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Representation of a gauge group as motions of a Hilbert space

Clara Lucía Aldana Domínguez

Abstract

This is a survey article based on the author's Master thesis[1] on affine representations of a gauge group. Most of the results presented here are well-known to differential geometers and physicists familiar with gauge theory. However, we hope this short systematic presentation offers a useful self-contained introduction to the subject.
In the first part we present the construction of the group of motions of a Hilbert space and we explain the way in which it can be considered as a Lie group. The second part is about the definition of the gauge group, $\mathfrak{G}_P$, associated to a principal bundle, P. In the third part we present the construction of the Hilbert space where the representation takes place. Finally, in the fourth part, we show the construction of the representation and the way in which this representation goes to the set of connections associated to P.

1 Group of Motions of a Hilbert Space

In this section we want to study the group of motions of a Hilbert space $\mathcal{H}$ considering it as a Banach Lie group. A general account of the theory of infinite dimensional Lie groups and their structure can be found in [13], [14], [16], [20] and [21]. The group of motions of a Hilbert space is defined as the semi-direct product of the group of translations of $\mathcal{H}$ and the Hilbert group $\mathrm{Hilb}(\mathcal{H})$ associated to $\mathcal{H}$, (if $\mathcal{H}$ is real $\mathrm{Hilb}(\mathcal{H})$ is the orthogonal group $O(\mathcal{H})$, if $\mathcal{H}$ is complex $\mathrm{Hilb}(\mathcal{H})$ is the unitary group $U(\mathcal{H})$).

Let $\mathcal{H}$ be a Hilbert space over $\mathbb{R}$ or $\mathbb{C}$. We consider $L(\mathcal{H},\mathcal{H})$, the set of all bounded linear operators defined on $\mathcal{H}$. It is well known that $L(\mathcal{H},\mathcal{H})$

[1] 'Group of Motions of a Hilbert Space and Unitary Representations of Gauge Groups'. Universidad de los Andes. Work partially supported by Colciencias NSF Grant 1101-05-11-445.

is a Banach Lie algebra with norm and Lie bracket given by the following equations:

$$\|A\| = \sup_{\|x\|=1} \|Ax\|$$

$$[A, B] = AB - BA$$

with $A, B \in L(\mathcal{H}, \mathcal{H})$ and $x \in \mathcal{H}$. Let $\mathrm{GL}(\mathcal{H})$ be the set of all invertible elements of $L(\mathcal{H}, \mathcal{H})$, from elementary functional analysis we know that it is a group. Since multiplication and inversion operations are smooth functions in $\mathrm{GL}(\mathcal{H})$ we have that it is a Banach Lie group, its exponential function $\exp : L(\mathcal{H}, \mathcal{H}) \to \mathrm{GL}(\mathcal{H})$ is given by the series

$$\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

that converges for all $A \in L(\mathcal{H}, \mathcal{H})$. Furthermore if $A \in L(\mathcal{H}, \mathcal{H})$ is close to the identity I, the series

$$\log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(A - I)^n}{n}$$

converges, see [15], chapter VII, section 2.

If $A \in \mathrm{Sk}(\mathcal{H})$, $\mathrm{Sk}(\mathcal{H})$ the skew-symmetric algebra of $\mathcal{H}$, then $\exp(A) \in \mathrm{Hilb}(\mathcal{H})$ and exp is an analytic diffeomorphism of the open unit ball of $0 \in \mathrm{Sk}(\mathcal{H})$ onto the open unit ball at $I \in \mathrm{Hilb}(\mathcal{H})$. Since $\mathrm{Hilb}(\mathcal{H})$ is a closed submanifold of $\mathrm{GL}(\mathcal{H})$, the last results allow us to consider $U(\mathcal{H})$ as a Banach Lie Group with associated Banach Lie algebra $\mathrm{Sk}(\mathcal{H})$.

Semi-direct Product of Lie Groups and its Lie Algebra. The general construction of the semi-direct product of two Lie groups and its corresponding Lie algebra can be found in [5], [11] and [17], in the infinite dimensional case of regular Lie groups it can be found in [13], section 38.9. Here we consider the semi-direct product in the particular case when $H = (\mathcal{H}, +)$, the additive group of the Hilbert space $\mathcal{H}$, and $G = (\mathrm{GL}(\mathcal{H}), \circ)$, where '$\circ$' denotes the composition of linear operators. Let $\eta : GL(\mathcal{H}) \to \mathrm{Aut}(\mathcal{H})$, $A \mapsto A$, where $\mathrm{Aut}(\mathcal{H})$ is the group of automorphism of $\mathcal{H}$ as an additive group. η can be thought of as an action $\eta : GL(\mathcal{H}) \times \mathcal{H} \to \mathcal{H}$, $(A, x) \mapsto Ax$. From the definition of the norm we have that

$$\|Ax\| \leq \|A\|\|x\|$$

for any $A \in L(\mathcal{H},\mathcal{H})$ and for all $x \in \mathcal{H}$, thus η is continuous and the semi-direct product

$$\mathrm{Aff}(\mathcal{H}) := \mathcal{H} \times_\eta \mathrm{GL}(\mathcal{H}) = \mathcal{H} \rtimes \mathrm{GL}(\mathcal{H})$$

is a Lie group. It is called the affine group, its elements are the functions $(x, A) : \mathcal{H} \to \mathcal{H}$ that are composition of invertible linear maps and translations, if $w \in \mathcal{H}$

$$(x, A)w = Aw + x.$$

The group operation in $\mathrm{Aff}(\mathcal{H})$ is given by

$$(x, A)(y, B) = (x + Ay, AB) \in \mathrm{Aff}(\mathcal{H}).$$

and it corresponds to the well known composition of affine transformations.

Definition 1.1: The group of motions $M(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is defined as the semi-direct product $\mathcal{H} \rtimes \mathrm{Hilb}(\mathcal{H})$, where $\mathrm{Hilb}(\mathcal{H})$ is the Hilbert group of $\mathcal{H}$.

$M(\mathcal{H})$ is a subgroup of the affine group, it is called the group of motions of $\mathcal{H}$ because its elements can be interpreted as the physical motions of rigid objects lying in $\mathcal{H}$; if there is a rigid object in $\mathcal{H}$ the classical way in which it can move is by rotations, translations and compositions of these two; these are precisely the elements of $M(\mathcal{H})$. Every element of $M(\mathcal{H})$ is an isometry of $\mathcal{H}$, but it is important to note that the whole set of isometries is not $M(\mathcal{H})$.

In the following part we describe the Lie algebra associated to $\mathrm{Aff}(\mathcal{H})$. In order to do that we identify $\mathcal{H}$ with its Lie algebra, $\mathfrak{h} = T_0(\mathcal{H})$, since $\mathcal{H}$ is an additive commutative Lie group its Lie algebra has a trivial Lie bracket, see [22], chapter 1, section 2.

In the general case, let G and H be Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$, respectively and let $e \in H$ denote the neutral element in H. The semi-direct product of G and H is determined by an action $\eta : G \to \mathrm{Aut}(H)$. Let $g \in G$, the derivative of $\eta(g) : H \to H$ at $e \in H$, $T\eta(g)_e : T_e(H) \to T_e(H)$, is an automorphism of $\mathfrak{h}$. If we denote $T\eta(g)_e$ by τ_g, we have that $\tau : G \to \mathrm{Aut}(\mathfrak{h})$, $g \mapsto \tau_g$ satisfies

$$\tau_{g_1 g_2} = \tau_{g_1} \circ \tau_{g_2}.$$

In this way, τ is a homomorphism of groups, where $\mathrm{Aut}(\mathfrak{h})$ is considered as:

$$\mathrm{Aut}(\mathfrak{h}) = \{A \in \mathrm{GL}(\mathfrak{h}) | A([Y_1, Y_2]) = [A(Y_1), A(Y_2)], \text{ for } Y_1, Y_2 \in \mathfrak{h}\}.$$

$\mathrm{Aut}(\mathfrak{h})$ is a group with the usual composition of linear operators. Therefore we have a representation of the group G on the vector space $\mathfrak{h}$. To this representation there corresponds a linear representation of the Lie algebra $\mathfrak{g}$ in $\mathfrak{h}$ given by the formula:

$$\tilde{\tau}_X(Y) = \frac{d}{dt}\, \tau_{\exp(tX)}(Y)\bigg|_{t=0} \tag{1.1}$$

with $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}$. Since $\tilde{\tau}_X$ satisfies

$$\begin{aligned} \tilde{\tau}_X([Y_1, Y_2]) &= [\tilde{\tau}_X(Y_1), Y_2] + [Y_1, \tilde{\tau}_X(Y_2)] && (1.2)\\ \tilde{\tau}_{[X_1,X_2]} &= \tilde{\tau}_{X_1} \circ \tilde{\tau}_{X_2} - \tilde{\tau}_{X_2} \circ \tilde{\tau}_{X_1} && (1.3) \end{aligned}$$

for $X, X_1, X_2 \in \mathfrak{g}$ and for $Y_1, Y_2 \in \mathfrak{h}$, we have that $\tilde{\tau}_X$ is a derivation. Thus $\tilde{\tau} : \mathfrak{g} \to \mathrm{Der}\,\mathfrak{h}$, where $\mathrm{Der}\,\mathfrak{h}$ is the set of all derivations of $\mathfrak{h}$, is a Lie algebra homomorphism.
The semi-direct product of Lie algebras can be defined in the following way. Let $\mathfrak{g}$ and $\mathfrak{h}$ be Lie algebras, and let $\pi : \mathfrak{g} \to \mathrm{Der}\,\mathfrak{h}$, be a Lie algebra homomorphism, the vector space $\mathfrak{h} \times \mathfrak{g}$ becomes a Lie algebra with structure given by the equation:

$$[(Y_1, X_1), (Y_2, X_2)] = ([Y_1, Y_2] + \pi(X_1)(Y_2) - \pi(X_2)(Y_1), [X_1, X_2]). \tag{1.4}$$

This Lie algebra is called the semi-direct product of the Lie algebras $\mathfrak{h}$ and $\mathfrak{g}$ by the homomorphism π and it is denoted by $\mathfrak{h} \times_\pi \mathfrak{g}$.

Proposition 1.2: *With the notation introduced above we have that the Lie algebra given by $\mathfrak{h} \times_{\tilde{\tau}} \mathfrak{g}$ is the Lie algebra associated to the Lie group $H \times_\eta G$. See [17], appendix 5, section 5, and [11], chapter I, section 12.*

In our particular case equations (1.2) and (1.3) take the form

$$A([y_1, y_2]) = [A(y_1), y_2] + [y_1, A(y_2)] = 0 + 0 = 0$$

$$[A_1, A_2](y) = A_1(A_2 y) - A_2(A_1 y),$$

for $y, y_1, y_2 \in \mathcal{H}$ and $A, A_1, A_2 \in L(\mathcal{H}, \mathcal{H})$. The Lie bracket in $\mathcal{H} \times L(\mathcal{H}, \mathcal{H})$, given by equation (1.4), is now

$$[(y_1, A_1), (y_2, A_2)] = (A_1(y_2) - A_2(y_1), [A_1, A_2]).$$

Thus, the Lie algebra associated to $\mathrm{Aff}(\mathcal{H})$ is $\mathcal{H} \rtimes L(\mathcal{H},\mathcal{H}) := L(\mathcal{H}.\mathcal{H}) \times_{\tilde{\tau}} \mathcal{H}$. In $\mathcal{H} \times L(\mathcal{H},\mathcal{H})$ we can also define a norm by

$$\|(x, A)\| = \sqrt{\|x\|^2 + \|A\|^2}.$$

With this norm $\mathcal{H} \times L(\mathcal{H},\mathcal{H})$ is a Banach space and $\mathcal{H} \rtimes L(\mathcal{H},\mathcal{H})$ is a Banach Lie Algebra. The group of motions $M(\mathcal{H})$ is a Lie group and its Lie algebra is given by $\mathrm{Sk}(\mathcal{H}) \times_{\tilde{\tau}} \mathcal{H}$.

2 The gauge group associated to a principal bundle

The theory of gauge groups has been widely studied and there are many references related to this topic, see for example [3], pages 539 and 546, [4], section 5.6, [13], chapter IX section 44, and [18], section 5. In this section we consider M a compact Riemannian manifold without boundary and G a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$.

2.1 Principal Bundles and Connections

Let M be a manifold and G be a Lie group with identity e. We consider here only the finite dimensional case. The theory presented here can be found, among others, in [12], sections II.5 and III.1, [4], chapter 4, [18], sections 2 and 3, and [19], chapter 3.

Definition 2.1: A differentiable principal fiber bundle over M with group G consists of a differentiable manifold P and a right action Θ of G on P that satisfy the following conditions:

1. The action of G on P is free, that is $\Theta(p, g) = p$ for some $p \in P$, implies that $g = e$.

2. M is the quotient space P/G and $\pi : P \to M$ is differentiable.

3. P is locally trivial. This is, for each $x \in M$ there exist U a neighborhood of x, $U \in \mathcal{V}_x$, and $\psi : \pi^{-1}(U) \to U \times G$ such that $\psi(p) = (\pi(p), \varphi(p))$ where $\varphi : \pi^{-1}(U) \to G$ satisfies $\varphi(p \cdot a) = \varphi(p) \cdot a$, for all $p \in \pi^{-1}(U)$, and $a \in G$.

A principal bundle is denoted by $P(M, G, \pi)$ or just by $P(M, G)$ or P. The manifold P is called the total space, M is the base space, G is the structural group and π is the projection. The fiber at $x \in M$, $\pi^{-1}(x)$, is denoted by P_x.

Example 2.2: The bundle of orthonormal linear frames on $\mathbb{R}^2$. We consider the set:

$$P = \{u_x | u_x \text{ is an orthonormal frame in } T_x(\mathbb{R}^2), x \in \mathbb{R}^2\},$$

and the projection $\pi : P \to \mathbb{R}^2$, $u_x \mapsto x$. $O(2)$ acts on the right of P as follows: if $u_x \in P$, $u_x = (v_1, v_2)$ with $v_1, v_2 \in \mathbb{R}^2$, let $A \in O(2)$, the action is given by

$$u_x \cdot A = (v_1\ v_2)A,$$

which is just the usual matrix multiplication. It is clear that $O(2)$ acts freely on P, if $u_x A = u_x$ then taking $u_x = (v_1, v_2)$ we have $(v_1\ v_2)A = (v_1\ v_2)$, then $(v_1\ v_2)^{-1}(v_1\ v_2)A = I$, $A = I$. The action is also transitive because given $u_1, u_2 \in P$, we take $A = (u_2^{-1})u_1$ and we have that $u_1 = u_2 A$. Note that $\mathbb{R}^2 = P/O(2)$. Then we have that $P(\mathbb{R}^2, O(2))$ is a principal bundle. If we take $U = B_1(x) \subseteq \mathbb{R}^2$ the trivializations are given by:

$$\psi : \pi^{-1}(U) \to U \times O(2), u_y \mapsto (y, (v_1\ v_2))$$

where $u_y = (v_1\ v_2)$. P is a manifold of dimension 3.

Let $x \in M$, $u \in P$, such that $\pi(u) = x$, $\pi_{*u} : T_u(P) \to T_x(M)$, and we define $V_u = \ker(\pi_{*u}) \subset T_u(P)$.

The Lie algebra $\mathfrak{g}$ can be considered either as the tangent space to G at the identity e or as the space of left invariant vector fields over G. If we consider $\mathfrak{g}$ as the space of left invariant vector fields, then $\exp(tX)$ is the one-parametric group associated to the left invariant vector field X, that is, $\exp(0) = e$ and $\frac{d}{dt}\exp(tX)\big|_{t=0} = X$.

Definition 2.3: For $A \in \mathfrak{g}$ and $p \in P$ we define the fundamental vector field over P corresponding to A as:

$$A^*(p) = \frac{d}{dt}\, p \cdot \exp(tA)\bigg|_{t=0}.$$

The application $\sigma : \mathfrak{g} \to \mathcal{X}(P), A \mapsto \sigma(A) = A^*$ is a homomorphism of Lie algebras. If $A \in \mathfrak{g}$, A^*_p is tangent to the fibre in each $p \in P$, that is, $\pi_*(A^*) = 0$. If A is not zero then A^* is also different from zero. Thus, we have that the function $\sigma : \mathfrak{g} \to V_u, A \mapsto A^*_u$ is a linear isomorphism that satisfies $(R_a)_* A^* = (\mathrm{Ad}(a^{-1})A)^*$.

When a principal bundle is given it carries a notion of verticality in its tangent space but there is not an intrinsic notion of horizontality associated to it. To give a connection on P is to give a horizontal subspace of the tangent space in each point of P.

Definition 2.4: A connection Γ on P is an assignment of a subspace H_u of $T_u(P)$ to each $u \in P$ such that

1. $T_u(P) = V_u \oplus H_u$ (direct sum).
2. $H_{u\cdot a} = (R_a)_* H_u$, for each $a \in G$.
3. H_u depends differentiably on u.

Definition 2.5: A connection form over P is a $\mathfrak{g}$-valued 1-form over P, $\omega : T(P) \to \mathfrak{g}$, that satisfies:

1. $(R_a)^*\omega = \mathrm{Ad}(a^{-1})\omega$, for each, ($\omega$ is equivariant), and
2. $\omega(A^*) = A$, for each $A \in \mathfrak{g}$.

It is well known that to have a connection on P is equivalent to have a connection form on P. See [12], chapter II, section 1. The correspondence between a connection Γ and its connection form ω is given as follows: Given Γ a connection on P, for each $u \in P$, $X \in T_u(P)$, $\omega(X) = A$, where A is the unique vector in $\mathfrak{g}$ such that $A^*_u = X^v$, X^v the vertical component of X. Conversely, given a connection form the distribution of horizontal spaces is defined by $H_u = \ker \omega_u \subseteq T_u(P)$.

Example 2.6: Let $P = \mathbb{R} \times S^1$ be a trivial principal bundle over $\mathbb{R}$ with fibre and group S^1, S^1 acts on itself by the usual multiplication. The Lie algebra associated to S^1 is $i\mathbb{R}$ and the exponential function is given by $\exp : i\mathbb{R} \to S^1$, $i\lambda \mapsto e^{i\lambda}$. The tangent space to P at the point (x, z) is given by

$$T_{(x,z)}P = T_t\mathbb{R} \oplus T_z S^1 \cong \mathbb{R} \oplus zi\mathbb{R} \tag{2.1}$$

because $T_zS^1 = L_{z*}(T_1S^1) = L_{z*}(i\mathbb{R})$. Then a tangent vector at (x,z) is given by $X_{(x,z)} = (r, izs)$ with $r, s \in \mathbb{R}$. From equation (2.1) it follows that the vertical space at $p = (x,z)$, determined by the structure of the principal bundle, is $V_{(x,z)} = zi\mathbb{R}$.

Given a vector A in the Lie algebra of S^1, $A = ia$ with $a \in \mathbb{R}$, the corresponding fundamental vector field in P is given by

$$A^*_{(x,z)} = \frac{d}{dt}\,(x,z)\cdot\exp(iat)\Big|_{t=0} = \frac{d}{dt}\,(x, z\exp(iat))\Big|_{t=0} = (0, zia).$$

Now we want to define a nontrivial connection on P. We can do it by defining first a distribution of horizontal spaces at each point. Let $X = (r, zis)$ be a tangent vector to P at (x,z), let us define the corresponding projections,

$$\begin{aligned} p^h : T_t\mathbb{R}\oplus T_zS^1 \to H_{(x,z)} \quad &, \quad p^h((r,zis)) = (r,zis)^h := (r, zir) \\ p^v : T_t\mathbb{R}\oplus T_zS^1 \to zi\mathbb{R} \quad &, \quad p^v((r,zis)) = (r,zis)^v = (0, zi(s-r)) \end{aligned}$$

clearly p^h and p^v are projections and $T_t\mathbb{R}\oplus T_zS^1 = H_{(x,z)} \oplus V_{(x,z)}$. We also have to check that the distribution is right invariant, that is, that it satisfies $R_{a*}(H_{(x,z)}) = H_{(x,za)}$ for all $a \in S^1$. Let $a \in S^1$ and $X \in T_{(x,z)}P$, then $R_{a*}(X^h) = R_{a*}((r,zir)) = (r,(za)ir) \in H_{(x,za)}$.
The connection form ω associated with the distribution is given by

$$\omega_{(x,z)} : T_{(x,z)}P \to \mathfrak{g}, (r, zis) \mapsto \omega_{(x,z)}(r, zis) = i(s-r),$$

since $(i(s-r))^*_{(x,z)} = (0, zi(s-r))$, the connection is well defined. In a similar way other connections on P can be obtained.

Given two connections ω, ω', their difference $\omega - \omega'$ is a horizontal equivariant 1-form with values in $\mathfrak{g}$, where by horizontal we mean $(\omega-\omega')(X) = 0$, for all X vertical.

2.2 The Group Bundle and the Adjoint Bundle

Let $P(M,G)$ be a principal bundle. In order to construct the desired representation and to define the gauge group associated to P we need to define the group and the adjoint bundles. These bundles are associated bundles to P, see [12], and are defined as follows:

Definition 2.7: Consider the left action of G on itself given by $\gamma_a(g) = a\ g\ a^{-1}$, for $a, g \in G$. This gives rise to the right action of G on $P \times G$, $\Theta : (P \times G) \times G \to (P \times G)$ given by

$$\Theta((u, g), a) = (u \cdot a, \gamma_{a^{-1}}(g)) = (u \cdot a, a^{-1}\ g\ a)$$

The group bundle $\mathbb{G}$ is the bundle $(P, M, G, \pi_{\mathbb{G}}, G)$ associated to P. It looks locally like $M \times G$ and its elements are the equivalence classes:

$$[(u, g)] = \{(u \cdot a, a^{-1}\ g\ a) | a \in G\}.$$

The set of smooth section of $\mathbb{G}$ is denoted by $\Gamma(\mathbb{G})$, that is

$$\Gamma(\mathbb{G}) = \{\sigma : M \to \mathbb{G} \mid \sigma \text{ is smooth and } \pi_{\mathbb{G}} \circ \sigma = id_M\}.$$

Definition 2.8: Consider the left action of G on $\mathfrak{g}$ given by $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$, $(a, X) \mapsto \mathrm{Ad}(a)X$. This gives rise to the action of G on the right of $P \times \mathfrak{g}$ given by

$$\widetilde{\Theta}((u, X), a) = (u \cdot a, \mathrm{Ad}(a^{-1})(X)) = (u \cdot a, (R_a)_*(X))$$

The adjoint bundle $P(\mathfrak{g})$ is the bundle $(P, M, \mathfrak{g}, \pi_{P(\mathfrak{g})}, G)$ associated to P. It looks locally like $M \times \mathfrak{g}$ and its elements are the equivalence classes:

$$[(u, X)] = \{(u \cdot a, \mathrm{Ad}(a^{-1})X) | a \in G\}.$$

The set of all smooth sections of $P(\mathfrak{g})$ over M is denoted by $\Gamma(P(\mathfrak{g}))$ or by $A^0(\mathfrak{g})$.

Example 2.9: If P is trivial, that is, if $P = M \times G$, then $\mathbb{G} \cong M \times G$ and $P(\mathfrak{g}) \cong M \times \mathfrak{g}$. The isomorphism in the last case is given by the map $[(x, a, X)] \mapsto (x, \mathrm{Ad}(a)X)$, for $x \in M$, $a \in G$ and $X \in \mathfrak{g}$.

2.3 Description of the Gauge Group

In this section we will construct the gauge group associated to a principal bundle P. As we mentioned before, some good references for this topic are [3], [4], [13], and [18].

Let $\mathrm{Aut}(P)$ be the group of all equivariant diffeomorphisms of P, that is,

$$\mathrm{Aut}(P) = \{\alpha \in \mathrm{diff}(P) \mid \alpha(p \cdot g) = \alpha(p) \cdot g, \text{ for all } p \in P, g \in G\},$$

where $\mathrm{diff}(P)$ denotes the set of all diffeomorphism of P as a manifold. Now we define:

Definition 2.10: The gauge group, $\mathfrak{G}_P$, of P is the group

$$\mathfrak{G}_P = \{\alpha \in \mathrm{Aut}(P) | \pi \circ \alpha = \pi\}.$$

If we consider the correspondence between $\mathrm{Aut}(P)$ and $\mathrm{diff}(M)$ given by the following commutative diagram:

$$\begin{array}{ccc} P & \xrightarrow{\alpha} & P \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi} \\ M & \xrightarrow{\tilde{\rho}(\alpha)} & M \end{array},$$

where $\tilde{\rho} : \mathrm{Aut}(P) \to \mathrm{diff}(M)$, we have that $\mathfrak{G}_P = \ker(\tilde{\rho}) = \tilde{\rho}^{-1}\{id_M\}$.

Let $C^\infty_G(P, G)$ be given by

$$C^\infty_G(P, G) = \{f \in C^\infty(P, G)| \ f(p \cdot g) = g^{-1} f(p) g, \text{ for } p \in P, \ g \in G \ \}.$$

Then we have the following proposition:

Proposition 2.11: *The groups $\mathfrak{G}_P$, $C^\infty_G(P, G)$, and $\Gamma(\mathbb{G})$ are isomorphic as groups.*

Proof: Let $\alpha \in \mathfrak{G}_P$ and $p \in P$. Since $\alpha(p) \in P_{\pi(p)}$, there exists $\widetilde{g} \in G$ such that $\alpha(p) = p \cdot \widetilde{g}$. Let us define $f : P \to G$ by $f(p) = \widetilde{g}$, where $\widetilde{g}$ satisfies $\alpha(p) = p \cdot \widetilde{g}$. f is well defined because the action of G on P is free and transitive in the fibres. For $\alpha \in \mathfrak{G}_P$ and $g \in G$,

$$\alpha(p \cdot g) = \alpha(p) \cdot g = p \cdot f(p) g.$$

On the other hand,

$$\alpha(p \cdot g) = (p \cdot g) \cdot (f(p \cdot g)) = p \cdot gf(p \cdot g).$$

Thus we have that

$$p \cdot f(p)g = p \cdot gf(p \cdot g),$$

therefore $f(p)g = gf(p \cdot g)$, which implies that, $f(p \cdot g) = g^{-1}f(p)g$. Thus, $f \in C^\infty_G(P, G)$.

To show that $C^\infty_G(P, G) \subseteq \Gamma(\mathbb{G})$, let $f \in C^\infty_G(P, G)$, $x \in M$ and $p \in P$ with $\pi(p) = x$. We define $\sigma(x) = [p, f(p)]$ in $\Gamma(\mathbb{G})$. To see that σ is well defined, let $q \in P$ such that $\pi(q) = x$, then $q = p \cdot g$, and $f(q) = f(p \cdot g) = g^{-1}f(p)g$. Therefore $[q, f(q)] = [p \cdot g, g^{-1}f(p)g] = [p, f(p)]$. Since f is smooth it follows that σ is smooth too and $\sigma \in \Gamma(\mathbb{G})$.

Finally, if $\sigma \in \Gamma(\mathbb{G})$ and $x \in M$, then $\sigma(x) = [p(x), g(x)]$ with $[p(x), g(x)] = [p(x) \cdot h, h^{-1}g(x)h]$, for every $h \in G$. But for every $q \in P_x$, there exists a unique $\tilde{h} \in G$ such that $q = p \cdot \tilde{h}$. Thus, we can define $f(q) = f(p \cdot \tilde{h}) = \tilde{h}^{-1}g(x)\tilde{h}$, then $f \in C^\infty_G(P, G)$. This function f induces an automorphism $\alpha : P \to P$, $\alpha(q) = q \cdot f(q) = q \cdot \tilde{h}^{-1}g(x)\tilde{h} = p \cdot g(x)\tilde{h}$. It is clear from definition that $\pi \circ \alpha = \alpha$. Therefore $\alpha \in \mathfrak{G}_P$.

We need to check now that these are group isomorphisms. Let $\alpha, \beta \in \mathfrak{G}_P$, $f, g \in C^\infty_G(P, G)$, and $\sigma, \varsigma \in \Gamma(\mathbb{G})$ be in the correspondence described above, that is, for $p \in P$, $x \in M$, $\pi(p) = x$, we have $\alpha(p) = p \cdot f(p)$, $\sigma(x) = [p, f(p)]$ and $\beta(p) = p \cdot g(p)$, $\varsigma(x) = [p, g(p)]$. Since the group operation in $\mathfrak{G}_P$ is given by composition we have

$$(\alpha \cdot \beta)(p) = \alpha(\beta(p)) = \alpha(p \cdot g(p)) = \alpha(p) \cdot g(p) = (p \cdot f(p)) \cdot g(p) = p \cdot (f(p)g(p)),$$

$$(\sigma \cdot \varsigma)(x) = [p, f(p)] \cdot [p, g(p)] := [p, f(p)g(p)].$$

Thus the isomorphisms are group isomorphisms, and the proof is complete.

□

Example 2.12: If P is trivial, since $\mathbb{G} \cong M \times G$, we have that the gauge group is $\Gamma(\mathbb{G}) \cong C^\infty(M, G)$.

We want to consider $\Gamma(\mathbb{G})$ with a topology that makes it into a Lie group, then we use the isomorphisms given in Proposition 2.11 to endow $\mathfrak{G}_P$ and $C^\infty_G(P, G)$ with topologies that allow us to consider these three spaces as the same. For this, let $\mathcal{S}$ be the set

$$\mathcal{S} = \{\{f \in C^k(M, \mathbb{G}) | j^l(f) \subset O\} |\ O \subset J^l(M, \mathbb{G}) \text{ is open for } 0 \leq l \leq k\}$$

where $J^k(M,\mathbb{G})$ is the jet bundle and $j^l(f): M \to J^l(M,\mathbb{G})$, $p \mapsto j^l(f)(p) \in J^l(M,\mathbb{G})_{p,f(p)}$. (See [23] page 55 and [4] page 77). The set $\mathcal{S}$ forms a subbasis for a topology in $C^k(M,\mathbb{G})$, this topology is called the C^k-Whitney topology. Now we assume that M is compact and we take

$$C^\infty(M,\mathbb{G}) = \bigcap_{k=0}^{\infty} C^k(M,\mathbb{G}),$$

the C^∞-Whitney topology over $\Gamma(\mathbb{G})$ is the coarsest topology among all topologies on $\Gamma(\mathbb{G})$ for which the canonical injections:

$$i^k : C^\infty(M,\mathbb{G}) \to C^k(M,\mathbb{G})$$

are continuous, $k = 0, 1, \ldots$

3 Construction of the Hilbert Space of the Representation

Let us consider the set of p-forms on M with values in $P(\mathfrak{g})$.

$$A^p(\mathfrak{g}) = \Omega^p(M, P(\mathfrak{g})) \cong \Gamma(\Lambda^p(M) \otimes_{\mathbb{R}} P(\mathfrak{g})) \tag{3.1}$$

where $\Omega(M)$ is the set of scalar p-forms over M. For $p = 0$ this notation agrees with our previous definition of $A^0(\mathfrak{g}) = \Gamma(P(\mathfrak{g}))$.
Since $P(\mathfrak{g})$ is a bundle of Lie algebras, $A^0(\mathfrak{g})$ becomes a Fréchet Lie algebra under pointwise operations, a Fréchet Lie algebra is a Lie algebra that also is Fréchet space, (we refer the reader to [13], [21], [25] and [26] for Fréchet structures). Let $\sigma_1, \sigma_2 \in A^0(\mathfrak{g})$, for $x \in M$, without lost generality we can consider $\sigma_1(x) = [p(x), X(x)]$, $\sigma_2(x) = [p(x), Y(x)]$, then the Lie bracket in $A^0(\mathfrak{g})$ is defined as

$$[\sigma_1, \sigma_2](x) = [[p(x), X(x)], [p(x), Y(x)]] = [p(x), [X(x), Y(x)]]$$

We can consider the elements of $A^p(\mathfrak{g})$ to be of the form $\gamma \otimes \sigma$, with $\gamma \in \Omega^p(M)$ and $\sigma \in A^0(\mathfrak{g})$ but is important to keep in mind that their real form is $\sum_{i=1}^n \gamma_i \otimes \sigma_i$, with $\gamma_i \in \Omega^p(M)$ and $\sigma_i \in A^0(\mathfrak{g})$, $1 \le i \le n$, $n \in \mathbb{N}$.

Using the facts that M is a Riemannian compact manifold and that G is a compact semi-simple Lie group we can define an inner product in $A^1(\mathfrak{g})$

in the following way: In the Lie algebra $\mathfrak{g}$ the Killing form is given by the equation:

$$B(X,Y) = Tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y)),$$

with $X, Y \in \mathfrak{g}$; since G is semi-simple, the Killing form is negative-definite; then the equation

$$\langle X, Y \rangle = -B(X,Y)$$

gives an inner product on $\mathfrak{g}$, see for example [6], sections XXI.5 and XXI.6. We can lift this inner product to $P(\mathfrak{g})$ fibrewise putting, for $[p, X], [p, Y] \in P(\mathfrak{g})$,

$$\langle [p, X], [p, Y] \rangle = \langle X, Y \rangle, \tag{3.2}$$

it follows from the invariance of Killing form that last equation determines a well defined inner product in $P(\mathfrak{g})$. Let $\varsigma_1, \varsigma_2 \in A^1(\mathfrak{g})$ be given by $\varsigma_1(x) = \alpha(x) \otimes [p(x), X(x)]$, $\varsigma_2(x) = \beta(x) \otimes [q(x), Y(x)] = \beta(x) \otimes [p(x), \mathrm{Ad}(g(x))Y(x)]$; let us define a function $\langle \varsigma_1, \varsigma_2 \rangle \in C^\infty(M)$ in the following way:

$$\langle \varsigma_1, \varsigma_2 \rangle(x) = \langle \alpha(x), \beta(x) \rangle \cdot \langle X(x), \mathrm{Ad}(g(x))Y(x) \rangle,$$

where the first term in the product of the right hand is the inner product in the cotangent space of M induced by the Riemannian metric of M. We can extend this operation to the whole space $A^1(\mathfrak{g}) \times A^1(\mathfrak{g})$ by bilinearly and then define an inner product in $A^1(\mathfrak{g})$ by the equation:

$$\langle \varsigma_1, \varsigma_2 \rangle = \int_M \langle \varsigma_1, \varsigma_2 \rangle(x) d\mu(x) = \int_M \langle \alpha(x), \beta(x) \rangle \cdot \langle X(x), \mathrm{Ad}(g(x))Y(x) \rangle \; d\mu(x) \tag{3.3}$$

where $\mu(x)$ is the measure in M induced by the Riemannian metric. Thus $A^1(\mathfrak{g})$ has a well defined inner product and it can be completed to a Hilbert space, this is the space that we will use as representation space of the affine representation of the gauge group $\mathfrak{G}_P$, we will denote the completation of $A^1(\mathfrak{g})$ by $\mathcal{H}$.

Example 3.1: Let $P = \mathbb{R} \times \mathrm{SU}(2)$ be a trivial principal bundle over $\mathbb{R}$ with fibre and group SU(2), SU(2) is a semi-simple Lie group and its Lie algebra is su(2). A basis of su(2), as real vector space, is:

$$A_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

For $X, Y \in \mathrm{su}(2)$, with $X = \sum_{i=1}^{3} x_i A_i$ and $Y = \sum_{i=1}^{3} y_i A_i$, the Killing form is given by the equation:

$$B(X,Y) = Tr(\mathrm{ad}(X) \circ \mathrm{ad}(Y)) = \sum_{i,j=1}^{3} x_i y_j B(A_i, A_j)$$

where $B(A_i, A_j) = Tr(\mathrm{ad}(A_i) \circ \mathrm{ad}(A_j))$, that is, $B(A_i, A_j)$ is the trace of the matrix whose k-th column, $k = 1, 2, 3$, is the vector whose components are the coefficients of

$$(\mathrm{ad}(A_i) \circ \mathrm{ad}(A_j))(A_k),$$

in the basis $\{A_1, A_2, A_3\}$. An easy computation shows that $B(A_i, A_j) = -8\delta_{ij}$. Thus the killing form in su(2) has the form:

$$B(X,Y) = -8 \sum_{i=1}^{3} x_i y_i.$$

It is clearly negative definite and gives rise to an inner product in $A^1(\mathrm{su}(2))$.

Since in this case P is a trivial principal bundle, we have that $P(\mathfrak{g}) \cong \mathbb{R} \times \mathrm{su}(2)$ and $A^1(\mathrm{su}(2)) \cong \Gamma(\Lambda^1(\mathbb{R}) \otimes_{\mathbb{R}} (\mathbb{R} \times \mathrm{su}(2))) \cong \mathrm{su}(2)$.

4 Construction of the Representation

The representation we are going to construct arose as a preliminary step in the construction of the energy representation of the group of mappings of a Riemannian manifold into a compact semi-simple Lie group studied in [8] and [2].

Definition 4.1: Let G be a group, an affine representation of G in a vector space E is a homomorphism $\Psi : G \to \mathrm{Aff}(E)$, i.e. an element of $Hom(G, \mathrm{Aff}(E))$.

We want to construct an affine representation of the gauge group $\mathfrak{G}_P$ in $\mathcal{H}$, the completation of $A^1(\mathfrak{g})$ respect to the inner product given by equation (3.3). We have a natural action of the gauge group on each set $A^p(\mathfrak{g})$, the description of this action is the following: Consider the map $\mathbb{G} \times_M P(\mathfrak{g}) \to P(\mathfrak{g})$ given by $([p, g], [p, X]) \mapsto [p, \mathrm{Ad}(g)X]$. The corresponding action of $\Gamma(\mathbb{G})$ on $A^0(\mathfrak{g}) = \Gamma(P(\mathfrak{g}))$, by the Ω-lemma (see [1], page 101), is given by

$$\begin{array}{rcl} \Gamma(\mathbb{G}) \times A^0(\mathfrak{g}) & \to & A^0(\mathfrak{g}) \\ (\sigma, \beta) & \mapsto & (\sigma \cdot \beta)(x) = [p(x), \mathrm{Ad}(g(x))X(x)] \end{array}$$

where $\sigma(x) = [p(x), g(x)]$, $\beta(x) = [p(x), X(x)]$. This action is a left action that can be easily generalized to $A^p(\mathfrak{g})$, for $p = 1, 2, \dots$, in the following way: If $\sigma \in \Gamma(\mathbb{G})$ is given by $\sigma(x) = [p(x), g(x)]$, and $\beta \in A^p(\mathfrak{g})$ is represented by $\beta(x) = \alpha(x) \otimes [p(x), X(x)]$, with $\alpha \in \Omega^p(M)$, that is, α is a scalar p-form over M, then the action

$$\Gamma(\mathbb{G}) \times A^p(\mathfrak{g}) \to A^p(\mathfrak{g})$$

is given by the equation:

$$(\sigma \cdot \beta)(x) = \alpha(x) \otimes [p(x), \mathrm{Ad}(g(x))X(x)]. \tag{4.1}$$

Let ω be a fixed connection form over P, we consider the operator

$$d_\omega : C^\infty_G(P, G) \to \Omega^1(P, \mathfrak{g})$$

that is defined as follows:

Definition 4.2: Given $f \in C^\infty_G(P, G)$, $X \in \mathcal{X}(P)$ and $p \in P$, the covariant left logarithmic derivative of f, $d_\omega(f)$, is a 1-form over P with values on $\mathfrak{g}$ and it is given by

$$d_\omega f_p(X) = L_{f^{-1}(p)*}(T_\omega f(X)) = L_{f^{-1}(p)*}(f_*(X^h))$$

where $T_\omega f(X) := f_*(X^h)$ and X^h is the ω-horizontal component of X.

If $f \in C^\infty_G(P, G)$, then $f : P \to G$ and for $p \in P$, $f_{*p} : T_p(P) \to T_{f(p)}(G)$, then for $X \in T_p(P)$, $T_\omega f(X) \in T_{f(p)}(G)$. As we want a $\mathfrak{g}$-valued 1-form over P we need to translate $T_\omega f(X)$ to $T_e(G) = \mathfrak{g}$, we do that by mean of $(L_{f^{-1}(p)*})_{f(p)}$, thus $d_\omega f_p(X) \in \mathfrak{g}$.

Proposition 4.3: *If $f \in C^\infty_G(P, G)$, then $d_\omega f$ is a horizontal and equivariant 1-form over P with values in $\mathfrak{g}$.*

Proof: Given $X \in T_p(P)$, if X is vertical we have that $X^h = 0$, therefore $d_\omega f(X) = L_{f^{-1}(p)*}(f_*(0)) = 0$. Then $d_\omega f$ is horizontal. Now, we have to see that $d_\omega f$ is equivariant, i.e. we have to prove that $(R_a)^* d_\omega f = \mathrm{Ad}(a^{-1})(d_\omega f)$, for every $a \in G$. To do that let $a \in G$, $X \in T_p(P)$ be ω-horizontal, and let

$\gamma(t)$ be such that $X = \frac{d}{dt}\ \gamma(t)\big|_{t=0}$ and $\gamma(0) = p$. Then

$$\begin{aligned}
((R_a)^*(d_\omega f)_{p\cdot a})_p(X_p) &= d_\omega f_{p\cdot a}(R_{a\ *\ p}X_p) \\
&= L_{f(p\cdot a)^{-1}\ *}\left(\frac{d}{dt}\ f(\gamma(t)\cdot a)\Big|_{t=0}\right) \\
&= L_{a^{-1}f(p)^{-1}a\ *}\left(\frac{d}{dt}\ a^{-1}f(\gamma(t))a\Big|_{t=0}\right) \\
&= L_{a^{-1}\ *}\left(L_{f(p)^{-1}\ *}\left(R_{a\ *}\left(\frac{d}{dt}f(\gamma(t))\Big|_{t=0}\right)\right)\right) \\
&= \mathrm{Ad}(a^{-1})\left(L_{f(p)^{-1}\ *}\left(\frac{d}{dt}f(\gamma(t))\Big|_{t=0}\right)\right) \\
&= \mathrm{Ad}(a^{-1})((d_\omega f)_p(X_p))
\end{aligned}$$

If X is any vector in P, we get the decomposition of X into its horizontal and vertical parts, $X = X^h + X^v$, therefore the statement above also works for X. □

Proposition 4.4: *The operator d_ω satisfies the equations:*

1. $d_\omega(fg) = \mathrm{Ad}(g^{-1})d_\omega(f) + d_\omega(g)$*; with* $f, g \in C^\infty_G(P,G)$.
2. $d_\omega(f^{-1}) = -\mathrm{Ad}(f)d_\omega(f) = -R_{f^{-1}\ *}T_\omega f$*; with* $f \in C^\infty_G(P,G)$.

Proof: Let $p \in P$, and $X \in T_p(P)$

$$\begin{aligned}
d_\omega(fg)(X) &= L_{(f(p)g(p))^{-1}*}((fg)_{*\ p}(X^h)) \\
&= L_{g(p)^{-1}f(p)^{-1}*}(R_{g(p)\ *}(f_*(X^h)) + L_{f(p)\ *}(g_*(X^h))) \\
&= (L_{g(p)^{-1}\ *}\circ L_{f(p)^{-1}\ *})(R_{g(p)\ *}(f_*(X^h)) + L_{f(p)\ *}(g_*(X^h))) \\
&= \mathrm{Ad}(g(p)^{-1})(L_{f(p)^{-1}\ *}(f_*(X^h))) + L_{g(p)^{-1}\ *}(g_*(X^h)) \\
&= \mathrm{Ad}(g(p)^{-1})d_\omega f(X) + d_\omega g(X).
\end{aligned}$$

For the second equation we take $f = f$ and $g = f^{-1}$ in the first one. Then we have $d_\omega(e) = \mathrm{Ad}(f)d_\omega(f) + d_\omega(f^{-1})$, where $e : P \to G, p \mapsto e$. Obviously $e(p\cdot g) = g^{-1}eg = e$, and $d_\omega(e) = 0$, therefore $d_\omega(f^{-1}) = -\mathrm{Ad}(f)d_\omega(f)$. □

Definition 4.5: Given a connection form ω on P, for $f \in C^\infty_G(P,G)$ we define the right logarithmic differential of f as,

$$\delta_\omega f = d_\omega(f^{-1}) = -R_{f^{-1}*}T_\omega f.$$

Proposition 4.6: *The right logarithmic differential satisfies the following equation:*
$$\delta_\omega(f_1 f_2) = \delta_\omega f_1 + \mathrm{Ad}(f_1)\delta_\omega f_2, \tag{4.2}$$
for $f_1, f_2 \in C^\infty_G(P, G)$.

Relation given by equation (4.2) is important because it allows us to make the representation of the gauge group in the affine group of $\mathcal{H}$; this equation has the form of a Maurer-Cartan cocycle, Maurer-Cartan cocycles were used by Gelfand, Graev, and Vershik in [8] and [9] to define representations of current groups in L_2 spaces.

Now we have a description of the gauge group as a subset of the set of all horizontal and equivariant $\mathfrak{g}$-valued 1-forms over P, but the Hilbert space that we have as representation space is generated by $P(\mathfrak{g})$-valued 1-forms over M. Fortunately we have the following theorem:

Theorem 4.7: *To every $\mathfrak{g}$-valued horizontal and equivariant 1-form $\widetilde{\beta}$ over P, there corresponds a unique $P(\mathfrak{g})$-valued 1-form, β over M ($\beta \in A^1(\mathfrak{g})$).*

Proof: Let $\widetilde{\beta} \in \Gamma(\Lambda^1(P) \otimes \mathfrak{g})$ be horizontal and equivariant. Let $x \in M$, we define $\beta : T(M) \to P(\mathfrak{g})$ by

$$\beta_x(Z) = [p, \widetilde{\beta}_p(\widetilde{Z})], \text{ where } \; p \in P, \; \pi(p) = x, \text{ and } (\pi_*)_p(\widetilde{Z}) = Z_x. \tag{4.3}$$

To check that β is well defined, let $\widetilde{Z}_1, \widetilde{Z}_2 \in T_p(P)$ be two lifts of Z to P, that is, $\pi_*(\widetilde{Z}_1) = \pi_*(\widetilde{Z}_2) = Z$, then $\pi_*(\widetilde{Z}_1 - \widetilde{Z}_2) = 0$. Thus $\widetilde{Z}_1 - \widetilde{Z}_2$ is vertical and $\widetilde{\beta}_p(\widetilde{Z}_1 - \widetilde{Z}_2) = 0$, therefore $\widetilde{\beta}_p(\widetilde{Z}_1) = \widetilde{\beta}_p(\widetilde{Z}_2)$. On the other hand,

$$\begin{aligned} \beta_x(Z) &= \beta_{\pi(p)}(Z) = [p, \widetilde{\beta}_p(\widetilde{Z})] = [p \cdot g, \mathrm{Ad}(g^{-1})(\widetilde{\beta}_p(\widetilde{Z}))] && (4.4)\\ &= [p \cdot g, ((R_g)^* \widetilde{\beta})_p(\widetilde{Z})] = [p \cdot g, \widetilde{\beta}_{p \cdot g}(\widetilde{Z})] && (4.5) \end{aligned}$$

therefore, β is well defined, does not depend of the choice of p, and satisfies the conditions required. Then $\beta \in A^1(\mathfrak{g})$.

Conversely, let $\beta \in A^1(\mathfrak{g})$, we need to define $\widetilde{\beta}$ a $\mathfrak{g}$-valued, horizontal and equivariant 1-form over P. We have that $\beta_x : T_x(M) \to \pi^{-1}_{P(\mathfrak{g})}(x) \subset P(\mathfrak{g})$ then

$$\beta_x(Z) = [p(x, Z), X(x, Z)] = [p(x, Z) \cdot g, \mathrm{Ad}(g^{-1}) X(x, Z)].$$

for all $g \in G$. Let $Y \in \mathcal{X}(P)$ and $q \in P$ with $\pi(q) = x$, we define $\widetilde{\beta} : T(P) \to \mathfrak{g}$ by

$$\widetilde{\beta}_q(Y) = X(\pi(q), \pi_{*\ q}(Y)).$$

We need to check that $\widetilde{\beta}$ is horizontal and equivariant. If Y is vertical, $\pi_*(Y) = 0$. Then, $X(\pi(q), \pi_{*\ q}(Y)) = X(x, \pi_{*\ q}(0)) = 0$. Thus $\widetilde{\beta}_p(Y) = 0$, i.e., $\widetilde{\beta}$ is horizontal. We have to see now that $\widetilde{\beta}$ is equivariant. Let $Y \in \mathcal{X}(P)$, $g \in G$, such that $\gamma(t)$ is a curve with $\gamma(0) = q$ and $\frac{d}{dt}\ \gamma(t)\big|_{t=0} = Y_q$.

$$\begin{aligned}
((R_g)^*\widetilde{\beta})_q(Y) &= \widetilde{\beta}_{q\cdot g}(R_{g\ *\ q}(Y)) = X(\pi(q\cdot g), \pi_{*\ q\cdot g}(R_{g\ *\ q}Y)) \\
&= X(x, \pi_{*\ q\cdot g}(R_{g\ *\ q}\left(\frac{d}{dt}\ \gamma(t)\bigg|_{t=0}\right))) \\
&= \operatorname{Ad}(g^{-1})X(x, \left(\frac{d}{dt}\ \pi(\gamma(t)\cdot g)\bigg|_{t=0}\right)) \\
&= \operatorname{Ad}(g^{-1})X(x, \left(\frac{d}{dt}\ \pi(\gamma(t))\bigg|_{t=0}\right)) \\
&= \operatorname{Ad}(g^{-1})X(\pi(q), \pi_{*\ q}(Y)) = \operatorname{Ad}(g^{-1})(\widetilde{\beta}_q(Y))
\end{aligned}$$

Then the correspondence between the horizontal and equivariant elements $\widetilde{\beta}$, $\widetilde{\beta} \in \Gamma(\Lambda^1(P) \otimes \mathfrak{g})$, and the 1-forms β, $\beta \in A^1(\mathfrak{g})$, is one to one and onto. $\square$

Thus, we can consider $\delta_\omega(f)$, the right logarithmic differential of $f \in C^\infty_G(P, G)$ (i.e. f in the gauge group), as an element in the Hilbert space $\mathcal{H}$ and we can finally define the affine representation of the gauge group: Let ω be a fixed connection form over P, consider the action of $C^\infty_G(P, G)$ on $A^1(\mathfrak{g})$ given by

$$\begin{aligned}
\Theta : C^\infty_G(P, G) \times A^1(\mathfrak{g}) &\to A^1(\mathfrak{g}) \\
(f, \beta) &\mapsto \Theta(f, \beta) = \operatorname{Ad}(f)\beta + \delta_\omega f.
\end{aligned}$$

Let us check that the action is well defined:

$$\begin{aligned}
\Theta(f_1, \Theta(f_2, \beta)) &= \Theta(f_1, \delta_\omega f_2 + \operatorname{Ad}(f_2)\beta) \\
&= \delta_\omega f_1 + \operatorname{Ad}(f_1)(\delta_\omega f_2 + \operatorname{Ad}(f_2)\beta) \\
&= \delta_\omega f_1 + \operatorname{Ad}(f_1)(\delta_\omega f_2) + \operatorname{Ad}(f_1)(\operatorname{Ad}(f_2)\beta) \\
&= \delta_\omega f_1 + \operatorname{Ad}(f_1)(\delta_\omega f_2) + \operatorname{Ad}(f_1)\operatorname{Ad}(f_2)\beta,
\end{aligned}$$

on the other hand we have

$$\begin{aligned}\Theta(f_1 f_2, \beta) &= \delta_\omega(f_1 f_2) + \mathrm{Ad}(f_1 f_2)\beta \\ &= \delta_\omega(f_1 f_2) + \mathrm{Ad}(f_1)\,\mathrm{Ad}(f_2)\beta\end{aligned}$$

Then, from Proposition 4.6 we have that $\delta_\omega(f_1 f_2) = \delta_\omega f_1 + \mathrm{Ad}(f_1)\delta_\omega f_2$, so that the action of $\mathfrak{G}_P$ on $A^1(\mathfrak{g})$ is well defined. If β is in $\mathcal{H} = \overline{A^1(\mathfrak{g})}$ but not in $A^1(\mathfrak{g})$, then the action of $f \in \mathfrak{G}_P$ on β is obtained extending the action of f on $A^1(\mathfrak{g})$ by continuity. Then the affine representation of $\mathfrak{G}_P$ in $\mathcal{H}$ is given by:

$$\Psi : C^\infty_G(P, G) \to \mathrm{Aff}(\mathcal{H}), \quad f \mapsto (\mathrm{Ad}(f), \delta_\omega f)$$

Since $\mathrm{Ad}(f)$ is product preserving it follows that $\Psi(C^\infty_G(P, G)) \subset M(\mathcal{H})$ and we get that the representation is in fact on the group of motions of $\mathcal{H}$, $M(\mathcal{H})$.

The affine representation of the gauge group in the Hilbert space $\mathcal{H}$ is interesting because it provides the first step for the construction of a unitary representation $\widetilde{\Psi}$ of the gauge group in the symmetric Fock space $\mathcal{F}_s(\mathcal{H})$ associated to $\mathcal{H}$. The representation $\widetilde{\Psi}$ is obtained as the composition of the affine representation described in this paper and a representation of the group of motions in $\mathcal{F}_s(\mathcal{H})$. Although the description of $\widetilde{\Psi}$ is not the purpose of this paper, let us say something about it: The representation of $\mathfrak{G}_P$ in $\mathcal{F}_s(\mathcal{H})$ is equivalent to a representation of $\mathfrak{G}_P$ in a space $L^2_\mu(\mathcal{H})$ where μ is a special measure, see for example [9] and [10]. As we already said, these kind of representations arose from physics and from the study of current groups. The study of the irreducibility of these representations involves a lot of work in measure theory and functional analysis. In the case when the gauge group is of the form $C^\infty(M, G)$ this study was developed by Gelfand, Graev and Vershik ([8] and [9]), Wallach ([27]), Pressley ([24]) and Albeverio ([2]), among others, during the 70s and the 80s and the results depend strongly on the dimension of the base manifold M: if $\dim M \geq 4$, then the representation is irreducible, see [9], but if we consider loop groups, that is, if $M = S^1$, then the representation is highly reducible, see [24], chapter 9. The question about the representation of the gauge group in the general case (when $\mathfrak{G}_P$ is not of the form $C^\infty(M, G)$) is still open and we think that the geometry involved in the representation of $\mathfrak{G}_P$ in $M(\mathcal{H})$ could help to solve this problem.

Representation of the gauge group in the set of connections. The representation of the gauge group in $M(\mathcal{H})$ described above can also be interpreted as an affine representation of the gauge group in the space of connections C, thinking of C as an affine space. This interpretation is important when Yang-Mills theories are considered and it has implications in this area, see [3], section 3, [7], and [18], section 6; other applications to physics can be found in [4] and [19]. Now we will describe the action of the gauge group on the set of connections: As we mentioned above, given two connections their difference is a horizontal, equivariant 1-form over P with values in the Lie algebra $\mathfrak{g}$. Then for a fixed connection ω, it follows from theorem 4.7 that C is isomorphic to the affine space $\omega + A^1(\mathfrak{g})$, and the action of $\mathfrak{G}_P$ on C

$$\begin{array}{rcl} \widetilde{\Theta} : \mathfrak{G}_P \times C & \rightarrow & C \\ (\alpha, \omega + \beta) & \mapsto & \widetilde{\Theta}(\alpha, \omega + \beta) \end{array}$$

is given by the following equations:

$$\begin{aligned} \widetilde{\Theta}(\alpha, \omega + \beta) &= \alpha^*(\omega + \beta) && (4.6) \\ &= \alpha^*\omega + \mathrm{Ad}(f^{-1})\beta && (4.7) \\ &= \omega + d_\omega f + \mathrm{Ad}(f^{-1})\beta && (4.8) \\ &= \omega + \mathrm{Ad}(f^{-1})\beta + \delta_\omega f^{-1} && (4.9) \\ &= \omega + \Theta(f^{-1}, \beta) && (4.10) \end{aligned}$$

where $\alpha(p) = p \cdot f(p)$, and $\beta \in A^1(\mathfrak{g})$.

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Representation of a gauge group

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