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Lahsen Aharouch, Youssef Akdim

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Existence of solutions of degenerated unilateral problems with L^1 data

Lahsen Aharouch
Youssef Akdim

Abstract

In this paper, we shall be concerned with the existence result of the Degenerated unilateral problem associated to the equation of the type

$$Au + g(x, u, \nabla u) = f - \mathrm{div}F,$$

where A is a Leray-Lions operator and g is a Carathéodory function having natural growth with respect to $|\nabla u|$ and satisfying the sign condition. The second term is such that, $f \in L^1(\Omega)$ and $F \in \Pi_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'})$.

1 Introduction

Let Ω be a bounded open set of $\mathbb{R}^N$, p be a real number such that $1 < p < \infty$ and $w = \{w_i(x),\ 0 \leq i \leq N\}$ be a vector of weight functions on Ω, i.e. each $w_i(x)$ is a measurable *a.e.* strictly positive function on Ω, satisfying some integrability conditions (see section 2). Now we consider the obstacle problem associated to the following differential equations

$$Au + g(x, u, \nabla u) = f - \mathrm{div}F. \tag{1.1}$$

Where A is a Leray-Lions operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined by $Au = -\mathrm{div}a(x, u, \nabla u)$ and where g is a nonlinear lower order term having natural growth (order p) with respect to $|\nabla u|$, with respect to $|u|$, we do not assume any growth restrictions, but we assume only the "sign-condition"

$$g(x, s, \xi)s \geq 0.$$

As regards the second member, we suppose that $f \in L^1(\Omega)$ and that

$$F \in \Pi_{i=1}^{N} L^{p'}(\Omega, w_i^{1-p'}).$$

In the case where $F = 0$, an existence theorem has been proved in [2] with $f \in W^{-1,p'}(\Omega, w)$, and in [3] with $f \in L^1(\Omega)$ and where the nonlinearity g satisfies further the following coercivity condition

$$|g(x,s,\xi)| \geq \beta \sum_{i=1}^{N} w_i |\xi_i|^p \quad \text{for} \quad |s| > \gamma. \tag{1.2}$$

Our purpose, in this paper, is to prove an existence result for degenerated unilateral problems associated to (1.1) in the case where $F \neq 0$ and without assuming the coercivity condition (1.2). So that, we generalize The previous results given in [3].

Let us point out that another work in the L^p case can be found in [6, 12] in the case of equation, and in [11] in the case of obstacle problems.

This paper is organized as follows, sections 2 containe some preliminaries and some technical lemmas, section 3 is concerned with main results and basic assumptions, in section 4, we prove main results and we study the stability and the positivity of solution.

2 Preliminaries

Let Ω be a bounded open subset of $\mathbb{R}^N (N \geq 1)$. Let $1 < p < \infty$, and let $w = \{w_i(x); i = 1, ..., N\}$,
$0 \leq i \leq N$ be a vector of weight functions i.e. every component $w_i(x)$ is a measurable function which is strictly positive a.e. in Ω. Further, we suppose in all our considerations that for $0 \leq i \leq N$

$$w_i \in L^1_{loc}(\Omega) \quad \text{and} \quad w_i^{-\frac{1}{p-1}} \in L^1_{loc}(\Omega). \tag{2.1}$$

We define the weighted space with weight γ in Ω as

$$L^p(\Omega, \gamma) = \{u(x) : u\gamma^{\frac{1}{p}} \in L^p(\Omega)\},$$

which is endowed with, we define the norm

$$\|u\|_{p,\gamma} = (\int_\Omega |u(x)|^p \gamma(x)\, dx)^{\frac{1}{p}}.$$

We denote by $W^{1,p}(\Omega, w)$ the space of all real-valued functions $u \in L^p(\Omega, w_0)$ such that the derivatives in the sense of distributions satisfy

$$\frac{\partial u}{\partial x_i} \in L^p(\Omega, w_i) \quad \text{for all} \quad i = 1, ..., N.$$

This set of functions forms a Banach space under the norm

$$\|u\|_{1,p,w} = \left(\int_\Omega |u(x)|^p w_0 \, dx + \sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x) \, dx \right)^{\frac{1}{p}}. \qquad (2.2)$$

To deal with the Dirichlet problem, we use the space

$$X = W_0^{1,p}(\Omega, w),$$

defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm (2.2). Note that, $C_0^\infty(\Omega)$ is dense in $W_0^{1,p}(\Omega, w)$ and $(X, \|.\|_{1,p,w})$ is a reflexive Banach space.
We recall that the dual space of the weighted Sobolev spaces $W_0^{1,p}(\Omega, w)$ is equivalent to $W^{-1,p'}(\Omega, w^*)$, where $w^* = \{w_i^* = w_i^{1-p'}\}, \quad i = 1, ..., N$ and p' is the conjugate of p i.e. $p' = \frac{p}{p-1}$. For more details we refer the reader to [10].
We now introduce the functional spaces we will need later.
For $p \in (1, \infty)$, $\tau_0^{1,p}(\Omega, w)$ is defined as the set of measurable functions $u : \Omega \to \mathbb{R}$ such that for $k > 0$ the truncated functions $T_k(u) \in W_0^{1,p}(\Omega, w)$.
We gives the following lemma this is a generalization of Lemma 2.1 [4] in weighted spaces.

Lemma 2.1: *For every $u \in \tau_0^{1,p}(\Omega, w)$, there exists a unique measurable function $v : \Omega \to \mathbb{R}$ such that*

$$\nabla T_k(u) = v\chi_{\{|v|<k\}}.$$

Lemma 2.2: *Let $\lambda \in \mathbb{R}$ and let u and v be two measurable functions defined on Ω which are finite almost everywhere, and which are such that $T_k(u)$, $T_k(v)$ and $T_k(u + \lambda v)$ belong to $W_0^{1,p}(\Omega, w)$ for every $k > 0$ then*

$$\nabla(u + \lambda v) = \nabla(u) + \lambda\nabla(v) \text{ a.e. in } \Omega$$

where $\nabla(u)$, $\nabla(v)$ and $\nabla(u + \lambda v)$ are the gradients of u, v and $u + \lambda v$ introduced in Lemma 2.1.

The proof of this lemma is similar to the proof of Lemma 2.12 [9] for the non weighted case.

Now, we state the following assumptions.
(H_1)-The expression

$$\|u\|_X = \left(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}}, \tag{2.3}$$

is a norm defined on X and is equivalent to the norm (2.2). (Note that $(X, \|u\|_X)$ is a uniformly convex (and reflexive) Banach space.
-There exist a weight function σ on Ω and a parameter q, $1 < q < \infty$, such that

$$1 < q < p + p' \tag{2.4}$$

and

$$\sigma^{1-q'} \in L^1_{loc}(\Omega). \tag{2.5}$$

with $q' = \frac{q}{q-1}$ and such that the Hardy inequality

$$\left(\int_\Omega |u|^q \sigma(x)\, dx \right)^{\frac{1}{q}} \leq C \left(\sum_{i=1}^{N} \int_\Omega |\frac{\partial u}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}}, \tag{2.6}$$

holds for every $u \in X$ with a constant $C > 0$ independent of u. Moreover, the imbeding

$$X \hookrightarrow L^q(\Omega, \sigma) \tag{2.7}$$

determined by the inequality (2.6) is compact.
Now, we state the following technical lemmas which are needed later.

Lemma 2.3:*[1] Let $g \in L^r(\Omega, \gamma)$ and let $g_n \in L^r(\Omega, \gamma)$, with $\|g_n\|_{\Omega,r} \leq c, 1 < r < \infty$. If $g_n(x) \to g(x)$ a.e. in Ω, then $g_n \rightharpoonup g$ weakly in $L^r(\Omega, \gamma)$.*

Lemma 2.4:*[1]: Assume that (H_1) holds. Let $F : \mathbb{R} \to \mathbb{R}$ be unifomly Lipschitzian, with $F(0) = 0$. Let $u \in W_0^{1,p}(\Omega, w)$. Then $F(u) \in W_0^{1,p}(\Omega, w)$. Moreover, if the set D of discontinuity points of F' is finite, then*

$$\frac{\partial (F \circ u)}{\partial x_i} = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & a.e. \quad in \quad \{x \in \Omega : u(x) \notin D\} \\ 0 & a.e. \quad in \quad \{x \in \Omega : u(x) \in D\}. \end{cases}$$

The previous lemma, we deduce the following.

Lemma 2.5: *[1]: Assume that* (H_1) *holds. Let* $u \in W_0^{1,p}(\Omega, w)$*, and let* $T_k(u), k \in \mathbb{R}^+$*, be the usual truncation then* $T_k(u) \in W_0^{1,p}(\Omega, w)$*. Moreover, we have*

$$T_k(u) \to u \quad \textit{strongly in} \quad W_0^{1,p}(\Omega, w).$$

3 Main results

Let A be a nonlinear operator from $W_0^{1,p}(\Omega, w)$ into its dual $W^{-1,p'}(\Omega, w^*)$ defined as

$$Au = -\mathrm{div}(a(x, u, \nabla u)),$$

where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying the following assumptions:
(H_2)

$$|a_i(x, s, \xi)| \leq w_i^{\frac{1}{p}}(x)[k(x) + \sigma^{\frac{1}{p'}}|s|^{\frac{q}{p'}} + \sum_{j=1}^{N} w_j^{\frac{1}{p'}}(x)|\xi_j|^{p-1}] \quad \text{for} \;\; i = 1, ..., N \tag{3.1}$$

$$[a(x, s, \xi) - a(x, s, \eta)](\xi - \eta) > 0 \quad \text{for all} \;\; \xi \neq \eta \in \mathbb{R}^N, \tag{3.2}$$

$$a(x, s, \xi)\xi \geq \alpha \sum_{i=1}^{N} w_i(x)|\xi_i|^p, \tag{3.3}$$

where $k(x)$ is a positive function in $L^{p'}(\Omega)$ and α is a positive constants.
(H_3) $g(x, s, \xi)$ is a Carathéodory function satisfying

$$g(x, s, \xi).s \geq 0 \tag{3.4}$$

$$|g(x, s, \xi)| \leq b(|s|)(\sum_{i=1}^{N} w_i(x)|\xi_i|^p + c(x)), \tag{3.5}$$

where $b : \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative increasing function and $c(x)$ is a positive function which in $L^1(\Omega)$.
Let

$$K_\psi = \{u \in W_0^{1,p}(\Omega, w) / \;\; u \;\geq \psi \; a.e. \;\text{ in } \;\Omega\},$$

where $\psi : \Omega \to \overline{\mathbb{R}}$ is a measurable function on Ω such that

$$\psi^+ \in W_0^{1,p}(\Omega, w) \cap L^\infty(\Omega). \tag{3.6}$$

Finally, we assume that

$$f \in L^1(\Omega), \tag{3.7}$$

and

$$F \in \Pi_{i=1}^N L^{p'}(\Omega, w_i^{1-p'}). \tag{3.8}$$

We defined, for s and k in $\mathbb{R}$, $k \geq 0$, $T_k(s) = max(-k, min(k, s))$.
For the nonlinear Dirichlet boundary value problem (1.1), we state our main result as follows.

Theorem 3.1:: *Assume that the assumption* $(H_1) - (H_3)$ *and* $(3.6) - (3.8)$ *hold, then, there exists at least one solution of* (1.1) *in the following sense:*

$$(P) \begin{cases} u \geq \psi, \quad g(x, u, \nabla u) \in L^1(\Omega) \\ T_k(u) \in W_0^{1,p}(\Omega, w), \\ \displaystyle\int_\Omega a(x, u, \nabla u)\nabla T_k(u - v)\, dx + \int_\Omega g(x, u, \nabla u) T_k(u - v)\, dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f T_k(u - v)\, dx + \int_\Omega F \nabla T_k(u - v)\, dx \\ \forall\ v \in K_\psi \cap L^\infty(\Omega)\ \forall\ k > 0. \end{cases}$$

Remarks 3.2:

1. We obtain the same results of our theorem if we suppose that the signe condition (3.4) is only near infinity.

2. The statement of Theorem 3.1 generalizes in weighted case the analogous one in [12] and [11].

4 Proof of main results

We recall the following lemma wich play an important rôle in the proof of our main result,

Lemma 4.1:*[1] Assume that (H_1) and (H_2) are satisfied, and let (u_n) be a sequence in $W_0^{1,p}(\Omega, w)$ such that $u_n \rightharpoonup u$ weakly in $W_0^{1,p}(\Omega, w)$ and*

$$\int_\Omega [a(x, u_n, \nabla u_n) - a(x, u_n, \nabla u)]\nabla(u_n - u)\, dx \to 0$$

then, $u_n \to u$ in $W_0^{1,p}(\Omega, w)$.

Proof of Theorem 3.1

For the prove of the existence theorem we proceed by steps.

Step 1. A priori estimates

Let us defined the following sequence of the unilaterals problems

$$\begin{cases} u_n \in K_\psi,\ g(x, u_n, \nabla u_n) \in L^1(\Omega),\ g(x, u_n, \nabla u_n)u_n \in L^1(\Omega) \\ \langle Au_n, u_n - v\rangle + \displaystyle\int_\Omega g(x, u_n, \nabla u_n)(u_n - v)\, dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f_n(u_n - v)\, dx + \int_\Omega F_n \nabla(u_n - v)\, dx, \\ \forall\ v \in K_\psi \cap L^\infty(\Omega). \end{cases} \tag{4.1}$$

where f_n and F_n are a regular functions such that f_n strongly converges to f in $L^1(\Omega)$ and F_n strongly converges to F in $\Pi_{i=1}^N L^{p'}(\Omega, w_i^{1-p'})$. By Theorem 3.1 of [2], there exists at least one solution of (4.1).

Taking $v \in K_\psi$ and choosing $h \geq \|\psi^+\|_\infty$ so as $\tilde{w} = T_h(u_n - T_k(u_n - v)) \in K_\psi \cap L^\infty(\Omega)$. The choice of $\tilde{w}$ as a test function in (4.1) and letting $h \to +\infty$, we obtain

$$(P_n) \begin{cases} \langle Au_n, T_k(u_n - v)\rangle + \displaystyle\int_\Omega g(x, u_n, \nabla u_n)T_k(u_n - v)\, dx \\ \qquad\qquad \leq \displaystyle\int_\Omega f_n T_k(u_n - v)\, dx + \int_\Omega F_n \nabla T_k(u_n - v)\, dx, \\ \forall\ v \in K_\psi. \end{cases}$$

The use of $v = \psi^+$ as test function in (P_n) gives

$$\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - \psi^+)\, dx + \int_\Omega g(x, u_n, \nabla u_n)T_k(u_n - \psi^+)\, dx$$
$$\leq \int_\Omega f_n T_k(u_n - \psi^+)\, dx + \int_\Omega F_n \nabla T_k(u_n - \psi^+)\, dx,$$

since $g(x, u_n, \nabla u_n) T_k(u_n - \psi^+) \geq 0$, then

$$\int_{\{|u_n-\psi^+|\leq k\}} a(x, u_n, \nabla u_n) \nabla T_k(u_n - \psi^+)\, dx \leq Ck + \int_{\{|u_n-\psi^+|\leq k\}} F_n \nabla T_k(u_n - \psi^+)\, dx$$

by Young's inequality and (3.3), one easily has

$$\frac{\alpha}{2} \int_\Omega \sum_{i=1}^{N} |\frac{\partial T_k(u_n)}{\partial x_i}|^p w_i(x)\, dx \leq c_1 k \quad \forall\, k > 1. \tag{4.2}$$

Now, as in [5], we prove that u_n converges to some function u locally in measure (and therefore, we can aloways assume that the convergence is a.e. after passing to a suitable subsequence). We shall show that u_n is a Cauchy sequence in measure in any ball B_R.
Let $k > 0$ large enough, we have

$$\begin{aligned} k\ meas(\{|u_n| > k\} \cap B_R) &= \int_{\{|u_n|>k\}\cap B_R} |T_k(u_n)|\, dx \leq \int_{B_R} |T_k(u_n)|\, dx \\ &\leq \left(\int_\Omega |T_k(u_n)|^q \sigma\, dx \right)^{\frac{1}{q}} \left(\int_{B_R} \sigma^{1-q'}\, dx \right)^{\frac{1}{q'}} \\ &\leq c_R \left(\int_\Omega \sum_{i=1}^{N} |\frac{\partial T_k(u_n)}{\partial x_i}|^p w_i(x)\, dx \right)^{\frac{1}{p}} \\ &\leq c_1 k^{\frac{1}{p}} \end{aligned}$$

which implies

$$meas(\{|u_n| > k\} \cap B_R) \leq \frac{c_1}{k^{1-\frac{1}{p}}} \quad \forall\, k > 1. \tag{4.3}$$

We have, for every $\delta > 0$,

$$\begin{aligned} meas(\{|u_n - u_m| > \delta\} \cap B_R) &\leq meas(\{|u_n| > k\} \cap B_R) \\ +meas(\{|u_m| > k\} \cap B_R) &+ meas\{|T_k(u_n) - T_k(u_m)| > \delta\}. \end{aligned} \tag{4.4}$$

Since $T_k(u_n)$ is bounded in $W_0^{1,p}(\Omega, w)$, there exists some $v_k \in W_0^{1,p}(\Omega, w)$, such that

$$\begin{array}{ll} T_k(u_n) \rightharpoonup v_k & \text{weakly in } W_0^{1,p}(\Omega, w) \\ T_k(u_n) \to v_k & \text{strongly in } L^q(\Omega, \sigma) \text{ and a.e. in } \Omega. \end{array}$$

Consequently, we can assume that $T_k(u_n)$ is a Cauchy sequence in measure in Ω.
Let $\varepsilon > 0$, then, by (4.3) and (4.4), there exists some $k(\varepsilon) > 0$ such that $meas(\{|u_n - u_m| > \delta\} \cap B_R) < \varepsilon$ for all $n, m \geq n_0(k(\varepsilon), \delta, R)$. This proves that (u_n) is a Cauchy sequence in measure in B_R, thus converges almost everywhere to some measurable function u. Then

$$\begin{array}{ll} T_k(u_n) \rightharpoonup T_k(u) & \text{weakly in } W_0^{1,p}(\Omega, w), \\ T_k(u_n) \to T_k(u) & \text{strongly in } L^q(\Omega, \sigma) \text{ and } a.e. \text{ in } \Omega. \end{array}$$

Which implies, by using (3.1), for all $k > 0$ there exists a function $h_k \in \Pi_{i=1}^N L^{p'}(\Omega, w_i)$, such that

$$a(x, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly in } \Pi_{i=1}^N L^{p'}(\Omega, w_i). \tag{4.5}$$

Step 2. Strong convergence of truncation
Let $k > 0$ large enough such that $k > \|\psi^+\|_\infty$, we consider the function $\phi(t) = te^{\gamma t^2}$, with $\gamma > (\frac{b(k)}{2\alpha})^2$ (this function is introduce by [7, 8]). Thanks to Lemma 1 of [8], we have the following inequality

$$\phi'(s) - \frac{b(k)}{\alpha}|\phi(s)| \geq \frac{1}{2} \tag{4.6}$$

hold for all $s \in \mathbb{R}$.
Here, we define $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ where $h > 2k > 0$. For $\eta = \exp(-4\gamma \mathrm{k}^2)$, we define the following function as

$$v_{n,h} = u_n - \eta\phi(w_n). \tag{4.7}$$

The use of $v_{n,h}$ as test function in (P_n), we obtain, for all $l > 0$

$$\begin{aligned} \langle A(u_n), T_l(\eta\phi(w_n))\rangle &+ \int_\Omega g(x, u_n, \nabla u_n) T_l(\eta\phi(w_n))\, dx \\ &\leq \int_\Omega f_n T_l(\eta\phi(w_n))\, dx + \int_\Omega F_n \nabla T_l(\eta\phi(w_n))\, dx, \end{aligned}$$

we take also l large enough, we have

$$\begin{aligned} \langle A(u_n), \phi(w_n)\rangle &+ \int_\Omega g(x, u_n, \nabla u_n)\phi(w_n)\, dx \\ &\leq \int_\Omega f_n \phi(w_n)\, dx + \int_\Omega F_n \nabla\phi(w_n)\, dx. \end{aligned} \tag{4.8}$$

Note that, $\nabla w_n = 0$ on the set where $\{|u_n| > h+4k\}$, therefore, setting $M = 4k + h$, and denoting by $\varepsilon_h^1(n), \varepsilon_h^2(n), \ldots$ various sequences of real numbers which converge to zero as n tends to infinity for any fixed value of h, we get, by (4.8),

$$\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \nabla w_n \phi'(w_n)\, dx + \int_\Omega g(x, u_n, \nabla u_n)\phi(w_n)\, dx \\ \leq \int_\Omega f_n \phi(w_n)\, dx + \int_\Omega F_n \nabla w_n \phi'(w_n)\, dx. \tag{4.9}$$

since $\phi(w_n) g(x, u_n, \nabla u_n) \geq 0$ on the subset $\{x \in \Omega : |u_n(x)| > k\}$, we deduce from (4.9) that

$$\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \nabla w_n \phi'(w_n)\, dx + \int_{\{|u_n| \leq k\}} g(x, u_n, \nabla u_n)\phi(w_n)\, dx \\ \leq \int_\Omega f_n \phi(w_n)\, dx + \int_\Omega F_n \nabla w_n \phi'(w_n)\, dx. \tag{4.10}$$

Splitting the first integral on the left hand side of (4.10) where $|u_n| \leq k$ and $|u_n| > k$, we can write, by using (3.3):

$$\int_\Omega a(x, T_M(u_n), \nabla T_M(u_n)) \nabla w_n \phi'(w_n)\, dx \\ \geq \int_\Omega a(x, T_k(u_n), \nabla T_k(u_n))[\nabla T_k(u_n) - \nabla T_k(u)]\phi'(w_n)\, dx \\ -C_k \int_{\{|u_n| > k\}} |a(x, T_M(u_n), \nabla T_M(u_n))||\nabla T_k(u)|\, dx, \tag{4.11}$$

where $C_k = \phi'(2k)$. Since, when n tends to infinity, we have for all $i = 1, \ldots, N$, $\frac{\partial (T_k(u))}{\partial x_i} \chi_{\{|u_n| > k\}}$ tends to 0 strongly in $L^p(\Omega, w_i)$ while, $(a_i(x, T_M(u_n), \nabla T_M(u_n)))_n$ is bounded in $L^{p'}(\Omega, w_i^{1-p'})$ hence the last term in the previous inequality tends to zero for every h fixed as n tends to infinity.

Now, observe that

$$\begin{aligned}\int_\Omega & a(x,T_k(u_n),\nabla T_k(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)\,dx\\ &= \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)\,dx\\ &+\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)\,dx.\end{aligned} \tag{4.12}$$

By the continuity of the Nymetskii operator, we have for all $i=1,...,N$

$$a_i(x,T_k(u_n),\nabla T_k(u))\phi'(w_n))\to a_i(x,T_k(u),\nabla T_k(u))\phi'(T_{2k}(u-T_h(u))$$

strongly in $L^{p'}(\Omega,w_i^{1-p'})$ and since $\frac{\partial(T_k(u_n))}{\partial x_i}\rightharpoonup\frac{\partial(T_k(u))}{\partial x_i}$ weakly in $L^p(\Omega,w_i)$, the second term of the right hand side of (4.12) tends to 0 as $n\to\infty$.
So that (4.11) yields

$$\begin{aligned}\int_\Omega & a(x,T_M(u_n),\nabla T_M(u_n))[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)\,dx\\ &\geq \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))]\\ &\qquad\times[\nabla T_k(u_n)-\nabla T_k(u)]\phi'(w_n)\,dx+\varepsilon_h^2(n).\end{aligned} \tag{4.13}$$

For the second term of the left hand side of (4.10), we can estimate as follows

$$\begin{aligned}\int_{\{|u_n|\leq k\}} & g(x,u_n,\nabla u_n)\phi(w_n)\,dx\\ &\leq \int_{\{|u_n|\leq k\}} b(k)(c(x)+\sum_{i=1}^N w_i|\frac{\partial T_k(u_n)}{\partial x_i}|^p|\phi(w_n)|\,dx\\ &\leq b(k)\int_\Omega c(x)|\phi(w_n)|\,dx\\ &+\tfrac{b(k)}{\alpha}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(w_n)|\,dx,\end{aligned} \tag{4.14}$$

remark that, we have

$$\begin{aligned}\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n)|\phi(w_n)|\,dx \\ = \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ \times[\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|\,dx \\ +\int_\Omega a(x,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u)|\phi(w_n)|\,dx \\ +\int_\Omega a(x,T_k(u_n),\nabla T_k(u))[\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|\,dx.\end{aligned} \tag{4.15}$$

By the Lebesgue's Theorem, we have

$$\nabla T_k(u)|\phi(w_n)| \to \nabla T_k(u)|\phi(T_{2k}(u - T_h(u)))| \text{ strongly in } \Pi_{i=1}^N L^p(\Omega, w_i).$$

Moreover, in view of (4.5) the second term of the right hand side of (4.15) tends to

$$\int_\Omega h_k \nabla T_k(u)|\phi(T_{2k}(u - T_h(u)))|\,dx.$$

The third term of the right hand side of (4.15) tends to 0 since for all $i = 1, ..., N$
$a_i(x, T_k(u_n), \nabla T_k(u))|\phi(w_n)| \to a_i(x, T_k(u), \nabla T_k(u))|\phi(T_{2k}(u - T_h(u)))|$ strongly in $L^{p'}(\Omega, w_i^{1-p'})$, while

$$\frac{\partial(T_k(u_n))}{\partial x_i} \rightharpoonup \frac{\partial(T_k(u))}{\partial x_i} \text{ weakly in } L^p(\Omega, w_i).$$

From (4.14) and (4.15), we obtain

$$\begin{aligned}\int_{\{|u_n|\le k\}} g(x,u_n,\nabla u_n)\phi(w_n)\,dx \\ \le \int_\Omega [a(x,T_k(u_n),\nabla T_k(u_n)) - a(x,T_k(u_n),\nabla T_k(u))] \\ \times[\nabla T_k(u_n) - \nabla T_k(u)]|\phi(w_n)|\,dx \\ +\varepsilon_h^3(n) + \int_\Omega h_k\nabla T_k(u)|\phi(T_{2k}(u - T_h(u)))|\,dx \\ +b(k)\int_\Omega c(x)|\phi(w_n)|\,dx.\end{aligned} \tag{4.16}$$

Now, by the strongly convergence of f_n and F_n and in fact that

$$w_n \rightharpoonup T_{2k}(u - T_h(u)) \text{ weakly in } W_0^{1,p}(\Omega, w) \text{ and weakly-}* \text{ in } L^\infty(\Omega), \tag{4.17}$$

moreover, combining (4.13) and (4.16), we conclude that

$$\begin{aligned}
\int_\Omega &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \\
&\qquad\qquad \times [\nabla T_k(u_n) - \nabla T_k(u)](\phi'(w_n) - \tfrac{b(k)}{\alpha}|\phi(w_n)|)\, dx \\
&\le \int_\Omega h_k \nabla T_k(u) |\phi(T_{2k}(u - T_h(u)))|\, dx + \varepsilon_h^5(n) \\
&\quad + b(k) \int_\Omega c(x) \phi(T_{2k}(u - T_h(u)))\, dx + \int_\Omega f \phi(T_{2k}(u - T_h(u)))\, dx \\
&\qquad + \int_\Omega F \nabla T_{2k}(u - T_h(u)) \phi'(T_{2k}(u - T_h(u)))\, dx.
\end{aligned}$$

which and using (4.6), implies that

$$\begin{aligned}
\int_\Omega &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx \\
&\le 2 \int_\Omega h_k \nabla T_k(u) |\phi(T_{2k}(u - T_h(u)))|\, dx + \varepsilon_h^5(n) \\
&\quad + 2b(k) \int_\Omega c(x) \phi(T_{2k}(u - T_h(u)))\, dx + 2 \int_\Omega f \phi(T_{2k}(u - T_h(u)))\, dx \\
&\qquad + 2 \int_\Omega F \nabla T_{2k}(u - T_h(u)) \phi'(T_{2k}(u - T_h(u)))\, dx.
\end{aligned}$$

Hence, passing to the limit over n, we get

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega &[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))][\nabla T_k(u_n) - \nabla T_k(u)]\, dx \\
&\le 2 \int_\Omega h_k \nabla T_k(u) |\phi(T_{2k}(u - T_h(u)))|\, dx \\
&\quad + 2b(k) \int_\Omega c(x) \phi(T_{2k}(u - T_h(u)))\, dx + 2 \int_\Omega f \phi(T_{2k}(u - T_h(u)))\, dx \\
&\qquad + 2 \int_\Omega F \nabla T_{2k}(u - T_h(u)) \phi'(T_{2k}(u - T_h(u)))\, dx.
\end{aligned} \tag{4.18}$$

It remains to show, for our purposes, that the all term on the right hand side of (4.18) converge to zero as h goes to infinity. The only difficulty that exists is in the last term. For the other terms it suffices to apply Lebesgue's

theorem.
We deal now with this term. Let us observe that, if we take $u_n-\eta\phi(T_{2k}(u_n-T_h(u_n)))$ as test function in (P_n), we obtain by using (3.3):

$$\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega g(x,u_n,\nabla u_n)\phi(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_{\{h\leq|u_n|\leq 2k+h\}}F_n\nabla u_n\phi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx.$$

Since $g(x,u_n,\nabla u_n)\phi(T_{2k}(u_n-T_h(u_n)))\geq 0$, We have

$$\alpha\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_{\{h\leq|u_n|\leq 2k+h\}}F_n\nabla u_n\phi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$+\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx,$$

which yields, thanks to Young's inequalities

$$\tfrac{\alpha}{2}\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u_n}{\partial x_i}|^p w_i\phi'(T_{2k}(u_n-T_h(u_n)))\,dx$$
$$\leq\int_\Omega f_n\phi(T_{2k}(u_n-T_h(u_n)))\,dx+c_k\int_{\{h\leq|u_n|\}}|w^{\frac{-1}{p}}F_n|^{p'}\,dx,$$

consequently, thanks to the strong convergence of $|w^{\frac{-1}{p}}F_n|^{p'}$ and f_n, in $L^1(\Omega)$ and letting firstly $n\to\infty$ after h tend to infinity, we obtain

$$\limsup_{h\to\infty}\int_{\{h\leq|u_n|\leq 2k+h\}}\sum_{i=1}^{N}|\frac{\partial u}{\partial x_i}|^p w_i\phi'(T_{2k}(u-T_h(u)))\,dx=0,$$

so that

$$\lim_{h\to\infty}\int_\Omega F\nabla T_{2k}(u-T_h(u))\phi'(T_{2k}(u-T_h(u)))\,dx=0.$$

Therefore by (4.18), letting h go to infinity, we conclude,

$$\lim_{n\to\infty}\int_\Omega \begin{array}{l}[a(x,T_k(u_n),\nabla T_k(u_n))-a(x,T_k(u_n),\nabla T_k(u))] \\ \cdot[\nabla T_k(u_n)-\nabla T_k(u)]\,dx=0,\end{array}$$

which implies that by using Lemma 4.1

$$T_k(u_n)\to T_k(u)\ \text{ strongly in }\ W_0^{1,p}(\Omega,w)\ \ \forall\ k>0. \tag{4.19}$$

Step 3. Passing to the limit
We take $v\in K_\psi\cap L^\infty(\Omega)$ as test function in (P_n), we can write

$$\begin{aligned}\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx& \\ +\int_\Omega g(x,u_n,\nabla u_n)T_k(u_n-v)\,dx& \\ \leq\int_\Omega f_nT_k(u_n-v)\,dx+\int_\Omega F_n\nabla T_k(u_n-v)\,dx.&\end{aligned} \tag{4.20}$$

By Fatou's lemma and in fact that

$$a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\rightharpoonup a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))$$

weakly in $\Pi_{i=1}^N L^{p'}(\Omega,w_i^{1-p'})$. It is easily see that

$$\begin{aligned}\int_\Omega a(x,T_{k+\|v\|_\infty}(u),\nabla T_{k+\|v\|_\infty}(u))\nabla T_k(u-v)\,dx& \\ \leq\liminf_{n\to\infty}\int_\Omega a(x,T_{k+\|v\|_\infty}(u_n),\nabla T_{k+\|v\|_\infty}(u_n))\nabla T_k(u_n-v)\,dx.&\end{aligned} \tag{4.21}$$

On the other hand, by using the strong convergence of F_n and

$$\nabla T_k(u_n-v)\rightharpoonup\nabla T_k(u-v)\ \text{ weakly in }\ \Pi_{i=1}^N L^p(\Omega,w_i),$$

we deduce that the integral

$$\int_\Omega F_n\nabla T_k(u_n-v)\,dx\to\int_\Omega F\nabla T_k(u-v)\,dx\ \text{ as }\ n\to\infty.$$

Now, we need to prove that

$$g(x,u_n,\nabla u_n)\to g(x,u,\nabla u)\ \text{ strongly in }\ L^1(\Omega), \tag{4.22}$$

in particular it is enough to prove the equiintegrable of $g(x, u_n, \nabla u_n)$. To this purpose, we take $T_{l+1}(u_n) - T_l(u_n)$ as test function in (P_n), we obtain

$$\int_{\{|u_n|>l+1\}} |g(x, u_n, \nabla u_n)|\, dx = \int_{\{|u_n|>l\}} |f_n|\, dx.$$

Let $\varepsilon > 0$. Then there exists $l(\varepsilon) \geq 1$ such that

$$\int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx < \varepsilon/2. \tag{4.23}$$

For any measurable subset $E \subset \Omega$, we have

$$\begin{aligned}\int_E |g(x, u_n, \nabla u_n)|\, dx \ \leq\ & \int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^N w_i |\frac{\partial (T_{l(\varepsilon)}(u_n))}{\partial x_i}|^p \right) dx \\ & + \int_{\{|u_n|>l(\varepsilon)\}} |g(x, u_n, \nabla u_n)|\, dx.\end{aligned}$$

In view of (4.19), there exists $\eta(\varepsilon) > 0$ such that

$$\int_E b(l(\varepsilon)) \left(c(x) + \sum_{i=1}^N w_i |\tfrac{\partial (T_{l(\varepsilon)}(u_n))}{\partial x_i}|^p \right) dx < \varepsilon/2 \tag{4.24}$$

for all E such that $|E| < \eta(\varepsilon)$.
Finally, by combining (4.23) and (4.24) one easily has

$$\int_E |g(x, u_n, \nabla u_n)|\, dx < \varepsilon \ \text{ for all } E \text{ such that } |E| < \eta(\varepsilon),$$

which allows us, by using (4.21) and (4.22), we can pass to the limit in (4.20). This completes the proof of Theorem 3.1.

Remark 4.2: Note that, we obtain the existence result withowt assuming the coercivity condition. However one can overcome this difficulty by introduced the function $w_n = T_{2k}(u_n - T_h(u_n) + T_k(u_n) - T_k(u))$ in the test function (4.7).

Corollary 4.3: *Let $1 < p < \infty$. Assume that the hypothesis $(H_1) - (H_3)$, (3.6) and (3.7) holds, let f_n any sequence of function in $L^1(\Omega)$ converge to f weakly in $L^1(\Omega)$ and let u_n the solution of the following unilateral* *problem*

$$(P'_n)\begin{cases} u_n \geq \psi \ a.e. \ \ in \ \ \Omega. \\ T_k(u_n) \in W_0^{1,p}(\Omega, w), \quad g(x, u_n, \nabla u_n) \in L^1(\Omega) \\ \displaystyle\int_\Omega a(x, u_n, \nabla u_n)\nabla T_k(u_n - v)\, dx + \int_\Omega g(x, u_n, \nabla u_n) T_k(u_n - v)\, dx \\ \qquad\qquad\qquad \leq \displaystyle\int_\Omega f_n T_k(u_n - v)\, dx, \\ \forall\ v \in K_\psi \cap L^\infty(\Omega),\ \forall\ k > 0. \end{cases}$$

Then, there exists a subsequence of u_n still denoted u_n such that u_n converges to u almost everywhere and $T_k(u_n) \rightharpoonup T_k(u)$ weakly in $W_0^{1,p}(\Omega, w)$, further u is a solution of the unilateral problem (P) (with $F = 0$).

Proof. We give the proof brievely.

Step 1. A priori estimates

We proceed as previous, we take $v = \psi^+$ as test function in (P'_n), we get

$$\int_\Omega \sum_{i=1}^N w_i \left|\frac{\partial T_k(u_n)}{\partial x_i}\right|^p dx \leq C_1. \tag{4.25}$$

Hence, by the same method used in the first step in the proof of Theorem 3.1 there exists a function u (with $T_k(u) \in W_0^{1,p}(\Omega, w)\ \forall\ k > 0$) and a subsequence still denoted by u_n such that

$$T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ W_0^{1,p}(\Omega, w),\ \forall\ k > 0.$$

Step 2. Strong convergence of truncation

The choice of $v = T_h(u_n - \eta\phi(w_n)),\ h > \|\psi^+\|_\infty$ as test function in (P'_n), we get, $\forall\ l > 0$

$$\begin{aligned} &\int_\Omega a(x, u_n, \nabla u_n)\nabla T_l(u_n - T_h(u_n - \eta\phi(w_n))) \\ &\qquad\qquad + \int_\Omega g(x, u_n, \nabla u_n) T_l(u_n - T_h(u_n - \eta\phi(w_n))\, dx \\ &\qquad\qquad \leq \int_\Omega f_n T_l(u_n - T_h(u_n - \eta\phi(w_n)))\, dx. \end{aligned}$$

Which implies that

$$\int_{\{|u_n-\eta\phi(w_n)|\leq h\}} a(x,u_n,\nabla u_n)\nabla T_l(\eta\phi(w_n))$$
$$+\int_\Omega g(x,u_n,\nabla u_n)T_l(u_n-T_h(u_n-\eta\phi(w_n)))\,dx$$
$$\leq \int_\Omega f_nT_l(u_n-T_h(u_n-\eta\phi(w_n)))\,dx.$$

Letting h tends to infinity and choosing l large enough, we deduce

$$\int_\Omega a(x,u_n,\nabla u_n)\nabla\phi(w_n)+\int_\Omega g(x,u_n,\nabla u_n)\phi(w_n)\,dx\leq\int_\Omega f_n\phi(w_n)\,dx,$$

the rest of the proof of this step is the same as in step 2 of the proof of Theorem 3.1.

Step 3. Passing to the limit

This step is similarly to the step 3 of the proof of Theorem 3.1, by using the Egorov's theorem in the last term of (P_n').

Remark 4.4: In the case where $F=0$, if we suppose that the second mumber are nonnegative, then we obtain a nonnegative solution.

Indeed. If we take $v=T_h(u^+)$ (with $h\geq\|\psi\|_\infty$) in (P), we have

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(u^+))\,dx+\int_\Omega g(x,u,\nabla u)T_k(u-T_h(u^+))\,dx$$
$$\leq\int_\Omega fT_k(u-T_h(u^+))\,dx.$$

Since $g(x,u,\nabla u)T_k(u-T_h(u^+))\geq 0$, we deduce

$$\int_\Omega a(x,u,\nabla u)\nabla T_k(u-T_h(u^+))\,dx\leq\int_\Omega fT_k(u-T_h(u^+))\,dx,$$

we remark also, using $f\geq 0$

$$\int_\Omega fT_k(u-T_h(u^+))\,dx\leq\int_{\{u\geq h\}} fT_k(u-T_h(u))\,dx.$$

On the other hand, by using (3.3), we conclude

$$\alpha\int_\Omega\sum_{i=1}^N w_i\left|\frac{\partial T_k(u^-)}{\partial x_i}\right|^p dx\leq\int_{\{u\geq h\}} fT_k(u-T_h(u))\,dx.$$

Letting h tend to infinity, we can easily conclude that, $T_k(u^-)=0,\ \forall\ k>0$, which implies $u\geq 0$.

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Lahsen Aharouch
Faculté des Sciences
Dhar-Mahraz
Dép. de Math. et Informatique
B.P 1796 Atlas Fès.
Fès
MAROC
lahrouche@caramail.com

Youssef Akdim
Faculté des Sciences
Dhar-Mahraz
Dép. de Math. et Informatique
B.P 1796 Atlas Fès.
Fès
MAROC
akdimyoussef@yahoo.fr