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Singular Perturbations for a Class of Degenerate Parabolic Equations with Mixed Dirichlet-Neumann Boundary Conditions

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Abstract

We establish a singular perturbation property for a class of quasilinear parabolic degenerate equations associated with a mixed Dirichlet-Neumann boundary condition in a bounded domain of $\mathbb{R}^p$, $1 \leq p < +\infty$. In order to prove the L^1-convergence of viscous solutions toward the entropy solution of the corresponding first-order hyperbolic problem, we refer to some properties of bounded sequences in L^∞ together with a weak formulation of boundary conditions for scalar conservation laws.

1 Introduction

This paper is devoted to the study of the singular limit, as ϵ goes to 0^+, for the class of second-order degenerate equations

$$\partial_t u_\epsilon + Div_x(\mathbf{B}(t,x)\varphi(u_\epsilon)) + \psi(t,x,u_\epsilon) = \epsilon\Delta\phi(u_\epsilon) \ in \ ]0,T[\times\Omega, \epsilon > 0, \quad (1.1)$$

where Ω is a bounded domain of $\mathbb{R}^p, 1 \leq p < +\infty$, with a Lipschitz boundary Γ and T a positive real. Moreover, $\mathbf{B}$ is a vector field on $]0,T[\times\Omega$, φ is a non-decreasing function and ϕ is a $\mathcal{C}^1$-class function with

$$\phi' \geq 0, \mathcal{L}_1(\{x \in \mathbb{R}, \phi'(x) = 0\}) = 0, \quad (1.2)$$

where $\mathcal{L}_p$ refers to the Lebesgue measure on $\mathbb{R}^p$, $p \geq 1$.

Equations (1.1) are associated with the mixed boundary conditions:

$$u_\epsilon = u_{\Gamma_e} \ on \ ]0,T[\times\Gamma_e, \nabla\phi(u_\epsilon).\nu = 0 \ on \ ]0,T[\times(\Gamma\backslash\Gamma_e), \quad (1.3)$$

where ν is the outward unit vector defined a.e. on Γ and Γ_e is a part of Γ with a positive $d\Gamma$-measure.

In petroleum engineering within the context of a secondary recovery process, the **Dead Oil** isothermal model leads us to consider a two-phase (oil-water) flow incompressible and immiscible through a porous medium (the reservoir rock Ω) and a partitioning of its boundary Γ in three open separated areas Γ_e (with a positive dΓ-measure), Γ_s (with a non-negative $d\Gamma$-measure) and Γ_L and corresponding respectively to the water injection wells, the oil production ones and the waterproof walls of the deposit with:

$$\Gamma = \Gamma_e \cup \Gamma_s \cup \Gamma_L \cup \partial\Gamma_L, \overline{\Gamma_e} \cap \overline{\Gamma_s} = \emptyset.$$

Then the oil-reduced saturation is the solution of an (1.1)-type equation reladocumentclassted to boundary conditions (1.3), where φ denotes the oil flood ratio, $(-\mathbf{B})$ is a pressure gradient, for example the *global pressure* P introduced by G.Chavent & P.Jaffré [6], which is the solution of a linear parabolic equation with the mixed Neumann-Robin boundary conditions

$$d\nabla P.\nu = f \ \ on\ \Gamma_e, \nabla P.\nu = 0 \ \ on\ \Gamma_L, d\nabla P.\nu = -\lambda P \ \ on\ \Gamma_s, \tag{1.4}$$

where d is a positive constant, f a given smooth stationary filtering velocity at the injection wells and λ the permeability coefficient at the production wells satisfying:

$$\begin{aligned} &f \geq 0 \ \ on\ \Gamma_e, d\Gamma - meas\{x \in \Gamma_e, f(x) > 0\} > 0,\\ &\lambda \geq 0 \ \ on\ \Gamma_s, d\Gamma - meas\{x \in \Gamma_s, \lambda(x) > 0\} > 0. \end{aligned}$$

A detailed presentation may be found in G.Gagneux & M.Madaune-Tort's book [10].

As (1.1) is written, the capillary pressure between the oil and water phases has been taken into account through the function ϕ. Usually, these capillary effects are negligible in favor of transport ones, meaning that ϵ is small; the case when ϵ is equal to zero signifying that the capillary effects are completely neglected. Hence, our aim is to compare both models whether the viscous parameter ϵ is positive (and (1.1) is a quasilinear degenerate parabolic equation) or nul (and (1.1) is an hyperbolic first-order quasilinear equation).

We can summarize the properties obtained in this work through the next theorem whose proof will be developed in the following sections:

Theorem 1.1:
(i) For any positive ϵ, the degenerate parabolic-hyperbolic equation (1.1) *associated with the mixed Dirichlet-Neumann boundary conditions* (1.3) *and the initial data u_0 has an unique solution u_ϵ.*
(ii) When ϵ goes to 0^+ the sequence $(u_\epsilon)_{\epsilon>0}$ gives an L^1-approximation of the weak entropy solution to the quasilinear first-order hyperbolic problem formally described by:

$$\partial_t u + Div_x(\boldsymbol{B}(t,x)\varphi(u)) + \psi(t,x,u) = 0 \text{ in }]0,T[\times\Omega, \tag{1.5}$$

$$u = u_{\Gamma_e} \text{ on an unknown part of }]0,T[\times\Gamma \text{ but including }]0,T[\times\Gamma_e, \tag{1.6}$$

$$u(0,.) = u_0 \text{ on } \Omega. \tag{1.7}$$

Numerous works have been achieved on the study of the behavior of the viscous sequence $(u_\epsilon)_{\epsilon>0}$ as ϵ goes to 0^+, especially when the diffusion term is linear [3, 18] or for an homogeneous Dirichlet boundary condition [12] and even for obstacle problems [15, 16, 17, 20]. However, few works have dealt with the particular framework considered here. In fact, this paper refers to the study in [13] when $\mathbf{B}$ is stationary and sufficiently smooth to obtain local *a priori* estimates for the sequence $(u_\epsilon)_{\epsilon>0}$ in the space of bounded functions with bounded variations on $]0,T[\times\Omega$, by using weight-functions nonincreasing along the characteristics of the linear operator $v \to \mathbf{B}.\nabla v$. But since these smoothness conditions are not generally satisfied by the pressure gradient ∇P, we try to release in this paper the assumptions on $\mathbf{B}$ bearing in mind that here $\mathbf{B}$ is time-and-space depending and considering that due to the concept of Young measures [7, 8, 18, 21] a uniform L^∞-estimate of approximate solutions is sufficient to characterize the ϵ-limit of (1.1, 1.3).

Thence our first objective is to prove the existence and uniqueness of the solution u_ϵ to (1.1, 1.3) associated with the initial datum u_0. With this view we introduce, for each value of the parameter δ in $]0,1]$, a regularized problem obtained by turning the nonlinearity ϕ into $\phi_\delta = \phi + \delta Id_{\mathbb{R}}$. We establish *a priori* estimates of $u_{\epsilon,\delta}$ that are independent from δ so as to provide an existence result for the degenerate problem. In fact, the mathematical context considered here only permits to obtain δ-uniform Hilbertian estimates for $\phi_\delta(u_{\epsilon,\delta})$ and in order to establish the uniqueness proof for $(1.1, 1.3)$ an assumption on the local behavior of $\varphi \circ \phi^{-1}$ is necessary. Namely, we assume that

$$\begin{aligned} &\varphi \circ \phi^{-1} \textit{ is Lipschitzian on } [\phi(-M(T)), \phi(M(T))], \\ &\qquad\qquad \textit{with a constant } M'_{\varphi\circ\phi^{-1}}. \end{aligned} \tag{1.8}$$

where $M(T)$ is defined by (1.9).

Our second objective is to pass to the limit when ϵ goes to 0^+. The maximum principle ensures that the sequence $(u_\epsilon)_{\epsilon>0}$ is bounded in $L^\infty(]0,T[\times\Omega)$. Accordingly, the behavior of bounded sequences in L^∞ and its consequences that we owe to R.Eymard, T.Gallouët & R.Herbin [8] permit to pass to the L^∞-weak star limit. This provides the desired singular perturbations property: as ϵ goes to 0^+ the sequence of viscous solutions $(u_\epsilon)_{\epsilon>0}$ gives an L^1-approximation of the solution u to the hyperbolic first-order problem (1.5,1.6,1.7).

*Remark:*The main feature of this paper is to deal with some mixed Dirichlet-Neumann boundary conditions on $]0,T[\times\partial\Omega$. The nonhomogeneous conditions are given by the physical model but they are not essential to comprehension.

Of course all the results exposed here still hold in the situation of Dirichlet data for the same kind of operator. Especially the singular perturbations property stated in theorem 3.1 applies with the same smoothness assumption on the boundary data and with the same mathematical tool. Concerning the existence and uniqueness of theorem 2.1, the hypothesis (1.8) falls given that that every weak solution in the sense of theorem 2.1 fulfills implicitly an entropy inequality which is chosen as the starting point for the uniqueness (see [10]). It is also possible to weaken (1.2) and consider a strong degenerate operator for which $\phi' \geq 0$, $\mathcal{L}_1(\{x \in \mathbb{R}, \phi'(x) = 0\}) \geq 0$, assuming thereby that ϕ^{-1} is not a function necessary. In this special context we need to refer to a weak entropy formulation for the second-order problem which is the state of the art at the moment (see [4],[5],[9],[19]...). At our knowledge there is no result concerning the existence and uniqueness of the weak entropy solution to a strongly degenerate parabolic-hyperbolic operator associated with mixed Dirichlet-Neumann boundary conditions.

1.1 Main notations and hypotheses on data

For any t of $]0,T]$, Q_t denotes the cylinder $]0,t[\times\Omega$ knowing that $Q_T = Q$. Similarly $\Sigma_t =]0,t[\times\Gamma$ and $\Sigma_T = \Sigma$. Thus, in the rest of this paper we assume the following hypotheses are fulfilled:

• $\mathbf{B}$ a is vector field of $(W^{1,+\infty}(Q))^p$, this regularity being actually given when one follows the Kruskov uniqueness method for first-order quasilinear equation [14]. Furthermore we suppose that $Div_x\mathbf{B} = 0$, which is not restrictive given that (1.1) involves a reaction term.

• Let us define:

$$\Gamma_- = \{\sigma \in \Sigma, \mathbf{B}(\sigma).\nu < 0\} \textit{ with } d\Gamma(\Gamma_-) > 0$$

and thus, as a result of (1.4) with $\nabla P = -\mathbf{B}$, Γ_e is an open part of Γ such that

$$\overline{\Gamma_-} \subseteq \Gamma_e \textit{ and } d\Gamma(\Gamma_e) > 0.$$

It is important to notice that Γ_-, which is the part of Γ corresponding to the inward characteristics for the linear hyperbolic operator $v \to \mathbf{B}.\nabla v$, may be considered independent from the time variable t. Indeed, the physical model considered shows that, due to (1.4),

$$\mathbf{B}.\nu = -f/d \textit{ on } \Gamma_e.$$

• u_0 is a measurable and bounded function Ω.

• u_{Γ_e} is smooth enough to reduce (1.3) to an homogeneous Dirichlet boundary condition on $]0,T[\times\Gamma_e$ by means of a translation procedure. Namely, it is sufficient to assume that u_{Γ_e} is the trace on $]0,T[\times\Gamma$ of a function $\overline{u_{\Gamma_e}}$ of $H^1(Q)\cap L^\infty(Q)$ such that $\partial_t\overline{u_{\Gamma_e}}$ and $\partial_t\phi(\overline{u_{\Gamma_e}})$ belong to $H^1(Q)$.

Let us observe that these assumptions ensure the existence of a time-depending function $M : t \to M(t)$ defined for any t of $[0, T]$ by

$$M(t) = max(\|u_{\Gamma_e}\|_{L^\infty(\Sigma)}, \|u_0\|_{L^\infty(\Omega)})e^{M'_\psi t} + \frac{e^{M'_\psi t} - 1}{M'_\psi}\|\psi(t, x, 0)\|_{L^\infty(Q)} \quad (1.9)$$

• φ is a non-decreasing Lipschitzian function on $[-M(T), M(T)]$ with a constant M'_φ.

• ϕ is an increasing $W^{1,+\infty}(-M(T), M(T))$-class function such that (1.2) holds. Let us note that these hypotheses only entail the continuity of ϕ^{-1}.

We remind that $\varphi \circ \phi^{-1}$ is Lipschitzian on $[\phi(-M(T)), \phi(M(T))]$ with a constant $M'_{\varphi\circ\phi^{-1}}$.

• ψ belongs to $W^{1,+\infty}(Q \times \mathbb{R})$, $M'_\psi = ess \sup_{(t,x,u)\in Q\times\mathbb{R}} |\partial_u \psi(t, x, u)|$

Eventually, we introduce "sgn_λ", the Lipschitzian and bounded approximation of the function "sgn", given for any positive parameter λ and for all positive real x by:

$$sgn_\lambda(x) = \min\left(\frac{x}{\lambda}, 1\right) \text{ and } sgn_\lambda(-x) = -sgn_\lambda(x).$$

2 The vanishing viscosity method

In order to take the ϵ-limit in (1.1), we seek *a priori* estimates for the sequence $(u_\epsilon)_{\epsilon>0}$. This is the purpose of the next theorem which is the first main result of this paper:

Theorem 2.1: *For any positive ϵ, the parabolic degenerate equation* (1.1) *has a unique solution u_ϵ associated with initial datum u_0 and boundary condition u_{Γ_e} on Γ_e. Precisely, u_ϵ belongs to $L^\infty(Q)$, $\phi(u_\epsilon)$ and $\partial_t u_\epsilon$ are respectively elements of $L^2(0, T; H^1(\Omega))$ and $L^2(0, T; V')$, and*

$$\begin{gathered} \text{for a.e. } t \text{ of }]0, T[\ u_\epsilon = u_{\Gamma_e} \text{ a.e. on } \Gamma_e, \\ ess \lim_{t\to 0^+} \int_\Omega |u_\epsilon(t, x) - u_0(x)| dx = 0. \end{gathered} \quad (2.1)$$

In addition, u_ϵ satisfies the variational equality for a.e. t of $]0,T[$ and for any v of V:

$$
\begin{aligned}
< \partial_t u_\epsilon, v > \quad &- \int_\Omega \varphi(u_\epsilon)\boldsymbol{B}(t,x).\nabla v dx + \int_\Gamma \varphi(u_\epsilon) v \boldsymbol{B}(t,\gamma).\nu d\gamma \\
&+ \quad \epsilon \int_\Omega \nabla\phi(u_\epsilon).\nabla v dx + \int_\Omega \psi(t,x,u_\epsilon) v dx = 0. \qquad (2.2)
\end{aligned}
$$

where $< .,. >$ stands for the duality brackets V'-V. What is more, for any positive ϵ,

$$\|u_\epsilon\|_{L^\infty(Q)} \leq M(T), \tag{2.3}$$

where $M(T)$ is defined through (1.9).

2.1 Proof of theorem 2.1: uniqueness property

The uniqueness proof for the solution of parabolic degenerate equations have been achieved by many authors, principally for the Cauchy-Dirichlet problem. Various techniques have been used to avoid the difficulties owing to the lack of regularity of first-order partial derivatives for a weak solution and to the nonlinear context considered. At the moment, the state-of-art consists in considering a *strongly* degenerate problems where ϕ is only non-decreasing. Following the original ideas of J.Carrillo [5] many works dealt with entropy formulation for parabolic degenerate problems (see [4, 9, 11, 19] and the corresponding references). Then the uniqueness proof refers to the Kruskov method [14] classically used to study the uniqueness of the weak entropy solution to quasilinear conservation laws by splitting the time and the space variables in two.

In this work, as we take into account a mixed Dirichlet-Neumann boundary condition, we have at our disposal the works of M.J.Jasor [13] where $\mathbf{B}$ is stationary and those of G.Gagneux & M.Madaune-Tort [10] for variational inequality. In each case, in order to deal with the convective term, the function $\varphi \circ \phi^{-1}$ is supposed Hölder-continuous with an exponent of at least 1/2. However, given that here $\mathbf{B}$ is time depending, we have been forced to assume that $\varphi \circ \phi^{-1}$ is Lipschitzian on a bounded interval of $\mathbb{R}$. In this weakly degenerate framework, the next statement holds:

Proposition 2.2: *Suppose that* (1.8) *holds and let* u_ϵ *and* v_ϵ *be two weak solutions to degenerate equation* (1.1) *corresponding respectively to the couple of boundary data* (u_0, u_{Γ_e}) *and* (v_0, u_{Γ_e}). *Then, for a.e.* t *of* $]0,T[$:

$$\int_\Omega (u_\epsilon(t,x) - v_\epsilon(t,x))^+ dx \le \int_\Omega (u_0 - \ v_0)^+ dx\, e^{M'_\psi t}.$$

Proof: The demonstration draws its inspiration from that presented in [10] by splitting only the time variable in two. Let us briefly describe the mathematical tools: let ζ be an element of $\mathcal{D}_+(0,T)$ and for any positive μ, let ρ_μ be the standard mollifier with support in $[-\mu, +\mu]$. For any t and $\tilde{t}$ of $]0,T[$, we consider

$$\zeta_\mu(t,\tilde{t}) = \zeta\left(\frac{t+\tilde{t}}{2}\right)\rho_\mu\left(\frac{t-\tilde{t}}{2}\right),$$

with μ sufficiently small such that ζ_μ belongs to $\mathcal{D}_+(]0,T[\times]0,T[)$.

In order to simplify the writing, we drop the index ϵ temporarily and we add a tilde superscript to any function with the variable $\tilde{t}$. Thus, in variational formulation (2.2) for u written in the variables (t,x) we may choose the test-function $sgn^+_\lambda(\phi(u) - \phi(\tilde{v}))\zeta_\mu$ and in the one satisfied by v and written in the variables $(\tilde{t},x)$, the test-function $-sgn^+_\lambda(\phi(u) - \phi(\tilde{v}))\zeta_\mu$. Then we integrate on the time variables over $]0,T[\times]0,T[$. By adding up and taking into account the positiveness of the diffusive terms, it comes:

$$\begin{aligned}
&\int_0^T\int_0^T < \partial_t u - \partial_{\tilde{t}}\tilde{v}, sgn^+_\lambda(\phi(u) - \phi(\tilde{v})) > \zeta_\mu dt d\tilde{t} \\
&- \int_0^T\int_Q \{\varphi(u)\mathbf{B} - \varphi(\tilde{v})\tilde{\mathbf{B}}\}.\nabla sgn^+_\lambda(\phi(u) - \phi(\tilde{v}))\zeta_\mu dx dt d\tilde{t} \\
&+ \int_0^T\int_\Sigma \{\varphi(u)\mathbf{B} - \varphi(\tilde{v})\tilde{\mathbf{B}}\}.\nu sgn^+_\lambda(\phi(u) - \phi(\tilde{v}))\zeta_\mu d\sigma dt d\tilde{t} \\
&\le - \int_0^T\int_Q \{\psi(t,x,u) - \psi(\tilde{t},x,\tilde{v})\} sgn^+_\lambda(\phi(u) - \phi(\tilde{v}))\zeta_\mu dx dt d\tilde{t}. \qquad (2.4)
\end{aligned}$$

In order to pass to the limit with respect to λ and then to μ in (2.4), let us develop the following transformations in the right-hand side:

• For the duality brackets V'-V, thanks to an adaptation of the Mignot-Bamberger Lemma [2], one gets an integration-by-parts formula which provides the next integral:

$$-\int_0^T \int_Q \left(\int_{\tilde{v}}^{u} sgn_\lambda^+(\phi(\tau) - \phi(\tilde{v}))d\tau \partial_t \zeta_\mu \right.$$
$$\left. + \int_{\tilde{v}}^{u} sgn_\lambda^+(\phi(u) - \phi(\tau))d\tau \partial_{\tilde{t}} \zeta_\mu \right) dxdtd\tilde{t}.$$

When the λ-limit is taken, it comes:

$$-\int_0^T \int_Q (u - \tilde{v})^+ (\partial_t \zeta_\mu + \partial_{\tilde{t}} \zeta_\mu) dxdtd\tilde{t}.$$

• For the second line:

$$-\int_0^T \int_Q \{\varphi(u) - \varphi(\tilde{v})\} \mathbf{B}.\nabla sgn_\lambda^+(\phi(u) - \phi(\tilde{v}))\zeta_\mu dxdtd\tilde{t}$$
$$-\int_0^T \int_Q \varphi(\tilde{v})\{\mathbf{B} - \tilde{\mathbf{B}}\}.\nabla sgn_\lambda^+(\phi(u) - \phi(\tilde{v}))\zeta_\mu dxdtd\tilde{t}.$$

Thus, thanks to hypothesis (1.8) and to the Sacks Lemma, the first term goes to 0 with λ. For the second one a Green formula is used since (1.8) ensures that $\varphi(\tilde{v})$ belongs to $L^2(0, T; H^1(\Omega))$.

Finally, when λ goes to 0^+, by expressing the time-partial derivative of ζ_μ and taking the monotonicity of ϕ into account, it comes:

$$\int_{]0,T[\times Q} \left\{ -(u - \tilde{v})^+ \zeta' \rho_\mu + \nabla\varphi(\tilde{v}).\{\mathbf{B} - \tilde{\mathbf{B}}\} sgn^+(u - \tilde{v})\zeta\rho_\mu \right\} dxdtd\tilde{t}$$

$$+ \int\limits_{]0,T[\times\Sigma} \{\varphi(u) - \varphi(\tilde{v})\}\mathbf{B}.\nu sgn^+(u-\tilde{v})\zeta\rho_\mu d\sigma d\tilde{t}$$
$$\leq M'_\psi \int\limits_{]0,T[\times Q} (u-\tilde{v})^+\zeta\rho_\mu dxdtd\tilde{t}.$$

Let us note that due to the partitioning of Γ and to the monotonicity of φ the boundary integral is non-negative. Then, by referring to the notion of a Lebesgue point for an integrable function, we may take the μ-limit in the previous inequality. As the second term in the left-hand side goes to 0, it follows:

$$-\int\limits_Q (u-v)^+\zeta' dxdt \leq M'_\psi \int\limits_Q (u-v)^+\zeta dxdt,$$

for any ζ of $\mathcal{D}_+(0,T)$ and by density, for any ζ of $W^{1,1}_+(0,T)$ with $\zeta(0) = \zeta(T) = 0$.

Now the conclusion is classical: it uses a piecewise linear approximation of $\mathbb{I}_{]0,t[}$, with t given outside a set of measure zero. Thanks to the definition of initial condition (2.1) for u and v, and to the Gronwall Lemma we complete the proof of theorem.

□

Remark: The Lipschitz condition for $\varphi\circ\phi^{-1}$ is only needed to transform the second line of inequality (2.4). When $\mathbf{B}$ is stationary an Hölder condition is sufficient to transform the convective term in (2.4) (see [10],[13],..)

2.2 Proof of theorem 2.1: existence property

The equation (1.1) being nonlinear and degenerated in the sense (1.2), we first introduce some auxiliary non-degenerate problems by turning ϕ into $\phi+\delta Id_{\mathbb{R}}$, for each value of the parameter δ in $]0,1]$. Thus we look for estimates of $u_{\epsilon,\delta}$ that are independent from δ (the latter will fix the regularity of u_ϵ) and from ϵ as soon as possible (the latter will specify the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$, as ϵ goes to 0^+). First and foremost, if we refer to G.Gagneux & M.Madaune-Tort's book [10], the next statement holds:

Lemma 2.3: *For any positive ϵ and δ the non-degenerate parabolic problem: find $u_{\epsilon,\delta}$ in $W(0,T;H^1(\Omega);L^2(\Omega))\cap L^\infty(0,T;H^1(\Omega))$ satisfying for any v in $H^1(\Omega)$, $v_{|\Gamma_e}=0$, the variational equality for a.e. t in $]0,T[$,*

$$\begin{aligned}
&\int_\Omega \left(\partial_t u_{\epsilon,\delta} + Div_x\{\boldsymbol{B}(t,x)\varphi(u_{\epsilon,\delta})\} + \psi(t,x,u_{\epsilon,\delta})\right) v dx \\
= \quad & -\epsilon \int_\Omega \nabla\phi_\delta(u_{\epsilon,\delta}).\nabla v dx,
\end{aligned}\tag{2.5}$$

and the boundary conditions,

$$u_{\epsilon,\delta} = u^\delta_{\Gamma_e} \ d\Sigma\text{-a.e. on } \Sigma_e, \tag{2.6}$$

$$u_{\epsilon,\delta}(0,.) = u^\delta_0 \text{ a.e. on } \Omega, \tag{2.7}$$

has a unique solution.

What is more, if $u_{\epsilon,\delta}$ and $v_{\epsilon,\delta}$ are two solutions associated with the boundary data $(u^\delta_0, u^\delta_{\Gamma_e})$ and $(v^\delta_0, v^\delta_{\Gamma_e})$, then for any t of $[0,T]$:

$$\begin{aligned}
&\int_\Omega (u_{\epsilon,\delta}(t,x) - v_{\epsilon,\delta}(t,x))^+ dx \\
\leq \quad & (\|\boldsymbol{B}\|_\infty M'_\varphi \int_{(\Sigma_e)_t} (u^\delta_{\Gamma_e} - v^\delta_{\Gamma_e})^+ d\sigma + \int_\Omega (u^\delta_0(x) - v^\delta_0(x))^+ dx) e^{M'_\psi t}.
\end{aligned}\tag{2.8}$$

In addition, $\phi_\delta(u_{\epsilon,\delta})$ belongs to $L^2(0,T;H^{3/2-\eta}(\Omega))$ for any positive η.

Lastly for any positive ϵ and δ and for any t in $[0,T]$:

$$|u_{\epsilon,\delta}(t,x)| \leq M_\delta(t) \text{ for a.e. } x \text{ in } \Omega,$$

where $M_\delta(t)$ is given by

$$M_\delta(t) = max(\|u^\delta_{\Gamma_e}\|_{L^\infty(\Sigma)}, \|u^\delta_0\|_{L^\infty(\Omega)}) e^{M'_\psi t} + \frac{e^{M'_\psi t} - 1}{M'_\psi} \|\psi(t,x,0)\|_{L^\infty(Q)}$$

In formulations (2.6, 2.7) of boundary conditions, the smooth data $u^\delta_{\Gamma_e}$ and u^δ_0 are defined by first introducing a.e. on Q the function u_{0,Γ_e} in the same spirit as in [18]:

$$\begin{cases} u_{0,\Gamma_e}(t, r + s\nu) = u_{\Gamma_e}(t, r) \ \textit{for}\ t \in]0, T[, r \in \Gamma, 0 \le s < min(t, \delta), \\ u_{0,\Gamma_e}(t, x) = \overline{u_{\Gamma_e}}(t, x) \ \textit{for}\ min(dist(x, \Gamma), \delta) < t < T, x \in \Omega, \\ u_{0,\Gamma_e}(t, x) = u_0(x) \ \textit{for}\ -\delta < t < min(dist(x, \Gamma), \delta), x \in \Omega, \\ u_{0,\Gamma_e}(t, x) = 0 \ \textit{elsewhere}\ . \end{cases}$$

Then let us define for p in Q:

$$u^\delta_{0,\Gamma_e}(p) = \int_{\mathbb{R}^{p+1}} u_{0,\Gamma_e}(\tilde{p})\rho_\delta(p - \tilde{p})d\tilde{p},$$

where ρ_δ is the usual mollifier. Let us now denote by $u^\delta_{\Gamma_e}$ and u^δ_0 the restriction of u^δ_{0,Γ_e} to Σ and $\{0\} \times \Omega$ respectively. Observe that

$$(u^\delta_{\Gamma_e})_{\delta>0} \ \textit{and}\ (u^\delta_0)_{\delta>0} \ \textit{are uniformly bounded in the respective}\ L^\infty\textit{-norm,}$$
$$\lim_{\delta\to 0^+} u^\delta_0 = u_0 \ \textit{in}\ L^1(\Omega).$$

Remark:
i) Owing to the smoothness of $u_{\epsilon,\delta}$, one has a.e. on Q:

$$\partial_t u_{\epsilon,\delta} + Div_x\{\mathbf{B}(t, x)\varphi(u_{\epsilon,\delta})\} + \psi(t, x, u_{\epsilon,\delta}) = \epsilon\Delta\phi_\delta(u_{\epsilon,\delta}). \tag{2.9}$$

In this way, $\Delta\phi_\delta(u_{\epsilon,\delta})$ belongs to $L^2(Q)$ and $\nabla\phi_\delta(u_{\epsilon,\delta})$ is an element of $L^2(0, T; H(div, \Omega))$. So, $\nabla\phi_\delta(u_{\epsilon,\delta}).\nu = 0$ in $L^2(]0, T[\times\Gamma\backslash\Gamma_e)$ and a.e. on $]0, T[\times\Gamma\backslash\Gamma_e$.

ii) It is essential to mention that by considering in the definition of $M_\delta(t)$ the respective L^∞-norm for the sequences $(u^\delta_{\Gamma_e})_{\delta>0}$ and $(u^\delta_0)_{\delta>0}$, the ϵ and δ uniform estimate holds for every t of $[0, T]$:

$$|u_{\epsilon,\delta}(t, x)| \le M(t) \ \textit{for a.e.}\ x \ \textit{in}\ \Omega, \tag{2.10}$$

where $M(t)$ is given by (1.9).

On the one hand, we assume in this work that the vector field $\mathbf{B}$ is unstationary. Consequently it is well-known that in this situation, an L^1-estimate of $\partial_t u_{\epsilon,\delta}$ (in terms of differential ratio or continuity modulus) is linked to an L^1-estimate of $\nabla u_{\epsilon,\delta}$, and conversely (see for example S.N.Kruskov [14] for the Cauchy problem or [3, 12] for the Dirichlet one). On the other hand, we release the smoothness assumptions on $\mathbf{B}$ and u_0 as much as possible; thus, even if $\mathbf{B}$ is independent from t, we may not refer to F.Mignot & J.P.Puel's weight-functions [20] associated with the operator $v \to \mathbf{B}.\nabla v$ in order to obtain, as in [13], an L^1-local estimate uniformly with respect to ϵ and δ for $\nabla u_{\epsilon,\delta}$. This explains why we solely focus on Hilbertian estimates for $\phi_\delta(u_{\epsilon,\delta})$ that are in fact based on an energy inequality. With this view, we introduce an arbitrary sequence $(\mu_n)_{n\in\mathbb{N}}$ in $]0,T]$ which converges to zero as n goes to $+\infty$. Thus, for a fixed n,

Lemma 2.4: *Then there exists a constant c_1, independent from n and δ in $]0,\mu_n/2[$, such that:*

$$\epsilon\|\nabla\Phi_\delta(u_{\epsilon,\delta})\|^2_{L^2(\mu_n,T;L^2(\Omega)^p)} \leq c_1, \tag{2.11}$$

$$\|\partial_t u_{\epsilon,\delta}\|_{L^2(\mu_n,T;V')} \leq c_1, \tag{2.12}$$

$$\epsilon\|\sqrt{t-\mu_n}\,\partial_t\Phi_\delta(u_{\epsilon,\delta})\|^2_{L^2(\mu_n,T;L^2(\Omega))} \leq c_1, \tag{2.13}$$

where $\Phi_\delta(r) = \int_0^r (\phi'_\delta(\tau))^{1/2}d\tau$ *and V denotes the Hilbert space*

$$V = \{v \in H^1(\Omega), v_{|\Gamma_e} = 0\},$$

used with the H^1-equivalent norm: $\|v\|_V = \left(\int_\Omega [\nabla v]^2 dx\right)^{1/2}.$

Some commentaries - These uniform *a priori* estimates are established by taking advantage of the regularity of $\overline{u_{\Gamma_e}}$.

i) As for (2.11), we choose the test-function $v = u_{\epsilon,\delta} - u^\delta_{0,\Gamma_e}$ in (2.5) and we integrate over $]\mu_n, T[$. We just mention that the convective term is written by developing the partial derivatives. Then if we denote $F(r) = \int_0^r \varphi'(\tau)\tau d\tau$, the Green formula, the partitioning of Γ and the monotonicity of φ ensure

$$\int_{]\mu_n,T[\times\Omega} \mathbf{B}.\nabla F(u_{\epsilon,\delta})dxdt \leq \int_{]\mu_n,T[\times\Gamma_e} \mathbf{B}.\nu\nabla F(u^{\delta}_{\Gamma_e})dxdt$$

ii) As for (2.13), we consider the $L^2(]\mu_n,T[\times\Omega)$-scalar product between (1.1) and $\epsilon(t-\mu_n)\partial_t\{\phi_\delta(u_{\epsilon,\delta})-\phi_\delta(u^{\delta}_{0,\Gamma_e})\}$ that is an element of $W(\mu_n,T,V,L^2(\Omega))$. To transform the diffusion term, we establish a Green formula thanks to the density of $\mathcal{D}([\mu_n,T];X)$ into $W(\mu_n,T;X;L^2(\Omega))$, where X is the Hilbert space

$$X=\{v\in V,\Delta v\in L^2(\Omega),\int_\Omega \Delta vwdx=-\int_\Omega \nabla v.\nabla wdx,\forall w\in V\},$$

used with the scalar product $((u,v))_X=\int_\Omega \nabla u.\nabla vdx+\int_\Omega \Delta u\Delta vdx$ and the associated norm.

The other terms are bounded thanks to a Cauchy inequality and estimates (2.10, 2.11).

iii) As for (2.12) we refer to the definition of the $L^2(\mu_n,T;V')$-norm and we use (2.10, 2.11).

*Remark:*When $\mathbf{B}$ is stationary then an L^1-uniform estimate of time-differentials ratio holds. Indeed, this particular context ensures - tanks to (2.8) - the existence of a constant $c(\delta)$ such that for any h of $]0,T[$ and any t of $[0,T-h]$,

$$\|u_{\epsilon,\delta}(t+h,.)-u_{\epsilon,\delta}(h,.)\|_{L^1(\Omega)}\leq c(\delta)h,$$

the dependence with respect to δ been released as soon as Δu_0 and $\Delta\phi(u_0)$ are bounded Radon measures on Ω.

The estimates of lemma 2.4 ensure that the family $(\sqrt{t-\mu_n}\phi_\delta(w_{\epsilon,\delta}))_{\delta\in]0,\frac{\mu_n}{2}[}$ remains at least in a bounded set of $W^{1,1}(]\mu_n,T[\times\Omega)$. The compactness embedding of the latter space into $L^1(Q)$ and the continuity of ϕ^{-1} provide the existence of a measurable function $u_{n,\epsilon}$ and a subsequence - still denoted $(u_{\epsilon,\delta})_{\delta\in]0,\frac{\mu_n}{2}[}$ - such that when δ goes to 0^+, $(u_{\epsilon,\delta})_{\delta\in]0,\frac{\mu_n}{2}[}$ goes to $u_{n,\epsilon}$ for every (t,x) in $(]\mu_n,T[\times\Omega)\backslash\mathcal{N}_n$ with $\mathcal{L}^{p+1}(\mathcal{N}_n)=0$

Let us denote $\mathcal{N} = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$ and let u_ϵ be the $L^\infty(Q)$ weak-$*$ limit of $(u_{\epsilon,\delta})_{\delta>0}$ up to a subsequence. Then, as δ goes to zero, $(u_{\epsilon,\delta})_{\delta>0}$ goes to u_ϵ for every (t,x) in $Q\backslash\mathcal{N}$, and so in $L^q(Q)$, $1 \leq q < +\infty$. What is more due to (2.11) and (2.12) and to the fact that c_1 does not depends on n, $(\nabla\phi_\delta(u_{\epsilon,\delta}))_{\delta>0}$ and $(\partial_t u_{\epsilon,\delta})_{\delta>0}$ weakly converge up to a subsequence respectively toward $\nabla\phi(u_\epsilon)$ in $L^2(0,T,L^2(\Omega)^p)$ and $\partial_t u_\epsilon$ in $L^2(0,T,V')$. Hence:

Proposition 2.5: *The degenerate problem* (1.1, 1.3) *has at least a weak solution u_ϵ associated with the initial data u_0 in the sense* (2.1). *This solution belongs to $L^\infty(Q)$, with $\phi(u_\epsilon)$ in $L^2(0,T;H^1(\Omega))$ and $\partial_t u_\epsilon$ in $L^2(0,T;V')$, and is characterized by the variational equality* (2.2).

Proof: There is no difficulty in establishing variational equality (2.2). We use the continuity of ϕ^{-1} and for the boundary term the continuity of the trace operator from $H^s(\Omega)$ into $L^2(\Gamma)$ for $1/2 < s < 1$. So let us focus on weak formulation (2.1) for the initial condition. The demonstration is inspired by that presented by F.Otto in [18] for the case of weak entropy solutions to quasilinear scalar conservation laws. Here, we take in (2.5) the test-function $v = sgn_\lambda(u_{\epsilon,\delta} - k)\alpha\zeta$ where α and ζ are respectively elements of $\mathcal{D}_+(]-\infty,T[)$ and $\mathcal{D}_+(\Omega)$ and k belongs to $\mathbb{R}$. By integrating with respect to t from 0 to T, it comes:

$$
\begin{aligned}
- & \int_Q \left(\int_k^{u_{\epsilon,\delta}} sgn_\lambda(\tau-k)d\tau\right)\zeta\partial_t\alpha dxdt - \int_Q F_\lambda(u_{\epsilon,\delta},k)\mathbf{B}.\nabla\zeta\alpha dxdt \\
+ & \int_Q \psi(t,x,u_{\epsilon,\delta})sgn_\lambda(u_{\epsilon,\delta}-k)\alpha\zeta dxdt \\
- & \epsilon\int_Q \left(\int_k^{u_{\epsilon,\delta}} \phi'_\delta(\tau)sgn_\lambda(\tau-k)d\tau\right)\Delta\zeta\alpha dxdt \\
\leq & \int_\Omega \left(\int_k^{u_0^\epsilon} sgn_\lambda(\tau-k)d\tau\right)\zeta\alpha(0)dx
\end{aligned}
$$

where,

$$F_\lambda(v,w) = \int\limits_w^v sgn_\lambda(\tau - w)\varphi'(\tau)d\tau.$$

Thus, when δ and then λ tend to 0^+, an integration by parts with respect to t provides :

$$-\int\limits_0^T \left(\int\limits_\Omega |u_\epsilon - k|\zeta dx + \theta_\epsilon(t) \right) \alpha'(t)dt \leq \int\limits_\Omega |u_0 - k|\zeta\alpha(0)dx, \qquad (2.14)$$

with:

$$\begin{aligned} \theta_\epsilon(t) &= \int\limits_\Omega (\int\limits_0^t [-|\varphi(u_\epsilon(\tau,x)) - \varphi(k)|\mathbf{B}(\tau,x).\nabla\zeta \\ &+ \psi(\tau,x,u_\epsilon(\tau,x))sgn(u_\epsilon(\tau,x)-k) - \epsilon|\phi(u_\epsilon(\tau,x)) - \phi(k)|\Delta\zeta]d\tau)dx. \end{aligned}$$

In this way, the time-depending function $t \to \int\limits_\Omega |u_\epsilon - k|\zeta dx + \theta_\epsilon(t)$ is identified a.e. with a non-increasing and bounded function, so it has a limit when t goes to 0^+, $t \in]0,T[\setminus\mathcal{O}, \mathcal{L}_1(\mathcal{O}) = 0$. As θ_ϵ goes to 0 with t, it comes

$$ess \lim_{t \to 0^+} \int\limits_\Omega |u_\epsilon - k|\zeta dx \leq \int\limits_\Omega |u_0 - k|\zeta dx,$$

for any function ζ of $\mathcal{D}_+(\Omega)$ and any real k. As a consequence, owing to F.Otto's reasoning in [18] we may announce:

$$ess \lim_{t \to 0^+} \int\limits_\Omega |u_\epsilon(t,x) - K(x)|dx \leq \int\limits_\Omega |u_0 - K(x)|dx$$

for any K of $L^\infty(\Omega)$. Condition (2.1) follows, which completes the proof.

$\square$

Remark:

i) Since $\phi(u_\epsilon)$ belongs to $L^2(0,T;H^1(\Omega))$ and $\partial_t\phi(u_\epsilon)$ to $L^2(\alpha,T;L^2(\Omega))$ for any positive α, then $\phi(u_\epsilon)$ is a function of $\mathcal{C}([\alpha,T];L^2(\Omega)) \cap \mathcal{C}_s([\alpha,T];H^1(\Omega))$. Thus we may define a trace for $u_\epsilon(t,.)$ on Γ, for every $t \in]0,T]$, through:

$$trace(u_\epsilon)(t,.) = \phi^{-1}(trace(\phi(u_\epsilon))(t,.)).$$

In this sense, $u_\epsilon = u_{\Gamma_e}$ on Σ_e.
ii) When $\mathbf{B}$ is independent from the time variable then the estimate of time-differential ratio pointed at the previous remark ensures, through the Ascoli Lemma, that the sequence $(u_{\epsilon,\delta})_{\delta>0}$ converges toward u_ϵ in $\mathcal{C}([0,T];L^1(\Omega))$. In this situation, the initial condition for u_ϵ can be strongly formulated under the form: $u_\epsilon(0,.) = u_0$ a.e. in Ω.
iii) The previous existence result for u_ϵ doest not require any smoothness assumption for $\varphi \circ \phi^{-1}$.

To complete this paragraph, let us remark that the uniqueness property for u_ϵ ensures that the whole sequence $(u_{\epsilon,\delta})_{\delta>0}$ gives an approximation of u_ϵ. Therefore, given the ϵ-uniform estimate (2.10) satisfied by $u_{\epsilon,\delta}$ one has (2.3) for u_ϵ.

Remark: If the vector field $\mathbf{B}$ is independent from t and if data are smooth enough (namely as soon as $\Delta\phi(u_0)$ and Δu_0 are bounded Radon measures on Ω) then the L^1-estimate of time-differential ratio pointed previously holds for u_ϵ uniformly with respect to δ, which gives an *a priori* estimate of the sequence $(u_\epsilon)_{\epsilon>0}$ in $BV(0,T;L^1(\Omega))$.

3 A singular perturbations property

We now focus on the behavior of the sequence $(u_\epsilon)_{\epsilon>0}$ as ϵ goes to 0^+ that is precisely to prove that the sequence of solutions to parabolic degenerate equations $(1.1)_{\epsilon>0}$ associated with the couple of boundary data (u_0, u_{Γ_e}) provides an L^1-approximation of the solution to the corresponding first-order problem $(1.5, 1.6, 1.7)$. This property of singular perturbations extends that of M.J.Jasor [13] obtained in the special situation when $\mathbf{B}$ is a smooth stationary vector field so as to reason within the framework of bounded functions with locally bounded variations on Q.

We first recall that a mathematical formulation for $(1.5, 1.6, 1.7)$ is provided by bearing in mind that for a general first-order quasilinear equation, it is classical to refer to an entropy criterium that warrants uniqueness. With this view we refer to the works of F.Otto in [18], chapter 2, which introduce a new formulation of boundary conditions for quasilinear hyperbolic equations, which generalize to $L^\infty(Q)$-solutions those of C.Bardos, A.Y.LeRoux & J.C.Nedelec [3], only aviable for solutions with bounded variation on Q.

Thus, we denote

$$F(u,v) = |\varphi(u) - \varphi(v)|,$$
$$\mathcal{L}(u,v,w) = |u-v|\partial_t w + F(u,v)\mathbf{B}.\nabla w - sgn(u-v)\psi(t,x,u)w.$$

We thus say:

Definition: A measurable function u is the entropy solution to $(1.5, 1.6, 1.7)$ if it satisfies:
$i)$ the inner entropy condition, for all ξ in $\mathcal{D}_+(]0,T[\times\Omega)$ and for any real k,

$$\int_Q \mathcal{L}(u,k,\xi)dxdt \geq 0, \tag{3.1}$$

$ii)$ the initial condition,

$$ess \lim_{t\to 0^+} \int_\Omega |u(t,x) - u_0(x)|dx = 0, \tag{3.2}$$

$iii)$ the boundary condition in the weak sense

$$ess \lim_{\tau\to 0^-} \int_\Sigma \mathcal{F}(u(\sigma+\tau\nu), u_{\Gamma_e}, k)\mathbf{B}(\sigma).\nu\zeta d\sigma \geq 0, \tag{3.3}$$

for any real k and any function ζ of $L^1_+(\Sigma)$ where, for any real a, b, c,

$$2\mathcal{F}(a,b,c) = F(a,b) - F(c,b) + F(a,c)$$

Remark: If it can be proved that for a.e. t of $]0,T[$, $u(t,.)$ has bounded variations on Ω, then (3.3) is equivalent to the classical formulation of boundary conditions owing to C.Bardos, A.Y.LeRoux & J.C.Nedelec [3], which is reduced in our particular context for a.e. t of $]0,T[$ to:

$$\forall k \in \mathcal{I}(u_{\Gamma_e}, \gamma_u), |\varphi(\gamma_u) - \varphi(k)|\mathbf{B}.\nu \geq 0 \ d\Gamma\text{-}a.e.,$$

where γ_u is the trace of u on Γ in the sense of bounded functions with bounded variations on Ω. Thus, by considering the definition of Γ_-, we may note that $\varphi(\gamma_u) = \varphi(u_{\Gamma_e})$ a.e. on Γ_- and since φ is non-decreasing, the Dirichlet condition is fulfilled as soon as φ is not constant.

Here the next statement holds:

Theorem 3.1: *When ϵ goes to 0^+, the family $(u_\epsilon)_{\epsilon>0}$ of solutions to degenerate equations $(1.1)_{\epsilon>0}$ associated with the couple of boundary conditions (u_0, u_{Γ_e}) strongly converges in $L^q(Q)$, $1 \leq q < +\infty$ and a.e. on Q toward the weak entropy solution u of first-order problem $(1.5, 1.6, 1.7)$.*

Before we must remind some properties of bounded sequences in L^∞.

3.1 Bounded sequences in L^∞

Let $\mathcal{O}$ be an open bounded subset of $\mathbb{R}^q$ $(q \geq 1)$ and let $(u_n)_{n>0}$ be a bounded sequence in $L^\infty(\mathcal{O})$. Clearly, for any continuous function h there exists $\bar{h}$ in $L^\infty(Q)$ such that for a subsequence,

$$h(u_n) \rightharpoonup \bar{h} \text{ weakly in } L^\infty(\mathcal{O}).$$

Since the works of L.Tartar [21] and J.M.Ball [1] one has been able to describe the composite limit $\bar{h}$. Actually, thanks to the properties of the weak-$*$ topology on the space of Radon measures, and to the properties of the generalized inverse of the distribution function linked to a probability measure, the next compacity result holds (see [8]):

Proposition 3.2: *Let $(u_n)_{n>0}$ be a sequence of measurable functions on $\mathcal{O}$ such that:*

$$\exists M > 0, \forall n > 0, \|u_n\|_{L^\infty(\mathcal{O})} \leq M.$$

Then, there exists a subsequence $(u_{f(n)})_{n>0}$ and a measurable function π in $L^\infty(]0,1[\times\mathcal{O})$ such that for all continuous and bounded functions h on $\mathcal{O}\times]-M, M[$,

$$\forall \xi \in L^1(\mathcal{O}), \lim_{n\to+\infty} \int_{\mathcal{O}} h(w, u_{f(n)})\xi dw = \int_{]0,1[\times\mathcal{O}} h(w, \pi(\alpha, w))d\alpha\xi dw.$$

Such a result - or the equivalent within the context of Young measure - has found its first application in the approximation through the artificial viscosity method of the Cauchy problem in $\mathbb{R}^p$ for scalar conservation laws, when only a uniform L^∞-control of approximate solutions holds (see [7, 21]). It has also been applied to the numerical analysis of transport equations since "Finite-Volume" schemes only give an L^∞-estimate uniformly with respect to the mesh length of the numerical solution (see e.g. [8, 9]...). Here, we refer to this concept when the approximating sequence is the sequence of solutions to viscous equations $(1.1)_{\epsilon>0}$ associated with data (u_0, u_{Γ_e}).

3.2 Proof of theorem 3.1

Owing to (2.3) there exists a subsequence extracted from $(u_\epsilon)_{\epsilon>0}$ - labelled $(u_\epsilon)_{\epsilon>0}$ - and a measurable and bounded function π on $]0,1[\times Q$ such that for any continuous and bounded function f on $Q\times]-M,M[$

$$\lim_{\epsilon\to 0^+}\int_Q f(t,x,u_\epsilon)\xi dxdt = \int_{]0,1[\times Q} f(t,x,\pi(\alpha,t,x))\xi d\alpha dxdt,$$

for any ξ of $L^1(Q)$.

In fact, in order to establish the proof of theorem 3.2 we are going to demonstrate that the function π is an *entropy process solution* to $(1.5, 1.6, 1.7)$, namely that π fulfills relations $(3.1, 3.2, 3.3)$, where the integrations with respect to the Lebesgue measure on Q, Ω and Σ are respectively turned into integrations with respect to the Lebesgue measure on $]0,1[\times Q$, $]0,1[\times\Omega$ and $]0,1[\times\Sigma$.

Indeed by following the F.Otto's works [18], we may prove that if π and ω are two entropy process solutions of $(1.5, 1.6, 1.7)$ in the sense of definition 3.1, then for a.e. t of $]0,T[$:

$$\int_0^t \int_{Q\times]0,1[^2} |\pi(\alpha,t,x)-\omega(\beta,t,x)| d\alpha d\beta dxdt = 0.$$

Thus, as mentioned in [8] the two processes π and ω are equal, for a.e. (t,x) on Q, to a common value $u(t,x)$, which does not depend on α or β in $]0,1[$. In addition, u is a measurable and bounded function on Q and is namely the weak entropy solution of $(1.5, 1.6, 1.7)$. Lastly, as a consequence of the uniqueness property, the whole sequence of approximate solutions strongly converges to u in $L^q(Q)$, $1 \leq q < +\infty$ and a.e. on Q (see e.g. [8] of [7] within the context of measure-valued solutions).

Remark: The uniqueness proof of an entropy process to $(1.5, 1.6, 1.7)$ refers to the classical Kruskov method [14]: it split the time and the space variables in two. As pointed out in the introduction, we then need some smoothness assumptions on $\mathbf{B}$ in order to pass to the limit with mollifiers at once in the convective term and in the boundary ones (by using Otto's techniques).

(i) Entropy inequality (3.1) and initial condition (3.2) for π

We take advantage of the approximation properties of u_ϵ through $(u_{\epsilon,\delta})_{\delta>0}$ to provide an entropy inequality for u_ϵ, in which we can pass to the limit with respect to ϵ. So, let us come back to the regularized problem $(2.5, 2.6, 2.7)$. By multiplying a.e. on Q equation (2.9) with the function $sgn_\lambda(u_{\epsilon,\delta} - k)$, $k \in \mathbb{R}$, and by taking into account that a.e. on Q

$$sgn_\lambda(u_{\epsilon,\delta} - k)\Delta\phi_\delta(u_{\epsilon,\delta}) \leq \Delta\left(\int_k^{u_{\epsilon,\delta}} sgn_\lambda(\tau - k)\phi_\delta'(\tau)d\tau\right)$$

and,

$$sgn_\lambda(u_{\epsilon,\delta} - k)Div_x(\varphi(u_{\epsilon,\delta})\mathbf{B}) = Div_x(F_\lambda(u_{\epsilon,\delta}, k)\mathbf{B})$$

where,

$$F_\lambda(v, w) = \int_w^v sgn_\lambda(\tau - w)\varphi'(\tau)d\tau.$$

Then, by letting $I_\lambda(u) = \int_k^u sgn_\lambda(\tau - k)d\tau$, one has a.e. on Q,

$$\begin{aligned}
&\partial_t I_\lambda(u_{\epsilon,\delta}) + Div_x(F_\lambda(u_{\epsilon,\delta}, k)\mathbf{B}) + \psi(t, x, u_{\epsilon,\delta})sgn_\lambda(u_{\epsilon,\delta} - k) \\
\leq\ & \epsilon\Delta\left(\int_k^{u_{\epsilon,\delta}} sgn_\lambda(\tau - k)\phi_\delta'(\tau)d\tau\right).
\end{aligned}$$

In this way, for any function ζ of $\mathcal{D}_+(]-\infty,T]\times\Omega)$,

$$-\int_Q \{I_\lambda(u_{\epsilon,\delta})\partial_t\zeta + F_\lambda(u_{\epsilon,\delta},k)\mathbf{B}.\nabla\zeta - \psi(t,x,u_{\epsilon,\delta})sgn_\lambda(u_{\epsilon,\delta}-k)\zeta\}\,dxdt$$

$$\leq \epsilon\int_Q\left(\int_k^{u_{\epsilon,\delta}} sgn_\lambda(\tau-k)\phi'_\delta(\tau)d\tau\right)\Delta\zeta dxdt + \int_\Omega I_\lambda(u_0^\epsilon)\zeta(0,x)dx.$$

Thence, we pass to the limit with respect to δ and then with respect to ϵ - the first term in the right hand side of the above inequality being bounded by ϵC^{st} thanks to (2.3). Lastly the λ-limit through the dominated convergence theorem gives

$$-\int_{]0,1[\times Q}\mathcal{L}(\pi,k,\zeta)d\alpha dxdt \leq \int_\Omega |u_0-k|\zeta(0,x)dx, \qquad (3.4)$$

for any ζ of $\mathcal{D}_+(]-\infty,T]\times\Omega)$. So, one gets (3.1) for π. Moreover F.Otto's ideas [18] ensure that if (3.4) holds, then for any function K of $L^\infty(\Omega)$,

$$ess\lim_{t\to 0^+} sup\int_{]0,1[\times\Omega}|\pi(\alpha,t,x)-K(x)|d\alpha dx \leq \int_\Omega |u_0(x)-K(x)|dx.$$

Initial condition (3.2) for π follows with $K=u_0$.

(ii) Dirichlet boundary condition (3.3) for π

The demonstration is based on F.Otto's original proof [18]. With this view, we introduce for any l of $\mathbb{N}$ the family of boundary entropy-entropy flux pairs (H_l,Q_l) defined by:

$$H_l(a,b,c) = H_l^b(a,c) = \left((dist(a,\mathcal{I}[b,c]))^2 + \left(\frac{1}{l}\right)^2\right)^{1/2} - \frac{1}{l}.$$

and

$$Q_l(a,b,c) = Q_l^b(a,c) = \int_c^a \partial_1 H_l^{\phi_\delta(b)}(\phi_\delta(\tau),\phi_\delta(c))\varphi'(\tau)d\tau,$$

hence $(Q_l)_{l\in\mathbb{N}}$ converges uniformly as l goes to $+\infty$, to $\mathcal{F}(a,b,c)$ for any non-negative δ. Again, we take advantage of the approximation properties of

u_ϵ through $(u_{\epsilon,\delta})_{\delta>0}$ to come back to regularized problem (2.5, 2.6, 2.7). By multiplying equation (2.9) with $\partial_1 H_l^{\phi_\delta(w)}(\phi_\delta(u_{\epsilon,\delta}),\phi_\delta(k))\zeta\xi$, where ζ belongs to $H^1_{0,+}(\Omega)$, ξ to $\mathcal{D}_+(]0,T[\times\mathbb{R}^p)$, w and k to $\mathbb{R}$, and by considering that a.e. on Q (to simplify the writing we drop the indexe ϵ temporarily)

$$\partial_1 H_l^{\phi(w)}(\phi(u),\phi(k))\Delta\phi(u) \leq Div_x[\partial_1 H_l^{\phi(w)}(\phi(u),\phi(k))\nabla\phi(u)]$$

and,

$$\partial_1 H_l^{\phi(w)}(\phi(u),\phi(k))Div_x(\varphi(u)\mathbf{B}) = Div_x(Q_l\mathbf{B})$$

it comes:

$$\begin{aligned}
& -\int_Q \Bigg\{ \Big(\int_k^{u_{\epsilon,\delta}} \partial_1 H_l^{\phi_\delta(w)}(\phi_\delta(\tau),\phi_\delta(k))d\tau\Big)\partial_t\xi\zeta + Q_l^w(u_{\epsilon,\delta},k)\mathbf{B}.\nabla\xi\zeta \\
& -\partial_1 H_l^{\phi_\delta(w)}(\phi_\delta(u_{\epsilon,\delta}),\phi_\delta(k))\psi(t,x,u_{\epsilon,\delta})\zeta\xi \Bigg\} dxdt \\
\leq\ & \int_Q Q_l^w(u_{\epsilon,\delta},k)\mathbf{B}.\nabla\zeta\xi dxdt - \epsilon\int_Q \nabla[H_l^{\phi_\delta(w)}(\phi_\delta(u_{\epsilon,\delta}),\phi_\delta(k))\xi].\nabla\zeta dxdt \\
+\ & 2\epsilon\int_Q H_l^{\phi_\delta(w)}(\phi_\delta(u_{\epsilon,\delta}),\phi_\delta(k))\nabla\xi.\nabla\zeta dxdt \\
+\ & \epsilon\int_Q H_l^{\phi_\delta(w)}(\phi_\delta(u_{\epsilon,\delta}),\phi_\delta(k))\zeta\Delta\xi dxdt. \qquad (3.5)
\end{aligned}$$

Owing to the converge properties of the sequence $(u_{\epsilon,\delta})_{\delta>0}$ toward u_ϵ, we may pass to the δ-limit in (3.5) and in order to take the ϵ-limit - that is to control the right-hand side of (3.5) - we consider the particular choice for the function ζ: for any positive ϵ,

$$\zeta_\epsilon(x) = 1 - \exp\left(-\frac{M'_{\varphi\circ\phi^{-1}} + \epsilon L}{\epsilon}s(x)\right) \qquad (3.6)$$

where for any positive parameter μ small enough,

$$s(x) = \begin{cases} \min(dist(x,\Gamma),\mu) \ for\ x\in\Omega \\ -\min(dist(x,\Gamma),\mu)\ for\ x\in\mathbb{R}^p\backslash\Omega, \end{cases}$$

with $L = \sup\limits_{0<s(x)<\mu} |\Delta s(x)|$. That way (see [18]), for any φ of $W^{1,1}_{+}(\mathbb{R}^p)$

$$M'_{\varphi\circ\phi^{-1}} \int\limits_{\Omega} |\nabla\zeta_\epsilon|\varphi dx \leq \epsilon \int\limits_{\Omega} \nabla\zeta_\epsilon.\nabla\varphi dx + (M'_{\varphi\circ\phi^{-1}} + L\epsilon) \int\limits_{\Gamma} \varphi d\sigma. \tag{3.7}$$

Therefore, considering that a.e. on Q

$$\begin{aligned} |Q^w_l(u_\epsilon, k)| &= |\int\limits_{\phi(k)}^{\phi(u_\epsilon)} \partial_1 H^{\phi(w)}_l(\tau, \phi(k))(\varphi\circ\phi^{-1})'(\tau)d\tau| \\ &\leq M'_{\varphi\circ\phi^{-1}} H^{\phi(w)}_l(\phi(u_\epsilon), \phi(k)), \end{aligned}$$

and using weak differential inequality (3.7) with $\varphi = H^{\phi(w)}_l(\phi(u_\epsilon), \phi(k))\xi$, we obtain a majoration of the right-hand side of (3.5) through:

$$\begin{aligned} &2\epsilon \int\limits_{Q} H^{\phi(w)}_l(\phi(u_\epsilon), \phi(k))\nabla\xi.\nabla\zeta_\epsilon dxdt + \epsilon \int\limits_{Q} H^{\phi(w)}_l(\phi(u_\epsilon), \phi(k))\zeta_\epsilon \Delta\xi dxdt \\ &+(M'_{\varphi\circ\phi^{-1}} + L\epsilon) \int\limits_{\Sigma} H^{\phi(w)}_l(\phi(u^\epsilon_{\Gamma_e}), \phi(k))\xi d\sigma. \end{aligned}$$

Thanks to (2.3) and as ζ_ϵ goes to 1 in $L^1(\Omega)$ and $\epsilon\nabla\zeta_\epsilon$ goes to 0 in $(L^1(\Omega))^p$ we may pass to the limit when ϵ tends to 0^+. For any function ξ of $\mathcal{D}_+(]0,T[\times\mathbb{R}^p)$, it comes:

$$\begin{aligned} &\int\limits_{]0,1[\times Q} \left\{ \int\limits_{k}^{\pi} \partial_1 H^{\phi(w)}_l(\phi(\tau), \phi(k))d\tau \partial_t\xi + Q^w_l(\pi, k)\mathbf{B}.\nabla\xi \right. \\ &\left. -\partial_1 H^{\phi(w)}_l(\phi(\pi), \phi(k))\psi(t,x,\pi)\xi \right\} d\alpha dxdt \\ &\geq -M'_{\varphi\circ\phi^{-1}} \int\limits_{\Sigma} H^{\phi(w)}_l(\phi(u_{\Gamma_e}), \phi(k))\xi d\sigma. \end{aligned}$$

When one refers to F.Otto's works [18], p. 115, lemma 7.34, and uses the smoothness of vector field $\mathbf{B}$, this inequality implies that for any ζ of $L^\infty(\Sigma)$

and ξ of $L^1_+(\Sigma)$,

$$\begin{aligned} & ess \lim_{\tau\to 0^-} \int\limits_{]0,1[\times\Sigma} Q_l(\pi(\alpha,\sigma+\tau\nu),\zeta,k)\mathbf{B}(\sigma).\nu\xi d\alpha d\sigma \\ \geq \quad & -M'_{\varphi o\phi^{-1}} \int\limits_{\Sigma} H_l(\phi(u_{\Gamma_e}),\phi(\zeta),\phi(k))\xi d\sigma. \end{aligned}$$

Boundary condition (3.3) for π follows with $\zeta = u_{\Gamma_e}$, which completes the proof of theorem 3.1.

Remark: In the special situation of an homogeneous boundary condition on Γ_e, the demonstration of boundary condition (3.3) for π can easily be established without referring to assumption (1.8). Indeed, in this particular context, for any ζ of $\mathcal{D}_+(]0,T[\times\mathbb{R}^p)$ and any real k, the function $\partial_1 H_l^0(u_{\epsilon,\delta},k)\zeta$ belongs to $W(0,T,V,L^2(\Omega))$; so it can be taken as a test-function in variational formulation (2.5) for $u_{\epsilon,\delta}$. We integrate with respect to t over $[0,T]$, we use the convexity of $z\to\partial_1 H_l^0(z,k)$ to have a majoration of the diffusive term and to transform the convective one, we note that:

$$\begin{aligned} \int\limits_Q Div_x\{\varphi(u_{\epsilon,\delta})\mathbf{B}\}\partial_1 H_l^0(u_{\epsilon,\delta},k)\zeta &= -\int\limits_Q Q_l^*(u_{\epsilon,\delta},k)\mathbf{B}.\nabla\zeta dxdt \\ &\quad + \int\limits_\Sigma Q_l^*(u_{\epsilon,\delta},k)\mathbf{B}.\nu\zeta d\sigma, \qquad (3.8) \end{aligned}$$

where

$$Q_l^*(a,b) = \int\limits_0^a \varphi'(\tau)H_l^0(\tau,b)d\tau.$$

Due to the partitioning of Γ and to the monotonicity of φ, we observe that the boundary integral in the right-hand side of (3.8) is non-negative. That way, passing to the limit with respect to δ and then with ϵ provides:

$$\int\limits_{]0,T[\times Q} \left(H_l^0(\pi,k)\partial_t\zeta + Q_l^*(\pi,k)\mathbf{B}.\nabla\zeta - \psi(t,x,\pi)\partial_1 H_l^0(\pi,k)\zeta\right) d\alpha dxdt \geq 0.$$

Thus, for the same reasons as before,

$$ess \lim_{\tau \to 0^-} \int\limits_{]0,1[\times\Sigma} Q_l^*(\pi(\alpha, \sigma + \tau\nu), k)\mathbf{B}(\sigma).\nu\xi d\alpha d\sigma \geq 0,$$

for any function ζ of $L^1_+(\Sigma)$. Boundary condition (3.3) for π follows when l goes to $+\infty$

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