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# Weak convergence to fractional Brownian motion in some anisotropic Besov space

M. Ait Ouahra[1]

**Abstract**

We give some limit theorems for the occupation times of 1-dimensional Brownian motion in some anisotropic Besov space. Our results generalize those obtained by Csaki et *al.* [4] in continuous functions space.

# 1 Introduction

The classical framework for weak convergence of a sequence $(X_n, n \geq 1)$ of stochastic processes is the Skorokhod space $\mathcal{D}([0,1])$ endowed with the Skorohod topology, for processes having jumps and a space $\mathcal{C}([0,1])$ of continuous functions equipped with the uniform convergence topology, for continuous processes, (See for instance Billingsley [2] and Jacod and Shiryaev [5]). Relative compactness in the space of probability measures is a key tool in the study of weak convergence. According to Prohorov's theorem, tightness is always a sufficient condition for relative compactness and is also necessary if the metric space $\mathcal{S}$ is separable and complet. In many usual cases, the paths of $X_n$ and of the limiting process $X$ offer more regularity than the bare continuity. For instance the Donsker Prokhorov's invariance principle establishes the $\mathcal{C}([0,1])$-weak convergence to the Brownian motion $B$ of the random polygonal lines $X_n$ interpolating the partial sums of an i.i.d. sequence. Here the paths of $B$ are almost surely of Hölder regularity $\alpha$ for any $\alpha < \frac{1}{2}$ and those of $X_n$ are of Hölder regularity 1. This remark was exploited by Lamperti [9] to prove the same invariance principle in the space $\mathcal{C}^{\alpha}(0,1)$ of $\alpha$-Hölder continuous functions for any $\alpha < \frac{1}{2}$. As pointed out by Lamperti, this is an improvement of the $\mathcal{C}([0,1])$-invariance principle since $\mathcal{C}^{\alpha}(0,1)$ is topologically imbeded in $\mathcal{C}([0,1])$.

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The aim of this paper is to provide a rather general framework to the study of Besov weak convergence of processes indexed by $\mathbb{R}^2$. To prove our main results, we need a characterization of Besov spaces in terms of the coefficients of the expansion of continuous functions in the Faber-Schauder basis. The first characterization of these spaces, in the $L^p$-norm with $p < +\infty$, by coefficients in the Faber-Schauder basis has been established in Ciesielski *et al.* [3]. The multiparameter case was considered by Kamont [6] et [7].

The structure of this paper is as follows. In section 2 we collecte necessary facts about anisotropic Besov spaces. We presente a regularity of Brownian local times and its fractional derivative in section 3. In section 4 we establich a tightness condition in anisotropic Besov spaces. The last section is devoted to the limit theorem for fractional derivative of Brownian local time.

Most of the estimates of this paper contain unspecified constants; we use the same notation of these constants, even when they vary from line to the next. We shall sometimes emphasize the dependance of these constants upon parameters.

## 2 The functional framework

Let $T = [0,1]^2$ and denote $L^p(T)$ the space of Lebesgue integrable functions with exponent $p$, $(1 \leq p < \infty)$. $\mathcal{C}(T)$ stands for the space of $\mathbb{R}$-valued continuous functions on $T$.

For any function $f : T \longrightarrow \mathbb{R}$, any $h \in \mathbb{R}$, and $i = 1,2$; the progressive difference in direction $e_i$ (where $e_i = (\delta_{1,i}, \delta_{2,i})$ denotes the $i^{th}$ coordinate vector in $\mathbb{R}^2$), is defined by:

$$\Delta_{h,i} f(z) = \begin{cases} f(z + h.e_i) - f(z) & \text{if } z, z + he_i \in T \\ 0 & \text{otherwise.} \end{cases}$$

For any $\bar{h} = (h_1, h_2) \in \mathbb{R}^2$, we set:

$$\Delta_{\bar{h}} f = \Delta_{h_1,1} \circ \Delta_{h_2,2} f.$$

Now, for any Borel function $f : T \longrightarrow \mathbb{R}$, such that $f \in L^p(T)$, $1 \leq p < \infty$ or $f \in \mathcal{C}(T)$ if $p = \infty$, one can measure its smoothness by its modulus of continuity computed in $L^p(T)$ norm. For this end let us define for any

$t \in [0,1]$ and $\bar{t} = (t_1, t_2) \in T$:

$$\begin{cases} \omega_{p,i}(f,t) = \sup_{|h|\le t} \|\Delta_{h,i} f\|_p , \quad i = 1, 2, \\ \omega_p(f, \bar{t}) = \sup_{|h_1|\le t_1, |h_2|\le t_2} \left\|\Delta_{\bar{h}} f\right\|_p . \end{cases}$$

For $\bar{\alpha} = (\alpha_1, \alpha_2)$, $0 < \alpha_1, \alpha_2 < 1$ and $\beta \in \mathbb{R}$, we will consider the real valued function $\omega_\beta^{\bar{\alpha}}(.)$ defined on $T$ by:

$$\omega_\beta^{\bar{\alpha}}(t_1, t_2) = t_1^{\alpha_1} t_2^{\alpha_2} (1 + \log \frac{1}{t_1 t_2})^\beta.$$

We are now going to consider some anisotropic generalized Hölder classes in $L^p$-norm.
Define the norm:

$$\|f\|_p^{\omega_\beta^{\bar{\alpha}}} := \|f\|_{L^p(T)} + \sup_{0<t\le 1} \frac{\omega_{p,1}(f,t)}{\omega_\beta^{\bar{\alpha}}(1,t)} + \sup_{0<t\le 1} \frac{\omega_{p,2}(f,t)}{\omega_\beta^{\bar{\alpha}}(t,1)} + \sup_{0<t_1,t_2\le 1} \frac{\omega_p(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}.$$

**Definition 2.1:** Let $1 \le p \le \infty$, $\bar{\alpha} = (\alpha_1, \alpha_2)$, $0 < \alpha_1, \alpha_2 < 1$ and $\beta \in \mathbb{R}$, the anisotropic Besov spaces are defined as follows

$$Lip_p(\bar{\alpha}, \beta) := \left\{ f \in L^p(T) : \|f\|_p^{\omega_\beta^{\bar{\alpha}}} < \infty \right\}.$$

$Lip_p(\bar{\alpha}, \beta)$ endowed with the norm $\|.\|_p^{\omega_\beta^{\bar{\alpha}}}$ is a non-separable Banach space. We will consider a separable Banach subspace of $Lip_p(\bar{\alpha}, \beta)$ defined by:

$$\begin{aligned} lip_p^*(\bar{\alpha}, \beta) := \quad & \{ f \in Lip_p(\bar{\alpha}, \beta) : \ \omega_{p,i}(f,t) = o(\omega_\beta^{\bar{\alpha}}(t,1)) \ \ i = 1, 2 \ \text{ as } t \to 0; \\ & \text{and} \qquad \omega_p(f, t_1, t_2) = o(\omega_\beta^{\bar{\alpha}}(t_1, t_2)) \text{ as } t_1 \wedge t_2 \to 0 \}, \end{aligned}$$

where $t_1 \wedge t_2 = \min(t_1, t_2)$.

About more on Besov spaces we refer to Peetre [12] or to Bergh and Löfström [1] for an introduction to one-parameter case. The first characterisation of these spaces in the $L^p$-norm with $p < \infty$ by coefficients in the

Faber-Schauder basis have been done in Ciesielski *et al.* [3]. The multiparameter case have been considered recently by Kamont [6] and [7]. To prove our main results, we need the characterization of the Besov spaces in terms of the coefficients of the expansion of a continuous function in the basis consisting of tensor products of Faber-Schauder functions. Let $f$ be a continuous function, let us note by $\{C_n(f), n \geq 0\}$ the coefficients of the decomposition of $f$ in the basis consisting of tensor products of Faber-Schauder functions. We refer to Kamont [7] for this characterization.
The family of Schauder functions on $[0,1]$ is defined by:

$$\begin{cases} \varphi_0(s) = \mathbf{1}_{[0,1]}(s); \;\; \varphi_1(s) = s\mathbf{1}_{[0,1]}(s) \\ for \;\; n = 2^j + k, \;\; j \in \mathbb{N} \;\; and \;\; k = 1, ..., 2^j \\ \varphi_n(s) = 2\, 2^{-\frac{j}{2}} \varphi(2^j s - k) \end{cases}$$

where $\varphi(u) = u\mathbf{1}_{[0,\frac{1}{2}]}(u) + (1-u)\mathbf{1}_{[\frac{1}{2},1]}(u)$. We know that for any continuous function on $[0,1]$ we have the decomposition:

$$f(s) = \sum_{n=0}^{\infty} C_n(f)\varphi_n(s).$$

This series is uniformly convergent and the coefficients $C_n(f)$ are given by:

$$\begin{cases} C_0(f) = f(0), \;\; C_1(f) = f(1) - f(0), \\ \text{and for } n = 2^j + k, \;\; j \geq 0 \text{ and } k = 1, ..., 2^j \\ C_n(f) = 2\, 2^{\frac{j}{2}} \{f(\frac{2k-1}{2^{j+1}}) - \frac{1}{2}\{f(\frac{2k}{2^{j+1}}) + f(\frac{2k-2}{2^{j+1}})\}\}. \end{cases}$$

Now for any continuous function $f \in \mathcal{C}(T)$, we have the following decomposition:

$$f(z) = \sum_{m=0}^{\infty} \sum_{\max(n,n')=m} C_{n,n'}(f)\varphi_n \otimes \varphi_{n'}(z), \quad z \in T,$$

where $C_{n,n'}(f) = C_n^1 \circ C_{n'}^2(f)$, with

$$\begin{cases} C_n^1(f) = C_n(f(.,t)) \\ C_n^2(f) = C_n(f(t,.)). \end{cases}$$

For $1 \leq p < \infty$, $j$, $j' \geq 0$, $(k,k') \in \{1,...,2^j\} \times \{1,...,2^{j'}\}$ and $0 < \alpha < 1$, we shall use the following notations:

$$A^1_{p,j,l} = \frac{2^{-j(\frac{1}{2}-\alpha_1+\frac{1}{p})}}{(1+j)^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} |C_{l,n}(f)|^p \right]^{\frac{1}{p}}, \quad l = 1,0.$$

$$A^2_{p,j,l'} = \frac{2^{-j(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j)^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} |C_{n,l'}(f)|^p \right]^{\frac{1}{p}}, \quad l' = 1,0.$$

$$A_{p,j,j'} = \frac{2^{-j(\frac{1}{2}-\alpha_1+\frac{1}{p})}2^{-j'(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j+j')^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} \sum_{n'=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(f)|^p \right]^{\frac{1}{p}}.$$

In order to state our main results we need the following characterization theorem (see Kamont [6] and [7]:

**Theorem 2.2:** *For $\bar{\alpha} = (\alpha_1, \alpha_2)$, $0 < \alpha_1, \alpha_2 < 1$, $\frac{1}{\alpha_i} < p$, $i = 1,2$ and $\beta \in \mathbb{R}$.*
*(1) The anisotropic Besov space $Lip_p(\bar{\alpha}, \beta)$ is a space of continuous functions linearly isomorphic to a sequence spaces, and we have the following equivalence of norms:*

$$\|f\|_p^{\omega_\beta^{\bar{\alpha}}} \sim \max\left\{ |C_{l,l'}(f)|, \ \sup_{j\geq 0} A^1_{p,j,l}(f), \ \sup_{j\geq 0} A^2_{p,j,l'}(f), \ \sup_{j,j'\geq 0} A^1_{p,j,j'}(f) \right\},$$

*with $l, l' = 0,1$.*
*(2) $f$ belong to $lip_p^*(\bar{\alpha}, \beta)$ if and only if:*

$$B(1) \qquad \lim_{j\to\infty} \frac{2^{-j(\frac{1}{2}-\alpha_1+\frac{1}{p})}}{(1+j)^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} |C_{l,n}(f)|^p \right]^{\frac{1}{p}} = 0, \qquad l = 0,1$$

$$B(2) \qquad \lim_{j\to\infty} \frac{2^{-j(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j)^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} |C_{n,l'}(f)|^p \right]^{\frac{1}{p}} = 0, \qquad l' = 0,1$$

$$B(3) \qquad \lim_{j\to\infty} \frac{2^{-j(\frac{1}{2}-\alpha_1+\frac{1}{p})}2^{-j'(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j+j')^\beta} \left[ \sum_{n=2^j+1}^{2^{j+1}} \sum_{n=2^{j'}+1}^{2^{j'+1}} |C_{n,n'}(f)|^p \right]^{\frac{1}{p}} = 0.$$

# 3 Brownian Local times and fractional derivatives

Throughout the paper, $\{B_t, t \geq 0\}$ will denote a real-valued Brownian motion, with a jointly continuous family of local times $\{L_t^x,\, x \in \mathbb{R},\, t \geq 0\}$. The local times satisfy the so called occupation density formula

$$\int_0^t f(B_s)ds = \int_{\mathbb{R}} f(x) L_t^x dx \quad ,$$

for any bounded or nonnegative Borel function $f$.
It is well known (see McKean [11] and Trotter [14]) that $x \longrightarrow L_t^x$ may be chosen to be Hölder continuous of order $\eta < \frac{1}{2}$, uniformly in $t$ varying in a compact interval.
This allows us to define the fractional derivative of order $\gamma$ for $x \longrightarrow L_t^x$ for all $0 < \gamma < \frac{1}{2}$.

**Definition 3.1:** Let $0 < \delta < 1$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function that belongs to $\mathcal{C}^\delta(\mathbb{R}) \cap L^1(\mathbb{R})$ where $\mathcal{C}^\delta(\mathbb{R})$ is the space of locally $\delta$-Hölder continuous functions on $\mathbb{R}$. For $0 < \gamma < \delta$ we define $D_\pm^\gamma f$ by (see e.g. Samko *et al.* [8])

$$D_\pm^\gamma f(x) := \frac{\gamma}{\Gamma(1-\gamma)} \int_0^{+\infty} \frac{f(x) - f(x \mp y)}{y^{1+\gamma}} dy \quad .$$

The operators $D_+^\gamma$ and $D_-^\gamma$ are called respectively, right–handed and left–handed Marchaud fractional derivatives of order $\gamma$.

They satisfy the (Switching identity)

$$\int_{\mathbb{R}} f(x) D_-^\gamma g(x) dx = \int_{\mathbb{R}} g(x) D_+^\gamma f(x) dx,$$

for any $f, g \in \mathcal{C}^\delta(\mathbb{R}) \cap L^1(\mathbb{R})$, and $0 < \gamma < \delta$.
We put $D^\gamma := D_+^\gamma - D_-^\gamma$.

**Definition 3.2:** Let $p \geq 1$, we define for any $f$ in $L^p(\mathbb{R})$, $D_\pm^0$ as

$$D_\pm^0 f(x) := \frac{1}{\pi} \int_0^{+\infty} \frac{f(x)\mathbf{1}_{]0,1]}(y) - f(x \mp y)}{y} dy \quad .$$

Set $D^0 = D^0_+ - D^0_-$. It is known that the operator $D^0$ maps $L^p(\mathbb{R})$ into $L^p(\mathbb{R})$ for $1 < p < \infty$. Moreover for any $f \in L^p(\mathbb{R})$, $p > 1$

$$\left\|D^0 f\right\|_{L^p(\mathbb{R})} \leq C \left\|f\right\|_{L^p(\mathbb{R})} \quad ,$$

(see e.g. Titchmarsh [13] Chap. V).

Therefore, for any $0 \leq \gamma < 1$ we have

$$D^\gamma f(x) = k(\gamma) \int_0^\infty \frac{f(x+y) - f(x-y)}{y^{1+\gamma}} dy,$$

where $k(\gamma) = \frac{\gamma}{\Gamma(1-\gamma)}$ for $0 < \gamma < 1$ and $k(0) = \frac{1}{\pi}$.
We can define again the fractional derivative as follows:

$$\frac{1}{k(\alpha-1)} D^{\alpha-1} L_t^\cdot(x) = H_t^\alpha(x) = \int_0^t \frac{ds}{(B_s - x)^\alpha} \qquad \text{where } \; y^\alpha := |y|^\alpha sgn(y), \tag{3.1}$$

for any $\alpha < \frac{3}{2}$.

From the work of Yamada [15] we know that in the process $(D^\gamma L_t^\cdot(x),\ t \geq 0,\ x \in \mathbb{R})$ satisfies the following Hölder conditions: for all $T > 0$ *a.s.*
$\forall 0 < \beta < \frac{1}{2} - \gamma$, $\exists C > 0$ such that $\forall 0 \leq t, s \leq T, x \in \mathbb{R}$

$$|D^\gamma L_t^\cdot(x) - D^\gamma L_s^\cdot(x)| \leq C|t-s|^\beta,$$

$\forall 0 < \beta < \frac{1}{2} - \gamma$, $\exists C > 0$ such that $\forall 0 \leq t \leq T, x, y \in \mathbb{R}$

$$|D^\gamma L_t^\cdot(x) - D^\gamma L_t^\cdot(y)| \leq C|x-y|^\beta.$$

# 4 Tightness in the anisotropic Besov space

In the sequel it is more convenient to work with $lip_p^*(\bar{\alpha}, \beta)$ instead of $Lip_p(\bar{\alpha}, \beta)$.
As the canonical injection of $lip_p^*(\bar{\alpha}, \beta)$ in $Lip_p(\bar{\alpha}, \beta)$ is continuous, weak convergence in the former implies weak convergence in the latter.
The following theorem due to Prohorov, plays a crucial role in the sequel.

**Theorem 4.1:** *Suppose a general metric space $\mathcal{S}$ is separable and complete. A family $\Pi$ of probability measures in $\mathcal{S}$ is relatively compact if and only if it is tight.*

A sufficient condition for the tightness in $lip_p^*(\bar{\alpha}, \beta)$ is given by

**Theorem 4.2:** *Let* $(X_{s,t}^n,\ (s,t) \in T)_{n \geq 1}$ *be a sequence of stochastic processes satisfying:*
$(i)$ $X_{.,0}^n = X_{0,.}^n = x \in \mathbb{R}$.
$(ii)$ $\forall p \geq 2$, *there exists a positive constant* $C_p$ *such that*

$$\mathbb{E}\left|X_{s,t}^n - X_{s',t}^n - X_{s,t'}^n + X_{s',t'}^n\right|^p \leq C_p |s-s'|^{\alpha_1 p} |t-t'|^{\alpha_2 p}.$$

*for some* $\bar{\alpha} = (\alpha_1, \alpha_2)$. *Then the family* $\{P^n\}$ *of law of* $X^n$ *is tight in the space* $lip_p^*(\bar{\alpha}, \beta)$ *for all* $\beta > \frac{2}{p}$ *and all* $p \geq 2$.

The proof of this theorem is based on the following lemmas:

**Lemma 4.3:** *Let* $\bar{\alpha} = (\alpha_1, \alpha_2)$, $0 < \alpha_1, \alpha_2 < 1$, $1 \leq p < \infty$ *and* $\beta > 0$. *A set* $F$ *of measurable functions* $f : T \longrightarrow \mathbb{R}$ *is relatively compact in* $lip_p^*(\bar{\alpha}, \beta)$ *if:*

(1) $\sup\limits_{f \in F} \|f\|_p^{\omega_\beta^{\bar{\alpha}}} < \infty$,

(2) $\lim\limits_{\delta \to 0} \sup\limits_{f \in F} K_\delta(f, \bar{\alpha}, \beta, p) = 0$,

*where* $K_\delta(f, \bar{\alpha}, \beta, p) := \sup\limits_{0<t\leq\delta} \frac{\omega_{p,1}(f,t)}{\omega_\beta^{\bar{\alpha}}(1,t)} + \sup\limits_{0<t\leq\delta} \frac{\omega_{p,2}(f,t)}{\omega_\beta^{\bar{\alpha}}(t,1)} + \sup\limits_{0<t_1,t_2\leq\delta} \frac{\omega_{p,1}(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}$.

**Proof of Lemma 4.3** Note that (1) implies in particular that $F$ is bounded in $L^p$. Recall that for $\alpha_i > \frac{1}{p}$, $i = 1, 2$ the Besov space is a space of continuous functions; by the Fréchet-Kolmogorov's theorem (See Brézis p.72), it is easy to check that $F$ is relatively compact in $L^p$. Hence, for any sequence $(f_n, n \geq 1)$ of $F$ there exists a subsequence (Also denoted by $(f_n)$) converging in $L^p$-norm to some function $f \in L^p$. To finish the proof it suffices to prove the two following conditions:
$(a)$ $f \in lip_p^*(\bar{\alpha}, \beta)$,
and
$(b)$ $(f_{n_k})_{k \geq 1}$ is a Cauchy's sequence in $lip_p^*(\bar{\alpha}, \beta)$.
We have:

$$\|f\|_p^{\omega_\beta^{\bar{\alpha}}} = \|f\|_{L^p} + \sup_{0<t\leq 1} \frac{\omega_{p,1}(f,t)}{\omega_\beta^{\bar{\alpha}}(1,t)} + \sup_{0<t\leq 1} \frac{\omega_{p,2}(f,t)}{\omega_\beta^{\bar{\alpha}}(t,1)} + \sup_{0<t_1,t_2\leq 1} \frac{\omega_p(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}.$$

Let us choose a subsequence of $(f_n)$ that converges almost surely to $f$. By Fatou's lemma we get

$$\|\Delta_h f\|_{L^p} \leq \liminf_{n\to\infty} \|\Delta_h f_n\|_{L^p} \leq \sup_{n\geq 0} \|\Delta_h f_n\|_{L^p}.$$

Therefore, for all $\bar{t}= (t_1, t_2) \in T$,

$$\omega_p(f, \bar{t}) \leq \sup_{n\geq 1} \omega_p(f_n, \bar{t}). \tag{4.1}$$

Then

$$\begin{aligned}
\sup_{0<t_1,t_2\leq 1} \frac{\omega_p(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)} &\leq \sup_{n\geq 1}\{ \sup_{0<t_1,t_2\leq 1} \frac{\omega_p(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}\} \\
&\leq \sup_{n\geq 1} \|f_n\|_p^{\omega_\beta^{\bar{\alpha}}} \\
&< \infty \quad \text{(by hypothesis (i))}.
\end{aligned}$$

In the same way we can prove that $\sup_{0<t\leq 1} \frac{\omega_{p,1}(f,t)}{\omega_\beta^{\bar{\alpha}}(t,1)} < \infty$ and $\sup_{0<t\leq 1} \frac{\omega_{p,2}(f,t)}{\omega_\beta^{\bar{\alpha}}(1,t)} < \infty$, this shows that $f \in Lip_p(\bar{\alpha}, \beta)$.
On the other hand, from (2), we have $\forall \varepsilon > 0$, $\exists \delta_0$ such that $\forall n \geq 1$ :

$$\sup_{t_1\wedge t_2} \sup_{0<t\leq\delta} \frac{\omega_p(f_n,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)} < \varepsilon.$$

As a consequence, condition (4.1) gives $\omega_p(f, \bar{t}) = \circ(\omega_\beta^{\bar{\alpha}}(t_1, t_2))$ as $t_1 \wedge t_2 \to 0$.

In the same way we prove that: $\omega_{p,i}(f, t) = \circ(\omega_\beta^{\bar{\alpha}}(t, 1))$ as $t \to 0$, $i = 1, 2$. This completes the proof of $(a)$.

To prove $(b)$; let $n, n' \geq 0$, we have

$$\begin{aligned}
\|f_n - f_{n'}\|_p^{\omega_\beta^{\bar{\alpha}}} &= \|f_n - f_{n'}\|_{L^p} + \sup_{0<t\leq 1} \frac{\omega_{p,1}(f_n - f_{n'}, t)}{\omega_\beta^{\bar{\alpha}}(1,t)} \\
&\quad + \sup_{0<t\leq 1} \frac{\omega_{p,2}(f_n - f_{n'}, t)}{\omega_\beta^{\bar{\alpha}}(t,1)} + \sup_{0<t_1,t_2\leq 1} \frac{\omega_p(f_n - f_{n'}, \bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}.
\end{aligned}$$

Note that $\lim\limits_{n,n'\to\infty} \|f_n - f_{n'}\|_{L^p} = 0$.

The estimate of the term $\sup\limits_{0<t_1,t_2\leq 1} \frac{\omega_p(f_n-f_{n'},\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}$.
Let $\delta_0 > 0$ to be small enough, it follows that

$$\begin{aligned}
\sup_{0<t_1,t_2\leq 1} \frac{\omega_p(f_n-f_{n'},\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)} &\leq \sup_{t_1\wedge t_2\leq\delta_0} \frac{\omega_p(f_n-f_{n'},\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)} + \sup_{t_1\wedge t_2\geq\delta_0} \frac{\omega_p(f_n-f_{n'},\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)} \\
&\leq K_{\delta_0}(f_n - f_{n'}, \bar{\alpha}, \beta, p) + \frac{4\|f_n-f_{n'}\|_{L^p}}{\omega_\beta^{\bar{\alpha}}(\delta_0,\delta_0)} \\
&\leq K_{\delta_0}(f_n, \bar{\alpha}, \beta, p) + K_{\delta_0}(f_{n'}, \bar{\alpha}, \beta, p) + \frac{4\|f_n-f_{n'}\|_{L^p}}{\omega_\beta^{\bar{\alpha}}(\delta_0,\delta_0)} \\
&\leq 3\varepsilon \quad \text{as } n, n' \to \infty \quad \text{by } (ii).
\end{aligned}$$

The two other terms can be estimated in the same way. This ends the proof of Lemma 4.3.

**Lemma 4.4:** *Let $B$ be a bounded set of $Lip_p(\bar{\alpha}, \beta)$. Then $B$ is relatively compact on $lip_p^*(\bar{\alpha}, \beta)$ $\forall\beta < \beta'$ and $\forall\alpha_i > \frac{1}{p}$ for $i = 1, 2$.*

**Proof of Lemma 4.4** Let $B$ be a bounded subset of $Lip_p(\bar{\alpha}, \beta)$. Lemma 4.4 is a consequence of (1) and (2) of the Lemma 4.3.

It is clear that if $\beta < \beta'$ then $\|f\|_p^{\omega_{\beta'}^{\bar{\alpha}}} \leq \|f\|_p^{\omega_\beta^{\bar{\alpha}}}$, which gives (1).
To show (2), we have

$$\begin{aligned}
K_\delta(f, \bar{\alpha}, \beta', p) &= \sup_{0<t\leq\delta} \frac{\omega_{p,1}(f,t)}{\omega_{\beta'}^{\bar{\alpha}}(1,t)} + \sup_{0<t\leq\delta} \frac{\omega_{p,2}(f,t)}{\omega_{\beta'}^{\bar{\alpha}}(t,1)} + \sup_{t_1\wedge t_2\leq\delta} \frac{\omega_p(f,\bar{t})}{\omega_{\beta'}^{\bar{\alpha}}(t_1,t_2)} \\
&\leq \sup_{0<t\leq\delta} \frac{\omega_{p,1}(f,t)}{\omega_\beta^{\bar{\alpha}}(1,t)}\omega^0_{\beta-\beta'}(1,\delta) + \sup_{0<t\leq\delta} \frac{\omega_{p,2}(f,t)}{\omega_\beta^{\bar{\alpha}}(t,1)}\omega^0_{\beta-\beta'}(1,\delta) \\
&\quad + \sup_{t_1\wedge t_2\leq\delta} \frac{\omega_p(f,\bar{t})}{\omega_\beta^{\bar{\alpha}}(t_1,t_2)}\omega^0_{\beta-\beta'}(1,\delta) \\
&\leq K_\delta(f, \bar{\alpha}, \beta, p)\{2\omega^0_{\beta-\beta'}(1,\delta) + \omega^0_{\beta-\beta'}(\delta,\delta)\} \\
&\leq \|f\|_p^{\omega_\beta^{\bar{\alpha}}} (2\omega^0_{\beta-\beta'}(1,\delta) + \omega^0_{\beta-\beta'}(\delta,\delta)).
\end{aligned}$$

Therefore,

$$\lim_{\delta\to 0} K_\delta(f,\bar{\alpha},\beta',p)=0 \quad \text{because } \beta-\beta'<0,$$

which gives (2). As a consequence $B$ is relatively compact on $lip_p^*(\bar{\alpha},\beta')$.
**Proof of Theorem 4.2** Observe that by the assumption $(i)$, we have $C_{0,0}(X_.^n)=x$, $C_{n,0}(X_.^n)=C_{0,n}(X_.^n)=0$, $\forall n\geq 1$ and $C_{1,1}(X_.^n)=X_{1,1}^n-X_{0,1}^n$. To prove the Theorem 4.2, we use the characterization Theorem 2.2, to show that there exists a constant $C_p>0$ for all $\lambda>0$, and $\frac{1}{p}<\beta'<\beta$, we have

$$\mathbb{P}\left[\|X_.^n\|_p^{\omega_{\beta'}^{\bar{\alpha}}}>\lambda\right]\leq C_p\lambda^{-p},$$

then, for $\lambda$ large enough, we get

$$\mathbb{P}\left[\|X_.^n\|_p^{\omega_{\beta'}^{\bar{\alpha}}}>\lambda\right]\leq \varepsilon, \qquad \forall\varepsilon>0.$$

As a consequence, $\forall\varepsilon>0$, $\exists\, A>0\ \forall\lambda\geq A$ we have

$$\mathbb{P}\left[X_.^n\in B_{Lip_p(\bar{\alpha},\beta')}(0,A)\right]>1-\varepsilon, \quad \text{for all } n\geq 1.$$

Applying the characterization Theorem, it suffices to show that

$$\mathbb{P}[\max\{|C_{0,0}(X_.^n)|,|C_{1,1}(X_.^n)|,\sup\nolimits_{j\geq 0}A_{p,j,1}^1(X_.^n),\sup\nolimits_{j\geq 0}A_{p,j,1}^2(X_.^n)$$

$$\sup\nolimits_{j,j'\geq 0}A_{p,j,j'}(X_.^n)\}>\lambda]\leq C_p\lambda^{-p}, \quad \text{for all } n\geq 1.$$

It is enough to check that

$$\mathbb{P}\left[\sup_{j,j'\geq 0}A_{p,j,j'}(X_.^n)>\lambda\right]\leq C_p\lambda^{-p}, \ \ \forall n\geq 1$$

since the remaining terms can be treated in the same manner.
Now by the Tchebyshev's inequality, we have

$$I=\mathbb{P}\left[\sup_{j,j'\geq 0}\frac{2^{-j(\frac{1}{2}-\alpha_1+\frac{1}{p})}2^{-j'(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j+j')^{\beta'}}[\textstyle\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}|C_{n,n'}(X_.^n)|^p]^{\frac{1}{p}}>\lambda\right]$$

$$\leq\sum_{j,j'\geq 0}\frac{2^{-jp(\frac{1}{2}-\alpha_1+\frac{1}{p})}2^{-j'p(\frac{1}{2}-\alpha_2+\frac{1}{p})}}{(1+j+j')^{\beta'p}}\textstyle\sum_{n=2^j+1}^{2^{j+1}}\sum_{n'=2^{j'}+1}^{2^{j'+1}}\mathbb{E}\,|C_{n,n'}(X_.^n)|^p\,\lambda^{-p}.$$

Note that for $n = 2^j + k$ and $n' = 2^{j'} + k'$, the coefficients $C_{n,n'}(X^n_.)$ are given by:

$$C_{n,n'}(X^n_.) = 2^{\frac{j+j'}{2}} \sum_{i=1}^{4} X^n(R^i_{n,n'}),$$

where $R^i_{n,n'}$ are dyadic rectangles of areas $4^{-1}2^{-(j+j')}$.
By virtue of assumption $(ii)$ we get:

$$I \leq \sum_{j,j' \geq 0} \tfrac{1}{(1+j+j')^{p\beta'}} C_p \lambda^{-p}$$

$$\leq C_p \lambda^{-p}, \text{ (because } \beta' p > 2).$$

Which ends the proof of Theorem 4.2.

# 5 Limit Theorems

We are concerned in this section with limit theorems for the occupation times of 1-dimensional Brownian motion in a suitable anisotropic Besov spaces.

Throughout we shall use the standard notation for fractional derivative (*cf.* Csaki *et al.* [4]).

**Definition 5.1:** If $\{\mathbb{B}(y,l),\ y \in \mathbb{R},\ l \geq 0\}$ denotes a Brownian sheet indexed by $\mathbb{R} \times \mathbb{R}_+$, independent of $\{B_t,\ t \geq 0\}$. We define

$$\mathbb{B}^{(\alpha)}(x,l) = \int_{\mathbb{R}} \frac{|y|^{1-\alpha} - |y-x|^{1-\alpha}}{\Gamma(2-\alpha)} d_y \mathbb{B}(y,l).$$

The process $\mathbb{B}^{(\alpha)}(x,l)$ is a fractional Brownian motion of order $3-2\alpha$ in the first variable and a Brownian motion in the second variable.

Here is the main results of this section.

**Theorem 5.2:** *Let* $1 < \alpha < \frac{3}{2}$ *. Then*

$$\left\{ B_t,\ \frac{H_t^{(\alpha)}(\varepsilon x) - H_t^{(\alpha)}(0)}{\Gamma(2-\alpha)\varepsilon^{\frac{3}{2}-\alpha}} \right\}$$

*converges weakly, as* $\varepsilon \to 0$ *to*

$$\left\{ B_t,\ \mathbb{B}^{(\alpha)}(x; L_t^0) \right\}.$$

*The convergence holds in the anisotropic Besov space* $lip_p^*(((\frac{3}{2}-\alpha)\frac{1}{2}, \frac{3}{2}-\alpha), \beta)$, $\forall p > \frac{1}{2\beta}$.

The proof of these results uses tension criterion established in Theorem 4.2.

**Proof.** We introduce the following notation:

$$A^\varepsilon(t,x) = \frac{H_t^{(\alpha)}(\varepsilon x) - H_t^{(\alpha)}(0)}{\Gamma(2-\alpha)\varepsilon^{\frac{3}{2}-\alpha}}.$$

The convergence of the finite-dimensional distributions follows from the results of Csaki *et al.* [4]. It suffices to prove that the sequence $A^\varepsilon(t,x)$ is tight in the anisotropic Besov space $lip_p^*(((\frac{3}{2}-\alpha)\frac{1}{2}, \frac{3}{2}-\alpha), \beta)$. It is enough to verify that the sequence satisfies conditions $(i)$ and $(ii)$ of Theorem 4.2.
By the relation (3.1), we have $H_0^{(\alpha)}(x) = 0$ then $A^\varepsilon(0,x) = 0$. It is clear that $A^\varepsilon(t,0) = 0$, which gives $(i)$.
Go to show $(ii)$.
It follows now from the Markov property and the additive property of fractional derivative of local time that
$\forall m \geq 1$, $(x,y) \in [0,1]^2$ and $(t,s) \in [0,1]^2$

$$\begin{aligned}
I &= \mathbb{E}\left|A^\varepsilon(t,x) - A^\varepsilon(s,x) - A^\varepsilon(t,y) + A^\varepsilon(s,y)\right|^{2m} \\
&= \frac{1}{\varepsilon^{(\frac{3}{2}-\alpha)2m}}\mathbb{E}\left|D^{\alpha-1}L_t(\varepsilon x) - D^{\alpha-1}L_s(\varepsilon x) - D^{\alpha-1}L_t(\varepsilon y) + D^{\alpha-1}L_s(\varepsilon y)\right|^{2m} \\
&= \frac{1}{\varepsilon^{(\frac{3}{2}-\alpha)2m}}\mathbb{E}\left|D^{\alpha-1}L_{t-s}(\varepsilon x) - D^{\alpha-1}L_{t-s}(\varepsilon y)\right|^{2m} \circ \theta_s \\
&= \frac{1}{\varepsilon^{(\frac{3}{2}-\alpha)2m}}\mathbb{E}\left[\mathbb{E}\left|D^{\alpha-1}L_{t-s}(\varepsilon x) - D^{\alpha-1}L_{t-s}(\varepsilon y)\right|^{2m} \circ \theta_s / B_s\right] \\
&= \frac{1}{\varepsilon^{(\frac{3}{2}-\alpha)2m}}\int_{\mathbb{R}} \mathbb{P}[B_s \in dz]\mathbb{E}\left|D^{\alpha-1}L_{t-s}(\varepsilon x - z) - D^{\alpha-1}L_{t-s}(\varepsilon y - z)\right|^{2m}.
\end{aligned}$$

Where $\theta$ denote the translation operator.
To finish the proof it suffices to prove that

$$\mathbb{E}\left|D^{\alpha-1}L_t(x) - D^{\alpha-1}L_t(y)\right|^{2m} \leq Ct^{(\frac{3}{2}-\alpha)\frac{1}{2}2m}|x-y|^{(\frac{3}{2}-\alpha)2m}.$$

By the definition of fractional derivative, we have

$$\begin{aligned} J &= \mathbb{E}\left|D^{\alpha-1}L_t(x) - D^{\alpha-1}L_t(y)\right|^{2m} \\ &= \mathbb{E}\left|\int_0^{+\infty}\frac{L_t^{x+z}-L_t^{x-z}}{z^\alpha}dz - \int_0^{+\infty}\frac{L_t^{y+z}-L_t^{y-z}}{z^\alpha}dz\right|^{2m} \\ &\leq J_1 + J_2. \end{aligned} \tag{5.1}$$

Where

$$J_1 = \mathbb{E}\left|\int_0^{b}\frac{L_t^{x+z}-L_t^{x-z}}{z^\alpha}dz - \int_0^{b}\frac{L_t^{y+z}-L_t^{y-z}}{z^\alpha}dz\right|^{2m},$$

and

$$J_2 = \mathbb{E}\left|\int_b^{+\infty}\frac{L_t^{x+z}-L_t^{x-z}}{z^\alpha}dz - \int_b^{+\infty}\frac{L_t^{y+z}-L_t^{y-z}}{z^\alpha}dz\right|^{2m}.$$

We estimate $J_1$ and $J_2$ separately.
The estimate of $J_1$:

$$J_1^{\frac{1}{2m}} \leq 2\int_0^b \frac{\left\|L_t^{x+z}-L_t^{x-z}\right\|_{2m}}{z^\alpha}dz,$$

where $\|.\|_{2m} = \left[\mathbb{E}\left|.\right|^{2m}\right]^{\frac{1}{2m}}$.
Using the Hölder condition of $x \longmapsto L_t^x$, we deduce

$$\begin{aligned} J_1^{\frac{1}{2m}} &\leq C\int_0^b z^{\frac{1}{2}-\alpha}dz \\ &= Cb^{\frac{3}{2}-\alpha} \text{ because } \alpha < \frac{3}{2}. \end{aligned} \tag{5.2}$$

Now, we deal with $J_2$.

$$\begin{aligned} J_2^{\frac{1}{2m}} &= \left(\mathbb{E}\left|\int_b^{+\infty}\frac{L_t^{x+z}-L_t^{x-z}}{z^\alpha}dz - \int_b^{+\infty}\frac{L_t^{y+z}-L_t^{y-z}}{z^\alpha}dz\right|^{2m}\right)^{\frac{1}{2m}} \\ &= \left(\mathbb{E}\left|\int_b^{+\infty}\frac{L_t^{x+z}-L_t^{y+z}}{z^\alpha}dz - \int_b^{+\infty}\frac{L_t^{x-z}-L_t^{y-z}}{z^\alpha}dz\right|^{2m}\right)^{\frac{1}{2m}} \\ &\leq 2\int_b^{+\infty}\frac{\left\|L_t^{x+z}-L_t^{y+z}\right\|_{2m}}{z^\alpha}dz. \end{aligned}$$

Using Lemma 3.3 in Marcus and Rosen [10], in the case of Brownian motion, we deduce that

$$\begin{aligned} J_2^{\frac{1}{2m}} &\leq C\int_b^{+\infty} \frac{t^{\frac{1}{4}}|x-y|^{\frac{1}{2}}}{z^{\alpha}}dz \\ &\leq Ct^{\frac{1}{4}}|x-y|^{\frac{1}{2}}b^{1-\alpha} \text{ because } 1<\alpha. \end{aligned} \tag{5.3}$$

Now, combining (5.1), (5.2) and (5.3), we get

$$J^{\frac{1}{2m}} \leq C(b^{\frac{3}{2}-\alpha} + t^{\frac{1}{4}}|x-y|^{\frac{1}{2}}b^{1-\alpha}).$$

By choosing $b = t^{\frac{1}{2}}|x-y|$, we obtain

$$J \leq Ct^{(\frac{3}{2}-\alpha)\frac{2m}{2}}|x-y|^{(\frac{3}{2}-\alpha)2m}.$$

As a consequence

$$I \leq C|t-s|^{(\frac{3}{2}-\alpha)\frac{2m}{2}}|x-y|^{(\frac{3}{2}-\alpha)2m}. \tag{5.4}$$

The tightness in the anisotropic Besov space $lip_p^*(((\frac{3}{2}-\alpha)\frac{1}{2}, \frac{3}{2}-\alpha), \beta)$ is a consequence of formula (5.4) and Theorem 4.2.

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