On $1$-cocycles induced by a positive definite function on a locally compact abelian group


<http://ambp.cedram.org/item?id=AMBP_2014__21_1_61_0>
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Abstract

For \( \varphi \) a normalized positive definite function on a locally compact abelian group \( G \), let \( \pi_\varphi \) be the unitary representation associated to \( \varphi \) by the GNS construction. We give necessary and sufficient conditions for the vanishing of 1-cohomology \( H^1(G, \pi_\varphi) \) and reduced 1-cohomology \( \overline{H}^1(G, \pi_\varphi) \). For example, \( \overline{H}^1(G, \pi_\varphi) = 0 \) if and only if either \( \text{Hom}(G, \mathbb{C}) = 0 \) or \( \mu_\varphi(1_G) = 0 \), where \( 1_G \) is the trivial character of \( G \) and \( \mu_\varphi \) is the probability measure on the Pontryagin dual \( \hat{G} \) associated to \( \varphi \) by Bochner’s Theorem. This streamlines an argument of Guichardet (see Theorem 4 in [7]).

Sur les 1-cocycles induits par une fonction de type positif sur un groupe abélien localement compact

Résumé

Soit \( \varphi \) une fonction de type positif normalisée sur un groupe localement compact abélien \( G \), et \( \pi_\varphi \) la représentation unitaire de \( G \) obtenue par construction GNS. Nous donnons des conditions nécessaires et suffisantes pour l’annulation de la 1-cohomologie \( H^1(G, \pi_\varphi) \) et de la 1-cohomologie réduite \( \overline{H}^1(G, \pi_\varphi) \). Par exemple, \( \overline{H}^1(G, \pi_\varphi) = 0 \) si et seulement si ou bien \( \text{Hom}(G, \mathbb{C}) = 0 \) ou bien \( \mu_\varphi(1_G) = 0 \), où \( 1_G \) est le caractère trivial de \( G \) et \( \mu_\varphi \) est la mesure de probabilité sur le dual de Pontryagin \( \hat{G} \) associée à \( \varphi \) par le théorème de Bochner. Cela simplifie un argument de Guichardet (Théorème 4 de [7]).

1. Introduction

The Gel’fand-Naimark-Segal construction (see [1]) provides a correspondence between positive definite functions \( \varphi \) on a locally compact
group $G$ and cyclic representations $\pi_\varphi$ on Hilbert space. This allows one to construct a dictionary between the functional-analytic and algebro-geometric pictures of $\varphi$ and $\pi_\varphi$. For example, $\varphi$ is an extreme point in the cone $P(G)$ of positive definite functions on $G$ if and only if $\pi_\varphi$ is an irreducible representation; or, there exists a constant $a > 0$ such that $\varphi - a$ is again positive definite if and only if $\pi_\varphi$ has nonzero fixed vectors (see [4]).

In view of their importance for rigidity questions and Kazhdan’s property (T), it seems natural to try to fit 1-cohomology and reduced 1-cohomology of $\pi_\varphi$ in that dictionary. Two examples show that the answer will depend heavily on the underlying group $G$: if $\varphi$ is the constant function 1, then $\pi_\varphi$ is the trivial 1-dimensional representation $1_G$ of $G$, and $H^1(G, 1_G) = \text{Hom}(G, \mathbb{C})$; the vanishing of this group depends on the structure of the abelianized group $G/[G, G]$ (e.g. if $G$ is compactly generated, then $\text{Hom}(G, \mathbb{C}) = 0$ if and only if $G/[G, G]$ is compact). For a second example, assume that $G$ is discrete, and let $\varphi = \delta_1$ be the Dirac measure at the identity; then $\pi_\varphi$ is the left regular representation $\lambda_G$ of $G$ on $\ell^2(G)$, and $H^1(G, \lambda_G) = 0$ if and only if $G$ is non-amenable with vanishing first $\ell^2$-Betti number (see [2]).

In this paper we consider the case where $G$ is a locally compact abelian group. This assumption makes the question more tractable and gives us a powerful tool, Bochner’s Theorem: $\varphi$ is the Fourier transform of a positive Borel measure $\mu_\varphi$ on the Pontryagin dual $\hat{G}$ (see [5]). Without relying on the cohomological machinery available in the literature (see [8, 1]), we achieve by completely elementary means the results of this paper, namely, we show that the existence of nontrivial 1-cohomology is determined by two factors: the existence of non-trivial homomorphisms from $G$ to $\mathbb{C}$ and the behavior of $\mu_\varphi$ near the trivial character $1_G$. This is all delineated in a precise way in Theorem 1.

Acknowledgements. The first author has been supported by a 2012 ThinkSwiss Research Scholarship of the Swiss Confederation and by a 2012-2013 Masters in Mathematics scholarship of the Graduiertenkolleg...
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2. Statement of results

For $G$ a locally compact group and $\pi$ a unitary representation of $G$ on a Hilbert space $\mathcal{H}_\pi$, recall that the space of 1-cocycles for $\pi$ is

$$Z^1(G, \pi) = \{ b : G \to \mathcal{H}_\pi : b \text{ continuous, } b(gh) = \pi(g)b(h) + b(g) \text{ for all } g, h \in G \}. $$

The space of 1-coboundaries for $\pi$ is:

$$B^1(G, \pi) = \{ b \in Z^1(G, \pi) : \exists v \in \mathcal{H}_\pi \text{ such that } b(g) = \pi(g)v - v \text{ for every } g \in G \}. $$

The 1-cohomology of $\pi$ is then the quotient

$$H^1(G, \pi) = Z^1(G, \pi)/B^1(G, \pi).$$

Endow $Z^1(G, \pi)$ with the topology of uniform convergence on compact subsets of $G$. The reduced 1-cohomology of $\pi$ is the quotient of the space of 1-cocycles by the closure of the space of 1-coboundaries, i.e.

$$\overline{H}^1(G, \pi) = Z^1(G, \pi)/\overline{B^1(G, \pi)}. $$

From now on, let $G$ be a locally compact abelian group, $\varphi$ a positive definite function on $G$. Excluding the zero function, we may without loss of generality take $\varphi$ to be normalized ($\varphi(e) = 1$). Let $\mu_\varphi$ be the probability measure on the Pontryagin dual $\hat{G}$ provided by Bochner’s Theorem, i.e. $\varphi(x) = \int_{\hat{G}} \xi(x) d\mu_\varphi(\xi)$ for $x \in G$. Let $(\pi_\varphi, \mathcal{H}_\varphi, \xi_\varphi)$ be the cyclic representation of $G$ associated to $\varphi$ through the GNS construction, so that the cyclic vector $\xi_\varphi \in \mathcal{H}_\varphi$ satisfies $\langle \pi_\varphi(x)\xi_\varphi | \xi_\varphi \rangle = \varphi(x)$. Let also $\rho_\varphi$ be the representation of $G$ on $L^2(\hat{G}, \mu_\varphi)$ given by $(\rho_\varphi(x)f)(\xi) = \xi(x)f(\xi)$ ($\xi \in \hat{G}$, $f \in L^2(\hat{G}, \mu_\varphi)$).

If $\lambda_G$ denotes the left regular representation of $G$, then $(\lambda_G(x)h)(\xi) = \int_{\hat{G}} h(x^{-1}g)\xi(g)dg = \overline{\xi(x)}h(\xi)$ for $h \in L^1(G)$. From Plancherel’s Theorem, it follows that the composition of the Fourier transform with conjugation is a unitary equivalence between the regular representation on $L^2(G)$ and the unitary representation defined by $\left( f(\xi) \mapsto \xi(x)f(\xi) \right)$ on $L^2(\hat{G})$. This, together with Bochner’s Theorem, intuits our introduction of $\rho_\varphi$, as well as the following proposition which we prove in Section 3.

**Proposition 1.** The representations $\pi_\varphi$ and $\rho_\varphi$ are unitarily equivalent.
We assume from now on that \( \varphi \) is not the constant function 1, so that \( \mu_\varphi \) is not the Dirac measure at the trivial character \( 1_G \) of \( G \). This is still equivalent to \( \mu_\varphi(1_G) < 1 \). Let \( \mu_\varphi^\perp \) be the probability measure on \( \hat{G} \) defined by 
\[
\mu_\varphi = \mu_\varphi(1_G)\delta_{1_G} + (1 - \mu_\varphi(1_G))\mu_\varphi^\perp.
\]

Let \( \pi_0^\varphi \) be the (trivial) subrepresentation of \( \pi_\varphi \) on the subspace \( \mathcal{H}_\varphi^0 \) of \( \pi_\varphi \)-fixed vectors, and \( \pi_\varphi^\perp \) be the subrepresentation on the orthogonal complement, so that \( \pi_\varphi = \pi_0^\varphi \oplus \pi_\varphi^\perp \). A simple computation in the \( \rho_\varphi \)-picture shows that \( \mathcal{H}_\varphi^0 \neq 0 \) if and only if \( \mu_\varphi(1_G) > 0 \), and in this case \( \mathcal{H}_\varphi^0 = \mathbb{C}\delta_{1_G} \). Moreover, the map 
\[
L^2(\hat{G}\backslash\{1_G\}, \mu_\varphi^\perp) \to L^2(\hat{G}, \mu_\varphi) : f \mapsto \frac{f}{\sqrt{1 - \mu_\varphi(1_G)}}
\]
is isometric and identifies \( \pi_\varphi^\perp \) with the restriction of \( \rho_\varphi \) to \( L^2(\hat{G}\backslash\{1_G\}, \mu_\varphi^\perp) \).

Our main result is:

**Theorem 1.** Let \( \varphi \) be a nonconstant, normalized positive definite function on a locally compact abelian group \( G \).

1) Consider the following statements:
   
   i) \( H^1(G, \pi_\varphi) = 0 \);
   
   ii) Both of the following properties are satisfied:
       
       a) \( \mu_\varphi(1_G) = 0 \) or \( \text{Hom}(G, \mathbb{C}) = 0 \);
       
       b) \( 1_G \notin \text{supp}(\mu_\varphi^\perp) \).

   Then (ii) \(\Rightarrow\) (i), and the converse holds if \( G \) is \( \sigma \)-compact.

2) The following are equivalent:
   
   i) \( \overline{H}^1(G, \pi_\varphi) = 0 \);
   
   ii) \( \mu_\varphi(1_G) = 0 \) or \( \text{Hom}(G, \mathbb{C}) = 0 \).

This result will be proved in Section 4. It is essentially equivalent to Theorem 4 in [7], but we emphasize the fact that our proof is direct and based on explicit construction of cocycles and coboundaries. Theorem 1 is sharp in the sense that the implication (i) \(\Rightarrow\) (ii) in Part 1 fails in general if \( G \) is not assumed \( \sigma \)-compact (see Example 1 below).

3. Proof of Proposition 1

**Lemma 1.** The constant function \( 1 \in L^2(\hat{G}, \mu_\varphi) \) is a cyclic vector for \( \rho_\varphi \).
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Proof: Consider the associated $L^1(G)$-representation of $\rho_\varphi$ acting on $L^2(\hat{G}, \mu_\varphi^\perp)$ and given by $\rho_\varphi(f) := \int_G f(x) \rho_\varphi(x) \, dx$, $f \in L^1(G)$. Then $(\rho_\varphi(f).1)(\xi) = \int_G f(x) \xi(x) \, dx = \hat{f}(\xi)$.

Denote by $C_0(\hat{G})$ the space of continuous functions vanishing at infinity on $\hat{G}$, and recall that $\hat{f} \in C_0(\hat{G})$ (the Riemann-Lebesgue Lemma). It is classical that the Fourier transform $L^1(G) \to C_0(\hat{G})$ is a continuous algebra homomorphism with dense image (a consequence of Stone-Weierstrass). Now compose this homomorphism with the continuous inclusion $C_0(\hat{G}) \to L^2(\hat{G}, \mu_\varphi) : h \mapsto h.1$. Since continuous functions with compact support are dense in $L^2(\hat{G}, \mu_\varphi)$, this inclusion has dense image. Since the map $L^1(G) \to L^2(\hat{G}, \mu_\varphi) : f \mapsto \rho_\varphi(f).1$ is the composition of the previous maps, it has dense image, meaning that 1 is a cyclic vector for the $L^1(G)$-representation $\rho_\varphi$. Then 1 is also a cyclic vector for the unitary $G$-representation $\rho_\varphi$ by a Dirac sequence argument (see [4]). □

Observe that $\langle \rho_\varphi(x).1 | 1 \rangle = \int_{\hat{G}} \xi(x) \, d\mu_\varphi(\xi) = \varphi(x)$, so Proposition 1 follows from Lemma 1 and the uniqueness statement of the GNS construction. □

4. Proof of Theorem 1

Since $\pi_\varphi = \pi_\varphi^0 \oplus \pi_\varphi^\perp$, we have $H^1(G, \pi_\varphi) = H^1(G, \pi_\varphi^0) \oplus H^1(G, \pi_\varphi^\perp)$ and analogously for $\overline{H}^1$. As $B^1(G, \pi_\varphi^0) = 0$, we see that $H^1(G, \pi_\varphi^0) = 0$ if and only if $\overline{H}^1(G, \pi_\varphi^0) = 0$, if and only if either $\mu_\varphi(1_G) = 0$ or $\text{Hom}(G, \mathbb{C}) = 0$: this proves the implications $(i) \Rightarrow (ii)(a)$ in part 1 of Theorem 1, and $(i) \Rightarrow (ii)$ in part 2 of Theorem 1; moreover, it reduces the main result to:

Theorem 2. Let $\varphi$ be a nonconstant, normalized positive definite function on a locally compact abelian group $G$.

1) If $1_G \notin \text{supp}(\mu_\varphi^\perp)$, then $H^1(G, \pi_\varphi^\perp) = 0$. The converse holds if $G$ is $\sigma$-compact.

2) $\overline{H}^1(G, \pi_\varphi^\perp) = 0$.

Remark: As was pointed out by the referee of a previous version, when $G$ is separable, combining Corollary 1 and Proposition 2 in [6], one gets an explicit description of $Z^1(G, \pi_\varphi^\perp)$: it identifies with the space of
measurable functions \( u \) on \( \hat{G} \) such that, for every \( g \in G \), the integral
\[
\int_{\hat{G}} |\xi(g) - 1|^2 |u(\xi)|^2 \, d\mu_\varphi(\xi)
\]
is finite, and it tends to zero for \( g \) tending to the identity of \( G \) (in this picture, \( B^1(G, \pi\perp_\varphi) \) identifies with \( L^2(\hat{G}, \mu\perp_\varphi) \)). We will, however, not need that result.

**Example 1.** In Part 1 of Theorems 1 and 2, the converse implications are false when \( G \) is not assumed to be \( \sigma \)-compact. Indeed, let \( G \) be an uncountable abelian group with the discrete topology, and take \( \varphi = \delta_1 \). Then \( \pi_\varphi \) is the left regular representation \( \lambda_G \) on \( \ell^2(G) \), while \( \mu_\varphi = \mu\perp_\varphi \) is the Haar measure on the compact group \( \hat{G} \). Since \( \mu_\varphi \) has full support, in particular \( 1_G \) lies in its support. On the other hand \( H^1(G, \lambda_G) = 0 \) by Proposition 4.13 in [3].

To prove the implication “\( \Rightarrow \)” in part 1 of Theorem 2, we will need:

**Lemma 2.** Let \( F \) be a closed subset of \( \hat{G} \), with \( 1_G \notin F \).

(a) There exists a regular Borel probability measure \( \nu_0 \) on \( G \) such that the Fourier transform \( \hat{\nu}_0 \) vanishes on \( F \).

(b) For every \( \varepsilon > 0 \), there exists a compactly supported regular Borel probability measure \( \nu \) on \( G \) such that \( |\hat{\nu}_0 - \hat{\nu}| < \varepsilon \) on \( F \).

**Proof:** (a) See Section 1.5.2 in [9].

(b) Let \( \nu_0 \) be a probability measure on \( G \) as in (a). Let \( \delta \) be a number \( 0 < \delta < 1 \), to be determined later. Let \( C \) be a compact subset of \( G \) such that \( \nu_0(C) > 1 - \delta \). Let \( \nu \) be the probability measure on \( G \) defined by \( \nu(B) = \frac{\nu_0(B \cap C)}{\nu_0(C)} \), for every Borel subset \( B \subseteq G \). By taking \( \delta \) small enough, the total variation distance \( |\nu_0 - \nu|(G) \) between \( \nu_0 \) and \( \nu \) can be made arbitrarily small. For any finite signed measure \( \mu \) on \( G \), we have the classical inequality \( |\int_G f(x) \, d\mu(x)| \leq \|f\|_\infty \|\mu\|(G) \); applied to \( \mu = \nu_0 - \nu \) and \( f(x) = \xi(x) \) with \( \xi \in \hat{G} \), it gives \( |\hat{\nu}_0(\xi) - \hat{\nu}(\xi)| \leq |\nu_0 - \nu|(G) \), so that \( \|\hat{\nu}_0 - \hat{\nu}\|_\infty < \varepsilon \) for \( \delta \) small enough. 

**Proof of “\( \Rightarrow \)” in part 1 of Theorem 2:** We assume that \( 1_G \) is not in the support of \( \mu_{\varphi} \), and prove that \( H^1(G, \pi_{\varphi}^\perp) = 0 \). Let \( b \in Z^1(G, \pi_{\varphi}^\perp) \) be a 1-cocycle. Expanding \( b(xy) = b(yx) \) using the cocycle relation, we get:
\[
(1 - \pi_{\varphi}^\perp(x))b(y) = (1 - \pi_{\varphi}(y))b(x) \quad (x, y \in G).
\]
In the realization of $\pi_{\varphi}^1$ on $L^2(\hat{G}, \mu_{\varphi}^1)$, this gives:

$$(1 - \xi(x))b(y)(\xi) = (1 - \xi(y))b(x)(\xi) \quad (4.1)$$

almost everywhere in $\xi$ (w.r.t. $\mu_{\varphi}^1$). By Lemma 2, we can find a compactly supported probability measure $\nu$ on $G$ such that $|1 - \hat{\nu}| \geq \frac{1}{2}$ on supp($\mu_{\varphi}^1$). Define an element $v \in L^2(\hat{G}, \mu_{\varphi}^1)$ by $v := \int_G b(y) d\nu(y)$: since $b$ is continuous and $\nu$ is compactly supported, the integral exists (in the weak sense) in $L^2(\hat{G}, \mu_{\varphi}^1)$. Integrating (4.1) w.r.t. $v$ in the variable $y$, we get:

$$(1 - \xi(x))v(\xi) = (1 - \hat{\nu}(\xi))b(x)(\xi) \quad (4.2)$$

almost everywhere in $\xi$. Since $|1 - \hat{\nu}| \geq \frac{1}{2}$ on supp($\mu_{\varphi}^1$), the function $w(\xi) := \frac{v(\xi)}{\nu(\xi) - 1}$ belongs to $L^2(\hat{G}, \mu_{\varphi}^1)$, and by (4.2) its coboundary is exactly $b$. □

**Lemma 3.** Let $H$ be a locally compact group. Let $(\sigma_n)_{n \geq 0}$ be a sequence of unitary representations of $H$ without nonzero fixed vectors, with $\sigma_n$ acting on a Hilbert space $\mathcal{H}_n$. Assume that, for each $n \geq 0$, there exists a unit vector $\eta_n \in \mathcal{H}_n$ such that the series $\sum_{n=0}^\infty \|\sigma_n(x)\eta_n - \eta_n\|^2$ converges uniformly on compact subsets of $H$. Set $\sigma = \bigoplus_{n=0}^\infty \sigma_n$. Then $b(x) := \bigoplus_{n=0}^\infty (\sigma_n(x)\eta_n - \eta_n)$ defines a nonzero element in $H^1(G, \sigma)$.

**Proof:** By assumption $b(x)$ belongs to $\bigoplus_{n=0}^\infty \mathcal{H}_n$ and the map $H \to \bigoplus_{n=0}^\infty \mathcal{H}_n$: $x \mapsto b(x)$ is continuous. Let $\eta \in \prod_{n=0}^\infty \mathcal{H}_n$ be defined as $\eta = (\eta_n)_{n \geq 0}$. Since $b$ is the formal coboundary of $\eta$, we have $b \in Z^1(H, \sigma)$. To prove that $b$ is not a coboundary, it suffices to show that the associated affine action $\alpha(x)v = \sigma(x)v + b(x)$ on $\bigoplus_{n=0}^\infty \mathcal{H}_n$ has no fixed point. But $\alpha(x)v = v$ translates into $\sigma_n(x)(v_n + \eta_n) = v_n + \eta_n$ for every $x \in H$ and $n \geq 0$. Since $\sigma_n$ has no nonzero fixed vector, we have $v_n + \eta_n = 0$ so $\|v_n\| = 1$, which contradicts $\sum_{n=0}^\infty \|v_n\|^2 < +\infty$. □

**Proof of “$\Leftarrow$” in part 1 of Theorem 2, assuming $G$ to be $\sigma$-compact:** We assume that $1_G$ is in the support of $\mu_{\varphi}^1$, and prove that $H^1(G, \pi_{\varphi}) \neq 0$. Let $(K_n)_{n \geq 0}$ be an increasing sequence of compact subsets of $G$, with $G = \bigcup_{n=0}^\infty K_n$, and $K_0 = \{1\}$. Define a basis $(U_k)_{k \geq 0}$ of open neighborhoods of $1_G$ in $\hat{G}$ by $U_k = \{\xi \in \hat{G} : \max_{g \in K_k} |\xi(g) - 1| < 2^{-k}\}$ (observe that $U_0 = \hat{G}$). Define a sequence $(k_n)_{n \geq 0}$ inductively by $k_0 = 0$ and $k_{n+1} = \min\{k : k > k_n, \mu_{\varphi}^1(U_k) < \mu_{\varphi}^1(U_{k_n})\}$. Define $1_G \{1_G\} = 0$ and $1_G$ is in the support of $\mu_{\varphi}^1$, this is well-defined). Set then $C_n := U_{k_n} \setminus U_{k_{n+1}}$.
for \( n \geq 0 \). For each \( n \geq 0 \) let \( \mathcal{H}_n \) be the space of functions in \( L^2(\hat{G}, \mu_{\varphi}^+) \) which are \( \mu_{\varphi}^+ \)-almost everywhere 0 on \( \hat{G}\setminus C_n \). Then \( \mathcal{H}_n \) is a closed, \( \rho_{\varphi} \)-invariant subspace of \( L^2(\hat{G}, \mu_{\varphi}^+) \). Denote by \( \sigma_n \) the restriction of \( \rho_{\varphi} \) to \( \mathcal{H}_n \), so that \( L^2(\hat{G}, \mu_{\varphi}^+) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n \) and \( \rho_{\varphi} = \bigoplus_{n=0}^{\infty} \sigma_n \). Let \( \eta_n = \frac{1}{\sqrt{\mu_{\varphi}^+(C_n)}} \) be the normalized characteristic function of \( C_n \). To appeal to Lemma 3, we still have to check that \( x \mapsto \sum_{n=0}^{\infty} \| \sigma_n(x) \eta_n - \eta_n \|^2 \) converges uniformly on every compact subset \( K \) of \( G \). Clearly we may assume \( K = K_\ell \). For \( n \geq \ell \) and \( x \in K_\ell \) and \( \xi \in C_n \), we have

\[
| \xi(x) - 1 | < 2^{-k_n},
\]

hence

\[
\max_{x \in K_\ell} \sum_{n=0}^{\infty} \| \sigma_n(x) \eta_n - \eta_n \|^2 \leq \sum_{n=0}^{\infty} 4^{-n} \leq \sum_{n=0}^{\infty} 4^{-n} = \frac{4}{3},
\]

and

\[
\max_{x \in K_\ell} \sum_{n=0}^{\infty} \| \sigma(x) \eta_n - \eta_n \|^2 \leq (\max_{x \in K_\ell} \sum_{n=0}^{\ell-1} \| \sigma(x) \eta_n - \eta_n \|) + \frac{4}{3}
\leq 4\ell + \frac{4}{3} < +\infty.
\]

So the result follows from Lemma 3. \( \square \)

**Proof of part 2 of Theorem 2:** Let \( b \in Z^1(G, \pi_{\varphi}^+) \) be a 1-cocycle. We must show that \( b \) is a limit of 1-coboundaries (uniformly on compact subsets of \( G \)). Since \( \mu_{\varphi}^+(1_G) = 0 \), by the regularity of \( \mu_{\varphi}^+ \), we can find a decreasing sequence of relatively compact open neighborhoods \( (V_n)_{n \geq 0} \) of \( 1_G \), such that \( \mu_{\varphi}^+(V_n) \to 0 \) for \( n \to \infty \).

Set \( \mathcal{H}_n := \{ f \in L^2(\hat{G}, \mu_{\varphi}^+) : f = 0 \) a.e. on \( V_n \} \); then \( \mathcal{H}_n \) is a closed, \( \rho_{\varphi} \)-invariant subspace, and the sequence \( (\mathcal{H}_n)_{n \geq 0} \) is increasing with dense union in \( L^2(\hat{G}, \mu_{\varphi}^+) \). Let \( \rho_n \) denote the restriction of \( \rho_{\varphi} \) to \( \mathcal{H}_n \), and let \( b_n \) denote the projection of \( b \) onto \( \mathcal{H}_n \). Then \( b_n \in Z^1(G, \rho_n) \), and \( \lim_{n \to \infty} b_n = b \) (uniformly on compact subsets of \( G \)). Since \( 1_G \) does not belong to the closed subset \( \hat{G}\setminus V_n \), by Lemma 2 and by repeating the proof of the forward direction of part 1 of Theorem 2, we obtain that \( b_n \) is a coboundary for each \( n \geq 0 \). \( \square \)
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References


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