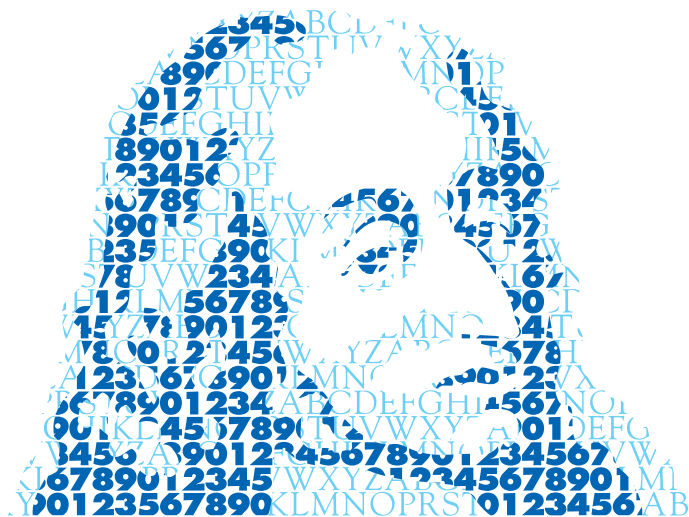


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# $L^p$ –boundedness of oscillating spectral multipliers on Riemannian manifolds

Michel Marias

## Abstract

We prove endpoint estimates for operators given by oscillating spectral multipliers on Riemannian manifolds with  $C^\infty$ -bounded geometry and nonnegative Ricci curvature.

KEYWORDS: spectral multipliers, wave equation, Riesz means

AMS SUBJECT CLASSIFICATION: 58G03

## 1 Introduction and statement of the results

Let  $M$  be an  $n$ –dimensional, complete, noncompact Riemannian manifold with nonnegative Ricci curvature and let us assume that it has  $C^\infty$ -bounded geometry, that is, the injectivity radius is positive and every covariant derivative of the curvature tensor is bounded (cf. [25]). Let  $d(., .)$  denote the Riemannian distance on  $M$ ,  $dx$  its volume element. Let us denote by  $B(x, r)$  the ball of radius  $r > 0$  centered at  $x \in M$  and by  $|B(x, r)|$  its volume. By the Bishop comparison theorem (cf. [5]), the assumption that  $M$  has nonnegative Ricci curvature implies that

$$\frac{|B(x, r)|}{|B(x, t)|} \leq \left(\frac{r}{t}\right)^n, \quad r \geq t > 0, \quad (1.1)$$

and hence

$$|B(x, 2r)| \leq 2^n |B(x, r)|, \quad r > 0.$$

This is the so called ‘doubling volume property’ and makes  $M$  a ‘space of homogeneous type’ in the sense of Coifman and Weiss [8]. Thus we can define the atomic Hardy space  $H^1(M)$  and  $BMO(M)$ , the space of functions of bounded mean oscillation, in the standard way (cf. [8]). Further, by Theorem B of [8],  $BMO(M)$  is the dual of  $H^1(M)$ .

Let  $L$  be the Laplace-Beltrami operator. It admits a selfadjoint extension on  $L^2(M)$ , also denoted by  $L$  and hence the spectral resolution

$$L = \int_0^\infty \lambda dE_\lambda.$$

Given a bounded measurable function  $m(\lambda)$ , we can define, by the spectral theorem, the operator

$$m(L) = \int_0^\infty m(\lambda) dE_\lambda.$$

This operator is bounded on  $L^2(M)$ . The function  $m(\lambda)$  is called multiplier.

Oscillating multipliers are multipliers of the type

$$m_{\alpha,\beta}(\lambda) = \psi(|\lambda|) |\lambda|^{-\beta/2} e^{i|\lambda|^{\alpha/2}}, \quad \alpha > 0, \beta \geq 0. \quad (1.2)$$

with  $\psi$  a smooth function which is 0 for  $|\lambda| \leq 1$  and 1 for  $|\lambda| \geq 2$ .

In this article we shall prove some endpoint results concerning the  $L^p$  boundedness of the family of operators

$$m_{\alpha,\beta}(L) = \int_0^\infty m_{\alpha,\beta}(\lambda) dE_\lambda.$$

We have the following:

**Theorem 1.1:** *Let  $m_{\alpha,\beta}$  be as above and let  $\alpha \in (0, 1)$ . The following hold:*

- (i). *If  $\beta = \frac{\alpha n}{2}$ , then  $m_{\alpha,\beta}(L)$  is bounded from  $H^1(M)$  to  $L^1(M)$ , on  $L^p(M)$ ,  $1 < p < \infty$  and from  $L^\infty(M)$  to  $BMO(M)$ .*
- (ii). *If  $0 \leq \beta < \frac{\alpha n}{2}$ , then  $m_{\alpha,\beta}(L)$  is bounded on  $L^p(M)$ , for  $\beta \geq \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|$ ,  $1 < p < \infty$ .*
- (iii). *If  $\beta > \frac{\alpha n}{2}$ , then  $m_{\alpha,\beta}(L)$  is bounded on  $L^p(M)$  for  $1 \leq p \leq \infty$ .*

Oscillating multipliers fall outside the scope of Calderón-Zygmund theory and they have been studied extensively. See for example [31, 14, 10, 11, 21, 22, 23, 28, 26] for  $\mathbb{R}^n$  and [9, 1, 20, 12] for more abstract settings.

The above result, in the context of  $\mathbb{R}^n$  and for  $0 \leq \beta \leq \alpha n/2$ , has been proved by Fefferman and Stein in [11]. In the context of Riemannian manifolds of nonnegative Ricci curvature, Alexopoulos [1], has proved that

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for any  $\alpha > 0$ ,  $m_{\alpha,\beta}(L)$  is bounded on  $L^p$  for  $\beta > \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|$ ,  $1 \leq p \leq \infty$ . According to [11], the results above, for  $0 \leq \beta \leq \alpha n/2$ , are optimal.

For the proof of the  $H^1 - L^1$  boundedness of  $m_{\alpha, \frac{\alpha n}{2}}(L)$ , we follow the strategy that Alexopoulos sketches at the end of the paper [1]. The idea, which is due to M. Taylor, is to express  $m_{\alpha,\beta}(L)$  in terms of the wave operator  $\cos t\sqrt{L}$  and then use the Hadamard parametrix method to get very precise estimates of its kernel near the diagonal. Away from the diagonal, we use the finite propagation speed property of  $\cos t\sqrt{L}$  and the fast decay of the multiplier at infinity to obtain that  $m_{\alpha,\beta}(L)$  is bounded on  $L^p$ ,  $p \geq 1$ .

To prove that the operator  $m_{\alpha,\beta}(L)$  is bounded on  $L^p$  for  $\beta = \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|$ ,  $1 < p < \infty$ , we compose  $m_{\alpha, \frac{\alpha n}{2}}(L)$  with the imaginary powers of the Laplacian, which are bounded on  $H^1$ , (cf. [19]), and then use the  $H^1 - L^1$  boundedness of  $m_{\alpha, \frac{\alpha n}{2}}(L)$  and complex interpolation.

We shall apply Theorem 1.1 in order to obtain similar results for the Riesz means associated with the Schrödinger type group  $e^{isL^{\alpha/2}}$  i.e. for the family of operators

$$I_{k,\alpha}(L) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isL^{\alpha/2}} ds, \quad 0 < \alpha < 1, \quad k > 0.$$

We have the following

**Theorem 1.2:** *For any  $\alpha \in (0, 1)$ , the following hold:*

- (i). *If  $k = \frac{n}{2}$ , then  $I_{k,\alpha}(L)$  is bounded from  $H^1(M)$  to  $L^1(M)$ , on  $L^p(M)$ ,  $1 < p < \infty$ , and from  $L^\infty(M)$  to  $BMO(M)$ .*
- (ii). *If  $k < \frac{n}{2}$ , then  $I_{k,\alpha}(L)$  is bounded on  $L^p(M)$ , for  $k \geq n \left| \frac{1}{p} - \frac{1}{2} \right|$ ,  $1 < p < \infty$ .*
- (iii). *If  $k > \frac{n}{2}$ , then  $I_{k,\alpha}(L)$  is bounded on  $L^p(M)$ ,  $1 \leq p \leq \infty$ .*

In the context of  $\mathbb{R}^n$ , the operators  $I_{k,\alpha}(L)$  are studied for example in [27] and [22]. According to [27], the results above, for  $k \leq n/2$ , are optimal. The operators  $I_{k,\alpha}(L)$  have also been studied in more abstract contexts, see for example [1, 2, 17, 18, 4, 6].

It is worth mentioning that our approach is valid only for  $\alpha \in (0, 1)$ . This is due to the fact that the estimates of the multiplier  $m_{\alpha,\beta}(\lambda)$  are available only for  $\alpha \in (0, 1)$ , (cf. [31] and Section 5).

The paper is organized as follows. In Section 2 we recall some known facts about the Hardy space  $H^1$  and  $BMO$  (Subsection 2.1), the wave operator and the construction of its parametrix (Subsection 2.2). In Section 3 the estimates of the Fourier transform of the derivatives of the multiplier  $m_{\alpha,\beta}(\lambda)$  are given. In Section 4 we give the estimates of the kernel of the operator  $m_{\alpha,\beta}(L)$  near the diagonal and in Section 5 we establish its  $L^p$ -boundedness when  $\beta > n/2$ . In Section 6 we prove the  $H^1 - L^1$  boundedness of the operator  $m_{\alpha,\frac{\alpha n}{2}}(L)$  and in Section 7 we finish the proofs of Theorems 1.1 and 1.2.

Throughout this article the different constants will always be denoted by the same letter  $c$ . When their dependence or independence is significant, it will be clearly stated.

## 2 Preliminaries

### 2.1 The Hardy space $H^1$ and $BMO$

Let us recall that a complex-valued function  $a$  on  $M$  is an atom if it is supported in a ball  $B(y_0, r)$  and satisfies

$$\|a\|_\infty \leq |B(y_0, r)|^{-1} \quad \text{and} \quad \int_M a(x) dx = 0.$$

A function  $f$  on  $M$  belongs to the Hardy space  $H^1(M)$  if there exist  $(\lambda_m)_{m \in \mathbb{N}} \in \ell^1$  and a sequence of atoms  $(a_m)_{m \in \mathbb{N}}$  such that

$$f = \sum_{m \in \mathbb{N}} \lambda_m a_m,$$

where the series converges in  $L^1(M)$ . The norm  $\|f\|_{H^1}$  is the infimum of  $\sum_{m \in \mathbb{N}} |\lambda_m|$  for all such decompositions of  $f$ .

A function  $f$  belongs to  $BMO(M)$ , if there exists a constant  $c > 0$  such that for all balls  $B(x, r)$ ,

$$\frac{1}{|B(x, r)|} \int_{B(x, r)} |f(y) - f_B| dy < c,$$

where

$$f_B = \frac{1}{|B(x, r)|} \int_{B(x, r)} f(y) dy.$$

The smallest of all such constants  $c$  is the *BMO* norm of  $f$ .

Finally we note that the dual of  $H^1(M)$  is *BMO*( $M$ ), (cf. [8], Theorem B, p. 593).

## 2.2 The wave operator

Let  $G_t(x, y)$  be the kernel of the wave operator  $\cos t\sqrt{L}$ . Note that  $G_t(x, y)$  is also the solution of the wave equation

$$\begin{aligned} (\partial_t^2 + L_y) u(t, x, y) &= 0, \\ u(0, x, y) &= \delta_x(y), \\ \partial_t u(0, x, y) &= 0. \end{aligned} \tag{2.1}$$

In this article we shall exploit the fact that  $G_t(x, y)$  propagates with finite propagation speed (cf. [7, 29]):

$$\text{supp}(G_t) \subseteq \{(x, y) : d(x, y) \leq |t|\}. \tag{2.2}$$

Next we shall recall some facts about the Hadamard parametrix construction for the kernel  $G_t(x, y)$ , (cf. [3, 4, 15]).

Let  $\delta \in (0, r_0)$ , to be fixed later, and let us consider, for every ball  $B(x, \delta)$ ,  $x \in M$ , the exponential normal coordinates centered at  $x$ . Let  $g_{ij}(x, y)$ ,  $y \in B(x, \delta)$ , be the metric tensor expressed in these coordinates and let us denote by  $(g^{ij}(x, y))$  its inverse matrix. We have the following Taylor expansion of  $g_{ij}$ :

$$\begin{aligned} g_{ij}(x, y) &= \delta_{ij} + {}^2A_{ijkl}(y_k - x_k)(y_l - x_l) \\ &\quad + {}^3A_{ijklm}(y_k - x_k)(y_l - x_l)(y_m - x_m) + \dots \end{aligned} \tag{2.3}$$

where the  ${}^kA_{ij\dots}$  are universal polynomials in the components of the curvature tensor and its first  $k-2$  covariant derivatives at the point  $x$ , (cf. [24], p. 85). By the term ‘‘universal’’ we mean that the coefficients of the polynomials  ${}^kA_{ij\dots}$  depend only on the dimension of the manifold.

It follows from (2.3) and the assumption of  $C^\infty$ -bounded geometry that for any multi-index  $\alpha$  there exists a positive constant  $c_\alpha$  such that

$$|\partial_y^\alpha g_{ij}(x, y)| \leq c_\alpha, \quad x \in M, \quad y \in B(x, \delta). \tag{2.4}$$

Since  $g_{ij}(x, x) = \delta_{ij}$ , there is  $c > 0$  and  $\delta \in (0, r_0)$  such that

$$c^{-1} \leq \det(g_{ij}(x, y)) \leq c. \quad (2.5)$$

for all  $x \in M$  and  $y \in B(x, \delta)$ .

In what follows, we shall fix a  $\delta \in (0, \min(1, r_0))$  such that (2.5) is satisfied.

From (2.4) and (2.5) we also have that there is  $c'_\alpha > 0$  such that

$$|\partial_y^\alpha g^{ij}(x, y)| \leq c'_\alpha. \quad (2.6)$$

for all  $x \in M$ ,  $y \in B(x, \delta)$ .

Let  $\Theta(x, y) = \det(g_{ij}(x, y))$ . Then, the Laplace-Beltrami operator  $L$  can be written as follows:

$$L = \frac{1}{(\Theta(x, y))^{1/2}} \sum_{i,j} \frac{\partial}{\partial y_i} (\Theta(x, y))^{1/2} g^{ij}(x, y) \frac{\partial}{\partial y_j}.$$

Note that by (2.4), (2.5) and (2.6), the Laplacian can also be written as

$$L = \sum_{|\alpha| \leq 2} c_\alpha(y) \partial_y^\alpha$$

with the coefficients satisfying

$$|\partial_y^\beta c_\alpha(y)| \leq c_{\alpha,\beta}, \quad (2.7)$$

for all  $x \in M$ ,  $y \in B(x, \delta)$  and any multi-index  $\beta$ .

Let us consider the following smooth functions:

$$U_0(x, y) = \Theta^{-1/2}(x, y)$$

and

$$U_{k+1}(x, y) = \Theta^{-1/2}(x, y) \int_0^1 s^k \Theta^{1/2}(x, y_s) L_2 U_k(x, y_s) ds,$$

where  $y_s$ ,  $s \in [0, 1]$ , is the geodesic from  $x$  to  $y$  and  $L_2$  denotes the Laplacian acting on the second variable. Note that  $U_0(x, x) = 1$ .

In what follows, we always assume that  $|t| \leq \delta$  and  $y \in B(x, \delta)$ ,  $x \in M$ .

Let us consider the kernels

$$E_N(t, x, y) = C_0 \sum_{k=0}^N (-1)^k U_k(x, y) |t| \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2}}}{4^k \Gamma(k - \frac{n-1}{2})}, \quad (2.8)$$

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where  $C_0$  is a normalizing constant.

They satisfy (cf. [3])

$$\begin{aligned} (\partial_t^2 + L_y) E_N(t, x, y) &= \frac{C_0(-1)^N}{4^N \Gamma(N - \frac{n-1}{2})} |t| (t^2 - d(x, y)^2)_+^{N - \frac{n+1}{2}} L_y U_N(x, y), \\ E_N(0, x, y) &= \delta_x(y), \\ \partial_t E_N(0, x, y) &= 0. \end{aligned} \tag{2.9}$$

Now, let us observe that by (2.4), (2.5) and (2.7) there exists a  $c > 0$  such that

$$|U_0(x, y)| \leq c_0 \quad \text{and} \quad |L_y U_0(x, y)| \leq c_1. \tag{2.10}$$

These also imply that for any  $k \in \mathbb{N}$  there is  $c > 0$  such that

$$|U_k(x, y)| \leq \frac{c_1^k}{k!}, \quad |L_y U_k(x, y)| \leq \frac{c_1^{k+1}}{k!} \quad \text{and} \quad \|\nabla_y U_k(x, y)\| \leq c \frac{c_1^k}{k!}, \tag{2.11}$$

for  $x \in M$  and  $y \in B(x, \delta)$ .

If  $k \geq \frac{n+1}{2}$ , then (2.11) and the fact that

$$\Gamma\left(k - \frac{n+1}{2}\right) \sim k!, \quad \text{as } k \rightarrow \infty,$$

imply that

$$\left| U_k(x, y) |t| \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2}}}{4^k \Gamma\left(k - \frac{n-1}{2}\right)} \right| \leq \frac{c_1^k}{k!} \delta \frac{\delta^{2k-(n+1)}}{4^k k!} \leq \frac{c_1^k}{k!} \frac{\delta^{2k-n}}{4^k k!}. \tag{2.12}$$

From (2.8) and (2.12) we get that  $E_N(t, x, y)$  converges uniformly as  $N \rightarrow \infty$  and (2.9), (2.11) and (2.1) that the limit is  $G_t(x, y)$ . Thus we have the expansion

$$G_t(x, y) = C_0 \sum_{k=0}^{\infty} (-1)^k U_k(x, y) |t| \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2}}}{4^k \Gamma\left(k - \frac{n-1}{2}\right)}, \tag{2.13}$$

the convergence being uniform for  $|t| \leq \delta$  and  $y \in B(y, \delta)$ .

### 3 Estimates of the multiplier and of its derivatives

In this section we shall give some estimates for the derivatives of the Fourier transform of the multiplier  $m_{\alpha, \beta}$ .



Let us consider the function

$$f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2) = \psi(t^2) |t|^{-\beta} e^{i|t|^\alpha}.$$

Let  $r_0$  be the injectivity radius of  $M$  and us fix  $\delta \in (0, r_0)$ . Let  $\chi_\delta(t)$  be a smooth and nonnegative function such that  $\chi_\delta(t) = 1$  for  $|t| \leq \delta/2$  and 0 for  $|t| \geq \delta$ . Set

$$\hat{f}_{\alpha,\beta}^0(t) = \hat{f}_{\alpha,\beta}(t)\chi_\delta(t), \quad \hat{f}_{\alpha,\beta}^\infty(t) = \hat{f}_{\alpha,\beta}(t)(1 - \chi_\delta(t)). \quad (3.1)$$

In this article we shall need the following:

**Lemma 3.1:** *Let  $\alpha \in (0, 1)$  and  $\beta = \frac{\alpha n}{2} + \varepsilon$ ,  $\varepsilon \geq 0$ . Then for all  $m, N \in \mathbb{N}$  and  $t \in \mathbb{R}$ ,*

$$\left| \partial_t^m \hat{f}_{\alpha,\beta}^0(t) \right| \leq c |t|^{-(1+m-\varepsilon-\frac{\alpha(n+1)}{2})/(1-\alpha)}, \quad (3.2)$$

and

$$\left| \partial_t^m \hat{f}_{\alpha,\beta}^\infty(t) \right| \leq c |t|^{-N}. \quad (3.3)$$

Before proceed to the proof of Lemma 3.1, let us recall the following estimates from Wainger [31], Theorem 9. For any  $\alpha \in (0, 1)$  and  $\varepsilon > 0$ , consider the function

$$f_{\varepsilon,\alpha,b}(x) = e^{-\varepsilon\|x\|} \psi(\|x\|^2) \|x\|^{-b} e^{i\|x\|^\alpha}, \quad x \in \mathbb{R}^k.$$

We have that

$$\hat{f}_{\varepsilon,\alpha,b}(\|x\|) = \|x\|^{\frac{2-k}{2}} \int_0^\infty e^{-\varepsilon u} \psi(u^2) u^{-b+\frac{k}{2}} e^{iu^\alpha} J_{\frac{k-2}{2}}(u\|x\|) du \quad (3.4)$$

where  $J_m(z)$  is the Bessel function.

Making use of this formula, Wainger proved that the limit

$$\hat{f}_{\alpha,b}(\|x\|) = \lim_{\varepsilon \rightarrow 0} \hat{f}_{\varepsilon,\alpha,b}(\|x\|)$$

exists and it is continuous for  $x \neq 0$ . Further, if  $b > k(1 - \frac{\alpha}{2})$ , then  $\hat{f}_{\alpha,b}$  is continuous also at  $x = 0$ , while if  $b \leq k(1 - \frac{\alpha}{2})$  and  $M \in \mathbb{N}$ , then

$$\begin{aligned} \hat{f}_{\alpha,b}(\|x\|) = & \|x\|^{-\left(k-b-\frac{\alpha k}{2}\right)/(1-\alpha)} e^{i\xi_\alpha\|x\|^{-\alpha/(1-\alpha)}} \sum_{m=0}^M a_m \|x\|^{m\alpha/(1-\alpha)} \\ & + O\left(\|x\|^{(M+1)\alpha/(1-\alpha)}\right) + C(\|x\|), \end{aligned} \quad (3.5)$$

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where  $a_0 \neq 0$ ,  $\xi_\alpha$  is real and  $\xi_\alpha \neq 0$ ;  $C$  is a continuous function.

Furthermore

$$\left| \hat{f}_{\alpha,b}(\|x\|) \right| = O(\|x\|^{-N}), \quad \text{as } \|x\| \rightarrow \infty, \quad (3.6)$$

for any  $N \in \mathbb{N}$ .

PROOF OF LEMMA 3.1: If  $m = 0$ , then (3.2) and (3.3) are an immediate consequence of (3.5), with  $k = 1$ , and (3.6).

If  $m = 2l$ ,  $l \geq 1$ , then  $\partial^{2l} \hat{f}_{\alpha,\beta}$  is the Fourier transform of the function

$$(-i\lambda)^{2l} f_{\alpha,\beta}(\lambda) = (-i)^{2l} \psi(|\lambda|^2) |\lambda|^{-\beta+2l} e^{i|\lambda|^\alpha} = (-i)^{2l} f_{\alpha,\beta-2l}(\lambda).$$

Hence (3.2) and (3.3) follow again from (3.5) and (3.6) with  $b = \beta - 2l$ .

If  $m = 2l + 1$ , then  $\partial^{2l+1} \hat{f}_{\alpha,\beta}$  is the Fourier transform of the function

$$\varphi(\lambda) = (-i)^{2l+1} \psi(|\lambda|^2) \lambda |\lambda|^{-\beta+2l} e^{i|\lambda|^\alpha}.$$

Since this function is odd, we have

$$\begin{aligned} \partial^{2l+1} \hat{f}_{\alpha,\beta}(t) &= -2i \int_0^{+\infty} \varphi(x) \sin(tx) dx \\ &= -2i \lim_{\epsilon \rightarrow 0} \int_0^{+\infty} e^{-\epsilon x} \varphi(x) \sin(tx) dx. \end{aligned}$$

Since

$$\sin x = \sqrt{\frac{\pi x}{2}} J_{\frac{1}{2}}(x),$$

we have

$$\begin{aligned} \partial^{2l+1} \hat{f}_{\alpha,\beta}(t) &= c\sqrt{2\pi t} \lim_{\epsilon \rightarrow 0} \int_0^\infty e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^\alpha} J_{\frac{1}{2}}(tx) dx \\ &= ct \lim_{\epsilon \rightarrow 0} \left\{ t^{-\frac{1}{2}} \int_0^\infty e^{-\epsilon x} \psi(x^2) x^{-\beta+2l+3/2} e^{ix^\alpha} J_{\frac{1}{2}}(tx) dx \right\}. \end{aligned}$$

The integral in brackets above is the same as the integral  $\hat{f}_{\epsilon,\alpha,b}(t)$  in formula (3.4), with  $k = 3$  and  $b = \beta - 2l$ . This gives, as  $\epsilon \rightarrow 0$ , the Fourier transform of the multiplier  $f_{\alpha,b}(\lambda)$  in  $\mathbb{R}^3$ . Therefore, the estimates  $\partial^{2l+1} \hat{f}_{\alpha,\beta}(t)$  follow again from (3.5) and (3.6).

## 4 The estimates of the kernel near the diagonal

Let us express the operator  $m_{\alpha,\beta}(L)$  in terms of the wave operator  $\cos t\sqrt{L}$ . If  $f_{\alpha,\beta}(t) = m_{\alpha,\beta}(t^2)$ , then  $m_{\alpha,\beta}(L) = f_{\alpha,\beta}(\sqrt{L})$  and since  $f_{\alpha,\beta}$  is an even function, by the Fourier inversion formula we have that

$$m_{\alpha,\beta}(L) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \hat{f}_{\alpha,\beta}(t) \cos t\sqrt{L} dt.$$

Let  $m_{\alpha,\beta}(x, y)$  be the kernel of  $m_{\alpha,\beta}(L)$ . Then by the finite propagation speed property (2.2)

$$m_{\alpha,\beta}(x, y) = (2\pi)^{-1/2} \int_{|t| \geq d(x,y)} \hat{f}_{\alpha,\beta}(t) G_t(x, y) dt.$$

This kernel is singular near the diagonal and integrable at infinity. We want to split  $m_{\alpha,\beta}(x, y)$  into these two parts and treat them separately. This can be done by considering the operators

$$m_{\alpha,\beta}^0(L) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}^0(t) \cos t\sqrt{L} dt$$

and

$$m_{\alpha,\beta}^{\infty}(L) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}_{\alpha,\beta}^{\infty}(t) \cos t\sqrt{L} dt,$$

where  $f_{\alpha,\beta}^0$  and  $f_{\alpha,\beta}^{\infty}$  are defined in (3.1). We have

$$m_{\alpha,\beta}(L) = m_{\alpha,\beta}^0(L) + m_{\alpha,\beta}^{\infty}(L).$$

Let  $m_{\alpha,\beta}^0(x, y)$  and  $m_{\alpha,\beta}^{\infty}(x, y)$  denote the kernels of  $m_{\alpha,\beta}^0(L)$  and  $m_{\alpha,\beta}^{\infty}(L)$ , respectively. Then

$$m_{\alpha,\beta}^0(x, y) = (2\pi)^{-1/2} \int_{\delta \geq |t| \geq d(x,y)} \hat{f}_{\alpha,\beta}^0(t) G_t(x, y) dt \quad (4.1)$$

and

$$m_{\alpha,\beta}^{\infty}(x, y) = (2\pi)^{-1/2} \int_{|t| > \delta} \hat{f}_{\alpha,\beta}^{\infty}(t) G_t(x, y) dt.$$

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In the present section we deal with the kernel  $m_{\alpha,\beta}^0(x, y)$ . This kernel contains the singular part of the kernel  $m_{\alpha,\beta}(x, y)$  and from (4.1) it follows that

$$\text{supp}(m_{\alpha,\beta}^0) \subset \{(x, y) \in M \times M : d(x, y) \leq \delta\}. \quad (4.2)$$

We shall obtain very good  $L^\infty$  estimates for  $m_{\alpha,\beta}^0(x, y)$  by using the Hadamard parametrix construction for  $G_t(x, y)$ . These estimates allow us to prove in Section 6 that  $m_{\alpha,\beta}(L)$  is bounded from  $H^1$  to  $L^1$  for  $\beta = n\alpha/2$ .

We have the following:

**Lemma 4.1:** *Let  $\alpha \in (0, 1)$ . Then for all  $\varepsilon \geq 0$ , there exists a constant  $c > 0$  such that for all  $x, y \in M$*

$$\left| m_{\alpha, \frac{n\alpha}{2} + \varepsilon}^0(x, y) \right| \leq cd(x, y)^{-n + \frac{\varepsilon}{1-\alpha}} \quad (4.3)$$

and

$$\left\| \nabla_y m_{\alpha, \frac{n\alpha}{2}}^0(x, y) \right\| \leq cd(x, y)^{-(n+1)+\alpha'}, \quad (4.4)$$

where  $\frac{1}{\alpha} + \frac{1}{\alpha'} = 1$ .

For  $\beta = \frac{n\alpha}{2} + \varepsilon$  and  $k = -1, 0, 1, \dots$ , we set

$$I_k(x, y) = \int_{\mathbb{R}} \hat{f}_{\alpha,\beta}^0(t) |t| \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2}}}{\Gamma(k - \frac{n-1}{2})} dt.$$

Lemma 4.1 is a consequence of the expansion (2.13) of  $G_t(x, y)$  and of the following:

**Lemma 4.2:** (i). *If  $0 \leq k \leq \frac{n+1}{2}$ , then there is a  $c > 0$  such that*

$$|I_k(x, y)| \leq cd(x, y)^{-n + \frac{\varepsilon}{1-\alpha}}, \quad \forall x, y \in M. \quad (4.5)$$

(ii). *If  $k > \frac{n+1}{2}$ , then there is a  $c > 0$  such that*

$$|I_k(x, y)| \leq c \frac{\delta^{2k}}{\Gamma(k - \frac{n-1}{2})}, \quad \forall x, y \in M. \quad (4.6)$$

(iii). *If  $k = -1$  and  $\varepsilon = 0$ , then there is a  $c > 0$  such that*

$$|I_k(x, y)| \leq cd(x, y)^{-(n+2)+\alpha'}, \quad \forall x, y \in M. \quad (4.7)$$

PROOF: The proof is given in steps. Let us set, for simplicity,  $d = d(x, y)$ .  
*Proof of (4.5) for  $n = 2p + 1$ .* This is the simpler case. If we put  $t = ud$ , then we have

$$\begin{aligned} I_k(x, y) &= d^{2k-n+1} \int_{\mathbb{R}} |u| \hat{f}_{\alpha, \beta}^0(ud) \frac{(u^2-1)_+^{k-\frac{n+1}{2}}}{\Gamma(k-\frac{n-1}{2})} du \\ &= d^{2k-n+1} \int_{\mathbb{R}} |u| \hat{f}_{\alpha, \beta}^0(ud) (u+1)^{k-p-1} \frac{(u-1)_+^{k-p-1}}{\Gamma(k-p)} du. \end{aligned}$$

Since

$$\frac{(u-1)_+^{k-p-1}}{\Gamma(k-p)} = \delta^{(p-k)}(u-1), \quad \text{for } k \leq p+1, \quad (4.8)$$

(cf. [13], p. 56), we have

$$\begin{aligned} I_k &= d^{2k-n+1} \left( \partial_u^{p-k} |u| \hat{f}_{\alpha, \beta}^0(ud) (u+1)^{k-p-1} \right) \Big|_{u=1} \\ &= d^{2k-n+1} \sum_{m=0}^{p-k} c_{m,p,k} \left( \partial_u^m \hat{f}_{\alpha, \beta}^0(ud) \partial_u^{p-k-m} \left( |u| (u+1)^{k-p-1} \right) \right) \Big|_{u=1} \\ &= d^{2k-n+1} \sum_{m=0}^{p-k} c'_{m,p,k} \left( \partial_u^m \hat{f}_{\alpha, \beta}^0(ud) \right) \Big|_{u=1}. \end{aligned}$$

Making use of Lemma 3.1, we get that for all  $m = 0, \dots, p-k$ ,

$$\begin{aligned} \left| \partial_u^m \hat{f}_{\alpha, \beta}^0(ud)_{u=1} \right| &\leq \frac{cd^m}{d^{(1+m-\varepsilon-\frac{n+1}{2}\alpha)/(1-\alpha)}} \\ &= \frac{cd^m d^{\varepsilon/(1-\alpha)}}{d^{(1+m-(p+1)\alpha)/(1-\alpha)}} \\ &= \frac{d^m d^{\varepsilon/(1-\alpha)}}{d^{d(m-p\alpha)/(1-\alpha)}} \\ &= cd^{-1} d^{\varepsilon/(1-\alpha)} d^{\alpha(p-m)/(1-\alpha)} \\ &\leq cd^{-1} d^{\varepsilon/(1-\alpha)} d^{\alpha k(1-\alpha)}. \end{aligned}$$

This implies that for all  $k \geq 0$ ,

$$|I_k| \leq cd^{2k-n+1} d^{-1} d^{\varepsilon/(1-\alpha)} d^{\alpha k(1-\alpha)} \leq cd^{-n} d^{\varepsilon/(1-\alpha)}$$

which proves (4.5), when  $n = 2p + 1$ .

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*Proof of (4.5), for  $n = 2p$ .* In this case we have

$$I_k(x, y) = \int_{\mathbb{R}} |t| \hat{f}_{\alpha, \beta}^0(t) \frac{(t^2 - d^2)_+^{k-p-\frac{1}{2}}}{\Gamma(k-p+\frac{1}{2})} dt.$$

The calculations now are more complicated because  $k - p - \frac{1}{2}$  is no more an integer. If we put  $t = du$  and  $v = u + 1$ , then

$$\begin{aligned} I_k &= cd^{2k-2p+1} \int_{|u|>1} |u| \hat{f}_{\alpha, \beta}^0(du) (u^2 - 1)_+^{k-p-\frac{1}{2}} du \\ &= cd^{2k-2p+1} \int_{u>1} u \hat{f}_{\alpha, \beta}^0(du) (u+1)^{k-p-\frac{1}{2}} (u-1)_+^{k-p-\frac{1}{2}} du \\ &\quad + cd^{2k-2p+1} \int_{u<-1} (-u) \hat{f}_{\alpha, \beta}^0(du) |u-1|^{k-p-\frac{1}{2}} (-(u+1))_+^{k-p-\frac{1}{2}} du \\ &= cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{\alpha, \beta}^0(d(v+1)) (v+2)^{k-p-\frac{1}{2}} v_+^{k-p-\frac{1}{2}} dv \\ &\quad + cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{\alpha, \beta}^0(-d(v+1)) (v+2)^{k-p-\frac{1}{2}} w_+^{k-p-\frac{1}{2}} dv. \end{aligned}$$

Since  $\hat{f}_{\alpha, \beta}^0$  is an even function

$$I_k = 2cd^{2k-2p+1} \int_{v>0} (v+1) \hat{f}_{\alpha, \beta}^0(d(v+1)) (v+2)^{k-p-\frac{1}{2}} v_+^{k-p-\frac{1}{2}} dv.$$

We shall only treat the term  $I_0$  which is the most singular near  $v = 0$ . The integrals  $I_k$ ,  $k > 0$ , can be treated similarly. We have

$$I_0 = cd^{-2p+1} \int_0^\infty (v+1) \hat{f}_{\alpha, \beta}^0(d(v+1)) (v+2)^{-p-\frac{1}{2}} v_+^{-p-\frac{1}{2}} dv. \quad (4.9)$$

By replacing the term  $(v+2)^{-p-\frac{1}{2}}$  by its Taylor's expansion at  $v = 0$ , we can see that the most singular part of  $I_0$  is the integral

$$J_0 := d^{-2p+1} \int_0^\infty \hat{f}_{\alpha, \beta}^0(d(v+1)) v_+^{-p-\frac{1}{2}} dv.$$

Let us observe that  $\hat{f}_{\alpha, \beta}^0(d(v+1))$  is the Fourier transform of the function

$$\frac{1}{d} f_{\alpha, \beta} \left( \frac{t}{d} \right) e^{it} = \frac{1}{d} \psi \left( \left| \frac{t}{d} \right|^2 \right) \left| \frac{t}{d} \right|^{-\varepsilon - \alpha n/2} e^{i \left| \frac{t}{d} \right|^\alpha} e^{it}.$$

Also, the Fourier transform of the distribution  $v_+^{-p-\frac{1}{2}}$  is equal to

$$i\Gamma\left(-p + \frac{1}{2}\right) \left[ e^{-i\frac{\pi}{2}(p+\frac{1}{2})} t_+^{p-\frac{1}{2}} - e^{+i\frac{\pi}{2}(p+\frac{1}{2})} t_-^{p-\frac{1}{2}} \right],$$

(cf. [13], p. 172). So,

$$\begin{aligned} J_0 &= d^{-2p+1} \int_{-\infty}^{\infty} \frac{1}{d} \psi\left(\left|\frac{t}{d}\right|^2\right) |t/d|^{-\varepsilon-\alpha n/2} e^{i|t/d|^\alpha} e^{it} \left[ c_1 t_+^{p-\frac{1}{2}} - c_2 t_-^{p-\frac{1}{2}} \right] dt \\ &= d^{-2p+1} d^{p-\frac{1}{2}} \int_{-\infty}^{\infty} \psi(u^2) |u|^{-\varepsilon-\alpha n/2} e^{i|u|^\alpha} e^{iud} \left[ c_1 u_+^{p-\frac{1}{2}} - c_2 u_-^{p-\frac{1}{2}} \right] du \\ &= J_{0,1} + J_{0,2}. \end{aligned} \tag{4.10}$$

We shall only treat  $J_{0,1}$ . The term  $J_{0,2}$  can be treated similarly. We have

$$\begin{aligned} J_{0,1} &= c_1 d^{-2p+1} d^{p-\frac{1}{2}} \int_0^{\infty} \psi(u^2) u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}} e^{iu^\alpha} \cos(ud) du \\ &\quad + i c_1 d^{-2p+1} d^{p-\frac{1}{2}} \int_0^{\infty} \psi(u^2) u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}} e^{iu^\alpha} \sin(ud) du \\ &= d^{-2p+1} d^{p-\frac{1}{2}} c_1 (L_1 + iL_2). \end{aligned} \tag{4.11}$$

Now  $L_1$  is the Fourier transform of the even function

$$f_{\alpha,b}(u) = \psi(|u|^2) |u|^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}} e^{i|u|^\alpha},$$

with  $b = \frac{\alpha n}{2} + \varepsilon - p + \frac{1}{2}$ . So, by (3.5), with  $k = 1$ , we get that

$$\begin{aligned} |L_1| &\leq cd^{-(1-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}-\frac{\alpha}{2})/(1-\alpha)} \\ &= cd^{-(\frac{1-\alpha}{2}+p(1-\alpha))/(1-\alpha)} d^{\frac{\varepsilon}{1-\alpha}} = d^{-p-\frac{1}{2}} d^{\frac{\varepsilon}{1-\alpha}}. \end{aligned} \tag{4.12}$$

By the formula  $\sin x = \sqrt{\frac{\pi x}{2}} J_{\frac{1}{2}}(x)$ , we have

$$\begin{aligned} L_2 &= \int_0^{\infty} \psi(u^2) u^{-\frac{\alpha n}{2}-\varepsilon+p-\frac{1}{2}} e^{iu^\alpha} \sin(ud) du \\ &= c\sqrt{d} \int_0^{\infty} \psi(u^2) u^{-\frac{\alpha n}{2}-\varepsilon+p} e^{iu^\alpha} J_{\frac{1}{2}}(ud) du \\ &= cd \lim_{\rho \rightarrow 0} \left\{ d^{-\frac{1}{2}} \int_0^{\infty} e^{-\rho u} \psi(u^2) u^{-\left(\frac{\alpha n}{2} + \varepsilon - p + \frac{3}{2}\right) + \frac{3}{2}} e^{iu^\alpha} J_{\frac{1}{2}}(ud) du \right\}. \end{aligned}$$

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The integral in the brackets above is the same as the integral  $\hat{f}_{\epsilon, \alpha, b}$  in (3.4) with  $k = 3$  and  $b = \frac{\alpha n}{2} + \epsilon - p + \frac{3}{2}$ . Therefore, by (3.5), with  $k = 3$ , we get that, for

$$\begin{aligned} |L_2| &\leq c d d^{-\left(3 - \frac{\alpha n}{2} - \epsilon + p - \frac{3}{2} - \frac{3\alpha}{2}\right)/(1-\alpha)} \\ &= c d d^{-\left(\frac{3}{2}(1-\alpha) + p(1-\alpha)\right)/(1-\alpha)} d^{\epsilon/(1-\alpha)} \\ &= c d d^{-\frac{3}{2}} d^{-p} d^{\epsilon/(1-\alpha)} = c d^{-\frac{1}{2}} d^{-p} d^{\epsilon/(1-\alpha)}. \end{aligned} \tag{4.13}$$

It follows from (4.11), (4.12) and (4.13) that

$$|J_{0,1}| \leq c d^{-2p+1} d^{p-\frac{1}{2}} d^{-p-\frac{1}{2}} = c d^{-n} d^{\frac{\epsilon}{(1-\alpha)}}. \tag{4.14}$$

Putting all together, from (4.9) to (4.14), we get

$$|I_k(x, y)| \leq c d^{-n} d^{\frac{\epsilon}{(1-\alpha)}}$$

which proves (4.5), for  $n = 2p$ .

*Proof of (4.6).* If  $k > \frac{n+1}{2}$ , then by (3.2) and (3.3) we get

$$\begin{aligned} |I_k(x, y)| &\leq c \int_{d \leq |t| \leq \delta} \left| \hat{f}_{\alpha, \beta}^0(t) \right| |t| \frac{(t^2 - d^2)_+^{k - \frac{n+1}{2}}}{\Gamma(k - \frac{n-1}{2})} dt \\ &\leq \frac{c}{\Gamma(k - \frac{n-1}{2})} \int_{d \leq |t| \leq \delta} |t|^{-\left(1 - \epsilon - \frac{\alpha(n+1)}{2}\right)/(1-\alpha)} |t|^{2k-n} dt. \end{aligned}$$

But, if  $k > \frac{n+1}{2}$ , then

$$2k - n - \frac{1 - \epsilon - \frac{\alpha(n+1)}{2}}{(1-\alpha)} \geq \frac{2\epsilon + \alpha(n-1)}{2(1-\alpha)} > 0,$$

so,

$$|I_k(x, y)| \leq c \frac{\delta^{2k-n+1 - \left(1 - \epsilon - \frac{\alpha(n+1)}{2}\right)/(1-\alpha)}}{\Gamma\left(k - \frac{n-1}{2}\right)} \leq c \frac{\delta^{2k}}{\Gamma\left(k - \frac{n-1}{2}\right)}.$$

*Proof of (4.7).* We shall only treat the case  $n = 2p + 1$ . The case  $n = 2p$  can be treated similarly. As in the proof of (4.5), we have to estimate the



integral

$$\begin{aligned}
 I_{-1}(x, y) &= \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(t) |t| \frac{(t^2 - d(x, y)^2)_+^{-1 - \frac{n+1}{2}}}{\Gamma(-1 - \frac{n-1}{2})} dt \\
 &= d^{-n-1} \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(du) |u| \frac{(u^2 - 1)_+^{-1 - \frac{n+1}{2}}}{\Gamma(-1 - \frac{n-1}{2})} du \\
 &= d^{-n-1} \int_{\mathbb{R}} \hat{f}_{\alpha, \beta}^0(du) |u| (u+1)^{-p-2} \frac{(u-1)_+^{-p-2}}{\Gamma(-p-1)} dt \\
 &= d^{-n-1} \partial_u^{p+1} \left( |u| \hat{f}_{\alpha, \beta}^0(ud) (u+1)^{-p-2} \Big|_{u=1} \right).
 \end{aligned}$$

So,

$$\begin{aligned}
 |I_{-1}(x, y)| &\leq cd^{-n-1} \sum_{m=0}^{p+1} c'_{m,p} \frac{d^m}{d^{(1+m-(p+1)\alpha)/(1-\alpha)}} \\
 &= cd^{-n-1} \sum_{m=0}^{p+1} c'_{m,p} \frac{d^m}{d^{(m-p\alpha)/(1-\alpha)}} \\
 &= cd^{-n-2} \sum_{m=0}^{p+1} c'_{m,p} d^{\frac{m-m\alpha-m+p\alpha}{1-\alpha}} \\
 &= cd^{-n-2} \sum_{m=0}^{p+1} c'_{m,p} d^{\frac{\alpha}{1-\alpha}(p-m)} \leq cd^{-n-2} d^{-\alpha/(1-\alpha)} = cd^{-n-2} d^{\alpha'}.
 \end{aligned}$$

□

PROOF OF LEMMA 4.1: (i). It is a consequence of (2.11) and Lemma 4.2.

(ii) Making use of (2.13), we have

$$\begin{aligned}
 \nabla_y G_t(x, y) &= \sum_{k=0}^{\infty} (-1)^k \nabla_y U_k(x, y) |t| \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2}}}{4^k \Gamma(k - \frac{n-1}{2})} \\
 &\quad - \sum_{k=0}^{\infty} U_k(x, y) |t| \left(k - \frac{n+1}{2}\right) \frac{(t^2 - d(x, y)^2)_+^{k - \frac{n+1}{2} - 1}}{4^k \Gamma(k - \frac{n-1}{2})} 2d \nabla_y(d) \\
 &= I + II.
 \end{aligned}$$

Now, it follows from (2.11) and the estimates (4.5), (4.6) for  $\varepsilon = 0$ , that

$$|I| \leq cd(x, y)^{-n}.$$

To deal with  $II$  we first note that  $\|\nabla_y d(x, y)\| \leq 1$  for  $d(x, y) \leq 1$ . Then, by (4.6) and (4.7) we have

$$|II| \leq cd(x, y)^{-(n+1)+\alpha'}.$$

## 5 The $L^p$ boundedness of $m_{\alpha, \beta}(L)$ for $\beta > \frac{\alpha n}{2}$

In this Section we prove claim (iii) of Theorem 1.1 which states that for all  $\alpha \in (0, 1)$  and  $\beta > \frac{\alpha n}{2}$ ,  $m_{\alpha, \beta}(L)$  is bounded on  $L^p$ ,  $p \geq 1$ .

We note that the  $L^p$  boundedness of  $m_{\alpha, \beta}^\infty(L)$  for  $\beta \geq \frac{\alpha n}{2}$ , can be extracted from [1]. We shall give below a simple proof of this result by adapting an argument from [29].

**Proposition 5.1:** *If  $\alpha \in (0, 1)$  and  $\beta \geq \frac{\alpha n}{2}$ , then  $m_{\alpha, \beta}^\infty(L)$  is bounded on  $L^p$ ,  $p \geq 1$ .*

PROOF: We have that

$$m_{\alpha, \beta}^\infty(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}_{\alpha, \beta}^\infty(t) \cos t\sqrt{\lambda} dt$$

and by the estimate (3.3) of  $\hat{f}_{\alpha, \beta}^\infty(t)$  we get that  $m_{\alpha, \beta}^\infty$  is bounded. Thus  $m_{\alpha, \beta}^\infty(L)$  is bounded on  $L^2$ . Therefore, the Proposition will be a consequence of the following:

$$\sup_{x \in M} \int_M |m_{\alpha, \beta}^\infty(x, y)| dy < \infty. \quad (5.1)$$

Let us first notice that the Dirac mass  $\delta_x$  at  $x$  can be written as  $\delta_x = L^k \varphi_x + \psi_x$ , where  $k = \left[ \frac{n}{4} \right] + 1$  and where the functions  $\varphi_x$  and  $\psi_x$  are in  $L^2(B(x, r_0))$ , with  $r_0$  the injectivity radius of  $M$  (cf. [29], p. 776). Also by the assumption of  $C^\infty$ -bounded geometry, we can assume that there is  $c > 0$  such that  $\|\varphi_x\|_2 \leq c$  and  $\|\psi_x\|_2 \leq c$  for all  $x \in M$ . We have

$$\begin{aligned} m_{\alpha, \beta}^\infty(x, y) &= m_{\alpha, \beta}^\infty(L)\delta_x(y) = L^k m_{\alpha, \beta}^\infty(L)\varphi_x(y) + m_{\alpha, \beta}^\infty(L)\psi_x(y) \\ &= (\sqrt{L})^{2k} f_{\alpha, \beta}^\infty(\sqrt{L})\varphi_x(y) + f_{\alpha, \beta}^\infty(\sqrt{L})\psi_x(y) \\ &= (-i)^{-2k} (2\pi)^{-1/2} \int_{-\infty}^{\infty} \partial^{2k} \hat{f}_{\alpha, \beta}^\infty(t) \cos t\sqrt{L}\varphi_x(y) dt \\ &\quad + (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}_{\alpha, \beta}^\infty(t) \cos t\sqrt{L}\psi_x(y) dt \\ &= I_1(x, y) + I_2(x, y). \end{aligned} \quad (5.2)$$

By the estimates (3.3) of  $\partial_t^m \hat{f}_{\alpha,\beta}^\infty(t)$  and the finite propagation speed property we have that

$$\begin{aligned}
 |I_1(x, y)| &\leq c \int_{-\infty}^{\infty} \left| \partial^{2k} \hat{f}_{\alpha,\beta}^\infty(t) \cos t\sqrt{L}\varphi_x(y) \right| dt \\
 &= c \sum_{j \geq 1} \int_{j \leq |t| \leq j+1} \left| \partial^{2k} \hat{f}_{\alpha,\beta}^\infty(t) \right| \left| \cos t\sqrt{L}\varphi_x(y) \right| dt \\
 &\leq c \sum_{j \geq 1} \frac{1}{j^N} \int_{j \leq |t| \leq j+1} \left| \mathbf{1}_{B(x, r_0+j+1)}(y) \cos t\sqrt{L}\varphi_x(y) \right| dt.
 \end{aligned} \tag{5.3}$$

By the Cauchy-Schwarz inequality

$$\begin{aligned}
 \int_M \left| \mathbf{1}_{B(x, R)}(y) \cos t\sqrt{L}\varphi_x(y) \right| dy &\leq |B(x, R)|^{\frac{1}{2}} \left\| \cos t\sqrt{L}\varphi_x \right\|_2 \\
 &\leq cR^{n/2} \left\| \cos t\sqrt{L} \right\|_2 \|\varphi_x\|_2 \\
 &\leq cR^{n/2}
 \end{aligned} \tag{5.4}$$

since  $\left\| \cos t\sqrt{L} \right\|_2 \leq 1$  and  $\|\varphi_x\|_2 \leq c$  for all  $x \in M$ .

Let  $N > 2 + \frac{n}{2}$ . Then, it follows from (5.2), (5.3) and (5.4) that

$$\int_M |I_1(x, y)| dy \leq c \sum_{j \geq 1} (r_0 + j + 1)^{\frac{n}{2}} \frac{1}{j^N} \int_{j \leq |t| \leq j+1} dt \leq c \sum_{j \geq 1} \frac{1}{j^{N-\frac{n}{2}}}$$

and hence

$$\sup_{x \in M} \int_M |I_1(x, y)| dy < \infty.$$

The term  $I_2(x, y)$  can be treated similarly.  $\square$

**Proposition 5.2:** *If  $\alpha \in (0, 1)$  and  $\beta > \frac{\alpha n}{2}$ , then  $m_{\alpha,\beta}^0(L)$  is bounded on  $L^p$ ,  $p \geq 1$ .*

PROOF: Since  $m_{\alpha,\beta}^0(L) = m_{\alpha,\beta}(L) - m_{\alpha,\beta}^\infty(L)$ , Proposition 5.1 implies that  $m_{\alpha,\beta}^0(L)$  is bounded on  $L^2$ . If  $\beta = \frac{\alpha n}{2} + \varepsilon$ ,  $\varepsilon > 0$ , then from (4.2) and (4.3) we have that

$$\begin{aligned}
 \sup_{x \in M} \int_M |m_{\alpha,\beta}^0(x, y)| dy &= \sup_{x \in M} \int_{B(x, \delta)} |m_{\alpha,\beta}^0(x, y)| dy \\
 &\leq c \sup_{x \in M} \int_{B(x, \delta)} d(x, y)^{-n+\frac{\varepsilon}{1-\alpha}} dy \\
 &= c \sup_{x \in M} \int_0^\delta r^{-n+\frac{\varepsilon}{1-\alpha}} r^{n-1} dr = c\delta^{\frac{\varepsilon}{1-\alpha}}
 \end{aligned}$$

and the Proposition follows.  $\square$

## 6 $H^1-L^1$ boundedness of the operator $m_{\alpha, \frac{\alpha n}{2}}(L)$

In this section we prove claim (i) of Theorem 1.1. By the duality of  $H^1$  with  $BMO$ , the  $H^1-L^1$  boundedness of  $m_{\alpha, \frac{\alpha n}{2}}(L)$  is a consequence of the following

**Proposition 6.1:** *If  $\alpha \in (0, 1)$ , then the operator  $m_{\alpha, \frac{\alpha n}{2}}(L)$  is bounded from  $L^\infty(M)$  to  $BMO(M)$ .*

The  $L^p$ -boundedness of  $m_{\alpha, \frac{\alpha n}{2}}(L)$  for  $p \in (1, \infty)$ , follows from the  $L^2$  boundedness and Proposition 6.1 by interpolation and duality.

The strategy of the proof of Proposition 6.1 is inspired from [11]. It is based on the following Lemmata.

**Lemma 6.2:** *There is a constant  $A > 0$  such that*

$$\int_{d(x, y_1) > 2d(y, y_1)^{1-\alpha}} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx < A, \quad (6.1)$$

for all  $y_1 \in M$  and  $y \in B(y_1, \delta)$ .

PROOF: Let us fix  $y_1 \in M$  and  $y \in B(y_1, \delta)$ . Let  $y(s)$ ,  $s \in [0, d(y, y_1)]$ , be the geodesic segment from  $y$  to  $y_1$ . Then

$$m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) = \int_0^{d(y, y_1)} \nabla_y m_{\alpha, \frac{\alpha n}{2}}^0(x, y(s)) ds.$$

By (4.4) and the mean value theorem, we get that

$$\left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| \leq c \frac{d(y, y_1)}{d(x, y^*)^{n+1-\alpha}}, \quad (6.2)$$

for some  $y^*$  on  $y(s)$ .

Let us set  $d = d(y, y_1)$ ,  $A_k = B(y_1, 2^{k+1}d^{1-\alpha}) \setminus B(y_1, 2^k d^{1-\alpha})$  and

$$I_k = \int_{A_k} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx.$$

Then

$$\begin{aligned} & \int_{d(x,y_1) > 2d(y,y_1)^{1-\alpha}} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx \\ &= \sum_{k \geq 1} \int_{A_k} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx = \sum_{k \geq 1} I_k. \end{aligned}$$

Since  $d \leq \delta \leq 1$ , we have

$$d(x, y^*) \geq 2^k d^{1-\alpha} - d \geq 2^{k-1} d^{1-\alpha}, \quad \forall x \in A_k, \quad \forall k \geq 1.$$

Now, by (6.2) and since  $(1 - \alpha)(1 - \alpha') = 1$ , we have

$$\begin{aligned} I_k &\leq c \int_{A_k} \frac{d(y, y_1) dx}{d(x, y^*)^{n+1-\alpha'}} \leq c \int_{A_k} \frac{ddx}{(2^{k-1} d^{1-\alpha})^{n+1-\alpha'}} \\ &\leq \frac{cd|A_k|}{(2^k d^{1-\alpha})^{n+1-\alpha'}} \leq \frac{cd(2^{k+1} d^{1-\alpha})^n}{(2^k d^{1-\alpha})^{n+1-\alpha'}} \\ &= \frac{cd}{(2^k)^{1-\alpha'} d^{(1-\alpha)(1-\alpha')}} = \frac{c}{(2^k)^{1-\alpha'}}. \end{aligned}$$

It follows that

$$\begin{aligned} & \int_{d(x,y_1) > d(y,y_1)^{1-\alpha}} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx \\ &= \sum_{k=1}^{\infty} I_k \leq c \sum_{k=1}^{\infty} \frac{1}{(2^{k-1})^{1-\alpha'}} < \infty \end{aligned}$$

since  $1 - \alpha' > 0$  for  $\alpha \in (0, 1)$ . □

The following Lemma is based on a local version of a generalization of Hardy-Littlewood-Sobolev theorem due to Varopoulos, (cf. [30], p. 12).

**Lemma 6.3:** *For any  $\alpha \in (0, 1)$ ,  $m_{\alpha, \frac{\alpha n}{2}}(L)$  is bounded from  $L^2$  to  $L^{\frac{2}{1-\alpha}}$ .*

PROOF: We write

$$\begin{aligned} m_{\alpha, \frac{\alpha n}{2}}(L) &= \psi(|L|) |L|^{-\alpha n/4} e^{i|L|^{\alpha/2}} \\ &= (1 + L)^{-\alpha n/4} \psi(|L|) |L|^{-\alpha n/4} (1 + L)^{\alpha n/4} e^{i|L|^{\alpha/2}} \\ &= (1 + L)^{-\alpha n/4} \Phi(L), \end{aligned}$$

where  $\Phi(\lambda) = \psi(|\lambda|) |\lambda|^{-\alpha n/4} (1 + \lambda)^{\alpha n/4} e^{i|\lambda|^{\alpha/2}}$ . Since  $\Phi(\lambda)$  is bounded, it suffices to show that the potential operator  $(1 + L)^{-\alpha n/4}$  is bounded from  $L^2$

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to  $L^{\frac{2}{1-\alpha}}$ . To this end, let  $q_t(x, y)$  be the kernel of the semigroup  $e^{-t(1+L)}$  and  $p_t(x, y)$  the heat kernel of  $M$ . Then

$$q_t(x, y) = e^{-t} p_t(x, y).$$

By the Li-Yau estimate of  $p_t$ :

$$p_t(x, y) \leq c \frac{e^{-d(x,y)^2/ct}}{|B(x, \sqrt{t})|},$$

for all  $t > 0$  and  $x, y \in M$ , (cf. [16]), it follows that

$$q_t(x, y) \leq \begin{cases} ct^{-n/2}, & \forall t \leq 1, \\ ce^{-t} \leq ct^{-n/2}, & \forall t \geq 1. \end{cases} \quad (6.3)$$

From (6.3) it follows that

$$\|e^{-t(1+L)} f\|_\infty \leq ct^{-n/2} \|f\|_1, \quad \forall f \in L^1, \quad \forall t > 0.$$

As it is shown by Varopoulos, (cf. [30], p. 12), this estimate implies that the operators  $(1+L)^{-\gamma/2}$ ,  $\gamma > 0$ , are bounded from  $L^p$  to  $L^q$  for  $\frac{1}{q} = \frac{1}{p} - \frac{\gamma}{n}$  and  $1 < p < \infty$ . The Lemma follows by taking  $\gamma = \alpha n/2$  and  $p = 2$ .  $\square$

**PROOF OF PROPOSITION 6.1:** In order to prove that  $m_{\alpha, \frac{\alpha n}{2}}(L)$  is bounded from  $L^\infty$  to  $BMO$  it enough to show that there is a constant  $c > 0$ , such that for every ball  $B(y_1, r) = B$  and every  $f \in C_0^\infty(M)$

$$\int_B |m_{\alpha, \frac{\alpha n}{2}}(L)f(x) - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B| dx \leq c \|f\|_\infty |B|, \quad (6.4)$$

where  $(m_{\alpha, \frac{\alpha n}{2}}(L)f)_B$  is the mean value of  $m_{\alpha, \frac{\alpha n}{2}}(L)f$  on  $B$ .

Let us then fix a ball  $B(y_1, r) = B$  and let us set, in order to simplify the notation,  $B_\alpha = B(y_1, 2r^{1-\alpha})$ . If  $f \in C_0^\infty(M)$ , then we shall write  $f = f\chi_{B_\alpha} + f\chi_{B_\alpha^c} := f_1 + f_2$ .

To prove (6.4), we shall show that

$$\int_B |m_{\alpha, \frac{\alpha n}{2}}(L)f_1(x)| dx \leq c \|f\|_\infty |B|, \quad (6.5)$$

and

$$\int_B |m_{\alpha, \frac{\alpha n}{2}}(L)f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B| dx \leq c \|f\|_\infty |B|. \quad (6.6)$$

*Proof of (6.5).* If  $r > 1$ , then  $r^{1-\alpha} \leq r$  and hence

$$\begin{aligned} \int_B |m_{\alpha, \frac{\alpha n}{2}}(L)f_1(x)| dx &\leq \|m_{\alpha, \frac{\alpha n}{2}}(L)f_1\|_2 |B|^{1/2} \leq c \|f_1\|_2 |B|^{1/2} \\ &= c \|f\chi_{B_\alpha}\|_2 |B|^{1/2} \leq c \|f\|_\infty |B_\alpha|^{1/2} |B|^{1/2} \\ &= c \|f\|_\infty |B(y_1, 2r^{1-\alpha})|^{1/2} |B|^{1/2} \\ &\leq c \|f\|_\infty |B(y_1, 2r)|^{1/2} |B|^{1/2} \leq c \|f\|_\infty |B|. \end{aligned}$$

In the case when  $r \leq 1$ , we proceed by arguing as in [11], Theorem 1, p. 143 (see also [9], Theorem 2.1). Let  $p = 2/(1 - \alpha)$  and let  $p'$  be its conjugate exponent. Then by Lemma 6.3 and Hölder's inequality

$$\begin{aligned} \int_B |m_{\alpha, \frac{\alpha n}{2}}f_1(x)| dx &\leq |B|^{1/p'} \|m_{\alpha, \frac{\alpha n}{2}}f_1\|_p \leq c |B|^{1/p'} \|f_1\|_2 \\ &\leq c |B|^{1/p'} \|f_1\|_2 = c |B|^{1/p'} \|f\chi_{B_\alpha}\|_2 \\ &\leq c |B|^{1/p'} \|f\|_\infty |B(y_1, 2r^{1-\alpha})|^{1/2} \\ &\leq c \|f\|_\infty r^{\frac{n}{p'} + (1-\alpha)\frac{n}{2}} = cr^n \|f\|_\infty \leq c |B| \|f\|_\infty, \end{aligned}$$

since  $\frac{n}{p'} + (1 - \alpha)\frac{n}{2} = \frac{n}{p'} + \frac{n}{p} = n$ . This completes the proof of (6.5).

*Proof of (6.6).* We have

$$\begin{aligned} &|m_{\alpha, \frac{\alpha n}{2}}(L)f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B| \\ &\leq \left| m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2)_B \right| \\ &\quad + \left| (m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2)_B - (m_{\alpha, \frac{\alpha n}{2}}(L)f)_B \right| + \left| m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f_2(x) \right|. \end{aligned} \tag{6.7}$$

We write

$$m_{\alpha, \frac{\alpha n}{2}}(L)f = m_{\alpha, \frac{\alpha n}{2}}^0(L)f_1 + m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2 + m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f,$$

and we recall that the operator  $m_{\alpha, \frac{\alpha n}{2}}^0(L)$  is bounded on  $L^2$  and that, by

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Proposition 5.1, the operator  $m_{\alpha, \frac{\alpha n}{2}}^\infty(L)$  is bounded on  $L^\infty$ . Therefore,

$$\begin{aligned}
 & \left| (m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2)_B - (m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f)_B \right| \\
 &= |B|^{-1} \left| \int_B m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2(x)dx - \int_B m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f(x)dx \right| \\
 &= |B|^{-1} \left| \int_B m_{\alpha, \frac{\alpha n}{2}}^0(L)f_1(x)dx + \int_B m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f(x)dx \right| \tag{6.8} \\
 &\leq |B|^{-1} \left\| m_{\alpha, \frac{\alpha n}{2}}^0(L)f_1 \right\|_2 |B|^{\frac{1}{2}} + |B|^{-1} \left\| m_{\alpha, \frac{\alpha n}{2}}^\infty(L)f \right\|_\infty |B| \\
 &\leq c|B|^{-1} \|f\|_\infty |B| + c\|f\|_\infty = c\|f\|_\infty.
 \end{aligned}$$

It follows from (6.7), (6.8) and the  $L^\infty$  boundedness of  $m_{\alpha, \frac{\alpha n}{2}}^\infty(L)$  that to prove (6.6), it is enough to show that

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2(x) - (m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2)_B \right| dx \leq c\|f\|_\infty |B|. \tag{6.9}$$

Let us set

$$c_B = \int_{B_\alpha^c} m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1)f_2(x)dx.$$

If  $y \in B(y_1, r)$ , then

$$m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2(y) - c_B = \int_{B_\alpha^c} \left\{ m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right\} f_2(x)dx.$$

Also, if  $x \in B(y_1, 2r^{1-\alpha})^c$  and  $y \in B(y_1, r)$ , then

$$d(x, y_1) > 2r^{1-\alpha} \geq 2d(y, y_1)^{1-\alpha}.$$

Therefore, by Lemma 6.2

$$\begin{aligned}
 & \left| m_{\alpha, \frac{\alpha n}{2}}^0(L)f_2(y) - c_B \right| \\
 &\leq \int_{B_\alpha^c} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| |f_2(x)| dx \\
 &\leq \|f\|_\infty \int_{d(x, y_1) > 2d(y, y_1)^{1-\alpha}} \left| m_{\alpha, \frac{\alpha n}{2}}^0(x, y) - m_{\alpha, \frac{\alpha n}{2}}^0(x, y_1) \right| dx \\
 &\leq A\|f\|_\infty.
 \end{aligned}$$



This implies that

$$\int_B \left| m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2(y) - c_B \right| dy \leq A |B| \|f\|_\infty. \quad (6.10)$$

By (6.10) we have

$$\begin{aligned} & \int_B \left| m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2(y) - (m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2)_B \right| dy \\ & \leq \int_B \left| m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2(y) - c_B \right| dy + \int_B \left| c_B - (m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2)_B \right| dy \\ & \leq A \|f\|_\infty |B| + |B| \left| c_B - (m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2)_B \right|. \end{aligned} \quad (6.11)$$

Finally, by using once more (6.10) we get

$$\begin{aligned} \left| (m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2)_B - c_B \right| &= |B|^{-1} \left| \int_{B(y_1, r)} m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2(y) dy - \int_B c_B dy \right| \\ &\leq |B|^{-1} \int_B \left| m_{\alpha, \frac{\alpha n}{2}}^0(L) f_2(y) - c_B \right| dy \leq A \|f\|_\infty. \end{aligned} \quad (6.12)$$

## 7 Proof of the results

In this Section we shall finish the proofs of Theorems 1.1 and 1.2.

PROOF OF THEOREM 1.1: The proof of claims (i) and (ii) of Theorem 1.1 are given in Sections 6 and 5 respectively. It remains to prove claim (ii). This will be done by complex interpolation as in Theorem 6 of [11]. Let us consider the analytic family of operators

$$T_z(L) = e^{z^2} L^{\frac{n\alpha}{4} z} m_{\alpha, \frac{\alpha n}{2}}(L), \quad \operatorname{Re} z \in [0, 1].$$

If  $t \in \mathbb{R}$ , then

$$T_{it}(L) = e^{-t^2} L^{i \frac{n\alpha t}{4}} m_{\alpha, \frac{\alpha n}{2}}(L).$$

But the imaginary powers of the Laplacian are bounded on  $H^1$  and

$$\|L^{i\gamma}\|_{H^1 \rightarrow H^1} \leq c \left(1 + \sqrt{|\gamma|} e^{\pi|\gamma|/2}\right), \quad \gamma \in \mathbb{R},$$

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(cf. [19]). So, if we combine with Theorem 1.1(i), we get that  $T_{it}(L)$  is bounded from  $H^1(M)$  to  $L^1(M)$  and

$$\|T_{it}(L)\|_{H^1 \rightarrow L^1} \leq ce^{-t^2} \left( c\sqrt{\pi} + \sqrt{\alpha n} |t| e^{\pi\alpha n |t|/8} \right),$$

for all  $t \in \mathbb{R}$ .

Also, the operators  $T_{1+it}(L)$  are bounded on  $L^2(M)$  and

$$\|T_{1+it}(L)\|_2 \leq ce^{-t^2}.$$

By complex interpolation between  $Rez = 0$  and  $Rez = 1$ , we obtain that for  $\theta \in (0, 1)$  and  $p \in (1, 2)$ , the operator  $T_\theta(L)$  is bounded on  $L^p$  for  $\frac{1}{p} = 1 - \frac{\theta}{2}$ . If we choose  $\theta = 1 - \frac{2\beta}{\alpha n}$ , then

$$T_\theta(L) = e^{\theta^2} L^{\frac{n\alpha}{4}} L^{-\frac{n\alpha}{4} \frac{2\beta}{\alpha n}} m_{\alpha, \frac{\alpha n}{2}}(L) = e^{\theta^2} m_{\alpha, \beta}(L)$$

and  $\frac{1}{p} - \frac{1}{2} = \frac{\beta}{\alpha n}$ . This is the desired result for  $p \in (1, 2)$ . The case  $p \in (2, \infty)$  is just the dual result.

PROOF OF THEOREM 1.2: As in [1], by replacing the operator  $L$  by  $L_1 = t^{2/\alpha}L$ , the operators

$$I_{k,\alpha}(L) = kt^{-k} \int_0^t (t-s)^{k-1} e^{isL^{\alpha/2}} ds, \quad 0 < \alpha < 1, k > 0,$$

can be written in the form

$$I_{k,\alpha}(L) = M_k(L_1^{\alpha/2}),$$

with

$$M_k(\lambda) = k \int_0^1 (1-s)^{k-1} e^{is|\lambda|} ds.$$

Further, the multiplier  $M_k(\lambda)$  can be written as

$$M_k(\lambda) = C_k \psi(\lambda) \lambda^{-k} e^{i\lambda} + \Omega(\lambda),$$

where  $\psi$  is as in (1.2) and  $\Omega(\lambda)$  satisfies

$$\partial_\lambda^N \Omega(\lambda) = O(\lambda^{-N-1}), \quad \text{as } \lambda \rightarrow \infty,$$

for all  $N \in \mathbb{N}$ , (cf. [1], [27], p. 336).

This implies that

$$\left| \hat{\Omega}(t) \right| \leq \frac{c(N, R)}{|t|^{N+1}}, \quad \text{for } |t| \geq R.$$

Making use of this and by arguing in exactly the same way as in Proposition 5.1 we can prove that the operator  $\Omega(L)$  is bounded on  $L^p$ ,  $p \geq 1$ . Furthermore, by Theorem 1.1(ii),  $C_k \psi(L_1) L_1^{-\alpha k/2} e^{iL_1^{\alpha/2}}$  is bounded on  $L^p$  for  $\alpha k \geq \alpha n \left| \frac{1}{p} - \frac{1}{2} \right|$  i.e. for  $k \geq n \left| \frac{1}{p} - \frac{1}{2} \right|$ ,  $1 < p < \infty$ . This proves the claim (ii) of Theorem 1.2. The claims (i) and (iii) can be deduced in a similar way from Theorem 1.1(i) and (iii).

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