Bérenger Akon Kpata, Ibrahim Fofana and Konin Koua

Necessary condition for measures which are \((L^q, L^p)\) multipliers


<http://ambp.cedram.org/item?id=AMBP_2009__16_2_339_0>
Necessary condition for measures which are $(L^q, L^p)$ multipliers

BÉRENGER AKON Kpata  
IBRAHIM FOFANA  
KONIN KOUA

Abstract

Let $G$ be a locally compact group and $\rho$ the left Haar measure on $G$. Given a non-negative Radon measure $\mu$, we establish a necessary condition on the pairs $(q, p)$ for which $\mu$ is a multiplier from $L^q(G, \rho)$ to $L^p(G, \rho)$. Applied to $\mathbb{R}^n$, our result is stronger than the necessary condition established by Oberlin in [14] and is closely related to a class of measures defined by Fofana in [7].

When $G$ is the circle group, we obtain a generalization of a condition stated by Oberlin [15] and improve on it in some cases.

Résumé

Soit $G$ un groupe localement compact et $\rho$ la mesure de Haar à gauche sur $G$. Etant donné une mesure de Radon positive $\mu$, nous établissons une condition nécessaire sur les couples $(q, p)$ pour lesquels $\mu$ est un multiplicateur de $L^q(G, \rho)$ dans $L^p(G, \rho)$. Appliqué à $\mathbb{R}^n$, notre résultat est plus fort que la condition nécessaire établie par Oberlin dans [14] et est très lié à une classe de mesures définie par Fofana dans [7].


1. Introduction

We suppose that $G$ is a locally compact group and $\rho$ is the left Haar measure on $G$.

For $1 \leq q < \infty$, a Radon measure $\mu$ on $G$ is said to be $L^q$-improving if there exists a real number $p > q$ such that

$$\mu * f \in L^p(G, \rho) \quad \text{and} \quad \|\mu * f\|_{L^p(G, \rho)} \leq c \|f\|_{L^q(G, \rho)}$$

Keywords: Cantor-Lebesgue measure, $L^q$-improving measure, non-negative Radon measure.

Math. classification: 43A05, 43A15.

339
for all $f \in L^q(G, \rho)$, where $c$ is a real number not depending on $f$.

Of course absolutely continuous measures with Radon-Nikodym derivatives with respect to $\rho$ in $L^r(G, \rho)$ with $\frac{1}{r} + \frac{1}{q} - 1 > 0$ are $L^q$-improving. But $L^q$-improving singular measures also exist.

Bonami \cite{2} showed that all tame Riesz products on the Walsh group are $L^q$-improving, and that was extended to all compact abelian groups by Ritter \cite{16}. Moreover it is well known that on the circle group $T = \mathbb{R}/\mathbb{Z}$, the Cantor-Lebesgue measure $\mu_2^3$ associated with the Cantor set of constant ratio of dissection $\delta > 2$ is $L^q$-improving for $1 < q < \infty$. (See Section 4 for a precise definition of this measure.) This result was proved by Oberlin \cite{12} for $\delta = 3$. Ritter \cite{17}, Beckner, Janson and Jerison \cite{1} proved the same for $\delta$ rational and Christ \cite{3} for $\delta$ irrational.

In fact, Christ has extended the result to Cantor-Lebesgue measures with variable but bounded ratios $2 < \delta_t \leq c$ of dissection.

In this note, we are interested in the following problem: given a non-negative Radon measure $\mu$ on $G$, determine the indices $1 < q < p < \infty$ for which there exists a non-negative constant $c(\mu, q, p)$ such that

$$
\|\mu \ast f\|_{L^p(G, \rho)} \leq c(\mu, q, p) \|f\|_{L^q(G, \rho)}, \quad f \in L^q(G, \rho).
$$

(1.1)

In \cite{15} Oberlin stated the following

**Proposition 1.1.** If the Cantor-Lebesgue measure $\mu_3^2$ associated to the middle third Cantor set satisfies (1.1), then

$$
\frac{1}{q} + \left(1 - \frac{\log 2}{\log 3}\right) \left(1 - \frac{1}{p}\right) \leq 1.
$$

(1.2)

Graham, Hare and Ritter obtained in \cite{9} the following

**Proposition 1.2.** Let $\mu$ be a measure on the circle group $T$ and $1 \leq q < 2$. If there exists a non-negative constant $c(\mu, q)$ such that

$$
\|\mu \ast f\|_{L^2(T)} \leq c(\mu, q) \|f\|_{L^3(T)}, \quad f \in L^q(T),
$$

then there exists a positive real number $K$ such that for any interval $I$ whose endpoints are $x$ and $x + h$, we have

$$
|\mu(I)| \leq K |h|^\frac{1}{q} - \frac{1}{2}.
$$

(1.3)
Measures which are \((L^q, L^p)\) multipliers

Inequality (1.3) means that \(\mu\) satisfies a Lipschitz condition of order \(\frac{1}{q} - \frac{1}{2}\).
Replacing \(\mathbb{T}\) by \(\mathbb{R}^n\), Oberlin proved a similar necessary condition (see the proof of Proposition 2 in [14]).

**Proposition 1.3.** If a non-negative Radon measure on \(\mathbb{R}^n\) satisfies (1.1), then there exists a positive real number \(K\) such that
\[
\mu(R) \leq K |R|^\frac{1}{q} - \frac{1}{p}
\]
for all rectangles \(R\) in \(\mathbb{R}^n\).

In the present paper, we establish the following necessary condition:

**Proposition 1.4.** Suppose that \(\mu\) is a non-negative Radon measure on \(G\) satisfying (1.1). Then for any subsets \(V\) and \(\{x_i / i \in I\}\) of \(G\) such that
i) \(V\) is relatively compact,
ii) \(I\) is countable and \((x_i V) \cap (x_j V) = \emptyset\) for \(i \neq j\),
we have
\[
\rho(V)^{\frac{1}{p}} \left( \sum_{i \in I} \mu(x_i V)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p) \rho\left( V^{-1} V \right)^{\frac{1}{q}}.
\]

We show that all the necessary conditions stated in Proposition 1.1, Proposition 1.2 and Proposition 1.3 follow from Proposition 1.4.

Moreover any non-negative Radon measure \(\mu\) on \(\mathbb{T}\) or \(\mathbb{R}^n\) satisfying the conclusion of Proposition 1.4 belongs to the space \(M^{p, \alpha}\), \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\) (see Notation 3.4 and Section 4 for the definition of \(M^{p, \alpha}\)). In [7], Fofana used these spaces of measures and their subspaces \((L^q, l^p)^\alpha\) to express a necessary condition for Fourier multipliers. He also obtained a generalization of Hausdorff-Young inequality. For other results related to these spaces see [6], [8] and [11].

Inequality (1.2) means exactly that \(\mu_3^2\) belongs to \(M^{p, \alpha}\) where \(\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}\) (see the comment after the proof of Proposition 4.2).

Applied to the Cantor-Lebesgue measure associated to the Cantor set of constant ratio of dissection \(\delta > 3\), Proposition 1.4 yields the following

**Proposition 1.5.** Let \(\delta > 3\) and \(1 < q < p < \infty\). Assume that
\[
\left\| \mu_3^2 \ast f \right\|_{L^p(\mathbb{T})} \leq c \left( \mu_3^2, p, q \right) \left\| f \right\|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}).
\]

341
Then

\[ p \leq \frac{\log \left( \frac{\delta}{2} \right)}{\log \left( \frac{\delta}{3} \right)} q \]  

(1.6)

and

\[ \frac{1}{q} + \left( 1 - \frac{\log 2}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1. \]  

(1.7)

Notice that (1.6) is stronger than (1.7) if \( q > \frac{\log 3}{\log 2} \).

The remainder of this paper is organized as follows: in Section 2 we prove Proposition 1.4 and apply it to \( G = \mathbb{R}^n \) in Section 3. In Section 4 we examine the case \( G = \mathbb{T} \).

2. Proof of Proposition 1.4

Proof. Let \( V \) be a relatively compact subset of \( G \). Then \( f = \chi_{V^{-1}V} \) belongs to \( L^q(G, \rho) \). We have, for all \( i \in I \) and all \( x \in x_iV \),

\[ \mu \ast f(x) = \int_G f(y^{-1}x) d\mu(y) \geq \int_{x_iV} f(y^{-1}x) d\mu(y), \]

\[ y \in x_iV \implies y^{-1}x \in V^{-1}V \quad \text{and} \quad f(y^{-1}x) = 1 \]

and therefore \( \mu \ast f(x) \geq \mu(x_iV) \). It follows that

\[ \int_G (\mu \ast f(x))^p d\rho(x) \geq \sum_{i \in I} \int_{x_iV} (\mu \ast f(x))^p d\rho(x) \geq \sum_{i \in I} \mu(x_iV)^p \rho(x_iV). \]

Therefore

\[ \rho(V)^{\frac{1}{p}} \left( \sum_{i \in I} \mu(x_iV)^p \right)^{\frac{1}{p}} \leq \| \mu \ast f \|_{L^p(G, \rho)} \]

\[ \leq c(\mu, q, p) \| f \|_{L^q(G, \rho)} \]

\[ = c(\mu, q, p) \rho \left( V^{-1}V \right)^{\frac{1}{q}}. \]

This completes the proof. \( \square \)
3. Case $G = \mathbb{R}^n$

Notation 3.1. Let $R$ be a rectangle in $\mathbb{R}^n$ with sides $a_i v_i$, $i = 1, \ldots, n$, where $(v_i)_{1 \leq i \leq n}$ is a direct orthonormal basis in $\mathbb{R}^n$ and $a_i > 0$, $i = 1, \ldots, n$.

For any $r > 0$ and $k = (k_1, \ldots, k_n) \in \mathbb{Z}^n$, set

$$R^r_k = \left\{ \sum_{i=1}^n (k_i r a_i + x_i) v_i / 0 \leq x_i < r a_i, i = 1, \ldots, n \right\}.$$ 

In other words, $R^r_k$ is a rectangle which $i$-th edge is parallel to the vector $v_i$ and of length $r a_i$. Notice that for $r > 0$, the family $\{R^r_k / k \in \mathbb{Z}^n\}$ is a partition of $\mathbb{R}^n$.

Proposition 3.2. Let $1 \leq q \leq p < \infty$. If a non-negative Radon measure $\mu$ on $\mathbb{R}^n$ satisfies (1.1), then for all rectangles $R$ in $\mathbb{R}^n$

$$\sup_{r > 0} (r^n |R|)^{\frac{1}{\alpha} - 1} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p) 2^n$$

where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$.

Proof. Let $r > 0$. Notice that for every $k \in \mathbb{Z}^n$ we have that $R^r_k = R_0 + u_k$, where $R_0 = \left\{ \sum_{i=1}^n x_i v_i / 0 \leq x_i < r a_i, i = 1, \ldots, n \right\}$ and $u_k = \sum_{i=1}^n k_i r a_i v_i$.

It follows from Proposition 1.4 that

$$|R_0 - R_0|^{-\frac{1}{q}} |R_0|^{\frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p).$$

Since $|R_0| = r^n |R|$, we have

$$2^{-\frac{n}{q}} (r^n |R|)^{-\frac{1}{q}} (r^n |R|)^{\frac{1}{p}} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p).$$

Hence

$$(r^n |R|)^{\frac{1}{\alpha} - 1} \left( \sum_{k \in \mathbb{Z}^n} \mu(R^r_k)^p \right)^{\frac{1}{p}} \leq c(\mu, q, p) 2^n.$$ 

The assertion follows. \qed
Remark 3.3. Proposition 1.3 is a direct consequence of Proposition 3.2. In fact, suppose that \( \mu \) satisfies (1.1) and let \( R \) be any rectangle. As \( \{ R_k \mid k \in \mathbb{Z}^n \} \) is a partition of \( \mathbb{R}^n \), \( R \subset \bigcup_{k \in M} R_k \), where \( M \) is a subset of \( \mathbb{Z}^n \) which number of elements does not exceed \( 2^n \). So \( \mu(R) \leq \sum_{k \in M} \mu(R_k) \) and by Hölder inequality we have

\[
|R|^{\frac{1}{p} - \frac{1}{q}} \mu(R) \leq 2^{n\left(1 - \frac{1}{p} + \frac{1}{q}\right)} c(\mu, q, p).
\]

where \( \frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p} \). Thus, by Proposition 3.2 we obtain

\[
|R|^{\frac{1}{p} - \frac{1}{q}} \mu(R) \leq 2^{n\left(1 - \frac{1}{p} + \frac{1}{q}\right)} c(\mu, q, p).
\]

Notation 3.4. For any \( k \in \mathbb{Z}^n \), \( x \in \mathbb{R}^n \) and \( r > 0 \), set

\[
I_r^k = \prod_{i=1}^n [k_i r, (k_i + 1) r] \quad \text{and} \quad J_r^x = \prod_{i=1}^n \left( x_i - \frac{r}{2}, x_i + \frac{r}{2} \right).
\]

Let \( M^0 \) denote the space of Radon measures (not necessarily non-negative) on \( \mathbb{R}^n \). For \( \mu \in M^0 \), \( |\mu| \) stands for its total variation. Let \( 1 \leq \alpha, p \leq \infty \). For \( \mu \in M^0 \) and \( r > 0 \), we set

\[
r \|\mu\|_p = \begin{cases} \left( \sum_{k \in \mathbb{Z}^n} |\mu|(I_k^r)^p \right)^{\frac{1}{p}} & \text{if } 1 \leq p < \infty, \\ \sup_{x \in \mathbb{R}^n} |\mu|(J_x^r) & \text{if } p = \infty \end{cases}
\]

and \( \|\mu\|_{p, \alpha} = \sup_{r > 0} r^n\left(\frac{1}{\alpha} - 1\right) r \|\mu\|_p \).

We define \( M^{p, \alpha}(\mathbb{R}^n) = \{ \mu \in M^0 \mid \|\mu\|_{p, \alpha} < \infty \} \).

Another consequence of Proposition 3.2 is the following
Corollary 3.5. Assume that $1 \leq q \leq p < \infty$ and $\mu$ satisfies (1.1). Then $\mu$ belongs to $M^{p, \alpha}(\mathbb{R}^n)$ where $\frac{1}{\alpha} = 1 - \frac{1}{q} + \frac{1}{p}$.

Proof. It follows by choosing $a_i = 1$ for $i \in \{1, \ldots, n\}$ and $(v_i)_{1 \leq i \leq n} = (e_i)_{1 \leq i \leq n}$ the usual basis of $\mathbb{R}^n$ in the definition of $R_k^r$ in Proposition 3.2. \qed

4. Case $G = \mathbb{T}$

In this section we suppose that $m \geq 2$ is an integer. Let us describe the construction of the Cantor set with variable ratios of dissection and its associated Cantor-Lebesgue measure. We take the interval $[0, 1)$ as a model for $\mathbb{T}$. Let $\delta_t > m$ for $t = 1, 2, \ldots$. Delete from $[0, 1)$, $(m - 1)$ left closed intervals of equal length $\frac{1}{m - 1} \left(1 - \frac{m}{\delta_t}\right)$ so that the $m$ remaining left closed intervals denoted by $E^1_l$, $1 \leq l \leq m$, are equally spaced and have the same length $\frac{1}{\delta_t}$. From each interval $E^1_l$, $1 \leq l \leq m$, delete $(m - 1)$ left closed intervals of equal length $\frac{1}{(m - 1)\delta_1} \left(1 - \frac{m}{\delta_1\delta_2}\right)$ so that the $m$ remaining left closed subintervals $E^2_l$, $1 \leq l \leq m^2$, are equally spaced and have the same length $\frac{1}{\delta_1\delta_2}$. At this stage, the remaining subset of $[0, 1)$ is $C^m_{(\delta_1, \delta_2)} = \bigcup_{l=1}^{m^2} E^2_l$. By iteration, we obtain a sequence of subsets $C^m_{(\delta_1, \delta_2, \ldots, \delta_j)} = \bigcup_{l=1}^{m^j} E^j_l$, where each $E^j_l$ is a left closed interval of length $r_j = \prod_{t=1}^{j} \delta_t^{-1}$. $C^m_{(\delta_t)}$ is the $(m, (\delta_t))$-Cantor set and the $\delta_t$ are called its ratios of dissection. Associated to $C^m_{(\delta_t)}$ in a natural way is a probability measure $\mu^m_{(\delta_t)}$ satisfying $\mu^m_{(\delta_t)}(E^j_l) = \frac{1}{m^j}$ for $j = 1, 2, \ldots$ and for $l = 1, 2, \ldots, m^j$. This measure is the Cantor-Lebesgue measure associated to the $(m, (\delta_t))$-Cantor set. When $\delta_t = \delta$, $t = 1, 2, \ldots$, we write $\mu^m_{(\delta_t)} = \mu^m_{\delta}$. It follows that $\mu^m_{\delta}$ is the usual Cantor-Lebesgue measure associated to the middle third Cantor set. For a detailed exposition on Cantor sets see Zygmund [19].

Notice that if $\mu$ is a non-negative Radon measure on $\mathbb{T}$, then in a natural way, we may identify $\mu$ with a non-negative Radon measure $\nu$ on $\mathbb{R}$ having support in the interval $[0, 1)$. In addition, we have the following result established by Ritter in [17].
Proposition 4.1. Let $1 \leq q \leq p < \infty$, and suppose there is a constant $K > 0$ such that
\[
\|\mu \ast f\|_{L^p(\mathbb{T})} \leq K \|f\|_{L^q(\mathbb{T})}, \quad f \in L^q(\mathbb{T}).
\]
Then there is a constant $K_0 > 0$ such that
\[
\|\nu \ast f\|_{L^p(\mathbb{R})} \leq K_0 \|f\|_{L^q(\mathbb{R})}, \quad f \in L^q(\mathbb{R}).
\]

Defining, for $1 \leq \alpha, p \leq \infty$,
\[
M_{p, \alpha}(\mathbb{T}) = \{\mu \in M_{p, \alpha}(\mathbb{R}) / \text{supp}(\mu) \subset [0, 1)\}
\]
where supp$(\mu)$ denotes the support of $\mu$, it is easy to see that Corollary 3.5 holds in this setting.

The following result gives a characterization of measures $\mu^m_{\delta_t}$ which belong to $M_{p, \alpha}(\mathbb{T})$.

Proposition 4.2. Let $\delta_t > m$, $t = 1, 2, \ldots$. Assume that $1 < \alpha \leq p < \infty$. Then $\mu^m_{\delta_t}$ belongs to $M_{p, \alpha}(\mathbb{T})$ if and only if there exists a constant $c > 0$ such that
\[
j \prod_{t=1}^j \delta_t \leq cm^{\alpha(p-1)/p}, \quad j = 1, 2, \ldots.
\]

In particular, the Cantor-Lebesgue measure $\mu^m_{\delta}$ of constant ratio of dissection $\delta$ belongs to $M_{p, \alpha}(\mathbb{T})$ if and only if
\[
1 - \frac{1}{\alpha} - \frac{\log m}{\log \delta} \left(1 - \frac{1}{p}\right) \leq 0. \quad (4.1)
\]

Proof. a) For all $r \geq 1$
\[
r^{\frac{1}{\alpha} - 1} \left\|\mu^m_{\delta_t}\right\|_p = r^{\frac{1}{\alpha} - 1} \leq 1.
\]
b) Let $j$ be a positive integer and $r_j = \prod_{t=1}^j \delta_t^{-1}$. Recall that for $l = 1, 2, \ldots, m^j$, $|E^j_l| = r_j$ and $\mu^m_{\delta_t}(E^j_l) = \frac{1}{m}$. For each fixed $l$, put $K_l = \{k \in \mathbb{N} / E^j_l \cap I^j_k \neq \emptyset\}$. Then $K_l$ has at most 2 elements. In the same way, for each fixed $k$ in $\mathbb{N}$ set $L_k = \{l \in \{1, 2, \ldots, m^j\} / E^j_l \cap I^j_k \neq \emptyset\}$. 

346
Measures which are \( (L^q, L^p) \) multipliers

Then the number of elements of \( L_k \) is at most 2. We have

\[
m^j m^{-jp} = \sum_{l=1}^{m^j} \mu_{(\delta_t)}^m (E_i^j)^p
\]

\[
= \sum_{l=1}^{m^j} \left( \sum_{k \in K_i} \mu_{(\delta_t)}^m (E_i^j \cap I_k^r) \right)^p
\]

\[
\leq 2^{p-1} \sum_{l \in L_k} \sum_{k \in K_i} \mu_{(\delta_t)}^m (E_i^j \cap I_k^r)^p
\]

\[
= 2^{p-1} \sum_{k \in N} \sum_{l \in L_k} \mu_{(\delta_t)}^m (E_i^j \cap I_k^r)^p
\]

\[
\leq 2^p \sum_{k \in N} \mu_{(\delta_t)}^m (I_k^r)^p.
\]

Then

\[
\left( r_j^{\frac{1}{\alpha} - 1} (m^p - 1) \right)^j = \left( \prod_{t=1}^{j} \delta_t \right)^{1 - \frac{1}{\alpha}} m^{-j(1 - \frac{1}{p})} \leq 2 r_j^{\frac{1}{\alpha} - 1} r_j \| \mu_{(\delta_t)}^m \|_p.
\]

\[
c) \text{ Let } r \in (0, 1). \text{ There exists an integer } j \geq 1 \text{ such that } r_j \leq r < r_{j-1}
\]

where \( r_0 = 1 \) and \( r_n = \prod_{t=1}^{n} \delta_t^{-1} \) for \( n \geq 1 \). Furthermore, each \( I_k^r \) intersects at most \( m \) intervals \( E_i^j \). So \( \mu_{(\delta_t)}^m (I_k^r) \leq m^{-j} m \). The number of \( I_k^r \) which intersect the intervals \( E_i^j \) is at most \( 2m^j \). It follows that

\[
\sum_{k \in N} \mu_{(\delta_t)}^m (I_k^r)^p \leq 2m^j(1-p)m^p.
\]

Hence

\[
r_j^{\frac{1}{\alpha} - 1} r \| \mu_{(\delta_t)}^m \|_p \leq 2 r_j^{\frac{1}{\alpha} - 1} m^j \left( \frac{1}{p} - 1 \right) m
\]

\[
\leq 2 r_j^{\frac{1}{\alpha} - 1} m^j \left( \frac{1}{p} - 1 \right) m
\]

\[
= 2 r_j^{\frac{1}{\alpha} - 1} m^j \left( \prod_{t=1}^{j} \delta_t \right)^{\frac{1}{\alpha} \left( \frac{1}{p} - 1 \right)} m^{\frac{1}{p} - 1}.
\]
Finally,
\[ \mu^m_{(\delta_t)} \in M^{p, \alpha}(T) \iff \sup_j \left( \frac{j^{\frac{1}{p}-1}}{m^{\frac{1}{\alpha}} \prod_{t=1}^{j} \delta_t} \right) < \infty \]
\[ \mu^m_{(\delta_t)} \in M^{p, \alpha}(T) \iff \frac{j^{\alpha(p-1)/p}}{m^{\alpha(p-1)/p}} \prod_{t=1}^{j} \delta_t \leq \frac{c m^{\alpha(p-1)/p}}{j^{\alpha(p-1)/p}}, \quad j = 1, 2, \ldots \]
where \( c \) is a positive constant not depending on \( j \).

d) Now, let \( \delta_t = \delta \) for all \( t \geq 1 \). From c) we know that:
\[ \mu^m_{\delta} \in M^{p, \alpha}(T) \iff \prod_{t=1}^{j} \delta_t \leq \frac{c m^{\alpha(p-1)/p}}{j^{\alpha(p-1)/p}}, \quad j = 1, 2, \ldots \]
where \( c \) is a positive constant not depending on \( j \). That means:
\[ \mu^m_{\delta} \in M^{p, \alpha}(T) \iff \log \delta \leq \frac{\alpha(p-1)}{p(\alpha-1)} \log m \]
\[ \mu^m_{\delta} \in M^{p, \alpha}(T) \iff 1 - \frac{1}{\alpha} - \log m \log \delta \left( 1 - \frac{1}{p} \right) \leq 0. \]

Notice that for \( 1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p} \), (4.1) reduces to (1.2) when \( m = 2 \) and \( \delta = 3 \).

**Proposition 4.3.** Let \( \mu^m_{(\delta_t)} \) be the Cantor-Lebesgue measure with variable ratios \( \delta_t > m \) of dissection. Let \( 1 < q < p < \infty \). Assume that
\[ \left\| \mu^m_{(\delta_t)} \ast f \right\|_{L^p(T)} \leq c \left( \mu^m_{(\delta_t)}, \, p, \, q \right) \left\| f \right\|_{L^q(T)}, \quad f \in L^q(T). \]
Then there exists a constant \( c > 0 \) such that
\[ \prod_{t=1}^{j} \delta_t \leq \frac{c m^{\alpha(p-1)/p}}{j^{\alpha(p-1)/p}}, \quad j = 1, 2, \ldots \]
In particular, if \( \delta_t = \delta \) for all \( t \geq 1 \), then
\[ \frac{1}{q} + \left( 1 - \frac{\log m}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1. \]

**Proof.** Let \( 1 - \frac{1}{\alpha} = \frac{1}{q} - \frac{1}{p} \). Then the desired result follows from Corollary 3.5 and Proposition 4.2.

Proposition 1.1 is obtained from Proposition 4.3 by taking \( m = 2 \) and \( \delta_t = 3 \) for all \( t \geq 1 \).
Measures which are \((L^q, L^p)\) multipliers

**Proof of Proposition 1.5.** We are in the case \(m = 2\) and \(\delta_t = \delta > 3\) for all \(t \geq 1\). Let \(j\) be a positive integer. Observe that for any non-negative integer \(k\), any \(l \in \{1, 2, \ldots, 2^{j+k}\}\), \(E_i^{j+k} = x_i^{j+k} + [0, \delta^{-j-k}]\) and \(\mu_\delta(E_i^{j+k}) = 2^{-j-k}\). Set \(A_0 = [0, \delta^{-j}]\) and \(B_0 = A_0 - A_0 = (-\delta^{-j}, \delta^{-j})\). From Proposition 1.4 we obtain

\[
|B_0|^{-\frac{1}{q}} |A_0|^\frac{1}{p} \left(\sum_{l=1}^{2^j} \mu_\delta(E_i^j)^p\right)^{\frac{1}{p}} \leq c \left(\mu_\delta^2, p, q\right).
\]

Observe that for fixed \(l\) in \(\{1, 2, \ldots, 2^j\}\), \(E_i^j\) contains two intervals \(E_i^{j+1}\) and \(E_i^{j+1}\) satisfying

\[
\mu_\delta^2(E_i^j) = \mu_\delta^2(E_1^{j+1} \cup E_2^{j+1})
\]

and

\[
E_1^{j+1} \cup E_2^{j+1} = x_i^{j+1} + \left([0, \delta^{-j-1}] \cup [\delta^{-j} - \delta^{-j-1}, \delta^{-j}]\right).
\]

Setting \(A_1 = [0, \delta^{-j-1}] \cup [\delta^{-j} - \delta^{-j-1}, \delta^{-j}]\) and applying Proposition 1.4 we obtain

\[
|A_1 - A_1|^{-\frac{1}{q}} |A_1|^\frac{1}{p} \left(\sum_{l=1}^{2^j} \mu_\delta^2(E_i^j)^p\right)^{\frac{1}{p}} \leq c \left(\mu_\delta^2, p, q\right).
\]

But each preceding interval \(E_i^{j+1}, i \in \{1, 2\}\), contains two intervals \(E_i^{j+2}\) and \(E_i^{j+2}\) such that

\[
\mu_\delta^2(E_i^{j+1}) = \mu_\delta^2(E_i^{j+2} \cup E_i^{j+2}) = \frac{1}{2^{j+1}}.
\]

Moreover \(\bigcup_{i=1}^{2} (E_i^{j+2} \cup E_i^{j+2}) = x_i^{j+1} + A_2\) where

\[
A_2 = [0, \delta^{-j-2}] \cup [\delta^{-j} - \delta^{-j-2}, \delta^{-j-1}] \cup [\delta^{-j-1} - \delta^{-j-2}, \delta^{-j-1} + \delta^{-j-2}] \cup [\delta^{-j} - \delta^{-j-2}, \delta^{-j}] + \delta^{-j-2} + \delta^{-j} + \delta^{-j-2} + \delta^{-j-2}.
\]

This remark enables us to apply again Proposition 1.4. Thus we obtain

\[
|A_2 - A_2|^{-\frac{1}{q}} |A_2|^\frac{1}{p} \left(\sum_{l=1}^{2^j} \mu_\delta^2(E_i^j)^p\right)^{\frac{1}{p}} \leq c \left(\mu_\delta^2, p, q\right).
\]

349
The iteration of the process leads us to two sequences of sets \((A_k)_{k \geq 0}\) and \((\widetilde{A}_k)_{k \geq 0}\) defined by:

\[
A_{k+1} = \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right), \quad \widetilde{A}_{k+1} = \frac{1}{\delta} \widetilde{A}_k \cup \left( \delta^{-j} - \frac{1}{\delta} A_k \right)
\]

with \(A_0 = [0, \delta^{-j}], \widetilde{A}_0 = (0, \delta^{-j}]\) and satisfying

\[
|B_k| \leq \frac{1}{\delta} |A_k| \left( \frac{2^j}{\mu^2_\delta} \right)^{1/2} \left( \sum_{l=1}^{2^j} E^j_l \right)^{1/2} \leq c \left( \mu^2_\delta, p, q \right), \tag{4.2}
\]

where \(B_k = A_k - \widetilde{A}_k\) for all \(k \geq 0\).

Notice that \(A_0 - A_0 = \widetilde{A}_0 - \widetilde{A}_0\) and \(|A_0| = |\widetilde{A}_0|\). Furthermore, for any \(k \geq 0\), clearly \(A_{k+1} - A_k = \widetilde{A}_{k+1} - \widetilde{A}_k\) and since \(\frac{1}{\delta} A_k \cap \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) = \emptyset = \frac{1}{\delta} \widetilde{A}_k \cap \left( \delta^{-j} - \frac{1}{\delta} A_k \right)\) we have \(|A_{k+1}| = |\widetilde{A}_{k+1}|\). Thus

\[
A_k - A_k = \widetilde{A}_k - \widetilde{A}_k \quad \text{and} \quad |A_k| = |\widetilde{A}_k|, \quad k \geq 0. \tag{4.3}
\]

Observe that: \(|A_0| = \delta^{-j}, |A_1| = 2\delta^{-j-1}\) and \(|A_2| = 2^2 \delta^{-j-2}\). Suppose that for some integer \(k \geq 0, |A_k| = 2^k \delta^{-j-k}\). By the preceding remarks we get \(|A_{k+1}| = \frac{1}{\delta} |A_k| + \frac{1}{\delta} |\widetilde{A}_k| = \frac{2}{\delta} |A_k| = 2^{k+1} \delta^{-j-(k+1)}\). We conclude that

\[
|A_k| = 2^k \delta^{-j-k}, \quad k \geq 0.
\]

Notice that \(A_0 + \widetilde{A}_0 = (0, \delta^{-j}) = \delta^{-j} + (-\delta^{-j}, \delta^{-j}) = \delta^{-j} - (A_0 - A_0) = \delta^{-j} - B_0\). Furthermore, for any \(k \geq 0\), on the one hand

\[
B_{k+1} = \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) - \frac{1}{\delta} A_k \cup \left( \delta^{-j} - \frac{1}{\delta} \widetilde{A}_k \right) = \frac{1}{\delta} (A_k - A_k) \cup \left( \frac{1}{\delta} \left( A_k + \widetilde{A}_k \right) - \delta^{-j} \right) \cup \\
\quad \cup \left( \delta^{-j} - \frac{1}{\delta} \left( \widetilde{A}_k + A_k \right) \right) \cup \frac{1}{\delta} \left( \widetilde{A}_k - A_k \right) = \frac{1}{\delta} (A_k - A_k) \cup \left( \frac{1}{\delta} \left( A_k + \widetilde{A}_k \right) - \delta^{-j} \right) \cup \left( \delta^{-j} - \frac{1}{\delta} \left( \widetilde{A}_k + A_k \right) \right)
\]

(because of (4.3))
Measures which are \((L^q, L^p)\) multipliers

and on the other hand

\[
A_{k+1} + \tilde{A}_{k+1} = \frac{1}{\delta} (A_k + \tilde{A}_k) \cup \left( \delta^{-j} + \frac{1}{\delta} (A_k - A_k) \right) \cup \\
\cup \left( \delta^{-j} + \frac{1}{\delta} (\tilde{A}_k - \tilde{A}_k) \right) \cup \left( 2\delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k) \right) \\
= \frac{1}{\delta} (A_k + \tilde{A}_k) \cup \left( \delta^{-j} + \frac{1}{\delta} (A_k - A_k) \right) \cup \\
\cup \left( 2\delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k) \right)
\]

(because of (4.3))

and so \(A_{k+1} + \tilde{A}_{k+1} = \delta^{-j} - B_{k+1}\). Thus

\[
\|A_k - A_k\| = \|A_k + \tilde{A}_k\|, \quad k \geq 0.
\]

(4.4)

Notice that for all \(k \geq 0\), the sets \(\frac{1}{\delta} (A_k - A_k), \frac{1}{\delta} (A_k + \tilde{A}_k) - \delta^{-j}\) and \(\delta^{-j} - \frac{1}{\delta} (\tilde{A}_k + A_k)\) form a partition of \(B_{k+1}\). Thus, by (4.4) we have \(|B_{k+1}| = \frac{3}{\delta} |A_k - A_k| = \frac{3}{\delta} |B_k|, k \geq 0\). As \(|B_0| = 2\delta^{-j}\), we conclude that for all \(k \geq 0\), \(|B_k| = \left( \frac{3}{\delta} \right)^k 2\delta^{-j}\).

Finally, using inequality (4.2) we get:

\[
\left( 2 \left( \frac{3}{\delta} \right)^k \delta^{-j} \right)^{-\frac{1}{q}} \left( 2^k \delta^{-j-k} \right)^{\frac{1}{p}} 2^{j\left(\frac{1}{p}-1\right)} \leq c \left( \mu_2^2, p, q \right), \quad k \geq 0, \ j \geq 1
\]

\[
2^{-\frac{1}{q}} \left( 3^{-\frac{1}{q}} \delta^{-\frac{1}{p}} \frac{1}{2^p} \right)^k \left( \delta^{-\frac{1}{p}} \frac{1}{2^p} \right)^j \leq c \left( \mu_2^2, p, q \right), \quad k \geq 0, \ j \geq 1
\]

\[
3^{-\frac{1}{q}} \delta^{-\frac{1}{p}} \frac{1}{2^p} \leq 1 \quad \text{and} \quad \delta^{-\frac{1}{p}} \frac{1}{2^p} \leq 1
\]

\[
p \leq \frac{\log \left( \frac{\delta}{2} \right)}{\log \left( \frac{\delta}{3} \right)} q \quad \text{and} \quad \frac{1}{q} + \left( 1 - \frac{\log 2}{\log \delta} \right) \left( 1 - \frac{1}{p} \right) \leq 1.
\]
References


352
Measures which are \((L^q, L^p)\) multipliers


BÉRENGER AKON KPAT A
UFR Mathématiques et Informatique
Université de Cocody
22 BP 582 Abidjan 22
kpata_akon@yahoo.fr

IBRAHIM FOFANA
UFR Mathématiques et Informatique
Université de Cocody
22 BP 582 Abidjan 22
fofana_ib_math_ab@yahoo.fr

KONIN KOUA
UFR Mathématiques et Informatique
Université de Cocody
22 BP 582 Abidjan 22
kroubla@yahoo.fr