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Local time and related sample paths of filtered white noises

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Abstract

We study the existence and the regularity of the local time of filtered white noises $X = \{X(t),\ t \in [0,1]\}$. We will also give Chung's form of the law of iterated logarithm for X, this shows that the result on the Hölder regularity, with respect to time, of the local time is sharp.

1. Introduction

The purpose of this paper is to investigate local times and some related sample paths properties for Filtered White Noises ([3][2], in short FWN). FWN are Gaussian processes with the following representation :

$$X(t) = \int_{\mathbb{R}} \frac{a(t,\lambda)(e^{it\lambda}-1)}{|\lambda|^{1/2+H}} dW(\lambda), \quad t \in [0,1],$$

where $0 < H < 1$ and $dW(\lambda)$ is the random Brownian measure on $L^2(\mathbb{R})$. When $a \equiv 1$, a FWN is a H-fractional Brownian motion (fBm). Through the paper we keep the same assumptions on $a(t,\lambda)$ as in [3] and [2]. Thus, we assume that $a(t;\lambda)$ is $C^2(\mathbb{R}^2;\mathbb{R})$, and that there exists a function $a_\infty(t) \neq 0$ such that $\lim_{|\lambda|\to\infty} a(t,\lambda) = a_\infty(t)$ and that $\sigma(t,\lambda) = a(t,\lambda) - a_\infty(t)$ satisfies :

$$\left| \frac{\partial^{i+j}\sigma(t,\lambda)}{\partial^i t \partial^j \lambda} \right| \leq \frac{C}{|\lambda|^{j+\eta}}, \tag{1.1}$$

for $i, j = 0, 1, 2$ and $\eta > 0$ such that $0 < H + \eta < 1$.

Keywords: Local time, Local nondeterminism, Chung's type law of iterated logarithm, Filtered white noises.
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The FWN was introduced by Priestley [13]. The dependence on t in the function $a(t, \lambda)$ was introduced to overcome the limitations, in the stationary increments processes, that they are not able to follow local modulations of the parameters: stationarity implies uniformity. Various properties of its trajectories have been already explored in the literature, related for instance to its regularity and the identification of its relevant parameters H and $a_\infty(t)$. More precisely, it is proved ([2]) that the FWN has a pointwise Hölder exponent equal to H almost surely. We will prove a Chung form of the law of iterated logarithm, Theorem 4.1, which links the pointwise regularity of the FWN, at each t, to the value of $a_\infty(t)$.

Recently, Boufoussi, Dozzi and Guerbaz [7] have studied the local time of another class of Gaussian processes possessing the same local fractal properties as the fBm, called the multifractional Brownian motion (mBm). This process extend the fBm in the sense that its Hurst parameter is no more constant, but a Hölder function of time.

Section 2 contains a brief review on local times of Gaussian processes and on Berman's concept of local nondeterminism. Section 3 is devoted to prove our result on local times. Chung's form of the law of iterated logarithm for FWN is obtained in Section 4, which is applied to derive a lower bound for the local moduli of continuity of the local times of FWN.

We will use $C, C_1, ...$ to denote unspecified positive finite constants which may not necessarily be the same at each occurrence.

2. Preliminaries

We recall some aspects of local times and we refer to the paper of Geman and Horowitz [9] for an insightful survey on local times. Let $X = (X(t),\ t \in \mathbb{R}^+)$ be a real valued separable random process with Borel sample functions. For any Borel set B of the real line, the occupation measure of X is defined as follows

$$\mu(A, B) \quad = \quad \lambda\{s \in A \,:\, X(s) \in B\} \qquad \forall\, A \in \mathcal{B}(\mathbb{R}^+),$$

and λ is the Lebesgue measure on $\mathbb{R}^+$. If $\mu(A, .)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}$, we say that X has local times on A and define its local time, L(A,.), as the Radon-Nikodym derivative of $\mu(A, .)$. Here x is the so-called space variable, and A is the time variable.

The existence of jointly continuous local time reveals information on the fluctuation of the sample paths of the process itself ([1], Chapter 8). There are several approaches for proving the joint continuity of local times, one of them is the Fourier analytic method developed by Berman to extend his earlier works on the local times of stationary Gaussian processes. The main tool used in Berman's approach [6] is the local nondeterminism. We give a brief review of the concept of local nondeterminism, more informations on this subject can be found in Berman [6]. Let J be an open interval on the t axis. Assume that $(X(t),\ t \in \mathbb{R}_+)$ is a zero mean Gaussian process which has no singularities in an interval of length δ, for some $\delta > 0$, nor does it have fixed zeros; that is, there exist $\delta > 0$ such that

$$(\mathcal{P}) \begin{cases} E(X(t) - X(s))^2 > 0, & \text{whenever } 0 < |s-t| < \delta; \\ E(X(t))^2 > 0, & \text{for all } t \in J. \end{cases}$$

To introduce the concept of local nondeterminism, Berman defined the relative conditioning error,

$$V_m = \frac{Var\{X(t_m) - X(t_{m-1})/X(t_1), ..., X(t_{m-1})\}}{Var\{X(t_m) - X(t_{m-1})\}} \tag{2.1}$$

where, for $m \geq 2$, $t_1, ..., t_m$ are arbitrary points in J ordered according to their indices, i.e. $t_1 < t_2 ... < t_m$. We say that the process X is locally nondeterministic (LND) on J if for every $m \geq 2$,

$$\liminf_{\substack{c \searrow 0^+ \\ 0 < t_m - t_1 \leq c}} \quad V_m > 0. \tag{2.2}$$

This condition means that a small increment of the process is not almost relatively predictable on the basis of a finite number of observations from the immediate past. Berman has proved, for Gaussian processes, that the local nondeterminism is characterized as follows.

Proposition 2.1. *X is LND if and only if for every integer $m \geq 2$, there exist positive constants C and δ (both may depend on m) such that*

$$Var\left(\sum_{j=1}^{m} u_j [X(t_j) - X(t_{j-1})]\right) \geq C_m \sum_{j=1}^{m} u_j^2 Var\left[X(t_j) - X(t_{j-1})\right], \tag{2.3}$$

for all ordered points $t_0 < t_1 < .... < t_m < 1$ with $t_m - t_1 < \delta$, $t_0 = 0$ and $(u_1, u_2, ..., u_m) \in \mathbb{R}^m$.

The proof of the previous proposition is given in [6], Lemmas 2.1 and 8.1.

3. **Local times**

The main result of this section reads as follows

Theorem 3.1. *The FWN has, almost surely, a jointly continuous local time $L(t,x)$, which has the following Hölder continuities. For any compact $K \subset \mathbb{R}$, we have*
(i) If $0 < \xi < 1 \wedge \frac{1-H}{2H}$, then for any $I \subset [0,1]$ with small length

$$\sup_{x,y \in K, x \neq y} \frac{|L(I,x) - L(I,y)|}{|x-y|^{\xi}} < +\infty, \ a.s.$$

(ii) If $0 < \delta < 1 - H$, then

$$\sup_{x \in K} \frac{|L(t+h,x) - L(t,x)|}{|h|^{\delta}} < +\infty, \ a.s.$$

where $|h| < \kappa$, κ being a small random variable, almost surely positive and finite.

We need some preliminaries lemmas for the proof of the theorem

Lemma 3.2. *The process $\{X_1(t),\ t \in [0,1]\}$ defined by*

$$X_1(t) = a_{\infty}(t) \int_{\mathbb{R}} \frac{e^{i\lambda t} - 1}{|\lambda|^{1/2+H}} dW(\lambda), \tag{3.1}$$

is LND on the interval $(0,1)$.

Proof. Observe that $X_1(t) = a_{\infty}(t) B^H(t)$ with B^H is, up to a multiplicative normalizing constant, a fBm with Hurst parameter H. Since $a_{\infty}(.)$ doesn't vanishes, then

$$\sigma(X_1(u),\ u \in A\ u \leq s) = \sigma(B^H(u),\ u \in A\ u \leq s),$$

for any $s \geq 0$ and any finite time set A. Hence

$$\begin{aligned} &Var\left(X_1(t+h)/X_1(u),\, u \in A,\, u \leq t\right) \\ &= Var\left(X_1(t+h)/\ B^H(u),\ u \in A\ u \leq t\right) \\ &\geq a_{\infty}^2(t+h) Var\left(B^H(t+h)/B^H(u),\ u \in A\ u \leq t\right). \end{aligned} \tag{3.2}$$

Moreover, according to Monrad and Rootzen [11], B^H is strongly nondeterministic, i.e.

$$Var\left(B^H(t+h)/B^H(u),\ u \in A\ u \leq t\right) \geq C_H h^{2H}$$

Therefore

$$Var\,(X_1(t+h)/X_1(u),\, u\in A,\, u\le t) \ge C_H \min_{u\in[0,1]} a_\infty^2(u)h^{2H} \tag{3.3}$$

We are now in position to prove that X_1 is LND on $(0,1)$. First, we have $E\left(X_1^2(t)\right) = a_\infty^2(t)t^{2H} > 0$, for all $t\in(0,1)$. Moreover, the inequality (3.3) implies that

$$\begin{aligned} E\,(X_1(t)-X_1(s))^2 &\ge Var\,(X_1(t)/X_1(u),\ u\in A\quad u\le s) \\ &\ge C_H\underline{a}(t-s)^{2H}. \end{aligned} \tag{3.4}$$

where $\underline{a} = \min_{u\in[0,1]}(a_\infty(u))^2$. Therefore the condition $(\mathcal{P})$ of LND holds. It remains to verify (2.2). By using the fact that $a_\infty(.)$ is C^1 and denoting the derivative of a by a', we obtain

$$\begin{aligned} E\,(X_1(t+h)-X_1(t))^2 &\le 2\Big[a_\infty^2(t+h)E\left(B^H(t+h)-B^H(t)\right)^2 \\ &\quad +(a_\infty(t+h)-a_\infty(t))^2E\left(B^H(t)\right)^2\Big] \\ &\le C[|h|^{2H}+|h|^2], \end{aligned} \tag{3.5}$$

where $C>0$ is a constant depending on H, $\max\limits_{t\in[0,1]} a_\infty(t)$ and $\max\limits_{t\in[0,1]} a'_\infty(t)$. Now, let us take $A=\{t_1,.....,t_{m-1}\}$ and $t_m = t_{m-1}+h$. Then, combining (3.3) and (3.5), we obtain that the relative prediction error V_m in (2.1) is at least equal to

$$\frac{C_H\underline{a}}{C}\times\frac{|h|^{2H}}{[|h|^{2H}+|h|^2]},$$

which is bounded away from 0, as h tends to 0, and the proof is complete. $\square$

Lemma 3.3. *The FWN satisfies the result of the Proposition 2.1.*

Proof. In the remainder of the paper we denote for simplicity :

$$X_1(t) = a_\infty(t)\int_{\mathbb{R}}\frac{e^{i\lambda t}-1}{|\lambda|^{1/2+H}}dW(\lambda),\ \text{ and } X_2(t) = \int_{\mathbb{R}}\frac{\sigma(t,\lambda)(e^{i\lambda t}-1)}{|\lambda|^{1/2+H}}dW(\lambda).$$

By using the elementary inequality $(x+y)^2 \geq \dfrac{x^2}{2} - y^2$ we obtain

$$\begin{aligned}
&Var\left(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]\right)\\
&\geq \frac{1}{2} Var\left(\sum_{j=1}^{m} u_j[X_1(t_j) - X_1(t_{j-1})]\right) - Var\left(\sum_{j=1}^{m} u_j[X_2(t_j) - X_2(t_{j-1})]\right).
\end{aligned}$$

Furthermore, since X_1 is LND then by Proposition 2.1, there exist δ_m and C_m such that for any $t_0 = 0 < t_1 < ... < t_m < 1$, with $t_m - t_1 < \delta_m$, we have

$$\begin{aligned}
&Var(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]) \qquad (3.6)\\
&\geq \frac{C_m}{2} \sum_{j=1}^{m} u_j^2 Var(X_1(t_j) - X_1(t_{j-1})) - m \sum_{j=1}^{m} u_j^2 Var(X_2(t_j) - X_2(t_{j-1}))
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&E\left(X_2(t) - X_2(t')\right)^2\\
&\leq \int_{\mathbb{R}} \frac{|\sigma(t,\lambda)|^2 |e^{i\lambda t} - e^{i\lambda t'}|^2}{|\lambda|^{1+2H}} d\lambda + \int_{\mathbb{R}} \frac{|\sigma(t,\lambda) - \sigma(t',\lambda)|^2 |e^{i\lambda t'} - 1|^2}{|\lambda|^{1+2H}} d\lambda
\end{aligned}$$

Then, according to (1.1), we have

$$\begin{aligned}
E\left(X_2(t) - X_2(t')\right)^2 &\leq C_{H,\eta}|t-t'|^{2H+2\eta} + C_{H,\eta}|t-t'|^2,\\
&\leq \widetilde{C}_{H,\eta}|t-t'|^{2H+2\eta}. \qquad (3.7)
\end{aligned}$$

This last inequality and (3.4), imply that (3.6) becomes

$$\begin{aligned}
&Var\left(\sum_{j=1}^{m} u_j[X(t_j) - X(t_{j-1})]\right)\\
&\geq \left(\frac{C_m C_H \underline{a}}{2} - m\delta_m^{2\eta}\widetilde{C}_{H,\eta}\right) \sum_{j=1}^{m} u_j^2 (t_j - t_{j-1})^{2H}.
\end{aligned}$$

In addition, combining (3.5) and (3.7), there exists a constant K such that

$$E\left(X(t) - X(s)\right)^2 \leq K|t-s|^{2H}, \text{ for all } t,s \text{ sufficiently close.} \qquad (3.8)$$

Therefore, it suffices now to chose

$$\widetilde{\delta}_m < \left(\frac{C_m C_H \underline{a}}{2m\widetilde{C}_{H,\eta}}\right)^{1/(2\eta)} \wedge \delta_m,$$

and to consider

$$\widetilde{C}_m = K\left(\frac{C_m C_H \underline{a}}{2} - m\widetilde{\delta}_m^{2\eta}\widetilde{C}_{H,\eta}\right),$$

and the lemma is proved. □

Now we are in position to prove the theorem, first we have the following existence result

Proposition 3.4. *The FWN has almost surely a local time $L(t,x)$, continuous in t for almost every $x \in \mathbb{R}$ and $L(t,x) \in \mathbb{L}^2(\mathbb{R})$.*

Proof. Combining (3.4) and (3.7) and the elementary inequality $(x+y)^2 \geq \frac{1}{2}x^2 - y^2$, we obtain

$$\begin{aligned} E\left(X(t)-X(s)\right)^2 &\geq \frac{1}{2}E\left(X_1(t)-X_1(s)\right)^2 - E\left(X_2(t)-X_2(s)\right)^2 \\ &\geq \left(\frac{C_H\underline{a}}{2} - \widetilde{C}_{H,\eta}|t-s|^{2\eta}\right)|t-s|^{2H} \\ &\geq C|t-s|^{2H}, \end{aligned} \tag{3.9}$$

for all $|t-s| < \delta$, for $\delta^{2\eta} < \dfrac{C_H\underline{a}}{2\widetilde{C}_{H,\eta}}$. Then, for any interval I of length smaller than δ, we have

$$\int_I \frac{1}{\sqrt{E\left(X(t)-X(s)\right)^2}}ds \leq \int_I \frac{1}{|t-s|^H}ds, \quad \text{for all } t \in I.$$

Since $0 < H < 1$, then last integral is finite and, according to Geman and Horowitz [9], the conclusion of the theorem hold for any interval I of small length. Since $[0,1]$ is finite interval one obtain the local time on $[0,1]$ by standard patch-up procedure, i.e. we partition $[0,1]$ into $\cup_{i=1}^n[\alpha_i,\alpha_{i+1}]$, such that $|\alpha_i - \alpha_{i-1}| < \delta$, and define $L([0,1],x) = \sum_{i=1}^n L([\alpha_i,\alpha_{i+1}],x)$, where $\alpha_0 = 0$ and $\alpha_n = 1$. □

In order to prove the joint continuity of L and the Hölder continuities stated in Theorem 3.1, we first establish appropriate upper bounds for the moments of the local time. According to Remark 3.4 in [7], the LND property can be used on the whole interval $[0,1]$ instead of $(0,1)$.

Lemma 3.5. *Let $\delta \in (0,1)$ be the constant such that the inequality (3.9) holds. Then, for any even integer $m \geq 2$ there exists a positive and finite constant C_m such that, for any $t \in [0,1]$, any $h \in (0,\delta)$, $x,y \in \mathbb{R}$ and any $\xi < 1 \wedge \frac{1-H}{2H}$ we have*

$$\mathbb{E}\left[L(t+h,x) - L(t,x)\right]^m \leq C_m \frac{h^{m(1-H)}}{\Gamma(1+m(1-H))}, \tag{3.10}$$

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \\ &\quad \leq C_m |y-x|^{\xi m} \frac{h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi)))}. \end{aligned} \tag{3.11}$$

Proof. We prove only (3.11), since (3.10) is easier and follows from similar arguments. It follows from (25.7) in Geman and Horowitz (1980) (see also (6) in Boufoussi and al. [7]) that for any $x,y \in \mathbb{R}$, $t,\ t+h \in [0,1]$ and for every even integer $m \geq 2$,

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \\ &= (2\pi)^{-m} \int_{[t,t+h]^m} \int_{\mathbb{R}^m} \prod_{j=1}^m \left[e^{-iyu_j} - e^{-ixu_j}\right] \\ &\quad \times \mathbb{E}\left(e^{i\sum_{j=1}^m u_j X(s_j)}\right) \prod_{j=1}^m du_j \prod_{j=1}^m ds_j. \end{aligned} \tag{3.12}$$

Using the elementary inequality $|1 - e^{i\theta}| \leq 2^{1-\xi}|\theta|^\xi$ for all $0 < \xi < 1$ and any $\theta \in \mathbb{R}$, we obtain

$$\begin{aligned} &\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \leq (2^\xi \pi)^{-m} m! |y-x|^{m\xi} \\ &\times \int_{t<t_1<\ldots<t_m<t+h} \int_{\mathbb{R}^m} \prod_{j=1}^m |u_j|^\xi \mathbb{E}\left[\exp\left(i\sum_{j=1}^m u_j X(t_j)\right)\right] \prod_{j=1}^m du_j \prod_{j=1}^m dt_j, \end{aligned} \tag{3.13}$$

where in order to apply the LND property of X, we have replaced the integration over the domain $[t,t+h]$ by the integration over the subset $t < t_1 < .... < t_m < t+h$.

We deal now with the inner multiple integral over the u's. Change the variables of integration by means of the transformation

$$u_j = v_j - v_{j+1},\ j = 1,....,m-1;\ u_m = v_m.$$

Then the linear combination in the exponent in (3.13) is transformed according to

$$\sum_{j=1}^{m} u_j X(t_j) = \sum_{j=1}^{m} v_j (X(t_j) - X(t_{j-1})),$$

where $t_0 = 0$. Since the FWN is a Gaussian process, the characteristic function in (3.13) has the form

$$exp\left(-\frac{1}{2}Var\left[\sum_{j=1}^{m} v_j [X(t_j) - X(t_{j-1})]\right]\right). \tag{3.14}$$

Since $|a-b|^\xi \leq |a|^\xi + |b|^\xi$ for all $0 < \xi < 1$, it follows that

$$\begin{aligned}\prod_{j=1}^{m} |u_j|^\xi &= \prod_{j=1}^{m-1} |v_j - v_{j+1}|^\xi |v_m|^\xi \\ &\leq \prod_{j=1}^{m-1} (|v_j|^\xi + |v_{j+1}|^\xi)|v_m|^\xi. \end{aligned} \tag{3.15}$$

Moreover, the last product is at most equal to a finite sum of terms each of the form $\prod_{j=1}^{m} |x_j|^{\xi\varepsilon_j}$, where $\varepsilon_j = 0, 1$, or 2 and $\sum_{j=1}^{m} \varepsilon_j = m$.
Let us write for simplicity $\sigma_j^2 = E\left(X(t_j) - X(t_{j-1})\right)^2$. Combining the result of Proposition 2.1, (3.14) and (3.15), we get that the integral in (3.13) is dominated by the sum over all possible choices of $(\varepsilon_1, ..., \varepsilon_m) \in \{0,1,2\}^m$ of the following terms

$$\int_{t<t_1<t_2...<t_m<t+h} \int_{\mathbb{R}^m} \prod_{j=1}^{m} |v_j|^{\xi\varepsilon_j} \exp\left(-\frac{C_m}{2}\sum_{j=1}^{m} v_j^2\sigma_j^2\right) \prod_{j=1}^{m} dt_j dv_j,$$

where C_m is the constant given in Proposition 2.1. The change of variable $x_j = \sigma_j v_j$ converts the last integral to

$$J(m,\xi) \times \int_{t<t_1...<t_m<t+h} \prod_{j=1}^{m} \sigma_j^{-1-\xi\varepsilon_j} dt_1...dt_m,$$

where we denote $J(m,\xi) = \int_{\mathbb{R}^m} \prod_{j=1}^m |x_j|^{\xi\varepsilon_j} \exp\left(-\frac{C_m}{2}\sum_{j=1}^m x_j^2\right) \prod_{j=1}^m dx_j$.

Consequently

$$\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m \leq$$
$$J(m,\xi) C_m |y-x|^{\xi m} \int_{t<t_1 \ldots <t_m<t+h} \prod_{j=1}^m \sigma_j^{-1-\xi\varepsilon_j} dt_1 \ldots dt_m. \tag{3.16}$$

According to (3.9), for h sufficiently small, namely $0 < h < \inf(\delta, 1)$, we have

$$\mathbb{E}[X(t_i) - X(t_{i-1})]^2 \geq C(t_i - t_{i-1})^{2H}, \tag{3.17}$$

It follows that the integral on the right hand side of (3.16) is bounded, up to a constant, by

$$\int_{t<t_1 \ldots <t_m<t+h} \prod_{j=1}^m (t_j - t_{j-1})^{-H(1+\xi\varepsilon_j)} dt_1 \ldots dt_m. \tag{3.18}$$

Since, $(t_j - t_{j-1}) < 1$, for all $j \in \{2, \ldots, m\}$, we have

$$(t_j - t_{j-1})^{-H(1+\xi\varepsilon_j)} \leq (t_j - t_{j-1})^{-H(1+2\xi)} \quad \forall\, \varepsilon_j \in \{0,1,2\}.$$

Since by hypothesis $\xi < \frac{1}{2H} - \frac{1}{2}$, the integral in (3.18) is finite. Moreover, by an elementary calculation (cf. Ehm, [8]), for all $m \geq 1$, $h > 0$ and $b_j < 1$,

$$\int_{t<s_1 \ldots <s_m<t+h} \prod_{j=1}^m (s_j - s_{j-1})^{-b_j} ds_1 \ldots ds_m = h^{m-\sum_{j=1}^m b_j} \frac{\prod_{j=1}^m \Gamma(1-b_j)}{\Gamma(1+k-\sum_{j=1}^m b_j)},$$

where $s_0 = t$. It follows that (3.18) is dominated by

$$C_m \frac{h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi))},$$

where we have used $\sum_{j=1}^m \varepsilon_j = m$. Consequently

$$\mathbb{E}\left[L(t+h,y) - L(t,y) - L(t+h,x) + L(t,x)\right]^m$$
$$\leq C_m \frac{|x-y|^{\xi m} h^{m(1-H(1+\xi))}}{\Gamma(1+m(1-H(1+\xi))}. \tag{3.19}$$

This completes the proof of the lemma. □

Proof of Theorem 3.1. Since $L(0, x) = 0$ for all $x \in \mathbb{R}$, hence if we replace t and $t + h$ by 0 and t respectively in (3.11), we obtain

$$\mathbb{E}\left[L(t, y) - L(t, x)\right]^m \leq \widetilde{C}_m |x - y|^{\xi m}. \tag{3.20}$$

The joint continuity of the local time is now straightforward from (3.10), (3.11) and (3.20) and classical two parameter Kolmogorov's theorem (c.f. Berman [4], Theorem 5.1).
The Hölder condition (i) of Theorem 3.1 follows from (3.11) and one parameter Kolmogorov's theorem (see also the proof Theorem 2 in [12], statements after (4.1)).

We turn out to the proof of (ii). According to Theorem 3.1 in [5], the inequalities (3.10), (3.11) and (3.20) implies that (ii) holds for any $\delta < 1 - H(1 + \xi)$, for all $\xi < 1 \wedge \frac{1-H}{2H}$. Letting ξ tends to zero, we obtain the desired result. □

As a classical consequence, we have the following result on the Hausdorff dimension of the level set. We refer to Adler [1] for definition and results for the fBm.

Proposition 3.6. *With probability one, for any interval $I \in [0, 1]$, we have*

$$dim\{t \in I / X(t) = x\} = 1 - H, \tag{3.21}$$

for all x such that $L(I, x) > 0$.

Proof. According to (3.8) and Kolmogorov's theorem, the FWN is β-Hölder continuous for every $\beta < H$. Moreover, the FWN has a jointly continuous local time, then Theorem 8.7.3 in Adler [1] completes the proof of the upper bound, i.e. $dim\{t \in I / X(t) = x\} \leq 1 - H,\ a.s.$
Now by (ii) of Theorem 3.1, the jointly continuous local time of the FWN satisfies an uniform Hölder condition of any order smaller than $1-H$. Then Theorem 8.7.4 of Alder implies that $dim\{t \in I / X(t) = x\} \geq 1 - H,\ a.s.$ for all x such that $L(I, x) > 0$. This completes the proof. □

4. Chung's law for the FWN and pointwise Hölder exponent of local time

The main result of this section is that the FWN satisfies the same form of Chung's law of iterated logarithm (LIL) as the fBm. For an excellent summary on LIL, we refer to the survey paper of Li and Shao [10].

Theorem 4.1. *Then the following Chung's type laws of iterated logarithm hold for the FWN :*

$$\liminf_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|X(t)-X(s)|}{(\delta/\log|\log(\delta)|)^H} = C(H)|a_\infty(t)|. \quad a.s. \tag{4.1}$$

where $C(H)$ is the constant appearing in the Chung law of the fBm.

Proof. Conserving the same notations as above, we can write

$$X(t)-X(s) = a_\infty(t)(B^H(t)-B^H(s))+(a_\infty(t)-a_\infty(s))B^H(s)+X_2(t)-X_2(s)$$

According to Monrad and Rootzen [11], the fBm satisfies (4.1) with $a_\infty \equiv 1$. Then (4.1) will be proved if we show that

$$\lim_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|\Lambda(t,s)|}{(\delta/\log|\log(\delta)|)^H} = 0, \quad a.s. \tag{4.2}$$

where we denote for simplicity

$$\Lambda(t,s) = (a_\infty(t)-a_\infty(s))B^H(s) + X_2(t) - X_2(s).$$

Combining (3.7) and the fact that the function $a_\infty(t)$ is C^1, we obtain

$$E\Lambda^2(t,s) \leq C(t-s)^{2(H+\eta)}, \text{ for all } s\in[t,t+\delta]. \tag{4.3}$$

The expression (4.3) correspond to the assumption (2.1) in [11]. Then, according to Theorem 2.1 of Monrad and Rootzen [11], we have

$$\mathbb{P}\left(\sup_{s\in[t,t+\delta]} |\Lambda(t,s)| > \varepsilon\right) \leq 1-\exp\left(-\frac{\delta}{K\varepsilon^{1/(H+\eta)}}\right),$$

for some constant $K>0$ depending on H and η only. Let us now consider, for example, $\delta_k = k^{-4(H+\eta)/\eta}$ and $\varepsilon_k = \delta_k^{H+\eta/2}$

$$\begin{aligned}
\sum_k \mathbb{P}\left(\sup_{s\in[t,t+\delta_k]} |\Lambda(t,s)| > \varepsilon_k\right) &\leq \sum_k \left(1-\exp\left(-\frac{\delta_k}{K\varepsilon_k^{1/(H+\eta)}}\right)\right)\\
&\sim \frac{1}{K}\sum_k \delta_k^{\frac{\eta}{2(H+\eta)}}\\
&= \frac{1}{K}\sum_k k^{-2} < \infty.
\end{aligned}$$

By Borel Cantelli lemma, it follows that $\sup_{s\in[t,t+\delta_k]} |\Lambda(t,s)| \leq \delta_k^{H+\eta/2}$ almost surely. Furthermore, for $\delta_{k+1} \leq \delta \leq \delta_k$, we have almost surely

$$\begin{aligned}
\sup_{s\in[t,t+\delta]} |\Lambda(t,s)| &\leq \sup_{s\in[t,t+\delta_k]} |\Lambda(t,s)| \\
&\leq \delta_k^{H+\eta/2} \\
&\leq \delta^{H+\eta/2}\left(\frac{\delta_k}{\delta_{k+1}}\right)^{H+\eta/2} \\
&\leq 2^{4(H+\eta)/\eta}\delta^{H+\eta/2}.
\end{aligned}$$

Consequently (4.2) is proved. This completes the proof of the theorem. □

Remark 4.2. The main interest of the previous proof is that it can be used to generalize many other LIL known for the fBm to the FWN, and always the constant is derived from the one corresponding to the fBm. For example, we can extend the following LIL given in Li and Shao ([10], equation (7.5)) for the fBm to the FWN as follows :

$$\limsup_{\delta\to 0} \sup_{s\in[t,t+\delta]} \frac{|X(t)-X(s)|}{\delta^H\left(\log|\log(\delta)|\right)^{1/2}} = C(H)|a_\infty(t)|, \quad a.s. \tag{4.4}$$

where $C(H) = \sqrt{\dfrac{2\pi}{H\Gamma(2H)\sin(\pi H)}}$.

The Chung laws are known to be linked to the optimality of the moduli of continuity of local times of stochastic processes. More precisely

Lemma 4.3. *The following lower bounds for the moduli of continuity of local times holds*

$$\frac{1}{2C(H)|a_\infty(t)|} \leq \limsup_{\delta\to 0} \sup_{x\in\mathbb{R}} \frac{L(t+\delta,x)-L(t,x)}{\delta^{1-H}(loglog(\delta^{-1}))^H} \qquad a.s. \tag{4.5}$$

Proof. Combining (4.1) and the following elementary computation

$$\begin{aligned}
\delta &= \int_{X([t,t+\delta])} L([t,t+\delta],x)dx \\
&\leq \sup_{x\in\mathbb{R}} L([t,t+\delta],x) \sup_{s,s'\in[t,t+\delta]} |X(s)-X(s')|, \\
&\leq 2\sup_{x\in\mathbb{R}} L([t,t+\delta],x) \sup_{s\in[t,t+\delta]} |X(s)-X(t)|,
\end{aligned}$$

we obtain the lemma. □

Remark 4.4. The upper bound in Lemma 4.3 need more fine properties, like the strong local nondeterminism of the FWN, which we are not able to derive with the method of this paper. We refer to the survey paper of Xiao [14] for an excellent summary on this subject.

Recall that the pointwise Hölder exponent of a stochastic process X at t_0 is defined by

$$\alpha_X(t_0,\omega) = \sup\left\{\alpha > 0,\ \lim_{\rho\to 0}\frac{X(t_0+\rho,\omega) - X(t_0,\omega)}{\rho^\alpha} = 0\right\}. \tag{4.6}$$

Proposition 4.5. *Consider the process* $L^*(t) = \{\sup_{x\in\mathbb{R}} L(t,x),\ t \in [0,1]\}$. *Then, the pointwise Hölder exponent* α_L *of* L^* *at each* t *satisfies*

$$\alpha_L(t) \geq 1 - H \quad a.s.$$

Proof. The local time $L(1,.)$ vanishes outside of the compact set $U = X([0,1])$, since the trajectories of X are a.s. continuous . Then it follows from Theorem 3.1 (ii) and the definition (4.6) that $\alpha_L(t) \geq 1-H.\ a.s.$ □

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