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Abstract

In this paper we obtain the $L^p$-boundedness of Riesz transforms for the Dunkl transform for all $1 < p < \infty$.

1. Introduction

On the Euclidean space $\mathbb{R}^N$, $N \geq 1$, the ordinary Riesz transform $R_j$, $j = 1, ..., N$ is defined as the multiplier operator

$$R_j(f)(\xi) = -i \frac{\xi_j}{\|\xi\|} \hat{f}(\xi).$$

It can also be defined by the principal value of the singular integral

$$R_j(f)(x) = d_0 \lim_{\varepsilon \to 0} \int_{\|x-y\| > \varepsilon} \frac{x_j - y_j}{\|x-y\|} f(y) dy$$

where $d_0 = 2^N \frac{\Gamma\left(\frac{N+1}{2}\right)}{\sqrt{\pi}}$. It follows from the general theory of singular integrals that Riesz transforms are bounded on $L^p(\mathbb{R}^N, dx)$ for all $1 < p < \infty$. What is done in this paper is to extend this result to the context of Dunkl theory where a similar operator is already defined.

Dunkl theory generalizes classical Fourier analysis on $\mathbb{R}^N$. It started twenty years ago with Dunkl’s seminal work [3] and was further developed by several mathematicians. See for instance the surveys [5, 6, 7, 9] and the
references cited therein. The study of the $L^p$-boundedness of Riesz transforms for Dunkl transform on $\mathbb{R}^N$ goes back to the work of S. Thangavelu and Y. Xu [10] where they established boundedness result only in a very special case of $N = 1$. It has been noted in [10] that the difficulty arises in the application of the classical $L^p$- theory of Caldéron-Zygmund, since Riesz transforms are singular integral operators. In this paper we describe how this theory can be adapted in Dunkl setting and gives an $L^p$-result for Riesz transforms for all $1 < p < \infty$. More precisely, through the fundamental result of M. Rösler [6] for the Dunkl translation of radial functions, we reformulate a Hörmander type condition for singular integral operators. The Riesz kernel is given by acting Dunkl operator on Dunkl translation of radial function.

This paper is organized as follows. In Section 2 we present some definitions and fundamental results from Dunkl’s analysis. The Section 3 is devoted to proving $L^p$-boundedness of Riesz transforms. As applications, we will prove a generalized Riesz and Sobolev inequalities. Throughout this paper $C$ denotes a constant which can vary from line to line.

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2. Preliminaries

In this section we collect notations and definitions and recall some basic facts. We refer to [5, 3, 6, 7, 9].

Let $G \subset O(\mathbb{R}^N)$ be a finite reflection group associated to a reduced root system $R$ and $k : R \rightarrow [0, +\infty)$ be a $G$-invariant function (called multiplicity function). Let $R_+$ be a positive root subsystem. We shall assume that $R$ is normalized in the sense that $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$ for all $\alpha \in R$, where $\langle , \rangle$ is the standard Euclidean scalar product on $\mathbb{R}^N$.

The Dunkl operators $T_\xi$, $\xi \in \mathbb{R}^N$ are the following $k$–deformations of directional derivatives $\partial_\xi$ by difference operators:

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha . x)}{\langle \alpha, x \rangle}, \quad x \in \mathbb{R}^N$$
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where $\sigma_\alpha$ denotes the reflection with respect to the hyperplane orthogonal to $\alpha$. For the standard basis vectors of $\mathbb{R}^N$, we simply write $T_j = T_{e_j}$.

The operators $\partial_\xi$ and $T_\xi$ are intertwined by a Laplace–type operator

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) \, d\mu_x(y),$$

associated to a family of compactly supported probability measures

$$\{ \mu_x \mid x \in \mathbb{R}^N \}.$$

Specifically, $\mu_x$ is supported in the the convex hull $\text{co}(G.x)$.

For every $\lambda \in \mathbb{C}^N$, the simultaneous eigenfunction problem,

$$T_\xi f = \langle \lambda, \xi \rangle f, \quad \xi \in \mathbb{R}^N$$

has a unique solution $f(x) = E_k(\lambda, x)$ such that $E_k(\lambda, 0) = 1$, which is given by

$$E_k(\lambda, x) = V_k(e^{(\lambda, \cdot)})(x) = \int_{\mathbb{R}^N} e^{(\lambda, y)} \, d\mu_x(y), \quad x \in \mathbb{R}^N.$$  

Furthermore $\lambda \mapsto E_k(\lambda, x)$ extends to a holomorphic function on $\mathbb{C}^N$.

Let $m_k$ be the measure on $\mathbb{R}^N$, given by

$$dm_k(x) = \prod_{\alpha \in R_+} |(\alpha, x)|^{2k(\alpha)} \, dx.$$  

For $f \in L^1(m_k)$ (the Lebesgue space with respect to the measure $m_k$) the Dunkl transform is defined by

$$\mathcal{F}_k(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x) E_k(-i \xi, x) dm_k(x), \quad c_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} \, dm_k(x).$$

This new transform shares many analogous properties of the Fourier transform.

(i) The Dunkl transform is a topological automorphism of $\mathcal{S}(\mathbb{R}^N)$ (Schwartz space).

(ii) (Plancherel Theorem) The Dunkl transform extends to an isometric automorphism of $L^2(m_k)$.

(iii) (Inversion formula) For every $f \in L^1(m_k)$ such that $\mathcal{F}_k f \in L^1(m_k)$, we have

$$f(x) = \mathcal{F}_k^2 f(-x), \quad x \in \mathbb{R}^N.$$
(iv) For all $\xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$
\[ F_k(T_\xi(f))(x) = <i\xi, x > F_k(f)(x), \quad x \in \mathbb{R}^N. \] (2.1)

Let $x \in \mathbb{R}^N$, the Dunkl translation operator $\tau_x$ is defined on $L^2(m_k)$ by,
\[ F_k(\tau_x(f))(y) = E_k(ix,y)F_kf(y), \quad y \in \mathbb{R}^N. \] (2.2)

If $f$ is a continuous radial function in $L^2(m_k)$ with $f(y) = \tilde{f}(\|y\|)$, then
\[ \tau_x(f)(y) = \int_{\mathbb{R}^N} \tilde{f}(\sqrt{\|x\|^2 + \|y\|^2 + 2 < y, \eta >})d\mu_x(\eta). \] (2.3)

This formula is first proved by M. Rösler [6] for $f \in \mathcal{S}(\mathbb{R}^N)$ and recently is extended to continuous functions by F. and H. Dai Wang [2].

We collect below some useful facts :

(i) For all $x, y \in \mathbb{R}^N$,
\[ \tau_x(f)(y) = \tau_y(f)(x). \] (2.4)

(ii) For all $x, \xi \in \mathbb{R}^N$ and $f \in \mathcal{S}(\mathbb{R}^N)$,
\[ T_\xi \tau_x(f) = \tau_x T_\xi(f). \] (2.5)

(iii) For all $x \in \mathbb{R}^N$ and $f, g \in L^2(m_k)$,
\[ \int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)dm_k(y) = \int_{\mathbb{R}^N} f(y)\tau_xg(-y)dm_k(y). \] (2.6)

(iv) For all $x \in \mathbb{R}^N$ and $1 \leq p \leq 2$, the operator $\tau_x$ can be extended to all radial functions $f$ in $L^p(m_k)$ and the following holds
\[ \| \tau_x(f) \|_{p,k} \leq \| f \|_{p,k}. \] (2.7)

$\| . \|_{p,k}$ is the usual norm of $L^p(m_k)$.


In Dunkl setting the Riesz transforms (see [10]) are the operators $\mathcal{R}_j$, $j=1...N$ defined on $L^2(m_k)$ by
\[ \mathcal{R}_j(f)(x) = d_k \lim_{\varepsilon \to 0} \int_{|y| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^p} dm_k(y), \quad x \in \mathbb{R}^N. \]
where
\[ d_k = 2^{p_k - 1} \frac{\Gamma(p_k/2)}{\sqrt{\pi}}; \quad p_k = 2\gamma_k + N + 1 \quad \text{and} \quad \gamma_k = \sum_{\alpha \in \mathbb{R}^+} k(\alpha). \]

It has been proved by S. Thangavelu and Y. Xu [10], that \( \mathcal{R}_j \) is a multiplier operator given by
\[ \mathcal{F}_k(\mathcal{R}_j(f))(\xi) = -i \frac{\xi_j}{\|\xi\|} \mathcal{F}_k(f)(\xi), \quad f \in \mathcal{S}(\mathbb{R}^N), \; \xi \in \mathbb{R}^N, \tag{3.1} \]

The authors state that if \( N = 1 \) and \( 2\gamma_k \in \mathbb{N} \) the operator \( \mathcal{R}_j \) is bounded on \( L^p(m_k) \), \( 1 < p < \infty \). In [1] this result is improved by removing \( 2\gamma_k \in \mathbb{N} \), where Riesz transform is called Hilbert transform. If \( \gamma_k = 0 \) \( (k = 0) \), this operator coincides with the usual Riesz transform \( R_j \) given by (1.1). Our interest is to prove the boundedness of this operator for \( N \geq 2 \) and \( k \geq 0 \). To do this, we invoke the theory of singular integrals. Our basic is the following,

**Theorem 3.1.** Let \( \mathcal{K} \) be a measurable function on \( \mathbb{R}^N \times \mathbb{R}^N \setminus \{(x,g.x); \; x \in \mathbb{R}^N, \; g \in G\} \) and \( S \) be a bounded operator from \( L^2(m_k) \) into itself, associated with a kernel \( \mathcal{K} \) in the sense that
\[ S(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x,y)f(y)dm_k(y), \tag{3.2} \]
for all compactly supported function \( f \) in \( L^2(m_k) \) and for a.e \( x \in \mathbb{R}^N \) satisfying \( g.x \notin \text{supp}(f) \), for all \( g \in G \). If \( \mathcal{K} \) satisfies
\[ \int_{\min_{g \in G} \|g.x-y\| > 2\|y-y_0\|} |\mathcal{K}(x,y) - \mathcal{K}(x,y_0)|dm_k(x) \leq C, \quad y, y_0 \in \mathbb{R}^N, \tag{3.3} \]
then \( S \) extends to a bounded operator from \( L^p(m_k) \) into itself for all \( 1 < p \leq 2 \).

**Proof.** We first note that \( (\mathbb{R}^N, m_k) \) is a space of homogenous type, that is, there is a fixed constant \( C > 0 \) such that
\[ m_k(B(x,2r)) \leq Cm_k(B(x,r)), \quad \forall x \in \mathbb{R}^N, \; r > 0 \tag{3.4} \]
where \( B(x,r) \) is the closed ball of radius \( r \) centered at \( x \) (see [8], Ch 1). Then we can adapt to our context the classical technic which consist to show that \( S \) is weak type \((1,1)\) and conclude by Marcinkiewicz interpolation theorem.
In fact, the Calderón-Zygmund decomposition says that for all \( f \in L^1(m_k) \cap L^2(m_k) \) and \( \lambda > 0 \), there exist a decomposition of \( f \), \( f = h + b \) with \( b = \sum_j b_j \) and a sequence of balls \((B(y_j, r_j))_j = (B_j)_j\) such that for some constant \( C \), depending only on the multiplicity function \( k \)

(i) \( \|h\|_\infty \leq C\lambda \);

(ii) \( \text{supp}(b_j) \subset B_j \);

(iii) \( \int_{B_j} b_j(x) dm_k(x) = 0 \);

(iv) \( \|b_j\|_{1,k} \leq C \lambda m_k(B_j) \);

(v) \( \sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda} \).

The proof consists in showing the following inequality hold for \( w = h \) and \( w = b \):

\[
\rho_\lambda(S(w)) = m_k(\{x \in \mathbb{R}^N; |S(w)(x)| > \frac{\lambda}{2}\}) \leq C \frac{\|f\|_{1,k}}{\lambda}. \tag{3.5}
\]

By using the \( L^2 \)-boundedness of \( S \) we get

\[
\rho_\lambda(S(h)) \leq \frac{4}{\lambda^2} \int_{\mathbb{R}^N} |S(h)(x)|^2 dm_k(x) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^N} |h(x)|^2 dm_k(x). \tag{3.6}
\]

From (i) and (v),

\[
\int_{\bigcup B_j} |h(x)|^2 dm_k(x) \leq C\lambda^2 \mu_k(\bigcup B_j) \leq C\lambda\|f\|_{1,k}. \tag{3.7}
\]

Since on \((\bigcup B_j)^c\), \( f(x) = h(x) \), then

\[
\int_{(\bigcup B_j)^c} |h(x)|^2 dm_k(x) \leq C\lambda\|f\|_{1,k}. \tag{3.8}
\]

From (3.6), (3.7) and (3.8), the inequality (3.5) is satisfied for \( h \).

Next we turn to the inequality (3.5) for the function \( b \). Consider

\[
B_j^* = B(y_j, 2r_j); \quad \text{and} \quad Q_j^* = \bigcup_{g \in G} g.B_j^*.
\]
Then

\[ \rho_\lambda(S(b)) \leq m_k \left( \bigcup_j Q_j^* \right) + m_k \{ x \in \left( \bigcup_j Q_j^* \right)^c ; |S(b)(x)| > \frac{\lambda}{2} \}. \]

Now by (3.4) and (v)

\[ m_k \left( \bigcup_j Q_j^* \right) \leq |G| \sum_j m_k(B_j^*) \leq C \sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda}. \]

Furthermore if \( x \notin Q_j^* \), we have

\[ \min_{g \in G} \|g.x - y_j\| > 2\|y - y_j\|, \quad y \in B_j. \]

Thus, from (3.2), (iii), (ii), (3.3), (iv) and (v)

\[ \int_{(\bigcup_j Q_j^*)^c} |S(b)(x)| dm_k(x) \]

\[ \leq \sum_j \int_{(Q_j^*)^c} |S(b_j)(x)| dm_k(x) \]

\[ = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} K(x, y)b_j(y) dm_k(y) \right| dm_k(x) \]

\[ = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} b_j(y) \left( K(x, y) - K(x, y_j) \right) dm_k(y) \right| dm_k(x) \]

\[ \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{(Q_j^*)^c} |K(x, y) - K(x, y_j)| dm_k(x) dm_k(y) \]

\[ \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{\min_{g \in G} \|g.x - y_j\| > 2\|y - y_j\|} |K(x, y) - K(x, y_j)| dm_k(x) dm_k(y) \]

\[ \leq C \sum_j \|b_j\|_{1,k} \]

\[ \leq C \|f\|_{1,k}. \]

Therefore,

\[ m_k \{ x \in \left( \bigcup_j Q_j^* \right)^c ; |S(b)(x)| > \frac{\lambda}{2} \} \]

\[ \leq \frac{2}{\lambda} \int_{(\bigcup_j Q_j^*)^c} |S(b)(x)| dm_k(x) \leq C \frac{\|f\|_{1,k}}{\lambda}. \]

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This achieves the proof of (3.5) for $b$. $\square$

Now, we will give an integral representation for the Riesz transform $R_j$. For this end, we put for $x, y \in \mathbb{R}^N$ and $\eta \in \text{co}(G.x)$

$$A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 < y, \eta >} = \sqrt{\|y - \eta\|^2 + \|x\|^2 - \|\eta\|^2}.$$ 

It is easy to check that

$$\min_{g \in G} \|g.x - y\| \leq A(x, y, \eta) \leq \max_{g \in G} \|g.x - y\|. \quad (3.9)$$

The following inequalities are clear

$$\left| \frac{\partial A^\ell}{\partial y_r}(x, y, \eta) \right| \leq C A^{\ell-1}(x, y, \eta),$$

$$\left| \frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x, y, \eta) \right| \leq C A^{\ell-2}(x, y, \eta) \quad (3.10)$$

and for $\alpha \in \mathbb{R}^+$,

$$\left| \frac{\partial A^\ell}{\partial y_r}(x, \sigma_\alpha.y, \eta) \right| \leq C A^{\ell-1}(x, \sigma_\alpha.y, \eta),$$

$$\left| \frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x, \sigma_\alpha.y, \eta) \right| \leq C A^{\ell-2}(x, \sigma_\alpha.y, \eta), \quad (3.11)$$

where $r, s = 1, \ldots, N$ and $\ell \in \mathbb{R}$.

Let us set

$$K^{(1)}_j(x, y) = \int_{\mathbb{R}^N} \frac{\eta_j - y_j}{A^{pk}(x, y, \eta)} d\mu_k(\eta),$$

$$K^{(\alpha)}_j(x, y) = \frac{1}{<y, \alpha>} \int_{\mathbb{R}^N} \left[ \frac{1}{A^{pk-2}(x, y, \eta)} - \frac{1}{A^{pk-2}(x, \sigma_\alpha.y, \eta)} \right] d\mu_k(\eta),$$

$$K_j(x, y) = d_k \left\{ K^{(1)}_j(x, y) + \sum_{\alpha \in \mathbb{R}^+} \frac{k(\alpha) \alpha_j}{p_k - 2} K^{(\alpha)}_j(x, y) \right\},$$

where $\alpha \in \mathbb{R}^+$.

**Proposition 3.2.** If $f \in L^2(m_k)$ with compact support, then for all $x \in \mathbb{R}^N$ satisfying $g.x \notin \text{supp}(f)$ for all $g \in G$, we have

$$R_j(f)(x) = \int_{\mathbb{R}^N} K_j(x, y) f(y) dm_k(y).$$

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Proof. Let \( f \in L^2(m_k) \) be a compact supported function and \( x \in \mathbb{R}^N \), such that \( g.x \notin \text{supp}(f) \) for all \( g \in G \). For \( 0 < \varepsilon < \min \min_{g \in G} \min_{y \in \text{supp}(f)} |g.x - y| \) and \( n \in \mathbb{N} \), we consider \( \tilde{\varphi}_{n, \varepsilon} \) a \( C^\infty \)-function on \( \mathbb{R} \), such that:

- \( \tilde{\varphi}_{n, \varepsilon} \) is odd .
- \( \tilde{\varphi}_{n, \varepsilon} \) is supported in \( \{ t \in \mathbb{R}; |t| \leq n + 1 \} \).
- \( \tilde{\varphi}_{n, \varepsilon} = 1 \) in \( \{ t \in \mathbb{R}; \varepsilon + \frac{1}{n} \leq t \leq n \} \).
- \( |\tilde{\varphi}_{n, \varepsilon}| \leq 1 \).

Let
\[
\tilde{\varphi}_{n, \varepsilon}(t) = \int_{-\infty}^{t} \tilde{\varphi}_{n, \varepsilon}(u) \frac{1}{|u|^{p_k-1}} du \quad \text{and} \quad \phi_{n, \varepsilon}(y) = \tilde{\varphi}_{n, \varepsilon}(|y|),
\]
for \( t \in \mathbb{R} \) and \( y \in \mathbb{R}^N \). Clearly, \( \phi_{n, \varepsilon} \) is a \( C^\infty \) radial function supported in the ball \( B(0, n + 1) \) and
\[
\lim_{n \to +\infty} \tilde{\varphi}_{n, \varepsilon}(|y|) = 1, \quad \forall \ y \in \mathbb{R}^N, \ |y| > \varepsilon.
\]
The dominated convergence theorem, (2.5) and (2.6) yield
\[
\int_{|y| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{|y|^{p_k}} dm_k(y)
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) \frac{y_j}{|y|^{p_k}} \tilde{\varphi}_{n, \varepsilon}(|y|) dm_k(y)
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) T_j(\phi_{n, \varepsilon})(y) dm_k(y)
\]
\[
= \lim_{n \to \infty} \int_{\mathbb{R}^N} f(y) T_j(\phi_{n, \varepsilon})(-y) dm_k(y).
\]
Now we have
\[
T_j \tau_x(\phi_{n, \varepsilon})(-y)
\]
\[
= \int_{\mathbb{R}^N} \left( \frac{(\eta_j - y_j) \tilde{\varphi}_{n, \varepsilon}(A(x, y, \eta))}{A^{p_k}(x, y, \eta)} \right) d\mu_x(\eta)
\]
\[
+ \sum_{\alpha \in \mathbb{R}^+} k(\alpha) \alpha_j \int_{\mathbb{R}^N} \frac{\tilde{\varphi}_{n, \varepsilon}(A(x, \sigma_\alpha y, \eta)) - \tilde{\varphi}_{n, \varepsilon}(A(x, y, \eta))}{< y, \alpha >} d\mu_x(\eta),
\]
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where from (3.9)

\[ \varepsilon < A(x, y, \eta); \quad \varepsilon < A(x, \sigma_\alpha y, \eta), \quad y \in \text{supp}(f), \quad \eta \in \text{co}(G.x). \]

Then with the aid of dominated convergence theorem

\[ \lim_{n \to \infty} T_j \tau_x (\phi_{n, \varepsilon})(-y) = \frac{1}{d_k} K_j(x, y), \]

and

\[ d_k \int_{\|y\| \geq \varepsilon} \tau_x (f)(-y) \frac{y_j}{\|y\|_p} dm_k(y) = \int_{\mathbb{R}^N} K_j(x, y) f(y) dm_k(y). \]

Letting \( \varepsilon \to 0 \), it follows that

\[ R_j(f)(x) = \int_{\mathbb{R}^N} K_j(x, y) f(y) dm_k(y), \]

which proves the result.

Now, we are able to state our main result.

**Theorem 3.3.** The Riesz transform \( R_j, j = 1 \ldots N \), is a bounded operator from \( L^p(m_k) \) into itself, for all \( 1 < p < \infty \).

**Proof.** Clearly, from (3.1) and Plancherel’s theorem \( R_j \) is bounded from \( L^2(m_k) \) into itself, with adjoint operator \( R_j^* = -R_j \). Thus, via duality it’s enough to consider the range \( 1 < p \leq 2 \) and apply Theorem 3.1. In view of Proposition 3.2 it only remains to show that \( K_j \) satisfies condition (3.3).

Let \( y, y_0 \in \mathbb{R}^N, y \neq y_0 \) and \( x \in \mathbb{R}^N \), such that

\[ \min_{g \in G} \|g.x - y\| > 2\|y - y_0\|. \quad (3.12) \]

By mean value theorem,
\[ |K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)| = \left| \sum_{i=0}^{N} (y_i - (y_0)_i) \int_0^1 \frac{\partial K_j^{(1)}}{\partial y_i}(x, y_t) \, dt \right| \]
\[ = \left| \sum_{i=0}^{N} (y_i - (y_0)_i) \int_0^1 \int_{\mathbb{R}^N} \left( \frac{\delta_{i,j}}{A^k(x, y_t, \eta)} + \frac{p_k((y_t)_i - \eta_i)(\eta_j - (y_t)_j)}{A^k + 2(x, y_t, \eta)} \right) d\mu_x(\eta) \right| \]
\[ \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} 1 \frac{1}{A^k + 2(x, y_t, \eta)} d\mu_x(\eta) dt. \]

where \(y_t = y_0 + t(y - y_0)\) and \(\delta_{i,j}\) is the Kronecker symbol.

In view of (3.9) and (3.12), we obtain
\[ \|y - y_0\| < A(x, y_t, \eta), \quad \eta \in \text{co}(G.x). \]

Therefore,
\[ |K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)| \]
\[ \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \frac{1}{\left( \|y - y_0\|^2 + A^2(x, y_t, \eta) \right)^{\frac{p_k}{2}}} d\mu_x(\eta) dt. \]
\[ \leq C \|y - y_0\| \int_0^1 \tau_x(\psi)(y_t) dt \]

where \(\psi\) is the function defined by
\[ \psi(z) = \frac{1}{\left( \|y - y_0\|^2 + \|z\|^2 \right)^{\frac{p_k}{2}}}, \quad z \in \mathbb{R}^N. \]

Using Fubini’s theorem, (2.4) and (2.7), we get
\[ \int_{\min{g \in G} \|g.x - y\| > 2\|y - y_0\|} |K_j^{(1)}(x, y) - K_j^{(1)}(x, y_0)| dm_k(x) \]
\[ \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \tau_{-y_t}(\psi)(x) dm_k(x) dt \]
\[ \leq C \|y - y_0\| \int_{\mathbb{R}^N} \psi(z) dm_k(z) = C \int_{\mathbb{R}^N} \frac{du}{(1 + u^2)^{\frac{p_k}{2}}} = C'. \]

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This established the condition (3.3) for $\mathcal{K}_j^{(1)}$.

To deal with $\mathcal{K}_j^{(\alpha)}$, $\alpha \in R_+$, we put for $x, y \in \mathbb{R}^N$, $\eta \in co(G.x)$ and $t \in [0, 1]$

\[
U(x, y, \eta) = A^{2p_k-4}(x, y, \eta), \\
V_\alpha(x, y, \eta) = A^{p_k-2} A^{p_k-2}_\alpha (A^{p_k-2} + A^{p_k-2}_\alpha), \\
h_{\alpha, t}(y) = y + t(\sigma_\alpha y - y) = y - t < y, \alpha>, \alpha > \alpha,
\]

By mean value theorem we have

\[
\mathcal{K}_j^{(\alpha)}(x, y) = \int_{\mathbb{R}^N} \frac{1}{<y, \alpha>} \frac{U(x, \sigma_\alpha y, \eta) - U(x, y, \eta)}{V_\alpha(x, y, \eta)} d\mu_x(\eta)
\]

\[
= -\int_{\mathbb{R}^N} \int_0^1 \frac{\partial_\alpha U(x, h_{\alpha, t}(.), \eta)}{V_\alpha(x, y, \eta)} dt d\mu_x(\eta)
\]

and

\[
\mathcal{K}_j^{(\alpha)}(x, y) - \mathcal{K}_j^{(\alpha)}(x, y_0)
\]

\[
= \int_{\mathbb{R}^N} \int_0^1 \int_0^1 \partial_{y-y_0} \left( \frac{\partial_\alpha U(x, h_{\alpha, t}(.), \eta)}{V_\alpha(x, ., \eta)} \right)(y_\theta) d\theta dt d\mu_x(\eta). \tag{3.13}
\]

Here the derivations are taken with respect to the variable $y$.

To simplify, let us denote by

\[
A = A(x, y_\theta, \eta); \quad A_\alpha = A(x, \sigma_\alpha y_\theta, \eta)
\]

Then using (3.10) and the fact

\[
\|\eta - h_{\alpha, t}(y_\theta)\| \leq \max(\|\eta - y_\theta\|, \|\eta - \sigma_\alpha(y_\theta)\|),
\]

we obtain

\[
\left| \frac{\partial U}{\partial y_r}(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C \left( A^{2p_k-5} + A^{2p_k-5}_\alpha \right)
\]

\[
\left| \frac{\partial^2 U}{\partial y_r \partial y_s}(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C \left( A^{2p_k-6} + A^{2p_k-6}_\alpha \right), \quad r, s = 1, ..., N.
\]

This gives us the following estimates

\[
\left| \partial_\alpha U(x, h_{\alpha, t}(y_\theta), \eta) \right| \leq C \left( A^{2p_k-5} + A^{2p_k-5}_\alpha \right), \tag{3.14}
\]

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By (3.10) and (3.11), we also have

\[
\left| \frac{\partial V}{\partial y_r} ((x, y_\theta, \eta)) \right| \leq CA_{p-3} A_{\alpha-3}^p (A_{p-2}^p + A_{\alpha-2}^p) (A + A_\alpha).
\]

The elementary inequality \( \frac{u + v}{u^\ell + v^\ell} \leq \frac{3}{u^{\ell-1} + v^{\ell-1}}, \ u, v > 0, \ \ell \geq 1, \) leads to

\[
\left| \frac{\partial y - y_0}{V_\alpha(x, y_\theta, \eta)} \right| \leq C \|y - y_0\| \frac{A_\alpha + A}{A_{p-1}^p A_{\alpha-1}^p (A_{p-2}^p + A_{\alpha-2}^p)} \leq C \|y - y_0\| \frac{1}{A_{p-1}^p A_{\alpha-1}^p (A_{p-3}^p + A_{\alpha-3}^p)}. \tag{3.16}
\]

Now (3.14), (3.15) and (3.16) yield

\[
\left| \frac{\partial y - y_0}{U(x, h_{\alpha, t}(\cdot), \eta)} \right| \leq C \|y - y_0\| \frac{A_{2p_k-6}^2 + A_{2p_k-6}^2}{A_{p-2}^p A_{\alpha-2}^p (A_{p-2}^p + A_{\alpha-2}^p)} + C \|y - y_0\| \frac{1}{A_{p-1}^p (A_{p-3}^p + A_{\alpha-3}^p)} \leq C \|y - y_0\| \frac{1}{A_{p-1}^p + A_{\alpha-1}^p} \leq C \|y - y_0\| \frac{1}{A_{p-1} + A_{\alpha-1}^p}.
\]

where in the last equality we have used the fact that \( \frac{1}{uv^{\ell-1}} \leq \frac{1}{u^\ell} + \frac{1}{v^{\ell}}, \ u, v > 0 \) and \( \ell \geq 1. \)
Thus, in view of (3.13),

\[
\left| K^{(\alpha)}_j(x,y) - K^{(\alpha)}_j(x,y_0) \right| \\
\leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \left[ \frac{1}{A_p(x,y_{\theta},\eta)} + \frac{1}{A_{p_k}(x,\sigma_{\alpha}y_{\theta},\eta)} \right] d\mu_x(\eta) d\theta.
\]

Then by same argument as for $K^{(1)}_j$ we obtain

\[
\int_{\min_{g \in G} \|g.x - y\| > 2\|y - y_0\|} |K^{(2)}_j(x,y) - K^{(2)}_j(x,y_0)| dm_k(x) \leq C,
\]

which established the condition (3.3) for the kernel $K^{(\alpha)}_j$ and furnishes the proof. □

As applications, we will prove a generalized Riesz and Sobolev inequalities

**Corollary 3.4** (Generalized Riesz inequalities). For all $1 < p < \infty$ there exists a constant $C_p$ such that

\[
\|T_rT_s(f)\|_{k,p} \leq C_p\|\Delta_k f\|_{k,p}, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N), \tag{3.17}
\]

where $\Delta_k$ is the Dunkl laplacian: $\Delta_k f = \sum_{r=1}^N T_r^2(f)$

**Proof.** From (2.1) and (3.1) one can see that

\[
T_rT_s(f) = \mathcal{R}_r\mathcal{R}_s(-\Delta_k)(f), \quad r, s = 1...N, \ f \in \mathcal{S}(\mathbb{R}^N).
\]

Then (3.17) is concluded by Theorem 3.3. □

**Corollary 3.5** (Generalized Sobolev inequality). For all $1 < p \leq q < 2\gamma(k) + N$ with $\frac{1}{q} = \frac{1}{p} - \frac{1}{2\gamma(k) + N}$, we have

\[
\|f\|_{q,k} \leq C_{p,q} \|\nabla_k f\|_{p,k} \tag{3.18}
\]

for all $f \in \mathcal{S}(\mathbb{R}^N)$. Here $\nabla_k f = (T_1 f, ..., T_N f)$ and $|\nabla_k f| = (\sum_{r=1}^N |T_r f|^2)^{\frac{1}{2}}$. 

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Riesz transforms for Dunkl transform

Proof. For all \( f \in \mathcal{S}(\mathbb{R}^N) \), we write

\[
\mathcal{F}_k(f)(\xi) = \frac{1}{\|\xi\|} \sum_{r=1}^{N} \frac{-i\xi_r}{\|\xi\|} \left( i\xi_r \mathcal{F}_k(f)(\xi) \right)
\]

\[
= \frac{1}{\|\xi\|} \sum_{r=1}^{d} \frac{-i\xi_r}{\|\xi\|} \left( \mathcal{F}_k(T_r f)(\xi) \right).
\]

This yields to the following identity

\[
f = I_k^1 \left( \sum_{j=1}^{N} \mathcal{R}_j(T_j f) \right),
\]

where

\[
I_k^\beta(f)(x) = (d_k^\beta)^{-1} \int_{\mathbb{R}^N} \frac{\tau_y f(x)}{\|\tau_y\|^{2\gamma(k)+N-\beta}} dm_k(y),
\]

here

\[
d_k^\beta = 2^{-\gamma(k)-N/2+\beta} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\gamma(k) + \frac{N-\beta}{2})}.
\]

Theorem 1.1 of [4] asserts that \( I_k^\beta \) a bounded operator from \( L^p(m_k) \) to \( L^q(m_k) \). Then (3.18) follows from Theorem 3.3.

References


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