in \( \beta_\alpha X \). Now \( f(x_\delta) \to f^{\beta_\alpha}(z) \) for all \( f \in H \). Since \( \pi(x_\delta) \to \pi(x) \), we have that \( d(x_\delta,x) \to 0 \), and so \( |f(x_\delta) - f(x)| \to 0 \) for all \( f \in H \). Thus, for \( f \in H \), we have \( f(x) = \lim f(x_\delta) = f^{\beta_\alpha}(z) \), and the result follows.

**Theorem 3.10.** If \( H \subset C(X) \) is equicontinuous, then the family

\[
H^{\theta_\alpha} = \{ f^{\theta_\alpha} : f \in H \}
\]

is equicontinuous on \( \theta_\alpha X \). Moreover, if \( H \) is pointwise bounded, then the same holds for \( H^{\theta_\alpha} \)

**Proof:** Define

\[
d : X^2 \to \mathbb{R}, \quad d(x,y) = \min\{1, \sup_{f \in H} |f(x) - f(y)| \}.
\]

Let \( \pi^{\theta_\alpha} : \theta_\alpha X \to X_d \) be the continuous extension of the quotient map \( \pi : X \to X_d \). Let \( z \in \theta_\alpha X \) and \( \epsilon > 0 \). There exists \( x \in X \) such that \( \pi^{\theta_\alpha}(z) = \pi(x) \). Let \((x_\delta)\) be a net in \( X \) converging to \( z \). Then \( \pi(x_\delta) \to \pi^{\theta_\alpha}(z) = \pi(x) \) and so \( d(x_\delta,x) \to 0 \). Thus, for \( f \in H \), we have \( f^{\theta_\alpha}(z) = \lim f(x_\delta) = f(x) \). The set \( W = \{ y \in X : d(x,y) \leq \epsilon \} \) is \( d \)-clopen (hence clopen) in \( X \) and so \( \overline{W}^{\theta_\alpha} = V \) is clopen in \( \theta_\alpha X \). Since \( x_\delta \in W \) eventually, it follows that \( z \in V \). Now, for \( f \in H \) and \( a \in V \), we have that \( |f^{\theta_\alpha}(a) - f^{\theta_\alpha}(z)| \leq \epsilon \). In fact, there exists a net \((y_\gamma)\) in \( W \) converging to \( a \). Thus

\[
|f^{\theta_\alpha}(a) - f^{\theta_\alpha}(z)| = |f(x) - f^{\theta_\alpha}(a)| = \lim \gamma |f(x) - f(y_\gamma)| \leq \epsilon.
\]

This proves that \( H^{\theta_\alpha} \) is equicontinuous on \( \theta_\alpha X \). The last assertion follows from the preceding Theorem.

**Theorem 3.11.** \( \mathcal{U}_c = \mathcal{U}_c^X \) is the uniformity \( \mathcal{U} \) generated by the family of all continuous ultrapseudometrics on \( X \).

**Proof:** Let \((A_i)\) be a clopen partition of \( X \) and let \( W = \bigcup A_i \times A_i \). Define

\[
d(x,y) = \sup_i |f_i(x) - f_i(y)|,
\]

where \( f_i = \chi_{A_i} \). Then \( d \) is a continuous ultrapseudometric on \( X \). Since

\[
W = \{(x,y) : d(x,y) < 1/2\},
\]

it follows that \( \mathcal{U}_c \) is coarser than \( \mathcal{U} \). Conversely, let \( d \) be a continuous ultrapseudometric on \( X \), \( \epsilon > 0 \) and \( D = \{(x,y) : d(x,y) \leq \epsilon\} \). If \( \alpha \) is the
clopen partition of $X$ corresponding to the equivalence relation $x \sim y$ iff $d(x, y) \leq \epsilon$, then $D = W_\alpha$ and the result follows.

**Theorem 3.12.** Let $(Y_i, f_i)_{i \in I}$ be the family of all pairs $(Y, f)$, where $Y$ is an ultrametric space and $f : X \to Y$ a continuous map. Then

$$\theta_\alpha X = \bigcap_{i \in I} (f_i^\beta_\alpha)^{-1}(Y_i).$$

**Proof:** It follows from Theorem 3.6.

**Theorem 3.13.** A Hausdorff zero-dimensional space $X$ is $\theta_\alpha$-complete iff it is homeomorphic to a closed subspace of a product of ultrametric spaces.

**Proof:** Every ultrametric space is $\theta_\alpha$-complete. Thus the sufficiency follows from Theorem 3.5. Conversely, assume that $X$ is $\theta_\alpha$-complete and let $(Y_i, f_i)_{i \in I}$ be as in the preceding Theorem. Then $X = \bigcap_i Z_i$, $Z_i = (f_i^\beta_\alpha)^{-1}(Y_i)$. Let $Y = \prod Y_i$ with its product topology. The map $u : X \to Y$, $u(x)_i = f_i(x)$, is one-to-one. Indeed, let $x \neq y$ and choose a clopen neighborhood $V$ of $x$ not containing $y$. Let $f = \chi_V$ and

$$d : X \times Y \to \mathbb{R}, \quad d(a, b) = |f(a) - f(b)|.$$ 

The quotient map $\pi : X \to X_d$ is continuous and $\pi(x) \neq \pi(y)$, which implies that $u(x) \neq u(y)$. Clearly $u$ is continuous. Also $u^{-1} : u(X) \to X$ is continuous. Indeed, let $V$ be a clopen subset of $X$ containing $x_\alpha$ and consider the pseudometric $d(x, y) = |\chi_V(x) - \chi_V(y)|$. Let $\pi : X \to X_d$ be the quotient map. There exists an $i \in I$ such that $Y_i = X_d$ and $f_i = \pi$. Then

$$f_i(V) = \pi(V) = \{ \pi(x) : d(\pi(x) - \pi(x_\alpha)) < 1 \}.$$ 

The set $\pi(V)$ is open in $Y_i = X_d$. Let $\pi_i : Y \to Y_i$ be the $i$th-projection map and $G = \pi_i^{-1}(\pi(V))$. If $x \in X$ is such that $u(x) \in G$, then $f_i(x) = \pi_i(x)_i = f_i(x)_i \in \pi(V)$ and so $d(x, x_\alpha) < 1$, which implies that $x \in V$ since $x_\alpha \in V$. This proves that $u : X \to u(X)$ is a homeomorphism. Finally, $u(X)$ is a closed subspace of $Y$. In fact, let $(x_\delta)$ be a net in $X$ with $u(x_\delta) \to y \in Y$. Then $f_i(x_\delta) \to y_i$ for all $i$. Going to a subnet if necessary, we may assume that $x_\delta \to z \in \beta_\alpha X$. Now $f_i(x_\delta) \to f_i^\beta_\alpha(z)$ in $\beta_\alpha Y_i$. But then $f_i^\beta_\alpha(z) = y_i \in Y_i$, for all $i$, and hence $z \in \theta_\alpha X = X$, by the preceding Theorem. Thus $y_i = f_i(z)$, for all $i$, and hence $y = u(z)$. This proves that $X$ is homeomorphic to a closed subspace of $Y$ and the result follows.
Corollary 3.14. Every Hausdorff ultraparacompact space is homeomorphic to a closed subspace of a product of ultrametric spaces.

Theorem 3.15. For a subset $A$ of $X$, the following are equivalent:

1. $A$ is bounding.
2. $A$ is $\mathcal{U}_c$-totally bounded.
3. For each continuous ultrapseudometric $d$ on $X$, $A$ is $d$-totally bounded.

Proof: In view of Theorem 3.11, (2) is equivalent to (3). Also, by [2, Theorem 4.6], (1) implies (2).

Proof: Let $f \in C(X)$,

$$A_1 = \{ x : |f(x)| \leq 1 \}, \quad A_{n+1} = \{ x : n < |f(x)| \leq n + 1 \}$$

for $n \geq 1$. Then $(A_n)$ is a clopen partition of $X$. Let $W = \bigcup_n A_n \times A_n$. By our hypothesis, there are $x_1, \ldots, x_N$ in $A$ such that $A \subset \bigcup_{1}^{N} W[x_k]$. For each $1 \leq k \leq N$, there exists $n_k$ such that $x_k \in A_{n_k}$. Then $A \subset \bigcup_{1}^{N} A_{n_k}$ and so

$$\|f\|_A \leq \max_{1 \leq k \leq N} n_k,$$

which proves that $A$ is bounding.

4. Polarly Barrelled Spaces of Continuous Functions

Definition 4.1. A Hausdorff locally convex space $E$ is called:

1. polarly barrelled if every bounded subset of $E'_\sigma = (E', \sigma(E', E))$ is equiconinuous.
2. polarly quasi-barrelled if every strongly bounded subset of $E'$ is equicontinuous.

We will denote by $C_c(X, E)$ the space $C(X, E)$ equipped with the topology of uniform convergence on compact subsets of $X$. By $M_c(X, E')$ we will denote the space of all $m \in M(X, E')$ with compact support. The dual space of $C_c(X, E)$ coincides with $M_c(X, E')$.

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Recall that a zero-dimensional Hausdorff topological space $X$ is called a $\mu_\sigma$-space (see [2]) if every bounding subset of $X$ is relatively compact. We
denote by \( \mu_o X \) the smallest of all \( \mu_o \)-subspaces of \( \beta_o X \) which contain \( X \). Then \( X \subset \mu_o X \subset \theta_o X \) and, for each bounding subset \( A \) of \( X \), the set \( \mathcal{A}^{\beta_o X} \) is contained in \( \mu_o X \) (see [2]). Moreover, if \( Y \) is another Hausdorff zero-dimensional space and \( f : X \to Y \), then \( f^{\beta_o}(\mu_o X) \subset \mu_o Y \) and so there exists a continuous extension \( f^{\mu_o} : \mu_o X \to \mu_o Y \) of \( f \).

**Theorem 4.2.** Assume that \( E' \neq \{0\} \) and let \( G = C_c(X,E) \). Then \( G \) is polarly barrelled iff \( X \) is a \( \mu_o \)-space and \( E \) polarly barrelled.

*Proof:* Assume that \( G \) is polarly barrelled.

I. \( E \) is polarly barrelled. Indeed, let \( \Phi \) be a \( w^* \)-bounded subset of \( E' \) and let \( x \in X \). For \( u \in E' \), let

\[ u_x : G \to \mathbb{K}, \quad u_x(f) = u(f(x)). \]

Let \( H = \{u_x : u \in \Phi\} \). For \( f \in C(X,E) \), we have

\[ \sup_{u \in \Phi} |u_x(f)| = \sup_{u \in \Phi} |u(f(x))| < \infty \]

and so \( H \) is a \( w^* \)-bounded subset of \( G' \). By our hypothesis, there exists \( p \in cs(E) \) and \( Y \) a compact subset of \( X \) such that

\[ \{f \in G : \|f\|_{Y,p} \leq 1\} \subset H^\circ. \]

But then \( \{s \in E : p(s) \leq 1\} \subset \Phi^\circ \) and so \( \Phi \) is equicontinuous.

II. \( X \) is a \( \mu_o \)-space. In fact, let \( A \) be a bounding subset of \( X \) and let \( x' \in E' \), \( x' \neq 0 \). Define \( p \) on \( E \) by \( p(x) = |x'(s)| \). Then \( p \in cs(E) \). The set

\[ D = \{f \in G : \|f\|_{A,p} \leq 1\} \]

is a polar barrel in \( G \) and so \( D \) is a neighborhood of zero in \( G \). Let \( Y \) a compact subset of \( X \) and \( q \in cs(E) \) be such that

\[ \{f \in G : \|f\|_{Y,p} \leq 1\} \subset D. \]

But then \( A \subset Y \) and so \( \overline{A} \) is compact.

Conversely, suppose that \( E \) is polarly barrelled and \( X \) a \( \mu_o \)-space. Let \( H \) be a \( w^* \)-bounded subset of the dual space \( M_c(X,E') \) of \( G \). Let \( s \in E \) and

\[ D = \{ms : m \in H\} \subset M(X). \]

For \( h \in C_{rc}(X) \), we have that

\[ \sup_{m \in H} |<ms, h>| = \sup_{m \in H} |<m, hs>| < \infty. \]
Thus, considering $M(X)$ as the dual of the Banach space $F = (C_\text{rc}(X), \tau_u)$, $D$ is $w^*$-bounded of $F'$ and so $\sup_{m \in H} \|ms\| = d_s < \infty$. Hence, $|m(V)| \leq d_s$ for all $V \in K(X)$. It follows that the set

$$M = \bigcup_{m \in H} m(K(X))$$

is a $w^*$-bounded subset of $E'$. Since $E$ is polarly barrelled, there exists $p \in cs(E)$ such that $|u(s)| \leq 1$ for all $u \in M$ and all $s \in E$ with $p(s) \leq 1$. Hence $\sup_{m \in H} \|m\|_p < \infty$. We may choose $p$ so that $\|m\|_p \leq 1$ for all $m \in H$. Let

$$Z = S(H) = \bigcup_{m \in H} \text{supp}(m).$$

Then $Z$ is bounding. In fact, assume that $Z$ is not bounding. Then, by [6, Proposition 6.6], there exists a sequence $(m_n)$ in $H$ and $f \in C(X,E)$ such that $<m_n,f> = \lambda^n$, for all $n$, where $|\lambda| > 1$, which contradicts the fact that $H$ is $w^*$-bounded. By our hypothesis now, $Z$ is compact. Since

$$\{f \in G : \|f\|_{Z,p} \leq 1\} \subset H^o,$$

the result follows.

**Corollary 4.3.** $C_c(X)$ is polarly barrelled iff $X$ is a $\mu_o$-space.

Let now $G, E$ be Hausdorff locally convex spaces. We denote by $L_s(G,E)$ the space $L(G,E)$ of all continuous linear maps, from $G$ to $E$, equipped with the topology of simple convergence.

**Theorem 4.4.** Assume that $E$ is polar and let $G$ be polarly barrelled. If $E$ is a $\mu_o$-space (e.g. when $E$ is metrizable or complete), then $L_s(G,E)$ is a $\mu_o$-space.

**Proof:** Let $\Phi$ be a bounding subset of $L_s(G,E)$. For $x \in G$, the set

$$\Phi(x) = \{\phi(x) : \phi \in \Phi\}$$

is a bounding subset of $E$ and hence its closure $M_x$ in $E$ is compact. $\Phi$ is a topological subspace of $E^G$ and it is contained in the compact set $M = \prod_{x \in G} M_x$. Since the closure of $\Phi$ in $E^G$ is compact, it suffices to show that this closure is contained in $L(G,E)$. To this end, we prove first that, given a polar neighborhood $W$ of zero in $E$, there exists a neighborhood
$U$ of zero in $G$ such that $\phi(U) \subset W$ for all $\phi \in \Phi$. In fact, for $\phi \in \Phi$, let $\phi'$ be the adjoint map. Let

$$Z = \bigcup_{\phi \in \Phi} \phi'(H),$$

where $H$ is the polar of $W$ in $E'$. If $x \in G$, then $\Phi(x)$ is a bounded subset of $E$ and hence $\Phi(x) \subset \alpha W$, for some $\alpha \in \mathbb{K}$. If now $\phi \in \Phi$ and $u \in H$, then

$$|\langle \phi'(u), x \rangle| = |\langle u, \phi(x) \rangle| \leq |\alpha|,$$

which proves that $Z$ is a $w^*$-bounded subset of $G'$. As $G$ is polarly barrelled, the polar $U = Z^o$, of $Z$ in $G$, is a neighborhood of zero and $\phi(U) \subset H^o = W$, for all $\phi \in \Phi$, which proves our claim. Let now $\phi \in E^G$ be in the closure of $\Phi$. Then $\phi$ is linear. There exists a net $(\phi_\delta)$ in $\Phi$ converging to $\phi$ in $E^G$. If $x \in U$, then $\phi(x) = \lim \phi_\delta(x) \in W$, which proves that $\phi$ is continuous. Hence the result follows.

**Corollary 4.5.** If $E$ is polarly barrelled, then the weak dual $E'_\sigma$ of $E$ is a $\mu_\sigma$-space.

**Theorem 4.6.** Suppose that $E$ is polar and $G$ polarly barrelled. For $f \in C(X, E)$, let $f^{\mu_\sigma} : \mu_\sigma X \to \hat{E}$ be its continuous extension. If $T : G \to C_c(X, E)$ is a continuous linear map, then the map

$$\tilde{T} : G \to C_c(\mu_\sigma X, \hat{E}), \quad s \mapsto (Ts)^{\mu_\sigma},$$

is continuous

**Proof:** Note that $\hat{E}$ is $\theta_\sigma$-complete and hence a $\mu_\sigma$-space. Let

$$\phi : X \to L_s(G, E), \quad \langle \phi(x), s \rangle = (Ts)(x).$$

Then $\phi$ is continuous. Since $L_s(G, \hat{E})$ is a $\mu_\sigma$-space, there exists a continuous extension

$$\phi^{\mu_\sigma} : \mu_\sigma X \to L_s(G, \hat{E}).$$

Let now $A$ be a compact subset of $\mu_\sigma X$ and $p$ a polar continuous seminorm on $E$. We denote also by $p$ the continuous extension of $p$ to all of $\hat{E}$. Let

$$V = \{g \in C(\mu_\sigma X, \hat{E}) : \|g\|_{A, p} \leq 1\}.$$

The set $\Phi = \phi^{\mu_\sigma}(A)$ is compact in $L_s(G, \hat{E})$. As in the proof of Theorem 4.4, there exists a neighborhood $U$ of zero in $G$ such that

$$\psi(U) \subset W = \{s \in \hat{E} : p(s) \leq 1\},$$

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for all $\psi \in \Phi$. Now, for $y \in A$ and $s \in U$, we have

$$p((\tilde{T}s)(y)) = p(<\phi^{\mu_0}(y), s>) \leq 1$$

and so $\tilde{T}s \in V$. This proves that $\tilde{T}$ is continuous and the result follows.

**Theorem 4.7.** Assume that $E$ is polar and polarly barrelled and let $\tau_o$ be the locally convex topology on $C(X, E)$ generated by the seminorms $f \mapsto \|f^{\mu_0}\|_{A,p}$, where $A$ ranges over the family of all compact subsets of $\mu_0X$ and $p \in \text{cs}(E)$. Then:

1. $(C(X, E), \tau_o)$ is polarly barrelled and $\tau_o$ is finer than $\tau_b$ (and hence finer than $\tau_c$).

2. If $\tau$ is any polarly barrelled topology on $C(X, E)$ which is finer than $\tau_c$, then $\tau$ is finer than $\tau_o$. Hence $\tau_o$ is the polarly barrelled topology associated with each of the topologies $\tau_b$ and $\tau_c$.

**Proof:** (1). Since $E$ is polarly barrelled, the same is true for $\hat{E}$. The space $F = C_c(\mu_0X, \hat{E})$ is polarly barrelled and the map

$$S : (C(X, E), \tau_o) \to F, \quad f \mapsto f^{\mu_0},$$

is a linear homeomorphism. Thus $\tau_o$ is polarly barrelled. Also, since for each bounding subset $B$ of $X$, its closure $\overline{B}^{\mu_0X}$ is compact, it follows that $\tau_o$ is finer than $\tau_b$.

(2). Let $\tau$ be a polarly barrelled topology on $C(X, E)$, which is finer than $\tau_c$, and let $G = (C(X, E), \tau)$. The identity map

$$T : G \to C_c(X, E)$$

is continuous and hence the map

$$\tilde{T} : G \to C_c(\mu_0X, \hat{E}), \quad f \mapsto f^{\mu_0},$$

is continuous. This proves that $\tau_o$ is coarser than $\tau$ and the Theorem follows.

**Theorem 4.8.** Suppose that $E$ is polar. Then $G = (C(X, E), \tau_b)$ is polarly barrelled iff $E$ is polarly barrelled and, for each compact subset $A$ of $\mu_0X$, there exists a bounding subset $B$ of $X$ such that $A \subset \overline{B}^{\mu_0X}$.
**Proof:** Assume that $G$ is polarly barrelled. It is easy to see that $E$ is polarly barrelled. In view of the preceding Theorem, $\tau_b = \tau_o$. Thus, for each compact subset $A$ of $\mu_o X$ and each non-zero $p \in cs(E)$, there exist a bounding subset $B$ of $X$ and $q \in cs(E)$ such that
\[
\{ f \in C(X,E) : \|f\|_{B,q} \leq 1 \} \subset \{ f : \|f^{\mu_o}\|_{A,p} \leq 1 \}.
\]
It follows easily that $A \subset \overline{B^{\mu_o} X}$. Conversely, suppose that the condition is satisfied. The condition clearly implies that $\tau_o$ is coarser than $\tau_b$ and hence $\tau_b = \tau_o$, which implies that $G$ is polarly barrelled by the preceding Theorem.

Let us say that a family $F$ of subsets of a set $Z$ is finite on a subset $F$ of $Z$ if the family of all members of $F$ which meet $F$ is finite.

**Definition 4.9.** A subset $D$, of a topological space $Z$, is said to be $w$-bounded if every family $F$ of open subsets of $Z$, which is finite on each compact subset of $Z$, is also finite on $D$. If this happens for families of clopen sets, then $D$ is said to be $w_o$-bounded. We say that $Z$ is a $w$-space (resp. a $w_o$-space ) if every $w$-bounded (resp. $w_o$-bounded) subset is relatively compact.

**Lemma 4.10.** A subset $D$, of a zero-dimensional topological space $Z$, is $w$-bounded iff it is $w_o$-bounded.

**Proof:** Assume that $D$ is not $w$-bounded. Then, there exists an infinite sequence $(x_n)$ of distinct elements of $D$ and a sequence $(V_n)$ of open sets such that $x_n \in V_n$ and $(V_n)$ is finite on each compact subset of $X$. By [5, Lemma 2.5], there exists a subsequence $(x_{n_k})$ and pairwise disjoint clopen sets $W_k$ with $x_{n_k} \in W_k$. We may choose $W_k \subset V_{n_k}$. Now $W_k$ is clearly finite on each compact subset of $X$, which implies that $D$ is not $w_o$-bounded. Hence the Lemma follows.

We easily get the following

**Lemma 4.11.** Every $w_o$-bounded subset of $X$ is bounding.

**Theorem 4.12.** Assume that $E' \neq \{0\}$. Then $G = C_c(X,E)$ is polarly quasi-barrelled iff $E$ is polarly quasi-barrelled and $X$ a $w_o$-space.

**Proof:** Suppose that $E$ is polarly quasi barrelled and $X$ a $w_o$-space. Let $H$ be a strongly bounded subset of the dual space $M_c(X,E)$ of $G$. We
show first that there exists \( p \in cs(E) \) such that \( \sup_{m \in H} \|m\|_p < \infty \). In fact, let \( B \) be a bounded subset of \( E \) and consider the set
\[
D = \{ ms : m \in H, s \in B \}.
\]
If \( h \in C_{rc}(X) \), then the set \( \{ hs : s \in B \} \) is a bounded subset of \( G \) and so
\[
\sup_{m \in H} \left| \int hs \, dm \right| = \sup_{m \in H} \left| \int h \, d(ms) \right| < \infty.
\]
Considering \( D \) a subset of the dual of the Banach space \( F = (C_{rc}(X), \tau_u) \), we see that \( D \) is a \( w^* \)-bounded subset of \( F' \) and hence equicontinuous. Thus
\[
d = \sup_{m \in H, s \in B} \|ms\| < \infty.
\]
Let
\[
\Phi = \bigcup_{m \in H} m(K(X)).
\]
Then for \( A \in K(X), s \in B, m \in H \), we have \( |m(A)s| \leq \|ms\| \leq d \).
Hence \( \Phi \) is a strongly bounded subset of \( E' \). By our hypothesis, \( \Phi \) is an equicontinuous subset of \( E' \). Thus, there exists \( p \in cs(E) \) such that \( |m(A)s| \leq 1 \) for all \( m \in H \) and all \( s \in E \) with \( p(s) \leq 1 \). It follows from this that \( \sup_{m \in H} \|m\|_p = r < \infty \). We may choose \( p \) so that \( r \leq 1 \). Let now
\[
Y = S(H) = \bigcup_{m \in H} \text{supp}(m).
\]
Then \( Y \) is \( w_o \)-bounded. Assume the contrary. Then, there exists a sequence \((V_n)\) of distinct clopen subsets of \( X \), such that \( V_n \cap Y \neq \emptyset \) for all \( n \) and \((V_n)\) is finite on each compact subset of \( X \). For each \( n \) there exists \( m_n \in H \) with \( V_n \cap \text{supp}(m_n) \neq \emptyset \). Then \( (m_n) \circ (V_n) > 0 \). There are a clopen subset \( W_n \) of \( V_n \) and \( s_n \in E \), with \( p(s_n) \leq 1 \), such that \( m(W_n)s_n = \gamma_n \neq 0 \). Let \( |\lambda| > 1 \) and take
\[
M = \{ \gamma_n^{-1} \lambda^n \chi_{W_n} s_n : n \in \mathbb{N} \}.
\]
Since \((W_n)\) is finite on each compact subset of \( X \), it follows that \( M \) is a bounded subset of \( G \) and so \( M \) is absorbed by \( H^o \). Let \( \lambda_o \neq 0 \) be such that \( M \subset \lambda_o H^o \). But then
\[
1 \geq |\lambda_o^{-1} \gamma_n^{-1} \lambda^n m_n(W_n)s_n| = |\lambda_o^{-1} \lambda^n|
\]
for all \( n \), which is a contradiction. So \( Y \) is \( w_o \)-bounded and hence compact by our hypothesis. Moreover

\[
\{ f \in G : \| f \|_{Y,p} \leq 1 \} \subset H^o.
\]

Indeed, let \( \| f \|_{Y,p} \leq 1 \). The set \( V = \{ x : p(f(x)) > 1 \} \) is disjoint from \( Y \) and hence \( m_p(V) = 0 \) for all \( m \in H \). Thus, for \( m \in H \), we have

\[
\left| \int_V f \, dm \right| \leq \| f \|_p \cdot m_p(V) = 0
\]

and so

\[
\left| \int f \, dm \right| = \left| \int_{V^c} f \, dm \right| \leq m_p(V^c) \leq 1.
\]

Conversely, suppose that \( G \) is polarly quasi-barrelled. Let \( \Phi \) be a strongly bounded subset of \( E' \) and let \( x \in X \). For \( u \in E' \), define \( u_x \) on \( G \) by

\[ u_x(f) = u(f(x)). \]

Then \( u_x \in G' \). The set \( H = \{ u_x : u \in \Phi \} \) is a strongly bounded subset of \( G' \). Indeed, let \( D \) be a bounded subset of \( G \). Since the set \( \{ f(x) : f \in D \} \) is a bounded subset of \( E \), we have that

\[
\sup_{f \in D, \ u \in \Phi} |u_x(f)| = \sup_{f \in D, \ u \in \Phi} |u(f(x))| < \infty.
\]

By our hypothesis, \( H \) is an equicontinuous subset of \( G' \). Thus, there exists a compact subset \( Y \) of \( X \) and \( p \in cs(E) \) such that

\[
\{ f \in G : \| f \|_{Y,p} \leq 1 \}.
\]

But then \( \{ s \in E : p(s) \leq 1 \} \subset \Phi^o \) and so \( \Phi \) is an equicontinuous subset of \( E' \), which proves that \( E \) is polarly quasi-barrelled. Finally, let \( A \) be a \( w_o \)-bounded subset of \( X \) and choose a non-zero element \( x' \) of \( E' \). Let \( p(s) = |x'(s)| \) and consider the set

\[ Z = \{ f \in G : \| f \|_{A,p} \leq 1 \}. \]

Then \( Z \) is a polar set. We will show that \( Z \) is bornivorous. So, suppose that there exists a bounded subset \( M \) of \( G \) which is not absorbed by \( Z \). Then, there exists a sequence \( (f_n) \) in \( M \), \( \| f_n \|_{A,p} > n \). Let

\[ V_n = \{ x : p(f_n(x)) > n \}. \]

Then \( V_n \) intersects \( A \). Since \( A \) is \( w_o \)-bounded, there exists a compact subset \( Y \) of \( X \) such that \( (V_n) \) is not finite on \( Y \), which is a contradiction since \( \sup_{f \in M} \| f \|_{Y,p} < \infty \). This contradiction shows that \( Z \) absorbs bounded
subsets of $G$. In view of our hypothesis, there exist a compact subset $Y$ of $X$ and $q \in cs((E))$ such that
\[ \{ f \in G : \| f \|_{Y,q} \leq 1 \}, \]
which implies that $A \subset Y$ and so $A$ is relatively compact. This clearly completes the proof.

**Corollary 4.13.** (1) $C_c(X)$ is polarly quasi-barrelled iff $X$ is a $w_o$-space.

(2) If $E' \neq \{0\}$, then $C_c(X, E)$ is polarly quasi-barrelled iff both $E$ and $C_c(X)$ are polarly quasi-barrelled.

**References**


P-adic Spaces of Continuous Functions I


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