ANNALES MATHÉMATIQUES

BLAISE PASCAL

Tibor Šalát, Vladimír Toma

A Classical Olivier's Theorem and Statistical Convergence

Volume 10, n°2 (2003), p. 305-313.

<http://ambp.cedram.org/item?id=AMBP_2003__10_2_305_0>

Publication éditée par le laboratoire de mathématiques
de l'université Blaise-Pascal, UMR 6620 du CNRS
Clermont-Ferrand — France

Annales Mathematiques Blaise Pascal **10**, 305-313 (2003)

A Classical Olivier's Theorem and Statistical Convergence

Tibor Šalát
Vladimír Toma

Abstract

L. Olivier proved in 1827 the classical result about the speed of convergence to zero of the terms of a convergent series with positive and decreasing terms. We prove that this result remains true if we omit the monotonicity of the terms of the series when the limit operation is replaced by the statistical limit, or some generalizations of this concept.

Résumé. L. Olivier démontrait en 1827 un résultat classique sur la vitesse de convergence vers zéro d'une série convergente à termes positifs décroissants. Nous démontrons que ce résultat reste valable si nous omettons la monotonie des termes de la série, en remplaçant l'opération limite par la limite statistique ou encore par des généralisations de ce concept.

1 Introduction

The above mentioned result of L. Olivier was published in [7] p. 39 (see also [5] p. 125) and claims that if $a_n \geq a_{n+1} > 0,\ (n = 1, 2, \dots)$ and $\sum\limits_{n=1}^{\infty} a_n < +\infty$ then $\lim\limits_{n\to\infty} na_n = 0$. Simple examples show that without the monotonicity condition $a_n \geq a_{n+1},\ (n = 1, 2, \dots)$, the sequence $(na_n)_{n\geq 1}$ need not converge to zero.

Example 1. Let $a_n = \frac{1}{n}$ if n is a square i.e. $n = k^2,\ (k = 1, 2, \dots)$, and $a_n = \frac{1}{n^2}$ otherwise. Then $a_n > 0\ (n = 1, 2, \dots)$, $\sum\limits_{n=1}^{\infty} a_n < +\infty$, but $\lim\limits_{n\to\infty} na_n \neq 0$ since $k^2 a_{k^2} = 1,\ (k = 1, 2, \dots)$.

Remark: As the unknown referee pointed out the example can be strengthened taking $a_n = \frac{\log n}{n}$ if $n = k^2$, in which case the sequence $(na_n)_{n\geq 1}$ is not bounded.

The notion which allows us to describe the behavior of the sequence $(na_n)_{n\geq 1}$ is the notion of statistical convergence introduced in paper [2], (see also [3], [10], and [9])

Definition: We say that a sequence $(x_n)_{n\geq 1}$ (of real or complex numbers) *statistically converges* to a number L, and we write lim-stat $x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) := \{n : |x_n - L| \geq \varepsilon\}$ has zero asymptotic density, i.e. the limit

$$d(A(\varepsilon)) := \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \chi_{A(\varepsilon)}(k)$$

exists and is equal to zero. Here χ_A is the characteristic function of a set A.

In what follows we will show that the sequence $(na_n)_{n\geq 1}$ statistically converges to 0 if $\sum_{n=1}^{\infty} a_n < +\infty,\ (a_n > 0, n = 1, 2, \dots)$ without the monotonicity assumption on the sequence $(a_n)_{n\geq 1}$.

The notion of statistical convergence was generalized using the concept of an *admissible ideal* $\Im$ of subsets of positive integers $\mathbb{N} = \{1, 2, \dots\}$, that is $\Im \subseteq \mathcal{P}(\mathbb{N})$ and $\Im$ is additive (i.e. $A, B \in \Im \Rightarrow A \cup B \in \Im$), hereditary (i.e. $B \subset A \in \Im \Rightarrow B \in \Im$), containing all singletons and not containing $\mathbb{N}$.

Definition: (See [6].) We say that a sequence $(x_n)_{n\geq 1}$ $\Im$*-converges* to a number L and we write $\Im$-lim $x_n = L$, if for each $\varepsilon > 0$ the set $A(\varepsilon) := \{n : |x_n - L| \geq \varepsilon\}$ belongs to the ideal $\Im$.

An admissible ideal is for example:

$$\Im_f := \{A \subseteq \mathbb{N} :\ A \text{ is finite set}\}.$$

Let us note that $\Im_f$-lim $x_n = L$ means the same as $\lim\limits_{n\to\infty} x_n = L$. Some admissible ideals can be obtained using various concepts of density of sets $A \subseteq \mathbb{N}$. Using the asymptotic density defined above we obtain the ideal

$$\Im_d := \{A \subseteq \mathbb{N} :\ d(A) = 0\}.$$

Obviously $\Im_d$-convergence means the statistical convergence. Another type of density is the logarithmic density defined by means of *lower* and *upper* *logarithmic density* of a set $A \subseteq \mathbb{N}$:

$$\underline{\delta}(A) := \liminf_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_A(k)\frac{1}{k}}{\log n}, \qquad \overline{\delta}(A) := \limsup_{n\to\infty} \frac{\sum_{k=1}^{n} \chi_A(k)\frac{1}{k}}{\log n}.$$

If $\underline{\delta}(A) = \overline{\delta}(A) =: \delta(A)$ the number $\delta(A)$ is called the *logarithmic density* of the set A. Using the logarithmic density we can define the ideal

$$\Im_\delta := \{A \subseteq \mathbb{N} : \ \delta(A) = 0\}.$$

A little bit more complicated is the concept of the uniform density. For any $A \subseteq \mathbb{N}, \ t \geq 0, \ s \geq 1$ denote by $A(t+1, t+s)$ the number of elements of the set $A \cap [t+1, t+s]$. Put

$$\beta_s := \liminf_{t\to\infty} A(t+1, t+s), \quad \beta^s := \limsup_{t\to\infty} A(t+1, t+s).$$

Then there exist

$$\underline{u}(A) := \lim_{s\to\infty} \frac{\beta_s}{s}, \quad \overline{u}(A) := \lim_{s\to\infty} \frac{\beta^s}{s}$$

called the *lower* and *upper uniform density* of A, respectively. We prove the existence of $\lim_{s\to\infty} \frac{\beta_s}{s}$ only, since the proof for $\lim_{s\to\infty} \frac{\beta^s}{s}$ is similar. Let us choose a fixed $p \in \mathbb{N}$. Then for any $s \in \mathbb{N}$ there exists $t_s \geq 1$ such that $\beta_s = A(t_s + 1, t_s + s)$ and simultaneously $\beta_p \leq A(t+1, t+p)$ for every $t \geq t_s$. For any $s \in \mathbb{N}$ there exist unique integer numbers $k_s, r_s \geq 0$ such that $s = k_s p + r_s$ with $0 \leq r_s \leq p - 1$. Then we have (with the convention $A(x, y) = 0$ if $y < x$)

$$\frac{\beta_s}{s} = \frac{1}{k_s p + r_s} A(t_s + 1, t_s + k_s p + r_s) =$$

$$= \frac{1}{k_s p + r_s}[\sum_{i=1}^{k_s} A(t_s + 1 + (i-1)p, t_s + ip) + A(t_s + k_s p + 1, t_s + k_s p + r_s)]$$

Hence $\frac{\beta_s}{s} \geq \frac{k_s \beta_p}{k_s p + r_s}$ and when $s \to \infty$ then also $k_s \to \infty$ and so for fixed p we have $\lim_{k_s\to\infty} \frac{k_s \beta_p}{k_s p + r_s} = \frac{\beta_p}{p}$. Therefore $\underline{u}(A) = \liminf\limits_{s\to\infty} \frac{\beta_s}{s} \geq \frac{\beta_p}{p}$ and consequently $\underline{u}(A) \geq \sup\limits_{p\geq 1} \frac{\beta_p}{p}$. Since obviously $\limsup\limits_{s\to\infty} \frac{\beta_s}{s} \leq \sup\limits_{p\geq 1} \frac{\beta_p}{p}$, we can conclude that there exists the limit $\lim_{s\to\infty} \frac{\beta_s}{s} = \sup\limits_{p\geq 1} \frac{\beta_p}{p}$. If $\underline{u}(A) = \overline{u}(A) =: u(A)$ then the common value $u(A)$ is called the uniform density of A (cf.[1]). Using the uniform density we can define the admissible ideal

$$\Im_u := \{A \subseteq \mathbb{N} : u(A) = 0\}.$$

To compare the above defined ideals let us remember the *lower* and *upper asymptotic density* defined by

$$\underline{d}(A) := \liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_A(k), \qquad \overline{d}(A) := \limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} \chi_A(k).$$

The following relations between these densities can be verified (cf.[1],[4]):

$$0 \le \underline{u}(A) \le \underline{d}(A) \le \underline{\delta}(A) \le \overline{\delta}(A) \le \overline{d}(A) \le \overline{u}(A) \le 1.$$

Consequently we get the chain of inclusions for the above defined ideals:

$$\Im_f \subseteq \Im_u \subseteq \Im_d \subseteq \Im_\delta. \tag{1.1}$$

The ideal which will play an important role in the main theorem is the following one

$$\Im_c := \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < +\infty\}.$$

It is well known (see [8]) that $\sum_{a\in A} a^{-1} < +\infty$ implies $d(A) = 0$. So the following inclusion holds

$$\Im_c \subseteq \Im_d. \tag{1.2}$$

2 Main Results

The above mentioned Olivier's result can be formulated in the terms of $\Im_f$-convergence as follows. If

$$a_n > 0 \ (n = 1, 2, \dots), \qquad \sum_{n=1}^{\infty} a_n < +\infty$$

and

$$a_1 \ge a_2 \ge \dots \ge a_n \ge a_{n+1} \ge \dots$$

then

$$\Im_f\text{-lim } na_n = 0.$$

In the sequel we are going to study the ideals $\Im$ with the following property:

$$\text{If } a_n > 0 \ (n = 1, 2, \dots), \ \sum_{n=1}^{\infty} a_n < +\infty \quad \text{then} \quad \Im\text{-lim } na_n = 0. \tag{2.1}$$

From Example 1 we can conclude that the ideal $\Im_f$ does not have the property 2.1. Let us denote by $\mathcal{S}(T)$ the class of all admissible ideals $\Im$, with the property 2.1. So we have that $\Im_f \notin \mathcal{S}(T)$. The following theorem claims a more useful fact.

Theorem 2.1: *Ideal $\Im_c$ is an element of $\mathcal{S}(T)$.*

Proof: We proceed by contradiction. Let

$$\Im_c := \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < +\infty\} \notin \mathcal{S}(T).$$

Then there exist numbers $a_n > 0$ $(n = 1, 2, \dots)$ with $\sum\limits_{n=1}^{\infty} a_n < +\infty$ such that the equality $\Im_c$-lim $na_n = 0$ does not hold. This means that there exists $\varepsilon_0 > 0$ for which $A(\varepsilon_0) = \{n : \ na_n \geq \varepsilon_0\} \notin \Im_c$. Hence from the definition of the ideal $\Im_c$ we get $\sum\limits_{n \in A(\varepsilon_0)} n^{-1} = +\infty$. For $n \in A(\varepsilon_0)$ we have $na_n \geq \varepsilon_0$ and so

$$a_n \geq \frac{\varepsilon_0}{n} \text{ for every } n \in A(\varepsilon_0).$$

Using this and the comparison criterion for infinite series we get

$$\sum_{n \in A(\varepsilon_0)} a_n \geq \varepsilon_0 \sum_{n \in A(\varepsilon_0)} n^{-1} = +\infty.$$

So it must be also $\sum\limits_{n=1}^{\infty} a_n = +\infty$ and this is a contradiction. □

The claim in the following lemma is a trivial fact about preservation of the limit.

Lemma 2.2: *Let $\Im_1, \Im_2$ be admissible ideals such that $\Im_1 \subseteq \Im_2$. If $\Im_1$-$\lim x_n = L$, then also $\Im_2$-$\lim x_n = L$.*

An obvious consequence of Lemma 2.2 is the following theorem.

Theorem 2.3: *If $\Im_1 \subseteq \Im_2$ are two admissible ideals and $\Im_1 \in \mathcal{S}(T)$ then $\Im_2 \in \mathcal{S}(T)$.*

Corollary 2.4: *If $\Im$ is an admissible ideal and $\Im \supseteq \Im_c$ then $\Im \in \mathcal{S}(T)$.*

Theorem 2.5: *The ideal $\Im_c$ is the smallest element in the class $\mathcal{S}(T)$ partially ordered by inclusion.*

Proof: Let $\Im \in \mathcal{S}(T)$. We prove that for any set $M = \{m_1 < m_2 < \dots\} \in \Im_c$ we have $M \in \Im$. We can suppose that M is an infinite set because if it were finite it belongs to $\Im$ as it is an admissible ideal, hence contains all finite sets. Since $M \in \Im_c$ following the definition of the ideal $\Im_c$ we have

$$\sum_{k=1}^{\infty} \frac{1}{m_k} < +\infty$$

Let us define numbers a_n $(n = 1, 2, \dots)$ as follows

$$\begin{aligned} a_{m_k} &= \frac{1}{m_k} \quad (k = 1, 2, \dots), \\ a_n &= \frac{1}{n^2 + n} \qquad \text{for } n \in \mathbb{N} \setminus M. \end{aligned}$$

Then obviously $a_n > 0$ $(n = 1, 2, \dots)$ and $\sum_{n=1}^{\infty} a_n < +\infty$ by the definition of numbers a_n. Since $\Im \in \mathcal{S}(T)$ we have

$$\Im\text{-lim } n a_n = 0.$$

This implies that for each $\varepsilon > 0$ we have

$$A(\varepsilon) = \{n : \ n a_n \geq \varepsilon\} \in \Im,$$

specifically $M = A(1) \in \Im$. □

The problem of characterization of the class of all ideals having the property 2.1 was formulated orally by the second author in a discussion in the Seminar on real functions theory in Bratislava. The above results allow us to give such a characterization.

Theorem 2.6: *The class $\mathcal{S}(T)$ consists of all admissible ideals $\Im \subseteq \mathcal{P}(\mathbb{N})$ such that $\Im \supseteq \Im_c$.*

Proof: 1) If $\Im \in \mathcal{S}(T)$ then after Theorem 2.5 we have $\Im \supseteq \Im_c$.
2) Let $\Im$ be an admissible ideal and $\Im_c \subseteq \Im$. Due to Theorem 2.1 we have $\Im_c \in \mathcal{S}(T)$ and the Corollary 2.4 yields $\Im \in \mathcal{S}(T)$. $\square$

Remark: Referring to inclusions 1.1 and 1.2 we can claim that $\mathcal{S}(T)$ contains as elements the ideals $\Im_d$ and $\Im_\delta$. Since the $\Im_d$-convergence is in fact the statistical convergence we get

$$\sum_{n=1}^{\infty} a_n < +\infty, \quad (a_n > 0) \Longrightarrow \text{lim-stat } na_n = 0.$$

In connection with Theorem 2.6 it is important to know how many ideals have the property 2.1 and how many don't have this property. We know yet that the ideal $\Im_f$ which yields usual convergence is not an element of $\mathcal{S}(T)$. We give some more such examples.

Example 2. For every integer $j \geq 0$ let $D_j := \{2^j(2n+1) : \ n = 0, 1, \dots\}$; then obviously $\bigcup_{j=0}^{\infty} D_j = \mathbb{N}$. Let $\Im^\sharp$ be the set of all $A \subseteq \mathbb{N}$ which intersect only finite number of the sets $D_0, D_1, \dots$. Then $\Im^\sharp$ is obviously an ideal which does not contain the set $A = \{1, 2, 2^2, \dots, 2^j, \dots\}$ since A meets every D_j but A is obviously an element of $\Im_c$. Hence $\Im_c \not\subseteq \Im^\sharp$.

Example 3. The ideal $\Im_u := \{A \subset \mathbb{N} : u(A) = 0\}$ where $u(A)$ is the uniform density of the set A, is not an element of the class $\mathcal{S}(T)$. To prove this let us choose $A = \bigcup_{k=1}^{\infty} A_k$ with $A_k = \{2^{2k}+1, \dots, 2^{2k}+2^k\}$, $(k = 1, 2, \dots)$. Obviously $A \in \Im_c$. To prove that $A \notin \Im_u$ let us determine the upper uniform density of the set A (see [1]). To this end, consider the sets $A \cap [t+1, t+s]$ with $t \geq 0, s \in \mathbb{N}$. If we take $t_k = 2^{2k}, s_k = 2^k$, $(k = 1, 2, \dots)$, then $A(t_k + 1, t_k + s_k) = s_k$, $(k = 1, 2, \dots)$ and $\beta^{(s_k)} := \limsup_{t \to \infty} A(t+1, t+s_k) = s_k$ and consequently $\overline{u}(A) = \lim_{k\to\infty} \frac{\beta^{(s_k)}}{s_k} = 1$. So we have proved that $A \notin \Im_u$ and hence $\Im_c \not\subseteq \Im_u$

The next two propositions give an idea of how many admissible ideals are appropriate for the analog of Olivier's theorem and how many can not be used to obtain this analog.

Proposition 2.7: *There are infinitely many admissible ideals which are not elements of the class $\mathcal{S}(T)$.*

Proof: For any infinite set $M \subseteq \mathbb{N}$ we enlarge the ideal $\Im_f$ by adjunction of the set M defining a new ideal $\Im_M := \{A \cup B : A \in \Im_f,\ B \subseteq M\}$. If we choose M such that $\mathbb{N} \setminus M$ is also infinite then $\Im_M$ is an admissible ideal. We can choose moreover M such that $\sum_{m \in M} m^{-1} < +\infty$ and then $\Im_M \subset \Im_c$. Applying Theorem 2.5 we conclude that $\Im_M$ is not an element of $\mathcal{S}(T)$. To see that there are infinitely many ideals of the type $\Im_M$ it is sufficient to observe that $\Im_M \neq \Im_{M'}$ if and only if the symmetric difference $M \triangle M'$ is an infinite set. □

Remark: P. Kostyrko observed that for each q, $0 < q < 1$, the admissible ideal $\Im_c^{(q)} := \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-q} < +\infty\}$ is a proper subset of the ideal $\Im_c$. So we get again by Theorem 2.5 infinitely many admissible ideals that do not belong to $\mathcal{S}(T)$.

Proposition 2.8: *There are infinitely many admissible ideals which are elements of the class $\mathcal{S}(T)$.*

Proof: Let us take a set M with $\sum\limits_{m \in M} m^{-1} = +\infty$ such that $\mathbb{N} \setminus M$ is infinite. Then the ideal

$$\Im_M^* = \{A \cup B : A \in \Im_c,\ B \subseteq M\}$$

is an admissible ideal such that $\Im_c \subset \Im_M^*$ and consequently $\Im_M^* \in \mathcal{S}(T)$. To see that there are infinitely many such ideals $\Im_M^*$ let us write $M = \{m_1 < m_2 < \dots\}$ with $\sum\limits_{k=1}^{\infty} m_k^{-1} = +\infty$. Then by comparison test we have also $\sum\limits_{k=1}^{\infty} m_{2k-1}^{-1} = +\infty = \sum\limits_{k=1}^{\infty} m_{2k}^{-1}$. If we take $M' = \{m_{2k-1} : k = 1, 2 \dots\}$ then $\Im_c \subset \Im_{M'}^* \subset \Im_M^*$ and in this way we construct a decreasing chain of infinitely many admissible ideals of the class $\mathcal{S}(T)$. □

References

[1] T. C. Brown and A. R. Freedman. The uniform density of sets of integers and Fermat's last theorem. *C. R. Math. Rep. Acad. Sci. Canada*, XII:1–6, 1990.

[2] H. Fast. Sur la convergence statistique. *Coll. Math.*, 2:241–244, 1951.

[3] J. A. Fridy. On statistical convergence. *Analysis*, 5:301–313, 1985.

[4] H. Halberstam and K. F. Roth. *Sequences I.* Oxford University Press, Oxford, 1966.

[5] K. Knopp. *Theorie und Anwendung der unendlichen Reihen 3. Aufl.* 1931.

[6] P. Kostyrko, T. Šalát, and W. Wilczński. $\mathcal{I}$-convergence. *Real Anal. Exch.*, 26:669–689, 2000-2001.

[7] L. Olivier. Remarques sur les séries infinies et leur convergence. *J. reine angew. Math.*, 2:31–44, 1827.

[8] B. J. Powel and T. Šalát. Convergence of subseries of the harmonic series and asymptotic densities of sets of positive integers. *Publ. Inst. Math.(Beograd)*, 50(64):60–70, 1991.

[9] I. J. Schoenberg. The integrability of certain functions and related summability methods. *Amer. Math. Monthly*, 66:361–375, 1959.

[10] T. Šalát. On statistically convergent sequences of real numbers. *Math. Slovaca*, 30:139–150, 1980.

Tibor Šalát
Comenius University
Department of Mathematics
Mlynská dolina
84248 Bratislava
SLOVAK REPUBLIC

Vladimír Toma
Comenius University
Department of Mathematics
Mlynská dolina
84248 Bratislava
SLOVAK REPUBLIC
toma@fmph.uniba.sk