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Abstract

An existence theorem is proved, for a quasilinear degenerated elliptic inequality involving nonlinear operators of the form $Au+g(x,u,\nabla u)$, where A is a Leray-Lions operator from $W_0^{1,p}(\Omega,w)$ into its dual, while $g(x,s,\xi)$ is a nonlinear term which has a growth condition with respect to ξ and no growth with respect to s, but it satisfies a sign condition on s, the second term belongs to $W^{-1,p'}(\Omega,w^*)$.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \le i \le N\}$ be a vector of weight functions on Ω, i.e. each $w_i(x)$ is a measurable *a.e.* strictly positive on Ω, satisfying some integrability conditions (see section 2). This paper is concerned with the existence of solution of unilateral degenerate problems associated to a nonlinear operator of the form

$$Au + g(x,u,\nabla u).$$

The principal part A is a differential operator of second order in divergence form of Leray-Lions type acting from $W_0^{1,p}(\Omega,w)$ into it's dual $W^{-1,p'}(\Omega,w^*)$, i.e.

$$Au = -\mathrm{div}(a(x,u,\nabla u)) \tag{1.1}$$

and g is a nonlinear lower order term having natural growth with respect to $|\nabla u|$, with respect to $|u|$ we do not assume any growth restrictions, but we assume the sign-condition. Bensoussan, Boccardo and Murat have proved in the first part of [3], the existence of a solution for the problem

$$Au + g(x,u,\nabla u) = f,$$

where $f \in W^{-1,p'}(\Omega)$. In the second part of [3], the authors have extended the last result to variational inequalities, more precisely, they have proved the existence of at least one solution of the following unilateral problem:

$$\begin{cases} \langle Au, v-u\rangle + \int_\Omega g(x,u,\nabla u)(v-u)\,dx \geq \langle f, v-u\rangle \\ \text{ for all } \ v \in K_\psi \\ u \in W_0^{1,p}(\Omega) \quad\ u \geq \psi \ a.e. \text{ in } \Omega \\ g(x,u,\nabla u) \in L^1(\Omega) \quad g(x,u,\nabla u)u \in L^1(\Omega), \end{cases}$$

where $K_\psi = \{v \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega), \quad v \geq \psi \ a.e. \text{ in } \Omega\}$, with ψ a measurable function on Ω such that $\psi^+ \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. The same result is also proved in [2] where $f \in L^1(\Omega)$.
It is our purpose in this paper, to study the variational degenerated inequalities. More precisely, we prove the existence of solution to the problem $(\mathcal{P})$ (see section 4), in the framework of weighted Sobolev space. We obtain the existence results by proving that the positive part u_ε^+ (resp. negative part u_ε^-) of u_ε strongly converges to u^+(resp. u^-) in $W_0^{1,p}(\Omega, w)$, where u_ε is a solution of the approximate problem $(\mathcal{P}_\varepsilon)$ (see section 4). Let us point out, that another work in this direction can be found in [6] and [1] in the case of equation.
Note that, this paper can be seen as a generalization of [3] in weighted case and as a continuation of [1] where the case of equation is treated. This paper is organized as follows: Section 2 contains some preliminaries, section 3 is concerned with the basic assumptions and some technical lemmas, in section 4 we state and prove main results.

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N$ $(N \geq 1)$, let $1 < p < \infty$, and let $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions, i.e. every component $w_i(x)$ is a measurable function which is strictly positive *a.e.* in Ω. Further, we suppose in all our considerations that

$$w_i \in L^1_{loc}(\Omega) \tag{2.1}$$

and

$$w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega) \tag{2.2}$$

for any $0 \leq i \leq N$.
We define the weighted space $L^p(\Omega, \gamma)$, where γ is a weight function on Ω by,

$$L^p(\Omega, \gamma) = \{u = u(x),\ u\gamma^{\frac{1}{p}} \in L^p(\Omega)\}$$

with the norm

$$\|u\|_{p,\gamma} = \left(\int_\Omega |u(x)|^p \gamma(x)\, dx\right)^{\frac{1}{p}}.$$

Now, we denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions fulfil

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \text{ for all } i = 1, ..., N,$$

which is a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0(x)\, dx + \sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}. \tag{2.3}$$

Since we shall deal with the Dirichlet problem, we shall use the space

$$X = W_0^{1,p}(\Omega, w) \tag{2.4}$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.3). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space.
We recall that the dual space of weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'},\ \forall i = 0, ..., N\}$, where p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$ (for more details we refer to [5]).

Definition: Let Y be a separable reflexive Banach space, the operator B from Y to its dual Y^* is called of the calculus of variations type, if B is bounded and is of the form,

$$B(u) = B(u, u), \tag{2.5}$$

where $(u, v) \longrightarrow B(u, v)$ is an operator from $Y \times Y$ into Y^* satisfying the following properties:

$$\begin{cases} \forall u \in Y,\ v \to B(u, v) \quad \text{is bounded hemicontinuous from } Y \text{ into } Y^* \\ \text{and } (B(u, u) - B(u, v), u - v) \geq 0, \end{cases} \tag{2.6}$$

$$\forall v \in Y,\ u \to B(u,v) \quad \text{is bounded hemicontinuous from } Y \text{ into } Y^*, \tag{2.7}$$

$$\begin{cases} \text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } (B(u_n,u_n) - B(u_n,u), u_n - u) \to 0 \\ \text{then, } \ B(u_n,v) \rightharpoonup B(u,v) \text{ weakly in } Y^*,\ \forall v \in Y, \end{cases} \tag{2.8}$$

$$\begin{cases} \text{if } u_n \rightharpoonup u \text{ weakly in } Y \text{ and if } B(u_n,v) \rightharpoonup \psi \text{ weakly in } Y^*, \\ \text{then, } \ (B(u_n,v),u_n) \to (\psi,u). \end{cases} \tag{2.9}$$

Definition: Let Y be a reflexive Banach space, a bounded mapping B from Y to Y^* is called pseudo-monotone if for any sequence $u_n \in Y$ with $u_n \rightharpoonup u$ weakly in Y and $\limsup_{n\to\infty} \langle Bu_n, u_n - u\rangle \leq 0$, one has

$$\liminf_{n\to\infty} \langle Bu_n, u_n - v\rangle \geq \langle Bu, u-v\rangle \qquad \text{for all } v \in Y.$$

3 Basic assumption and some technical lemmas

We start by the following assumptions.
Assumption (H_1)
The expression

$$|||u|||_X = \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}$$

is a norm defined on X and it's equivalent to the norm (2.3).
And there exist a weight function σ on Ω and a parameter q, such that

$$1 < q < p + p', \tag{3.1}$$

and

$$\sigma^{1-q'} \in L^1_{loc}(\Omega), \tag{3.2}$$

with $q' = \frac{q}{q-1}$ and that the Hardy inequality,

$$\left(\int_\Omega |u(x)|^q \sigma(x)\, dx\right)^{\frac{1}{q}} \leq c \left(\sum_{i=1}^N \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx\right)^{\frac{1}{p}}, \tag{3.3}$$

holds for every $u \in X$ with a constant $c > 0$ independent of u, and moreover, the imbedding

$$X \hookrightarrow\hookrightarrow L^q(\Omega, \sigma), \tag{3.4}$$

expressed by the inequality (3.3) is compact.
Notice that $(X, |||.|||_X)$ is a uniformly convex (and thus reflexive) Banach space.

Remark: If we assume that $w_0(x) \equiv 1$ and in addition the integrability condition:

$$\text{there exists } \nu \in]\frac{N}{p}, \infty[\cap [\frac{1}{p-1}, \infty[\text{ such that } w_i^{-\nu} \in L^1(\Omega) \quad \forall i = 1, ..., N,$$

which is stronger than (2.2). Then

$$|||u|||_X = \left(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u(x)}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}},$$

is a norm defined on $W_0^{1,p}(\Omega, w)$ and it's equivalent to (2.3), moreover

$$W_0^{1,p}(\Omega, w) \hookrightarrow\hookrightarrow L^q(\Omega),$$

for all $1 \leq q < p_1^*$ if $p\nu < N(\nu + 1)$ and for all $q \geq 1$ if $p\nu \geq N(\nu + 1)$, where $p_1 = \frac{p\nu}{\nu+1}$ and $p_1^* = \frac{Np_1}{N-p_1} = \frac{Np\nu}{N(\nu+1)-p\nu}$ is the Sobolev conjugate of p_1 (see [5]). Thus the hypotheses (H_1) are verified for $\sigma \equiv 1$ and for all $1 < q < \min\{p_1^*, p + p'\}$ if $p\nu < N(\nu + 1)$ and for all $1 < q < p + p'$ if $p\nu \geq N(\nu + 1)$.

Let A be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by (1.1), i.e.,

$$Au = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory vector-function satisfying the following assumptions:

Assumption (H_2)

$$|a_i(x, s, \xi)| \leq \beta w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}} |s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \text{ for all } i = 1, ..., N, \tag{3.5}$$

$$[a(x,s,\xi)-a(x,s,\eta)](\xi-\eta)>0, \quad \text{for all } \xi\neq\eta\in\mathbb{R}^N, \tag{3.6}$$

$$a(x,s,\xi).\xi \geq \alpha\sum_{i=1}^{N} w_i|\xi_i|^p, \tag{3.7}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α, β are strictly positive constants.

Assumption (H_3)
Let $g(x,s,\xi)$ be a Carathéodory function satisfying the following assumptions:

$$g(x,s,\xi)s \geq 0 \tag{3.8}$$

$$|g(x,s,\xi)| \leq b(|s|)\left(\sum_{i=1}^{N} w_i|\xi_i|^p + c(x)\right), \tag{3.9}$$

where $b:\mathbb{R}^+\longrightarrow\mathbb{R}^+$ is a continuous increasing function and $c(x)$ is a positive function which lies in $L^1(\Omega)$.
We consider,

$$f\in W^{-1,p'}(\Omega,w^*). \tag{3.10}$$

Now we recall some lemmas introduced in [1] which will be used later.

Lemma 3.1: *(cf. [1]) Let $g\in L^r(\Omega,\gamma)$ and let $g_n\in L^r(\Omega,\gamma)$, with $\|g_n\|_{r,\gamma}\leq c \quad (1<r<\infty)$. If $g_n(x)\longrightarrow g(x)$ a.e. in Ω, then $g_n\rightharpoonup g$ weakly in $L^r(\Omega,\gamma)$, where γ is a weight function on Ω.*

Lemma 3.2: *(cf. [1]) Assume that (H_1) holds. Let $F:\mathbb{R}\longrightarrow\mathbb{R}$ be uniformly Lipschitzian, with $F(0)=0$. Let $u\in W_0^{1,p}(\Omega,w)$. Then $F(u)\in W_0^{1,p}(\Omega,w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial(F\circ u)}{\partial x_i}=\begin{cases} F'(u)\frac{\partial u}{\partial x_i} & a.e. \quad in\ \{x\in\Omega:\ \ u(x)\notin D\} \\ 0 & a.e. \quad in\ \{x\in\Omega:\ \ u(x)\in D\}. \end{cases}$$

Lemma 3.3: *(cf. [1]) Assume that (H_1) holds. Let $u\in W_0^{1,p}(\Omega,w)$, and let $T_k(u)$, $k\in\mathbb{R}^+$, be the usual truncation then $T_k(u)\in W_0^{1,p}(\Omega,w)$. Moreover, we have*

$$T_k(u)\longrightarrow u \text{ strongly in } W_0^{1,p}(\Omega,w).$$

Lemma 3.4: *(cf. [1]) Assume that (H_1) holds. Let $u \in W_0^{1,p}(\Omega, w)$, then $u^+ = \max(u, 0)$ and $u^- = \max(-u, 0)$ lie in $W_0^{1,p}(\Omega, w)$. Moreover, we have*

$$\frac{\partial(u^+)}{\partial x_i} = \begin{cases} \frac{\partial u}{\partial x_i}, & \text{if } u > 0 \\ 0, & \text{if } u \le 0 \end{cases}$$

$$\frac{\partial(u^-)}{\partial x_i} = \begin{cases} 0, & \text{if } u \ge 0 \\ -\frac{\partial u}{\partial x_i}, & \text{if } u < 0. \end{cases}$$

Lemma 3.5: *(cf. [1]) Assume that (H_1) holds. Let (u_n) be a sequence of $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$. Then, $u_n^+ \rightharpoonup u^+$ weakly in $W_0^{1,p}(\Omega, w)$ and $u_n^- \rightharpoonup u^-$ weakly in $W_0^{1,p}(\Omega, w)$.*

Lemma 3.6: *(cf. [1]) Assume that (H_1) and (H_2) are satisfied, and let (u_n) be a sequence of $W_0^{1,p}(\Omega, w)$ such that*

$$u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega, w)$$

and

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\, dx \longrightarrow 0.$$

Then,

$$u_n \longrightarrow u \text{ strongly in } W_0^{1,p}(\Omega, w).$$

4 Main general result

Let ψ be a measurable function with values in $\mathbb{R}$ such that,

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{4.1}$$

Set

$$K_\psi = \{v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \quad v \ge \psi \text{ a.e. in } \Omega\}. \tag{4.2}$$

Remark that (4.1) implies that $K_\psi \ne \emptyset$.
Consider the nonlinear problem with Dirichlet boundary condition,

$$(\mathcal{P}) \begin{cases} \langle Au, v - u\rangle + \int_\Omega g(x, u, \nabla u)(v - u)\, dx \ge \langle f, v - u\rangle \\ \text{for all } v \in K_\psi \\ u \in W_0^{1,p}(\Omega, w) \quad u \ge \psi \text{ a.e. in } \Omega \\ g(x, u, \nabla u) \in L^1(\Omega) \quad g(x, u, \nabla u)u \in L^1(\Omega), \end{cases}$$

here u is the solution of the problem $(\mathcal{P})$.
Our main result is the following.

Theorem 4.1: *Assume that the assumption* $(H_1)-(H_3),(3.10)$ *and* (4.1) *hold, then, there exists at least one solution of* $(\mathcal{P})$.

Remarks:

1) The statement of Theorem 4.1, generalizes in weighted case the analogous one in [3].

2) If we take $\psi=-\infty$, we obtain the existence result for the equation case (see [1]).

Proof of Theorem 4.1
Step (1) The approximate problem and a priori estimate.
Let Ω_ε be a sequence of compact subsets of Ω such that Ω_ε increase to Ω as $\varepsilon\to 0$.
We consider the sequence of approximate problems:

$$(\mathcal{P}_\varepsilon)\begin{cases}\langle Au_\varepsilon, v-u_\varepsilon\rangle+\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)(v-u_\varepsilon)\,dx\ge\langle f,v-u_\varepsilon\rangle\\ v\in W_0^{1,p}(\Omega,w)\quad\ v\ge\psi\ a.e.\text{ in }\Omega\\ u_\varepsilon\in W_0^{1,p}(\Omega,w)\quad\ u_\varepsilon\ge\psi\ a.e.\text{ in }\Omega,\end{cases}$$

where

$$g_\varepsilon(x,s,\xi)=\frac{g(x,s,\xi)}{1+\varepsilon|g(x,s,\xi)|}\chi_{\Omega_\varepsilon}(x),$$

and where $\chi_{\Omega_\varepsilon}$ is the characteristic function of Ω_ε.
Note that $g_\varepsilon(x,s,\xi)$ satisfies the following conditions,

$$g_\varepsilon(x,s,\xi)s\ge 0,\quad |g_\varepsilon(x,s,\xi)|\le|g(x,s,\xi)|\quad\text{and } |g_\varepsilon(x,s,\xi)|\le\frac{1}{\varepsilon}.$$

We define the operator $G_\varepsilon:\ X\longrightarrow X^*$ by,

$$\langle G_\varepsilon u,v\rangle=\int_\Omega g_\varepsilon(x,u,\nabla u)v\,dx.$$

Thanks to Hölder's inequality we have for all $u \in X$ and $v \in X$,

$$\begin{aligned} |\int_\Omega g_\varepsilon(x,u,\nabla u)v\,dx| &\le \left(\int_\Omega |g_\varepsilon(x,u,\nabla u)|^{q'}\sigma^{-\frac{q'}{q}}\,dx\right)^{\frac{1}{q'}}\left(\int_\Omega |v|^q\sigma\,dx\right)^{\frac{1}{q}} \\ &\le \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon}\sigma^{1-q'}\,dx\right)^{\frac{1}{q'}}\|v\|_{q,\sigma} \\ &\le c_\varepsilon|||v|||. \end{aligned} \tag{4.3}$$

The last inequality is due to (3.2) and (3.4).

Lemma 4.2: *The operator $B_\varepsilon = A + G_\varepsilon$ from X into its dual X^* is pseudo-monotone. Moreover, B_ε is coercive, in the following sense:*

$$\begin{cases} \text{there exists } v_0 \in K_\psi \text{ such that} \\ \frac{\langle B_\varepsilon v, v-v_0\rangle}{|||v|||} \to +\infty \text{ if } |||v||| \to \infty, \quad v \in K_\psi. \end{cases}$$

This lemma will be proved below.
In view of lemma 4.2, $(\mathcal{P}_\varepsilon)$ has a solution by the classical result (cf. theorem 8.2 chapter 2 [7]).
Let $v = \psi^+$ as test function in $(\mathcal{P}_\varepsilon)$, we easily deduce that
$\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)(u_\varepsilon - \psi^+) \ge 0$, then, $\langle Au_\varepsilon, u_\varepsilon\rangle \le \langle f, u_\varepsilon - \psi^+\rangle + \langle Au_\varepsilon, \psi^+\rangle$, i.e.

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\,dx \le \langle f, u_\varepsilon - \psi^+\rangle + \sum_{i=1}^N \int_\Omega a_i(x,u_\varepsilon,\nabla u_\varepsilon)\frac{\partial \psi^+}{\partial x_i}\,dx,$$

then,

$$\begin{aligned} \alpha\sum_{i=1}^N\int_\Omega w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p\,dx &= \alpha|||u_\varepsilon|||^p \le \|f\|_{X^*}(|||u_\varepsilon||| + |||\psi^+|||) + \\ &+ \sum_{i=1}^N\left(\int_\Omega |a_i(x,u_\varepsilon,\nabla u_\varepsilon)|^{p'}w_i^{1-p'}\,dx\right)^{\frac{1}{p'}}\left(\int_\Omega |\frac{\partial\psi^+}{\partial x_i}|^p w_i\,dx\right)^{\frac{1}{p}} \\ &\le \|f\|_{X^*}(|||u_\varepsilon||| + |||\psi^+|||) + \\ &+ c\sum_{i=1}^N\left(\int_\Omega (k^{p'} + |u_\varepsilon|^q\sigma + \sum_{j=1}^N|\frac{\partial u_\varepsilon}{\partial x_j}|^p w_j)\,dx\right)^{\frac{1}{p'}}|||\psi^+|||. \end{aligned}$$

Using (3.4) the last inequality becomes

$$\alpha |||u_\varepsilon|||^p \le c_1 |||u_\varepsilon||| + c_2 |||u_\varepsilon|||^{\frac{q}{p'}} + c_3 |||u_\varepsilon|||^{p-1} + c_4$$

where c_i are various positive constants. Then thanks to (3.1), we can deduce that u_ε remains bounded in $W_0^{1,p}(\Omega, w)$, *i.e.*

$$|||u_\varepsilon||| \le \beta_0, \tag{4.4}$$

where β_0 is a positive constant.
Extracting a subsequence (still denoted by u_ε) we get

$$u_\varepsilon \rightharpoonup u \text{ weakly in } X \text{ and } a.e. \text{ in } \Omega.$$

Note that $u \ge \psi$ *a.e.* in Ω.
Step(2) We study the convergence of the positive part of u_ε.
Let $k > 0$. Define $u_k^+ = \min\{u^+, k\}$.
We shall fix k, and use the notation,

$$z_\varepsilon = u_\varepsilon^+ - u_k^+. \tag{4.5}$$

Assertion(i) We claim that,

$$\limsup_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, \nabla u_\varepsilon^+) - a(x, u_\varepsilon, \nabla u_k^+)] \nabla (u_\varepsilon^+ - u_k^+)^+ \, dx \le Q_k, \tag{4.6}$$

where $Q_k \to 0$, if $k \to +\infty$.
Indeed, Consider the test function $v_\varepsilon = u_\varepsilon - z_\varepsilon^+$. By lemma 3.3 and lemma 3.4, we have $z_\varepsilon \in W_0^{1,p}(\Omega, w)$ and $z_\varepsilon^+ \in W_0^{1,p}(\Omega, w)$. And since $k \longrightarrow \infty$ and $\psi^+ \in L^\infty(\Omega)$ we can assume that $k \ge \psi$ *a.e.* in Ω, by the choice of k, the above test function is admissible for $(\mathcal{P}_\varepsilon)$. Multiplying $(\mathcal{P}_\varepsilon)$ by v_ε we obtain,

$$\langle Au_\varepsilon, z_\varepsilon^+ \rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) z_\varepsilon^+ \, dx \le \langle f, z_\varepsilon^+ \rangle. \tag{4.7}$$

If $z_\varepsilon^+ > 0$, we have $u_\varepsilon > 0$ and from (3.8), $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \ge 0$, then,

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon) \nabla z_\varepsilon^+ \, dx \le \langle f, z_\varepsilon^+ \rangle.$$

Since $u_\varepsilon = u_\varepsilon^+$ in $\{x \in \Omega \;/\; z_\varepsilon^+ > 0\}$, then,

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon^+) \nabla z_\varepsilon^+ \, dx \le \langle f, z_\varepsilon^+ \rangle,$$

which implies that,

$$\int_\Omega [a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u_k^+)]\nabla(u_\varepsilon^+ - u_k^+)^+\,dx \leq$$
$$\leq -\int_\Omega a(x,u_\varepsilon,\nabla u_k^+)\nabla(u_\varepsilon^+ - u_k^+)^+\,dx + \langle f, z_\varepsilon^+\rangle. \tag{4.8}$$

As $\varepsilon \to 0$, we have $z_\varepsilon^+ \longrightarrow (u^+ - u_k^+)^+ a.e.$ in Ω, moreover z_ε^+ is bounded in $W_0^{1,p}(\Omega,w)$, hence, we have,

$$z_\varepsilon^+ \rightharpoonup (u^+ - u_k^+)^+ \text{ weakly in } W_0^{1,p}(\Omega,w).$$

Since $a(x,u_\varepsilon,\nabla u_k^+) \longrightarrow a(x,u,\nabla u_k^+)$ in $\prod_{i=1}^N L^{p'}(\Omega,w_i^*)$, we obtain by passing to the limit in ε in (4.8) the inequality (4.6) with Q_k defined by,

$$Q_k = -\int_\Omega a(x,u,\nabla u_k^+)\nabla(u^+ - u_k^+)^+\,dx + \langle f,(u^+ - u_k^+)^+\rangle. \tag{4.9}$$

Because, $(u^+ - u_k^+)^+ \longrightarrow 0$ in $W_0^{1,p}(\Omega,w)$ as $k \longrightarrow \infty$, we have $Q_k \longrightarrow 0$ as $k \longrightarrow \infty$.
Assertion(ii) Let us show that,

$$\limsup_{\varepsilon\to 0}\int_\Omega -[a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u_k^+)]\nabla(u_\varepsilon^+ - u_k^+)^-\,dx \leq 0. \tag{4.10}$$

Indeed. We consider for that, the test function $v_\varepsilon = u_\varepsilon + \varphi_\lambda(z_\varepsilon^-)$, where $\varphi_\lambda(s) = se^{\lambda s^2}$. We have, $0 \leq z_\varepsilon^- \leq k$, i.e., $z_\varepsilon^- \in L^\infty(\Omega)$ and since $z_\varepsilon^- \in W_0^{1,p}(\Omega,w)$, hence using lemma 3.2 we have $v_\varepsilon \in W_0^{1,p}(\Omega,w)$, then clearly, v_ε is an admissible test function.
Multiplying $(\mathcal{P}_\varepsilon)$ by v_ε we obtain,

$$\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla z_\varepsilon^-\varphi_\lambda'(z_\varepsilon^-)\,dx + \int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx \geq \langle f,\varphi_\lambda(z_\varepsilon^-)\rangle. \tag{4.11}$$

Define,

$$E_\varepsilon = \{x \in \Omega,\ u_\varepsilon^+(x) \leq u_k^+(x)\} \text{ and } F_\varepsilon = \{x \in \Omega,\ 0 \leq u_\varepsilon(x) \leq u_k^+(x)\}.$$

Since $\varphi_\lambda(z_\varepsilon^-) = 0$ in E_ε^c, we have,

$$\int_\Omega g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx = \int_{E_\varepsilon} g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx. \tag{4.12}$$

When $u_\varepsilon \le 0$, we have $g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon) \le 0$ and since $\varphi_\lambda(z_\varepsilon^-) \ge 0$, we obtain

$$\begin{aligned}\int_{E_\varepsilon} g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx &\le \int_{F_\varepsilon} g_\varepsilon(x,u_\varepsilon,\nabla u_\varepsilon)\varphi_\lambda(z_\varepsilon^-)\,dx \\ &\le \int_{F_\varepsilon} b(|u_\varepsilon|)[\sum_{i=1}^N w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p + c(x)]\varphi_\lambda(z_\varepsilon^-)\,dx \\ &\le b(k)\int_{F_\varepsilon}[\sum_{i=1}^N w_i|\frac{\partial u_\varepsilon}{\partial x_i}|^p + c(x)]\varphi_\lambda(z_\varepsilon^-)\,dx\end{aligned}$$

$$\le \frac{b(k)}{\alpha}\int_{F_\varepsilon} a(x,u_\varepsilon,\nabla u_\varepsilon)\nabla u_\varepsilon\varphi_\lambda(z_\varepsilon^-)\,dx + b(k)\int_{F_\varepsilon} c(x)\varphi_\lambda(z_\varepsilon^-)\,dx. \quad (4.13)$$

As in the proof of theorem 1.1 in [3], we can show that,

$$\begin{aligned}&-\frac{1}{2}\int_\Omega[a(x,u_\varepsilon,\nabla u_\varepsilon^+) - a(x,u_\varepsilon,\nabla u_k^+)]\nabla(u_\varepsilon^+ - u_k^+)^-\,dx \le \\ &\le \int_\Omega[a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla u_\varepsilon^+)]\nabla u_k^+\varphi_\lambda'(u_k^+)\,dx + \langle -f,\varphi_\lambda(z_\varepsilon^-)\rangle \\ &+\int_\Omega a(x,u_\varepsilon,\nabla u_k^+)\nabla z_\varepsilon^-\varphi_\lambda'(z_\varepsilon^-)\,dx + \frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,\nabla u_\varepsilon^+)\nabla u_k^+\varphi_\lambda(z_\varepsilon^-)\,dx+ \\ &+\frac{b(k)}{\alpha}\int_\Omega a(x,u_\varepsilon,\nabla u_k^+)\nabla(u_\varepsilon^+ - u_k^+)\varphi_\lambda(z_\varepsilon^-)\,dx + b(k)\int_\Omega c(x)\varphi_\lambda(z_\varepsilon^-)\,dx,\end{aligned} \quad (4.14)$$

for $\lambda = \frac{b(k)^2}{4\alpha^2}$.
For short notation, we rewrite the above inequality as,

$$I_{\varepsilon k} \le I_{\varepsilon k}^1 + I_{\varepsilon k}^2 + I_{\varepsilon k}^3 + I_{\varepsilon k}^4 + I_{\varepsilon k}^5. \quad (4.15)$$

Extracting a subsequence such that,

$$\begin{cases} a(x,u_\varepsilon,\nabla u_\varepsilon) \rightharpoonup \gamma_1 \text{ weakly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^*) \\ \text{ and} \\ a(x,u_\varepsilon,\nabla u_\varepsilon^+) \rightharpoonup \gamma_2 \text{ weakly in } \prod_{i=1}^N L^{p'}(\Omega,w_i^*). \end{cases} \quad (4.16)$$

Lemma 4.3: *(cf. [1]) For k fixed and letting $\varepsilon \to 0$, we claim that,*
1) $I_{\varepsilon k}^1 \longrightarrow I_k^1 = \int_\Omega[\gamma_1 - \gamma_2]\nabla u_k^+\varphi_\lambda'(u_k^+)\,dx + \langle -f,\varphi_\lambda((u^+ - u_k^+)^-)\rangle.$
2) $I_{\varepsilon k}^2 \longrightarrow I_k^2 = \int_\Omega a(x,u,\nabla u_k^+)\nabla((u^+ - u_k^+)^-)\varphi_\lambda'((u^+ - u_k^+)^-)\,dx.$

3) $I^3_{\varepsilon k} \longrightarrow I^3_k = \dfrac{b(k)}{\alpha} \displaystyle\int_\Omega \gamma_2 \nabla u^+_k \varphi_\lambda((u^+ - u^+_k)^-)\, dx.$
4) $I^4_{\varepsilon k} \longrightarrow I^4_k = \dfrac{b(k)}{\alpha} \displaystyle\int_\Omega a(x,u,\nabla u^+_k)\nabla(u^+ - u^+_k)\varphi_\lambda((u^+ - u^+_k)^-)\, dx.$
5) $I^5_{\varepsilon k} \longrightarrow I^5_k = b(k) \displaystyle\int_\Omega c(x)\varphi_\lambda((u^+ - u^+_k)^-)\, dx.$

In view of lemma 4.3 and $(u^+ - u^+_k)^- = 0$ and $\varphi_\lambda(0) = 0$ we have,

$$\limsup_{\varepsilon\to 0} I_{\varepsilon k} \leq I^1_k + I^2_k + I^3_k + I^4_k + I^5_k = \int_\Omega [\gamma_1(x) - \gamma_2(x)]\nabla u^+_k \varphi'_\lambda(u^+_k)\, dx.$$

Moreover, if $u_\varepsilon < 0$ we have $(u_\varepsilon)^+_k = 0$, hence,

$$(a(x,u_\varepsilon,\nabla u_\varepsilon) - a(x,u_\varepsilon,\nabla u^+_\varepsilon))(u_\varepsilon)^+_k = 0 \ a.e. \text{ in } \Omega$$

which implies that $(\gamma_1(x) - \gamma_2(x))u^+_k = 0$, and so that,

$$\limsup_{\varepsilon\to 0} I_{\varepsilon k} \leq 0,$$

thus, (4.10) follows.
Assertion(iii) Let us show that,

$$u^+_\varepsilon \longrightarrow u^+ \text{ strongly in } W^{1,p}_0(\Omega,w). \tag{4.17}$$

From (4.6) and (4.10) we have (as in the proof of theorem 1.1 in [3]),

$$\limsup_{\varepsilon\to 0} \int_\Omega [a(x,u_\varepsilon,\nabla u^+_\varepsilon) - a(x,u_\varepsilon,\nabla u^+)]\nabla(u^+_\varepsilon - u^+)\, dx \leq$$
$$\leq Q_k + \int_\Omega [\gamma_2(x) - a(x,u,\nabla u^+_k)]\nabla(u^+_k - u^+)\, dx.$$

Now letting $k \longrightarrow \infty$ and using lemma 3.6 we obtain (4.17).
Step(3) We study the convergence of the negative part of u_ε
Similarly to the preceding step, we shall prove that

$$u^-_\varepsilon \longrightarrow u^- \ \text{ strongly in } W^{1,p}_0(\Omega,w). \tag{4.18}$$

Assertion (j) Let us show that,

$$\limsup_{\varepsilon\to 0} \int_\Omega -[a(x,u_\varepsilon,-\nabla u^-_\varepsilon) - a(x,u_\varepsilon,-\nabla u^-_k)]\nabla(u^-_\varepsilon - u^-_k)^+\, dx \leq \tilde{Q}_k, \tag{4.19}$$

where $\tilde{Q}_k \to 0$, if $k \to +\infty$.
Indeed. We define

$$u_k^- = \min\{u^-, k\} \quad \text{and} \quad y_\varepsilon = u_\varepsilon^- - u_k^-.$$

Consider the test function $v_\varepsilon = u_\varepsilon + y_\varepsilon^+$ in $(\mathcal{P}_\varepsilon)$, it is clearly admissible. Multiplying $(\mathcal{P}_\varepsilon)$ by v_ε, we have,

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla y_\varepsilon^+ \, dx + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) y_\varepsilon^+ \, dx \geq \langle f, y_\varepsilon^+ \rangle.$$

Since $y_\varepsilon^+ > 0$ implies $u_\varepsilon < 0$, then from (3.8), we have $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \leq 0$, hence $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) y_\varepsilon^+ \leq 0$ *a.e.* in Ω, then,

$$\int_\Omega a(x, u_\varepsilon, \nabla u_\varepsilon)\nabla y_\varepsilon^+ \, dx \geq \langle f, y_\varepsilon^+ \rangle.$$

Since $u_\varepsilon = -u_\varepsilon^-$ on the set $\{x \in \Omega, \ y_\varepsilon^+ > 0\}$ we can also write $\int_\Omega a(x, u_\varepsilon, -\nabla u_\varepsilon^-)\nabla y_\varepsilon^+ \, dx \geq \langle f, y_\varepsilon^+ \rangle$, which implies that,

$$-\int_\Omega [a(x, u_\varepsilon, -\nabla u_\varepsilon^-) - a(x, u_\varepsilon, -\nabla u_k^-)]\nabla (u_\varepsilon^- - u_k^-)^+ \, dx \leq \int_\Omega a(x, u_\varepsilon, -\nabla u_k^-)\nabla (u_\varepsilon^- - u_k^-)^+ \, dx - \langle f, y_\varepsilon^+ \rangle.$$

As $\varepsilon \to 0$ we have $y_\varepsilon^+ \longrightarrow (u^- - u_k^-)^+ a.e.$ in Ω, and since y_ε^+ is bounded in $W_0^{1,p}(\Omega, w)$, then $y_\varepsilon^+ \rightharpoonup (u^- - u_k^-)^+$ weakly in $W_0^{1,p}(\Omega, w)$ (for k fixed). Passing to the limit in ε we obtain (4.19), with $\tilde{Q}_k$ defined by,

$$\tilde{Q}_k = \int_\Omega a(x, u, -\nabla u_k^-)\nabla (u^- - u_k^-)^+ \, dx - \langle f, (u^- - u_k^-)^+ \rangle.$$

Because $(u^- - u_k^-)^+ \longrightarrow 0$ strongly in $W_0^{1,p}(\Omega, w)$ as $k \longrightarrow \infty$ we obtain that $\tilde{Q}_k \longrightarrow 0$ as $k \longrightarrow \infty$.
Assertion (jj) Let us show that,

$$\limsup_{\varepsilon \to 0} \int_\Omega [a(x, u_\varepsilon, -\nabla u_\varepsilon^-) - a(x, u_\varepsilon, -\nabla u_k^-)]\nabla (u_\varepsilon^- - u_k^-)^- \, dx \leq 0. \quad (4.20)$$

Indeed. Considering the following test function $v_\varepsilon = u_\varepsilon - \delta_\varepsilon \varphi_\lambda(y_\varepsilon^-)$ where $\delta_\varepsilon > 0$ such that $\delta_\varepsilon e^{\lambda(y_\varepsilon^-)^2} \leq 1$ this function is admissible (cf. [3]), then,

$$\langle Au_\varepsilon, -\delta_\varepsilon \varphi_\lambda(y_\varepsilon^-) \rangle - \delta_\varepsilon \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(y_\varepsilon^-) \, dx \geq -\langle f, \delta_\varepsilon \varphi_\lambda(y_\varepsilon^-) \rangle,$$

i.e.,

$$\langle Au_\varepsilon, \varphi_\lambda(y_\varepsilon^-)\rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)\varphi_\lambda(y_\varepsilon^-)\,dx \le \langle f, \varphi_\lambda(y_\varepsilon^-)\rangle,$$

with this choice (4.20) follows as in (4.10).
Finally combining (4.19) and (4.20), we deduce as in (4.17) the assertion (4.18).
Step (4) Convergence of u_ε.
From (4.17) and (4.18) we deduce that for a subsequence

$$u_\varepsilon \longrightarrow u \text{ strongly in } W_0^{1,p}(\Omega, w) \text{ and } a.e. \text{ in } \Omega \tag{4.21}$$

$$\nabla u_\varepsilon \longrightarrow \nabla u \ a.e. \text{ in } \Omega, \tag{4.22}$$

which implies that,

$$\begin{cases} g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow g(x, u, \nabla u) \ a.e \text{ in } \Omega \\ g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \longrightarrow g(x, u, \nabla u)u \ a.e. \text{ in } \Omega. \end{cases} \tag{4.23}$$

On the other hand, multiplying $(\mathcal{P}_\varepsilon)$ by u_ε and using (3.7), (3.8), (4.3), (4.4), we obtain

$$0 \le \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon\,dx \le \tilde{\beta}, \tag{4.24}$$

where $\tilde{\beta}$ is some positive constant.
For any measurable subset E of Ω and any $m > 0$ we have,

$$\int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\,dx = \int_{E\cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\,dx + \int_{E\cap Y_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\,dx$$

where

$$\begin{cases} X_m^\varepsilon = \{x \in \Omega : \quad |u_\varepsilon(x)| \le m\} \\ Y_m^\varepsilon = \{x \in \Omega : \quad |u_\varepsilon(x)| > m\}. \end{cases} \tag{4.25}$$

From (3.9), (4.24), (4.25) we have,

$$\begin{aligned} \int_E |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\,dx &\le \int_{E\cap X_m^\varepsilon} |g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)|\,dx + \frac{1}{m}\int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon\,dx \\ &\le b(m)\int_E \Big(\sum_{i=1}^N w_i \Big|\frac{\partial u_\varepsilon}{\partial x_i}\Big|^p + c(x)\Big)\,dx + \tilde{\beta}\frac{1}{m}. \end{aligned} \tag{4.26}$$

Since the sequence (∇u_ε) strongly converges in $\prod_{i=1}^{N} L^p(\Omega, w_i)$, then (4.26) implies the equi-integrability of $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)$.
Thanks to (4.23) and Vitali's theorem one easily has

$$g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \longrightarrow g(x, u, \nabla u) \text{ strongly in } L^1(\Omega). \tag{4.27}$$

Moreover, since $g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)u_\varepsilon \geq 0$ *a.e.* in Ω, then by using (4.23), (4.24) and Fatou's lemma we have

$$g(x, u, \nabla u)u \in L^1(\Omega). \tag{4.28}$$

From (4.21) and (4.27) we can pass to the limit in

$$\langle Au_\varepsilon, v - u_\varepsilon\rangle + \int_\Omega g_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon)(v - u_\varepsilon)\, dx \geq \langle f, v - u_\varepsilon\rangle$$

and we obtain,

$$\begin{cases} \langle Au, v - u\rangle + \displaystyle\int_\Omega g(x, u, \nabla u)(v - u)\, dx \geq \langle f, v - u\rangle \\ \text{for any } v \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega) \ v \geq \psi \ p.p. \text{ in } \Omega. \end{cases} \tag{4.29}$$

Proof of lemma 4.2

By the proposition 2.6 chapter 2 [7], it is sufficient to show that B_ε is of the calculus of variations type. Indeed put,

$$b_1(u, v, \tilde{w}) = \sum_{i=1}^{N} \int_\Omega a_i(x, u, \nabla v)\nabla\tilde{w}\, dx$$

$$b_2(u, \tilde{w}) = \int_\Omega g_\varepsilon(x, u, \nabla u)\tilde{w}\, dx.$$

The form $\tilde{w} \longrightarrow b_1(u, v, \tilde{w}) + b_2(u, \tilde{w})$ is continuous in X. Then,

$$b_1(u, v, \tilde{w}) + b_2(u, \tilde{w}) = b(u, v, \tilde{w}) = \langle B_\varepsilon(u, v), \tilde{w}\rangle, \quad B_\varepsilon(u, v) \in W^{-1,p'}(\Omega, w^*)$$

and we have

$$B_\varepsilon(u, u) = B_\varepsilon u.$$

Using (3.5) and Hölder's inequality we can show that A is bounded [4], and thanks to (4.3), B_ε is bounded. Then, it is sufficient to check (2.6) − (2.9).

Show that (2.6) and (2.7) are true.
By (3.6) we have,

$$(B_\varepsilon(u,u) - B_\varepsilon(u,v), u-v) = b_1(u,u,u-v) - b_1(u,v,u-v) \geq 0.$$

The operator $v \to B_\varepsilon(u,v)$ is bounded hemicontinuous. Indeed, we have

$$a_i(x,u,\nabla(v_1+\lambda v_2)) \longrightarrow a_i(x,u,\nabla v_1) \text{ strongly in } L^{p'}(\Omega, w_i^*) \text{ as } \lambda \to 0. \tag{4.30}$$

On the other hand, $(g_\varepsilon(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)))_\lambda$ is bounded in $L^{q'}(\Omega,\sigma^{1-q'})$ and $g_\varepsilon(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \longrightarrow g_\varepsilon(x,u_1,\nabla u_1)$ *a.e.* in Ω hence lemma 3.1 gives

$$g_\varepsilon(x,u_1+\lambda u_2,\nabla(u_1+\lambda u_2)) \rightharpoonup g_\varepsilon(x,u_1,\nabla u_1) \text{ weakly in } L^{q'}(\Omega,\sigma^{1-q'}) \text{ as } \lambda \to 0. \tag{4.31}$$

Using (4.30) and (4.31), we can write

$$b(u,v_1+\lambda v_2,\tilde{w}) \longrightarrow b(u,v_1,\tilde{w}) \quad \text{as } \lambda \to 0 \;\; \forall u, v_i, \tilde{w} \in X.$$

Similarly we can prove (2.7).
Proof of assertion (2.8). Assume that $u_n \rightharpoonup u$ weakly in X and $(B(u_n,u_n) - B(u_n,u), u_n-u) \to 0$. We have: $(B(u_n,u_n) - B(u_n,u), u_n-u)$

$$= \sum_{i=1}^{N} \int_\Omega (a_i(x,u_n,\nabla u_n) - a_i(x,u_n,\nabla u))\, \nabla(u_n-u)\, dx \to 0,$$

then, by lemma 3.6 we have,

$$u_n \longrightarrow u \text{ strongly in } X,$$

which gives

$$b(u_n,v,\tilde{w}) \longrightarrow b(u,v,\tilde{w}) \quad \forall \tilde{w} \in X,$$

i.e.,

$$B_\varepsilon(u_n,v) \rightharpoonup B_\varepsilon(u,v) \text{ weakly in } X^*.$$

It remains to prove (2.9). Assume that

$$u_n \rightharpoonup u \;\text{ weakly in } X \tag{4.32}$$

and that

$$B(u_n,v) \rightharpoonup \psi \text{ weakly in } X^*. \tag{4.33}$$

Thanks to (3.4), (3.5) and (4.32), we obtain

$$a_i(x, u_n, \nabla v) \to a_i(x, u, \nabla v) \text{ strongly in } L^{p'}(\Omega, w_i^*) \text{ as } n \to \infty,$$

then,

$$b_1(u_n, v, u_n) \longrightarrow b_1(u, v, u). \tag{4.34}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} |b_2(u_n, u_n - u)| &\leq \left(\int_\Omega |g_\varepsilon(x, u_n, \nabla u_n)|^{q'} \sigma^{\frac{-q'}{q}}\, dx\right)^{\frac{1}{q'}} \left(\int_\Omega |u_n - u|^q \sigma\, dx\right)^{\frac{1}{q}} \\ &\leq \frac{1}{\varepsilon}\left(\int_{\Omega_\varepsilon} \sigma^{\frac{-q'}{q}}\, dx\right)^{\frac{1}{q'}} \|u_n - u\|_{L^q(\Omega,\sigma)} \to 0 \text{ as } n \to \infty, \end{aligned}$$

i.e.,

$$b_2(u_n, u_n - u) \longrightarrow 0 \text{ as } n \to \infty, \tag{4.35}$$

but in view of (4.33) and (4.34), we obtain

$$b_2(u_n, u) = (B_\varepsilon(u_n, v), u) - b_1(u_n, v, u) \longrightarrow (\psi, u) - b_1(u, v, u)$$

and from (4.35) we have,

$$b_2(u_n, u_n) \longrightarrow (\psi, u) - b_1(u, v, u).$$

Then,

$$(B_\varepsilon(u_n, v), u_n) = b_1(u_n, v, u_n) + b_2(u_n, u_n) \longrightarrow (\psi, u).$$

Now, let us show that B_ε is coercive. Let $v_0 \in K_\psi$.
From Hölder's inequality, the growth condition (3.5) and the compact imbedding (3.4) we have,

$$\begin{aligned} \langle Av, v_0\rangle &= \sum_{i=1}^N \int_\Omega a_i(x, v, \nabla v) \frac{\partial v_0}{\partial x_i}\, dx \\ &\leq \sum_{i=1}^N \left(\int_\Omega |a_i(x, v, \nabla v)|^{p'} w_i^{\frac{-p'}{p}}\, dx\right)^{\frac{1}{p'}} \left(\int_\Omega |\frac{\partial v_0}{\partial x_i}|^p w_i\, dx\right)^{\frac{1}{p}} \\ &\leq c_1 |||v_0||| \left(\int_\Omega k(x)^{p'} + |v|^q \sigma + \sum_{j=1}^N |\frac{\partial v}{\partial x_j}|^p w_j\, dx\right)^{\frac{1}{p'}} \\ &\leq c_2(c_3 + |||v|||^{\frac{q}{p'}} + |||v|||^{p-1}), \end{aligned}$$

where c_i are various constants,
thanks to (3.7) we obtain,

$$\frac{\langle Av, v\rangle}{|||v|||} - \frac{\langle Av, v_0\rangle}{|||v|||} \geq \alpha |||v|||^{p-1} - |||v|||^{p-2} - |||v|||^{\frac{q}{p'}-1} - \frac{c}{|||v|||}.$$

In view of (3.1) we have $p - 1 > \frac{q}{p'} - 1$. Then,

$$\frac{\langle Av, v - v_0\rangle}{|||v|||} \longrightarrow \infty \text{ as } |||v||| \longrightarrow \infty,$$

since $\langle G_\varepsilon v, v\rangle \geq 0$ and $\langle G_\varepsilon v, v_0\rangle$ is bounded we have

$$\frac{\langle B_\varepsilon v, v - v_0\rangle}{|||v|||} \geq \frac{\langle Av, v - v_0\rangle}{|||v|||} - \frac{\langle G_\varepsilon v, v_0\rangle}{|||v|||} \longrightarrow \infty \text{ as } |||v||| \longrightarrow \infty.$$

Remark: The assumption (3.1) appears necessary, in order to prove the boundedness of $(u_\varepsilon)_\varepsilon$ in $W_0^{1,p}(\Omega, w)$ and the coercivity of the operator B_ε. While the assumption (3.2) is necessary to prove the boundedness of G_ε in $W_0^{1,p}(\Omega, w)$. Thus, when $g \equiv 0$, we don't need to suppose (3.2).

References

[1] Y. Akdim, E. Azroul, and A. Benkirane. Existence of solutions for quasilinear degenerated elliptic equations. *Electronic J. Diff. Eqns.*, 2001(71):1–19, 2001.

[2] A. Benkirane and A. Elmahi. Strongly nonlinear elliptic unilateral problem having natural growth terms and L^1 data. *Rendiconti di Matematica*, 18:289–303, 1998.

[3] A. Bensoussan, L. Boccardo, and F. Murat. On a non linear partial differential equation having natural growth terms and unbounded solution. *Ann. Inst. Henri Poincaré*, 5(4):347–364, 1988.

[4] P. Drabek, A. Kufner, and V. Mustonen. Pseudo-monotonicity and degenerated or singular elliptic operators. *Bull. Austral. Math. Soc.*, 58:213–221, 1998.

[5] P. Drabek, A. Kufner, and F. Nicolosi. *Quasilinear elliptic equations with degenerations and singularities.* De Gruyter Series in Nonlinear Analysis and Applications, New York, 1997.

[6] P. Drabek and F. Nicolosi. Existence of bounded solutions for some degenerated quasilinear elliptic equations. *Annali di Mathematica pura ed applicata*, CLXV:217–238, 1993.

[7] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod, Paris, 1969.

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