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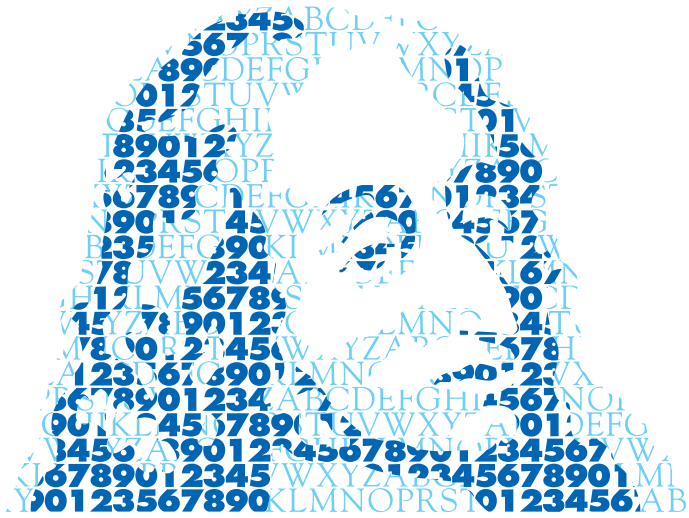

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Left-Garside categories, self-distributivity, and braids

Patrick Dehornoy

Abstract

In connection with the emerging theory of Garside categories, we develop the notions of a left-Garside category and of a locally left-Garside monoid. In this framework, the relationship between the self-distributivity law LD and braids amounts to the result that a certain category associated with LD is a left-Garside category, which projects onto the standard Garside category of braids. This approach leads to a realistic program for establishing the Embedding Conjecture of [Dehornoy, Braids and Self-distributivity, Birkhaüser (2000), Chap. IX].

The notion of a Garside monoid emerged at the end of the 1990's [23, 19] as a development of Garside's theory of braids [32], and it led to many developments [2, 4, 5, 6, 8, 7, 14, 13, 15, 31, 33, 36, 43, 42, 45, 46, 47, ...]. More recently, Bessis [3], Digne–Michel [27], and Krammer [39] introduced the notion of a Garside category as a further extension, and they used it to capture new nontrivial examples and improve our understanding of their algebraic structure. The concept of a Garside category is also used in [34], and it is already implicit in [25] and [35], and maybe in many diagrams of [18].

In this paper we describe and investigate a new example of (left)-Garside category, namely a certain category $\mathcal{LD}^+$ associated with the left self-distributivity law

$$x(yz) = (xy)(xz). \tag{LD}$$

The interest in this law originated in the discovery of several nontrivial structures that obey it, in set theory [16, 41] and in low-dimensional topology [37, 30, 44]. A rather extensive theory was developed in the decade 1985-95 [18]. Investigating self-distributivity in the light of Garside categories seems to be a good idea. It turns out that a large part of the theory developed so far can be summarized into one single statement, namely

Keywords: Garside category, Garside monoid, self-distributivity, braid, greedy normal form, least common multiple, LD-expansion.
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The category $\mathcal{LD}^+$ is a left-Garside category,

(this is the first part of Theorem 6.1 below).

The interest of the approach should be at least triple. First, it gives an opportunity to restate a number of previously unrelated properties in a new language that should make them more easily understandable. In particular, the connection between self-distributivity and braids is now expressed in the simple statement:

There exists a right-lcm preserving surjective functor of $\mathcal{LD}^+$ to the Garside category of positive braids,

(second part of Theorem 6.1). This result allows one to recover most of the usual algebraic properties of braids as a direct application of the properties of $\mathcal{LD}^+$: roughly speaking, Garside's theory of braids is the emerged part if an iceberg, namely the algebraic theory of self-distributivity.

Second, a direct outcome of the current approach is a realistic program for establishing the Embedding Conjecture. The latter is the most puzzling open question involving free self-distributive systems. Among others, it says that the equivalence class of any bracketed expression under self-distributivity is a semilattice, *i.e.*, any two expressions admit a least upper bound with respect to a certain partial ordering. Many equivalent forms of the conjecture are known [18, Chapter IX]. At the moment, no complete proof is available, but we establish the following new result

Unless the left-Garside category $\mathcal{LD}^+$ is not regular, the Embedding Conjecture is true,

(Theorem 6.2). This result reduces a possible proof of the conjecture to a (long) sequence of verifications.

Third, the category $\mathcal{LD}^+$ seems to be a seminal example of a left-Garside category, quite different from all previously known examples of Garside categories. In particular, being strongly asymmetric, $\mathcal{LD}^+$ is not a Garside category. The interest of investigating such objects *per se* is not obvious, but the existence of a nontrivial example such as $\mathcal{LD}^+$ seems to be a good reason, and a help for orientating further research. In particular, our approach emphasizes the role of *locally left-Garside monoids*[1]: this is a

[1]This is not the notion of a locally Garside monoid in the sense of [27]; we think that the name "preGarside" is more relevant for that notion, which involves no counterpart of any Garside element or map, but is only the common substratum of all Garside structures.

monoid M that fails to be Garside because no global element Δ exists, but nevertheless possesses a family of elements Δ_x that locally play the role of the Garside element and are indexed by a set on which the monoid M partially acts. Most of the properties of left-Garside monoids extend to locally left-Garside monoids, in particular the existence of least common multiples and, in good cases, of the greedy normal form.

Our definition of a left-Garside category is borrowed from [27] (up to a slight change in the formal setting, see Remark 2.6). Several proofs in Sections 2 and 3 use arguments that are already present, in one form or another, in [1, 48, 28, 29, 12, 19, 35] and now belong to folklore. Most appear in the unpublished paper [27] by Digne and Michel, and are implicit in Krammer's paper [39]. Our reasons for including such arguments here is that adapting them to the current weak context requires some useful polishing, and is necessary to explain our two main new notions, namely locally Garside monoids and regular left-Garside categories.

The paper is organized in two parts. The first one (Sections 1 to 3) contains those general results about left-Garside categories and locally left-Garside monoids that will be needed in the sequel, in particular the construction and properties of the greedy normal form. The second part (Sections 4 to 8) deals with the specific case of the category $\mathcal{LD}^+$ and its connection with braids. Sections 4 and 5 review basic facts about the self-distributivity law and explain the construction of the category $\mathcal{LD}^+$. Section 6 is devoted to proving that $\mathcal{LD}^+$ is a left-Garside category and to showing how the results of Section 3 might lead to a proof of the Embedding Conjecture. In Section 7, we show how to recover the classical algebraic properties of braids from those of $\mathcal{LD}^+$. Finally, we explain in Section 8 some alternative solutions for projecting $\mathcal{LD}^+$ to braids. In an appendix, we briefly describe what happens when the associativity law replaces the self-distributivity law: here also a left-Garside category appears, but a trivial one.

We use $\mathbb{N}$ for the set of all positive integers.

1. Left-Garside categories

We define left-Garside categories and describe a uniform way of constructing such categories from so-called locally left-Garside monoids, which are monoids with a convenient partial action.

1.1. **Left-Garside monoids**

Let us start from the now classical notion of a Garside monoid. Essentially, a Garside monoid is a monoid in which divisibility has good properties, and, in addition, there exists a distinguished element Δ whose divisors encode the whole structure. Slightly different versions have been considered [23, 19, 26], the one stated below now being the most frequently used. In this paper, we are interested in one-sided versions involving left-divisibility only, hence we shall first introduce the notion of a *left-Garside monoid*.

Throughout the paper, if a, b are elements of a monoid—or, from Section 1.2, morphisms of a category—we say that a *left-divides* b, or that a is a *left-divisor* of b, denoted $a \preccurlyeq b$, if there exists c satisfying $ac = b$. The set of all left-divisors of a is denoted by $\mathrm{Div}(a)$. If $ac = b$ holds with $c \neq 1$, we say that a is a *proper* left-divisor of b, denoted $a \prec b$.

We shall always consider monoids M where $\preccurlyeq$ is a partial ordering. If two elements a, b of M admit a greatest lower bound c with respect to $\preccurlyeq$, the latter is called a *greatest common left-divisor*, or *left-gcd*, of a and b, denoted $c = \gcd(a, b)$. Similarly, a $\preccurlyeq$-least upper bound d is called a *least common right-multiple*, or *right-lcm*, of a and b, denoted $d = \mathrm{lcm}(a, b)$. We say that M *admits local right-lcm's* if any two elements of M that admit a common right-multiple admit a right-lcm.

Finally, if M is a monoid and S, S' are subsets of M, we say that S *left-generates* S' if every nontrivial element of S' admits at least one proper left-divisor belonging to S.

Definition 1.1. A monoid M is called *left-preGarside* if

- (LG_0) for each a in M, every $\prec$-increasing sequence in $\mathrm{Div}(a)$ is finite,
- (LG_1) M is left-cancellative,
- (LG_2) M admits local right-lcm's.

An element Δ of M is called a *left-Garside element* if

- (LG_3) $\mathrm{Div}(\Delta)$ left-generates M, and $a \preccurlyeq \Delta$ implies $\Delta \preccurlyeq a\Delta$.

A monoid M is called *left-Garside* if it is left-preGarside and possesses at least one left-Garside element.

Using "generates" instead of "left-generates" in (LG_3) would make no difference, by the following trivial remark—but the assumption (LG_0) is crucial, of course.

Lemma 1.2. *Assume that M is a monoid satisfying* (LG_0). *Then every subset S left-generating M generates M.*

Proof. Let a be a nontrivial element of M. By definition there exist $a_1 \neq 1$ in S and a' satisfying $a = a_1 a'$. If a' is trivial, we are done. Otherwise, there exist $a_2 \neq 1$ in S and a'' satisfying $a' = a_2 a''$, and so on. The sequence 1, a_1, $a_1 a_2$, ... is $\prec$-increasing and it lies in $\mathrm{Div}(a)$, hence it must be finite, yielding $a = a_1 ... a_d$ with $a_1, ..., a_d$ in S. □

Right-divisibility is defined symmetrically: *a right-divides b* if $b = ca$ holds for some c. Then the notion of a right-(pre)Garside monoid is defined by replacing left-divisibility by right-divisibility and left-product by right-product in Definition 1.1.

Definition 1.3. A monoid M is called *Garside with Garside element* Δ if M is both left-Garside with left-Garside element Δ and right-Garside with right-Garside element Δ.

The equivalence of the above definition with that of [26] is easily checked. The seminal example of a Garside monoid is the braid monoid B_n^+ equipped with Garside's fundamental braid Δ_n, see for instance [32, 29]. Other classical examples are free abelian monoids and, more generally, all spherical Artin–Tits monoids [10], as well as the so-called dual Artin–Tits monoids [9, 3]. Every Garside monoid embeds in a group of fractions, which is then called a *Garside group*.

Let us mention that, if a monoid M is left-Garside, then mild conditions imply that it is Garside: essentially, it is sufficient that M is right-cancellative and that the left- and right-divisors of the left-Garside element Δ coincide [19].

1.2. **Left-Garside categories**

Recently, it appeared that a number of results involving Garside monoids still make sense in a wider context where categories replace monoids [3, 27, 39]. A category is similar to a monoid, but the product of two elements is defined only when the target of the former is the source of the latter. In the case of Garside monoids, the main benefit of considering categories is that it allows for relaxing the existence of the global Garside element Δ into a weaker, local version depending on the objects of the category, namely a map from the objects to the morphisms.

We refer to [40] for some basic vocabulary about categories—we use very little of it here.

Notation 1.4. Throughout the paper, composition of morphisms is denoted by a multiplication on the right: fg means "f then g". If f is a morphism, the source of f is denoted $\partial_0 f$, and its target is denoted $\partial_1 f$. In all examples, we shall make the source and target explicit: morphisms are triples (x, f, y) satisfying

$$\partial_0(x, f, y) = x, \qquad \partial_1(x, f, y) = y.$$

A morphism f is said to be nontrivial if $f \neq 1_{\partial_0 f}$ holds.

We extend to categories the terminology of divisibility. So, we say that a morphism f is a *left-divisor* of a morphism g, denoted $f \preccurlyeq g$, if there exists h satisfying $fh = g$. If, in addition, h can be assumed to be nontrivial, we say that $f \prec g$ holds. Note that $f \preccurlyeq g$ implies $\partial_0 f = \partial_0 g$. We denote by $\mathrm{Div}(f)$ the collection of all left-divisors of f.

The following definition is equivalent to Definition 2.10 given by F. Digne and J. Michel in [27] (see Remark 2.6 below).

Definition 1.5. A category $\mathcal{C}$ is called *left-preGarside* if

(LG_0) for each f in $\mathcal{H}om(\mathcal{C})$, every $\prec$-ascending sequence in $\mathrm{Div}(f)$ is finite,

(LG_1) $\mathcal{H}om(\mathcal{C})$ admits left-cancellation,

(LG_2) $\mathcal{H}om(\mathcal{C})$ admits local right-lcm's.

A map $\Delta : \mathcal{O}bj(\mathcal{C}) \to \mathcal{H}om(\mathcal{C})$ is called a *left-Garside map* if, for each object x, we have $\partial_0\Delta(x) = x$ and

(LG_3) $\mathrm{Div}(\Delta(x))$ left-generates $\mathrm{Hom}(x, -)$, and $f \preccurlyeq \Delta(x)$ implies $\Delta(x) \preccurlyeq f\Delta(\partial_1 f)$.

A category $\mathcal{C}$ is called *left-Garside* if it is left-preGarside and possesses at least one left-Garside map.

Example 1.6. Assume that M is a left-Garside monoid with left-Garside element Δ. One trivially obtains a left-Garside category $\mathcal{C}(M)$ by putting

$$\mathcal{O}bj(\mathcal{C}(M)) = \{1\}, \quad \mathcal{H}om(\mathcal{C}(M)) = \{1\} \times M \times \{1\}, \quad \Delta(1) = \Delta.$$

Another left-Garside category $\widehat{\mathcal{C}}_M$ can be attached with M, namely taking

$$\mathcal{O}bj(\widehat{\mathcal{C}}(M)) = M, \quad \mathcal{H}om(\widehat{\mathcal{C}}(M)) = \{(a, b, c) \mid ab = c\}, \quad \Delta(a) = \Delta.$$

It is natural to call $\widehat{\mathcal{C}}(M)$ the *Cayley category* of M since its graph is the Cayley graph of M (defined provided M is also right-cancellative).

The notion of a right-Garside category is defined symmetrically, exchanging left and right everywhere and exchanging the roles of source and target. In particular, the map Δ and Axiom (LG_3) are to be replaced by a map ∇ satisfying $\partial_1\nabla(x) = x$ and, using $b \mathrel{\widetilde{\preccurlyeq}} a$ for "a right-divides b",

(LG_3) $\nabla(y)$ right-generates $\mathrm{Hom}(-, y)$, and $\nabla(y) \mathrel{\widetilde{\preccurlyeq}} f$ implies $\nabla(\partial_0 y)f \mathrel{\widetilde{\preccurlyeq}} \nabla(y)$.

Then comes the natural two-sided version of a Garside category [3, 27].

Definition 1.7. A category $\mathcal{C}$ is called *Garside with Garside map* Δ if $\mathcal{C}$ is left-Garside with left-Garside map Δ and right-Garside with right-Garside map ∇ satisfying $\Delta(x) = \nabla(\partial_1(\Delta(x))$ and $\nabla(y) = \Delta(\partial_0(\nabla(y))$ for all objects x, y.

It is easily seen that, if M is a Garside monoid, then the categories $\mathcal{C}(M)$ and $\widehat{\mathcal{C}}(M)$ of Example (1.6) are Garside categories. Insisting that the maps Δ and ∇ involved in the left- and right-Garside structures are connected as in Definition 1.7 is crucial: see Appendix for a trivial example where the connection fails.

1.3. **Locally left-Garside monoids**

We now describe a general method for constructing a left-Garside category starting from a monoid equipped with a partial action on a set. The trivial examples of Example 1.6 enter this family, and so do the two categories $\mathcal{LD}^+$ and $\mathcal{B}^+$ investigated in the second part of this paper.

We start with a convenient notion of partial action of a monoid on a set. Several definitions could be thought of; here we choose the one that is directly adapted to the subsequent developments.

Definition 1.8. Assume that M is a monoid. A not necessarily everywhere defined function $\alpha : M \to (X \to X)$ is called a *partial (right) action* of M on X if, writing $x \bullet a$ for $\alpha(a)(x)$,

(*i*) $x \bullet 1 = x$ holds for each x in X,

(*ii*) $(x \bullet a) \bullet b = x \bullet ab$ holds for all x, a, b, this meaning that either both terms are defined and they are equal, or neither is defined,

(*iii*) for each finite subset S in M, there exists x in X such that $x \bullet a$ is defined for each a in S.

In the above context, for each x in X, we put

$$M_x = \{\ a \in M \mid\ x \bullet a \text{ is defined }\}. \tag{1.1}$$

Then Conditions (i), (ii), (iii) of Definition 1.8 imply

$$1 \in M_x, \quad ab \in M_x \Leftrightarrow (a \in M_x\ \&\ b \in M_{x \bullet a}), \quad \forall \,\text{finite}\, S\ \exists x (S \subseteq M_x).$$

A monoid action in the standard sense, *i.e.*, an everywhere defined action, is a partial action. For a more typical case, consider the n-strand Artin braid monoid B_n^+. We recall that B_n^+ is defined for $n \leqslant \infty$ by the monoid presentation

$$B_n^+ = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{lll} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for} & |i-j| \geqslant 2 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{for} & |i-j| = 1 \end{array} \right\rangle^+. \tag{1.2}$$

Then we obtain a partial action of B_∞^+ on $\mathbb{N}$ by putting

$$n \bullet a = \begin{cases} n & \text{if } a \text{ belongs to } B_n^+, \\ \text{undefined} & \text{otherwise.} \end{cases} \tag{1.3}$$

A natural category can then be associated with every partial action of a monoid.

Definition 1.9. For α a partial action of a monoid M on a set X, the category *associated with* α, denoted $\mathcal{C}(\alpha)$, or $\mathcal{C}(M, X)$ if the action is clear, is defined by

$$\mathcal{O}bj(\mathcal{C}(M,X)) = X, \quad \mathcal{H}om(\mathcal{C}(M,X)) = \{(x, a, x \bullet a) \mid x \in X, a \in M\}.$$

Example 1.10. We shall denote by $\mathcal{B}^+$ the category associated with the action (1.3) of B_∞^+ on $\mathbb{N}$, *i.e.*, we put

$$\mathcal{O}bj(\mathcal{B}^+) = \mathbb{N}, \quad \mathcal{H}om(\mathcal{B}^+) = \{(n, a, n) \mid n \in B_n^+\}.$$

Define $\Delta : \mathcal{O}bj(\mathcal{B}^+) \to \mathcal{H}om(\mathcal{B}^+)$ by $\Delta(n) = (n, \Delta_n, n)$. Then the well known fact that B_n^+ is a Garside monoid for each n [32, 38] easily implies that $\mathcal{B}^+$ is a Garside category, as will be formally proved in Proposition 1.12 below.

The example of $\mathcal{B}^+$ shows the benefit of going from a monoid to a category. The monoid B_∞^+ is not a (left)-Garside monoid, because it is of infinite type and there cannot exist a global Garside element Δ. However, the partial action of (1.3) enables us to restrict to subsets B_n^+ (submonoids in the current case) for which Garside elements exist: with the notation of (1.1), B_n^+ is $(B_\infty^+)_n$. Thus the categorical context enables us to capture

the fact that B_∞^+ is, in some sense, *locally* Garside. It is easy to formalize these ideas in a general setting.

Definition 1.11. Let M be a monoid with a partial action α on a set X. A sequence $(\Delta_x)_{x\in X}$ of elements of M is called a *left-Garside sequence* for α if, for each x in X, the element $x \bullet \Delta_x$ is defined and

($\mathsf{LG}_3^{\ell oc}$) $\mathrm{Div}(\Delta_x)$ left-generates M_x and $a \preccurlyeq \Delta_x$ implies $\Delta_x \preccurlyeq a\, \Delta_{x\bullet a}$.

The monoid M is said to be *locally left-Garside with respect to* α if it is left-preGarside and there is at least one left-Garside sequence for α.

A typical example of a locally left-Garside monoid is B_∞^+ with its action (1.3) on $\mathbb{N}$. Indeed, the sequence $(\Delta_n)_{n\in\mathbb{N}}$ is clearly a left-Garside sequence for (1.3).

The next result should appear quite natural.

Proposition 1.12. *Assume that M is a locally left-Garside monoid with left-Garside sequence $(\Delta_x)_{x\in X}$. Then $\mathcal{C}(M,X)$ is a left-Garside category with left-Garside map Δ defined by $\Delta(x) = (x, \Delta_x, x \bullet \Delta_x)$.*

Proof. By definition, $(x,a,y) \preccurlyeq (x',a',y')$ in $\mathcal{C}(M,X)$ implies $x' = x$ and $a \preccurlyeq a'$ in M. So the hypothesis that M satisfies (LG_0) implies that $\mathcal{C}(M,X)$ does.

Next, $(x,a,y)(y,b,z) = (x,a,y)(y,b',z')$ implies $ab = ab'$ in M, hence $b = b'$ by (LG_1), and, therefore, $\mathcal{C}(M,X)$ satisfies (LG_1).

Assume $(x,a,y)(y,b',x') = (x,b,z)(z,a',x')$ in $\mathcal{H}om(\mathcal{C}(M,X))$. Then $ab' = ba'$ holds in M. By (LG_2), a and b admit a right-lcm c, and we have $a \preccurlyeq c$, $b \preccurlyeq c$, and $c \preccurlyeq ab'$. By hypothesis, $x \bullet ab'$ is defined, hence so is $x \bullet c$, and it is obvious to check that $(x, c, x \bullet c)$ is a right-lcm of (x,a,y) and (x,b,z) in $\mathcal{H}om(\mathcal{C}(M,X))$. Hence $\mathcal{C}(M,X)$ satisfies (LG_2).

Assume that (x,a,y) is a nontrivial morphism of $\mathcal{H}om(\mathcal{C}(M,X))$. This means that a is nontrivial, so, by ($\mathsf{LG}_3^{\ell oc}$), some left-divisor a' of Δ_x is a left-divisor of a. Then $(x, a', x \bullet a') \preccurlyeq \Delta(x)$ holds, and $\Delta(x)$ left-generates $\mathrm{Hom}(x,-)$.

Finally, assume $(x,a,y) \preccurlyeq \Delta(x)$ in $\mathcal{H}om(\mathcal{C}(M,X))$. This implies $a \preccurlyeq \Delta_x$ in M. Then ($\mathsf{LG}_3^{\ell oc}$) in M implies $\Delta_x \preccurlyeq a\Delta_y$. By hypothesis, $y \bullet \Delta_y$ is defined, and we have $(x,a,y)\Delta(y) = (x, a\Delta_y, x \bullet a\Delta_y)$, of which $(x, \Delta_x, x \bullet \Delta_x)$ is a left-divisor in $\mathcal{H}om(\mathcal{C}(M,X))$. So ($\mathsf{LG}_3$) is satisfied in $\mathcal{C}(M,X)$. □

It is not hard to see that, conversely, if M is a left-preGarside monoid, then $\mathcal{C}(M, X)$ being a left-Garside category implies that M is a locally left-Garside monoid. We shall not use the result here.

If M has a total action on X, *i.e.*, if $x \bullet a$ is defined for all x and a, the sets M_x coincide with M, and Condition $(\mathsf{LG}_3^{\ell oc})$ reduces to (LG_3). In this case, each element Δ_x is a left-Garside element in M, and M is a left-Garside monoid. A similar result holds for each set M_x that turns out to be a submonoid (if any).

Proposition 1.13. *Assume that M is a locally left-Garside monoid with left-Garside sequence $(\Delta_x)_{x\in X}$, and x is such that M_x is closed under product and $\Delta_y = \Delta_x$ holds for each y in M_x. Then M_x is a left-Garside submonoid of M.*

Proof. By definition of a partial action, $x \bullet 1$ is defined, so M_x contains 1, and it is a submonoid of M. We show that M_x satisfies $(\mathsf{LG}_0), (\mathsf{LG}_1), (\mathsf{LG}_2)$, and (LG_3). First, a counter-example to (LG_0) in M_x would be a counter-example to (LG_0) in M, hence M_x satisfies (LG_0). Similarly, an equality $ab = ab'$ with $b \neq b'$ in M_x would also contradict (LG_1) in M, so M_x satisfies (LG_1). Now, assume that a and b admit in M_x, hence in M, a common right-multiple c. Then a and b admit a right-lcm c' in M. By hypothesis, $x \bullet c$ is defined, and $c' \preccurlyeq c$ holds. By definition of a partial action, $x \bullet c'$ is defined as well, *i.e.*, c' lies in M_x, and it is a right-lcm of a and b in M_x. So M_x satisfies (LG_2), and it is left-preGarside.

Next, Δ_x is a left-Garside element in M_x. Indeed, let a be any nontrivial element of M_x. By $(\mathsf{LG}_3^{\ell oc})$, there exists a nontrivial divisor a' of a satisfying $a' \preccurlyeq \Delta_x$. By definition of a partial action, $x \bullet a'$ is defined, so it belongs to M_x, and Δ_x left-generates M_x. Finally, assume $a \preccurlyeq \Delta_x$. As Δ_x belongs to M_x, this implies $a \in M_x$, hence $\Delta_x \preccurlyeq a\Delta_{x\bullet a}$ by $(\mathsf{LG}_3^{\ell oc})$, *i.e.*, $\Delta_x \preccurlyeq a\Delta_x$ since we assumed that the sequence Δ is constant on M_x. So Δ_x is a left-Garside element in M_x. □

2. **Simple morphisms**

We return to general left-Garside categories and establish a few basic results. As in the case of Garside monoids, an important role is played by the divisors of Δ, a local notion here.

2.1. **Simple morphisms and the functor ϕ**

Hereafter, we always use Δ as the default notation for the (left)-Garside map with left-Garside map Δ involved in the considered structure.

Definition 2.1. Assume that $\mathcal{C}$ is a left-Garside category. A morphism f of $\mathcal{C}$ is called *simple* if it is a left-divisor of $\Delta(\partial_0 f)$. In this case, we denote by f^* the unique simple morphism satisfying $f\, f^* = \Delta(\partial_0 f)$. The family of all simple morphisms in $\mathcal{C}$ is denoted by $\mathcal{H}om^{sp}(\mathcal{C})$.

By definition, every identity morphism 1_x is a left-divisor of every morphism with source x, hence in particular of $\Delta(x)$. Therefore 1_x is simple.

Definition 2.2. Assume that $\mathcal{C}$ is a left-Garside category. We put $\phi(x) = \partial_1(\Delta(x))$ for x in $\mathcal{O}bj(\mathcal{C})$, and $\phi(f) = f^{**}$ for f in $\mathcal{H}om^{sp}(\mathcal{C})$.

Although straightforward, the following result is fundamental—and it is the main argument for stating (LG_3) in the way we did.

Lemma 2.3. *Assume that $\mathcal{C}$ is a left-Garside category.*
(i) If f is a simple morphism, so are f^* and $\phi(f)$.
(ii) Every right-divisor of a morphism $\Delta(x)$ is simple.

Proof. (*i*) By (LG_3), we have $f f^* = \Delta(\partial_0 f) \preccurlyeq f\Delta(\partial_1 f)$, hence $f^* \preccurlyeq \Delta(\partial_1 f)$ by left-cancelling f. Therefore, f^* is simple. Applying the result to f^*, one shows that $\phi(f)$—as well as $\phi^k(f)$ for each positive k—are simple.

(*ii*) Assume that g is a right-divisor of $\Delta(x)$. This means that there exists f satisfying $fg = \Delta(x)$, hence $g = f^*$ by (LG_1). Then g is simple by (*i*). □

Lemma 2.4. *Assume that $\mathcal{C}$ is a left-Garside category.*
(i) The morphisms 1_x are the only left- or right-invertible morphisms in $\mathcal{C}$.
(ii) Every morphism of $\mathcal{C}$ is a product of simple morphisms.
(iii) There is a unique way to extend ϕ into a functor of $\mathcal{C}$ into itself.
(iv) The map Δ is a natural transformation of the identity functor into ϕ, i.e.*, for each morphism f, we have*

$$f\,\Delta(\partial_1 f) = \Delta(\partial_0 f)\ \phi(f). \tag{2.1}$$

Proof. (*i*) Assume $fg = 1_x$ with $f \neq 1_x$ and $g \neq 1_{\partial_1 f}$. Then we have

$$1_x \prec f \prec fg \prec f \prec fg \prec ...,$$

an infinite $\prec$-increasing sequence in $\mathrm{Div}(1_x)$ that contradicts (LG_0).

(*ii*) Let f be a morphism of $\mathcal{C}$, and let $x = \partial_0 f$. If f is trivial, then it is simple, as observed above. We wish to prove that simple morphisms generate $\mathcal{H}om(\mathcal{C})$. Owing to Lemma 1.2, it is enough to prove that simple morphisms left-generate $\mathcal{H}om(\mathcal{C})$, *i.e.*, that every nontrivial morphism with source x is left-divisible by a simple morphism with source x, in other words by a left-divisor of $\Delta(x)$. This is exactly what the first part of Condition (LG_3) claims.

(*iii*) Up to now, ϕ has been defined on objects, and on simple morphisms. Note that, by construction, (2.1) is satisfied for each simple morphism f. Indeed, applying Definition 2.1 for f and f^* gives the relations

$$ff^* = \Delta(\partial_0 f) \quad \text{and} \quad f^* f^{**} = \Delta(\partial_0 f^*) = \Delta(\partial_1 f),$$

whence

$$f\Delta(\partial_1 f) = ff^* f^{**} = \Delta(\partial_0 f) f^{**} = \Delta(\partial_0 f)\phi(f).$$

Applying this to $f = 1_x$ gives $\Delta(x) = \Delta(x)\phi(1_x)$, hence $\phi(1_x) = 1_{\phi(x)}$ by (LG_1).

Let f be an arbitrary morphism of $\mathcal{C}$, and let $f_1 ... f_p$ and $g_1 ... g_q$ be two decompositions of f as a product of simple morphisms, which exist by (*ii*). Repeatedly applying (2.1) to $f_p, ..., f_1$ and $g_q, ..., g_1$ gives

$$\begin{aligned} f\Delta(\partial_1 f) &= f_1 ... f_p \Delta(\partial_1 f) = \Delta(\partial_0 f)\phi(f_1)...\phi(f_p) \\ &= g_1 ... g_q \Delta(\partial_1 f) = \Delta(\partial_0 f)\phi(g_1)...\phi(g_q). \end{aligned}$$

By (LG_1), we deduce $\phi(f_1)...\phi(f_p) = \phi(g_1)...\phi(g_q)$, and therefore there is no ambiguity in defining $\phi(f)$ to be the common value. In this way, ϕ is extended to all morphisms in such a way that ϕ is a functor and (2.1) always holds. Conversely, the above definition is clearly the only one that extends ϕ into a functor.

(*iv*) We have seen above that (2.1) holds for every morphism f, so nothing new is needed here. See Figure 1 for an illustration. □

Lemma 2.5. *Assume that $\mathcal{C}$ is a left-Garside category. Then, for each object x and each simple morphism f, we have*

$$\phi(\Delta(x)) = \Delta(\phi(x)) \quad \textit{and} \quad \phi(f^*) = \phi(f)^*. \tag{2.2}$$

Proof. By definition, the source of $\Delta(x)$ is x and its target is $\phi(x)$, hence applying (2.1) with $f = \Delta(x)$ yields $\Delta(x)\Delta(\phi(x)) = \Delta(x)\phi(\Delta(x))$, hence $\Delta(\phi(x)) = \phi(\Delta(x))$ after left-cancelling $\Delta(x)$.

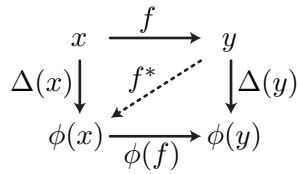


FIGURE 1. Relation (2.1): the Garside map Δ viewed as a natural transformation from the identity functor to the functor ϕ.

On the other hand, let $x = \partial_0 f$. Then we have $ff^* = \Delta(x)$, and $\partial_0(\phi(f)) = \phi(x)$. Applying ϕ and the above relation, we find

$$\phi(f)\phi(f^*) = \phi(\Delta(x)) = \Delta(\phi(x)) = \Delta(\partial_0(\phi(f))) = \phi(f)\phi(f)^*.$$

Left-cancelling $\phi(f)$ yields $\phi(f^*) = \phi(f)^*$. □

Remark 2.6. We can now see that Definition 1.5 is equivalent to Definition 2.10 of [27]: the only difference is that, in the latter, the functor ϕ is part of the definition. Lemma 2.4(*iv*) shows that a left-Garside category in our sense is a left-Garside category in the sense of [27]. Conversely, the hypothesis that ϕ and Δ satisfy (2.1) implies that, for $f : x \to y$, we have $\Delta(x)\phi(y) = f\Delta(y)$, whence $\Delta(x) \preccurlyeq f\Delta(y)$ and $f^*\phi(y) = \Delta(y)$, which, by (LG_1), implies $\phi(f) = f^{**}$. So every left-Garside category in the sense of [27] is a left-Garside category in the sense of Definition 1.5.

2.2. **The case of a locally left-Garside monoid**

We now consider the particular case of a category $\mathcal{C}(M, X)$ associated with a partial action of a monoid M.

Lemma 2.7. *Assume that M is a locally left-Garside monoid with left-Garside sequence $(\Delta_x)_{x\in X}$. Then $\Delta_x \preccurlyeq a\Delta_{x\bullet a}$ holds whenever $x \bullet a$ is defined, and, defining $\phi_x(a)$ by $\Delta_x\phi_x(a) = a\Delta_{x\bullet a}$, we have*

$$\phi(x) = x \bullet \Delta_x, \quad \phi((x, a, y)) = (\phi(x), \phi_x(a), \phi(y)). \tag{2.3}$$

Proof. Assume that $x \bullet a$ is defined. Lemma 2.4 (*ii*) implies that the morphism $(x, a, x \bullet a)$ of $\mathcal{C}(M, X)$ can be decomposed into a finite product of simple morphisms (x_0, a_1, x_1), ... , (x_{d-1}, a_d, x_d). This implies $a = a_1...a_d$ in M. The hypothesis that each morphism (x_{i-1}, a_i, x_i) is simple implies

$\Delta_{x_{i-1}} \preccurlyeq a_i \Delta_{x_i}$ for each i, whence

$$\Delta_x \preccurlyeq a_1 \Delta_{x_1} \preccurlyeq a_1 a_2 \Delta_{x_2} \preccurlyeq ... \preccurlyeq a_1 ... a_d \Delta_{x_d} = a \Delta_{x \bullet a}.$$

Hence, for each element a in M_x, there exists a unique element a' satisfying $\Delta_x a' = a \Delta_{x \bullet a}$, and this is the element we define to be $\phi_x(a)$. Then, $x \bullet a = y$ implies

$$(x, a, y)(y, \Delta_y, \phi(y)) = (x, \Delta_x, \phi(x))(\phi(x), \phi_x(a), \phi(y)).$$

By uniqueness, we deduce $\phi((x, a, y)) = (\phi(x), \phi_x(a), \phi(y))$. □

2.3. **Greatest common divisors**

We observe, or rather recall, that left-gcd's always exist in a left-preGarside category. We begin with a standard consequence of the noetherianity assumption (LG_0).

Lemma 2.8. *Assume that $\mathcal{C}$ is a left-preGarside category and $\mathcal{S}$ is a subset of $\mathcal{H}om(\mathcal{C})$ that contains the identity-morphisms and is closed under right-lcm. Then every morphism has a unique maximal left-divisor that lies in $\mathcal{S}$.*

Proof. Let f be an arbitrary morphism. Starting from $f_0 = 1_{\partial_0 f}$, which belongs to $\mathcal{S}$ by hypothesis, we construct a $\prec$-increasing sequence $f_0, f_1, ...$ in $\mathcal{S} \cap \mathrm{Div}(f)$. As long as f_i is not $\preccurlyeq$-maximal in $\mathcal{S} \cap \mathrm{Div}(f)$, we can find f_{i+1} in S satisfying $f_i \prec f_{i+1} \preccurlyeq f$. Condition ($\mathsf{LG}_0$) implies that the construction stops after a finite number d of steps. Then f_d is a maximal left-divisor of f lying in $\mathcal{S}$.

As for uniqueness, assume that g' and g'' are maximal left-divisors of f that lie in $\mathcal{S}$. By construction, g' and g'' admit a common right-multiple, namely f, hence, by (LG_2), they admit a right-lcm g. By construction, g is a left-divisor of f, and it belongs to $\mathcal{S}$ since g' and g'' do. The maximality of g and g' implies $g' = g = g''$. □

Proposition 2.9. *Assume that $\mathcal{C}$ is a left-preGarside category. Then any two morphisms of $\mathcal{C}$ sharing a common source admit a unique left-gcd.*

Proof. Let $\mathcal{S}$ be the family of all common left-divisors of f and g. It contains $1_{\partial_0 f}$, and it is closed under lcm. A left-gcd of f and g is a maximal left-divisor of f lying in $\mathcal{S}$. Lemma 2.8 gives the result. □

2.4. **Least common multiples**

As for right-lcm, the axioms of left-Garside categories only demand that a right-lcm exists when a common right-multiple does. A necessary condition for such a common right-multiple to exist is to share a common source. This condition is also sufficient. Again we begin with an auxiliary result.

Lemma 2.10. *Assume that $\mathcal{C}$ is a left-Garside category. Then, for $f = f_1...f_d$ with $f_1, ..., f_d$ simple and $x = \partial_0 f$, we have*

$$f \preccurlyeq \Delta(x)\,\Delta(\phi(x))\,...\,\Delta(\phi^{d-1}(x)). \tag{2.4}$$

Proof. We use induction on d. For $d = 1$, this is the definition of simplicity. Assume $d \geqslant 2$. Put $y = \partial_1 f_1$. Applying the induction hypothesis to $f_2...f_d$, we find

$$\begin{aligned} f = f_1(f_2...f_d) &\preccurlyeq f_1\,\Delta(y)\,\Delta(\phi(y))\,...\,\Delta(\phi^{d-2}(y)) \\ &= \Delta(x)\,\Delta(\phi(x))\,...\,\Delta(\phi^{d-2}(x))\,\phi^{d-1}(f_1) \\ &\preccurlyeq \Delta(x)\,\Delta(\phi(x))\,...\,\Delta(\phi^{d-2}(x))\,\Delta(\phi^{d-1}(x)). \end{aligned}$$

The second equality comes from applying (2.1) $d-1$ times, and the last relation comes from the fact that $\phi^{d-1}(f_1)$ is simple with source $\phi^{d-1}(x)$. □

Proposition 2.11. *Assume that $\mathcal{C}$ is a left-Garside category. Then any two morphisms of $\mathcal{C}$ sharing a common source admit a unique right-lcm.*

Proof. Let f, g be any two morphisms with source x. By Lemma 2.4, there exists d such that f and g can be expressed as the product of at most d simple morphisms. Then, by Lemma 2.10, $\Delta(x)\,\Delta(\phi(x))\,...\,\Delta(\phi^{d-1}(x))$ is a common right-multiple of f and g. Finally, (LG_2) implies that f and g admit a right-lcm. The uniqueness of the latter is guaranteed by Lemma 2.4(i). □

In a general context of categories, right-lcm's are usually called *push-outs* (whereas left-lcm's are called *pull-backs*). So Proposition 2.11 states that every left-Garside category admits push-outs.

Applying the previous results to the special case of categories associated with a partial action gives analogous results for all locally left-Garside monoids.

Corollary 2.12. *Assume that M is a locally left-Garside monoid with respect to some partial action of M on X.*

(*i*) *Any two elements of M admit a unique left-gcd and a unique right-lcm.*

(*ii*) *For each x in X, the subset M_x of M is closed under right-lcm.*

Proof. (*i*) As for left-gcd's, the result directly follows from Proposition 2.9 since, by definition, M is left-preGarside.

As for right-lcm's, assume that M is locally left-Garside with left-Garside sequence $(\Delta_x)_{x\in X}$. Let a, b be two elements of M. By definition of a partial action, there exists x in X such that both $x \bullet a$ and $x \bullet b$ are defined. By Proposition 2.11, $(x, a, x \bullet a)$ and $(x, b, x \bullet b)$ admit a right-lcm (x, c, z) in the category $\mathcal{C}(M, X)$. By construction, c is a common right-multiple of a and b in M. As M is assumed to satisfy (LG_2), a and b admit a right-lcm in M.

(*ii*) Fix now x in X, and let a, b belong to M_x, *i.e.*, assume that $x \bullet a$ and $x \bullet b$ are defined. Then $(x, a, x \bullet a)$ and $(x, b, x \bullet b)$ are morphisms of $\mathcal{C}(M, X)$. As above, they admit a right-lcm, which must be $(x, c, x \bullet c)$ where c is the right-lcm of a and b. Hence c belongs to M_x. □

3. **Regular left-Garside categories**

The main interest of Garside structures is the existence of a canonical normal form, the so-called greedy normal form [29]. In this section, we adapt the construction of the normal form to the context of left-Garside categories—this was done in [27] already—and of locally left-Garside monoids. The point here is that studying the computation of the normal form naturally leads to introducing the notion of a regular left-Garside category, crucial in Section 6.

3.1. **The head of a morphism**

By Lemma 2.4(*ii*), every morphism in a left-Garside category is a product of simple morphisms. Generally the decomposition does not need to be unique in general, and the first step for constructing a normal form consists in isolating a particular simple morphism that left-divides the considered morphism. It will be useful to develop the construction in a general framework where the distinguished morphisms do not need to be the simple ones.

Notation 3.1. Let us recall that, for f, g in $\mathcal{H}om(\mathcal{C})$, where $\mathcal{C}$ is a left-preGarside category, $\mathrm{lcm}(f, g)$ is the right-lcm of f and g, when it exists. In this case, we denote by $f\backslash g$ the unique morphism that satisfies

$$f \cdot f\backslash g = \mathrm{lcm}(f, g). \tag{3.1}$$

We use a similar notation in the case of a (locally) left-Garside monoid.

Definition 3.2. Assume that $\mathcal{C}$ is a left-preGarside category and $\mathcal{S}$ is included in $\mathcal{H}om(\mathcal{C})$. We say that $\mathcal{S}$ is a *seed* for $\mathcal{C}$ if

(*i*) $\mathcal{S}$ left-generates $\mathcal{H}om(\mathcal{C})$,

(*ii*) $\mathcal{S}$ is closed under the operations lcm and $\backslash$,

(*iii*) $\mathcal{S}$ is closed under left-divisor.

In other words, $\mathcal{S}$ is a seed for $\mathcal{C}$ if (*i*) every nontrivial morphism of $\mathcal{C}$ is left-divisible by a nontrivial element of $\mathcal{S}$, (*ii*) for all f, g in $\mathcal{S}$, the morphisms $\mathrm{lcm}(f, g)$ and $f\backslash g$ belong to $\mathcal{S}$ whenever they exist, and (*iii*) for each f in $\mathcal{S}$, the relation $h \preccurlyeq f$ implies $h \in \mathcal{S}$.

Lemma 3.3. *If $\mathcal{C}$ is a left-Garside category, then $\mathcal{H}om^{sp}(\mathcal{C})$ is a seed for $\mathcal{C}$.*

Proof. First, $\mathcal{H}om^{sp}(\mathcal{C})$ left-generates $\mathcal{H}om(\mathcal{C})$ by Condition (LG_3).

Next, assume that f, g are simple morphisms sharing the same source x. By Proposition 2.11, the morphisms $\mathrm{lcm}(f, g)$ and $f\backslash g$ exist. By definition, we have $f \preccurlyeq \Delta(x)$ and $g \preccurlyeq \Delta(x)$, hence $\mathrm{lcm}(f, g) \preccurlyeq \Delta_x$. Hence $\mathrm{lcm}(f, g)$ is simple. Let h satisfy $\mathrm{lcm}(f, g)h = \Delta(x)$. This is also $f\,(f\backslash g)\,h = \Delta(x)$. By Lemma 2.3(*ii*), $(f\backslash g)\,h$, which is a right-divisor of $\Delta(x)$, is simple, and, therefore, $f\backslash g$, which is a left-divisor of $(f\backslash g)\,h$, is simple as well by transitivity of $\preccurlyeq$.

Finally, $\mathcal{H}om^{sp}(\mathcal{C})$ is closed under left-divisor by definition. □

Lemma 2.8 guarantees that, if $\mathcal{S}$ is a seed for $\mathcal{C}$, then every morphism f of $\mathcal{C}$ has a unique maximal left-divisor g lying in $\mathcal{S}$, and Condition (*i*) of Definition 3.2 implies that g is nontrivial whenever f is.

Definition 3.4. In the context above, the morphism g is called the *$\mathcal{S}$-head* of f, denoted $H_{\mathcal{S}}(f)$.

In the case of $\mathcal{H}om^{sp}(\mathcal{C})$, it is easy to check, for each f in $\mathcal{H}om(\mathcal{C})$, the equality

$$H_{\mathcal{H}om^{sp}(\mathcal{C})}(f) = \gcd(f, \Delta(\partial_0 f)); \tag{3.2}$$

in this case, we shall simply write $H(f)$ for $H_{\mathcal{H}om^{sp}(\mathcal{C})}(f)$.

3.2. **Normal form**

The following result is an adaptation of a result that is classical in the framework of Garside monoids.

Proposition 3.5. *Assume that $\mathcal{C}$ is a left-preGarside category and $\mathcal{S}$ is a seed for $\mathcal{C}$. Then every nontrivial morphism f of $\mathcal{C}$ admits a unique decomposition*

$$f = f_1 ... f_d, \tag{3.3}$$

where $f_1, ..., f_d$ lie in $\mathcal{S}$, f_d is nontrivial, and f_i is the $\mathcal{S}$-head of $f_i ... f_d$ for each i.

Proof. Let f be a nontrivial morphism of $\mathcal{C}$, and let f_1 be the $\mathcal{S}$-head of f. Then f_1 belongs to $\mathcal{S}$, it is nontrivial, and we have $f = f_1 f'$ for some unique f'. If f' is trivial, we are done, otherwise we repeat the argument with f'. In this way we obtain a $\prec$-increasing sequence $1_{\partial_0 f} \prec f_1 \prec f_1 f_2 \prec ...$. Condition ($\mathsf{LG}_0$) implies that the construction stops after a finite number of steps, yielding a decomposition of the form (3.3).

As for uniqueness, assume that $(f_1, ..., f_d)$ and $(g_1, ..., g_e)$ are decompositions of f that satisfy the conditions of the statement. We prove $(f_1, ..., f_d) = (g_1, ..., g_e)$ using induction on $\min(d, e)$. First, $d = 0$ implies $e = 0$ by Lemma 2.4(i). Otherwise, the hypotheses imply $f_1 = H_{\mathcal{S}}(f) = g_1$. Left-cancelling f_1 gives two decompositions $(f_2, ..., f_d)$ and $(g_2, ..., g_e)$ of $f_2 ... f_d$, and we apply the induction hypothesis. □

Definition 3.6. In the context above, the sequence $(f_1, ..., f_d)$ is called the *$\mathcal{S}$-normal form* of f.

When $\mathcal{S}$ turns out to be the family $\mathcal{H}om^{sp}(\mathcal{C})$, the $\mathcal{S}$-normal form will be simply called the *normal form*. The interest of the $\mathcal{S}$-normal form lies in that it is easily characterized and easily computed. First, one has the following local characterization of normal sequences.

Proposition 3.7. *Assume that $\mathcal{C}$ is a left-preGarside category and $\mathcal{S}$ is a seed for $\mathcal{C}$. Then a sequence of morphisms $(f_1, ..., f_d)$ is $\mathcal{S}$-normal if and only if each length two subsequence (f_i, f_{i+1}) is $\mathcal{S}$-normal.*

This follows from an auxiliary lemma.

Lemma 3.8. *Assume that (f_1, f_2) is $\mathcal{S}$-normal and g belongs to $\mathcal{S}$. Then $g \preccurlyeq ff_1f_2$ implies $g \preccurlyeq ff_1$.*

Proof. The hypothesis implies that f and g have the same source. Put $g' = f\backslash g$. The hypothesis that $\mathcal{S}$ is closed under $\backslash$ and an easy induction on the length of the $\mathcal{S}$-normal form of f show that g' belongs to $\mathcal{S}$. By hypothesis, we have both $g \preccurlyeq ff_1f_2$ and $f \preccurlyeq ff_1f_2$, hence $\mathrm{lcm}(f, g) = fg' \preccurlyeq ff_1f_2$ whence $g' \preccurlyeq f_1f_2$ by left-cancelling f. As g belongs to $\mathcal{S}$ and (f_1, f_2) is normal, this implies $g' \preccurlyeq f_1$, and finally $g \preccurlyeq fg' \preccurlyeq ff_1$. □

Proof of Proposition 3.7. It is enough to consider the case $d = 2$, from which an easy induction on d gives the general case. So we assume that (f_1, f_2) and (f_2, f_3) are $\mathcal{S}$-normal, and aim at proving that (f_1, f_2, f_3) is $\mathcal{S}$-normal. The point is to prove that, if g belongs to $\mathcal{S}$, then $g \preccurlyeq f_1f_2f_3$ implies $g \preccurlyeq f_1$. So assume $g \preccurlyeq f_1f_2f_3$. As (f_2, f_3) is $\mathcal{S}$-normal, Lemma 3.8 implies $g \preccurlyeq f_1f_2$. As (f_1, f_2) is $\mathcal{S}$-normal, this implies $g \preccurlyeq f_1$. □

3.3. **A computation rule**

We establish now a recipe for inductively computing the $\mathcal{S}$-normal form, namely determining the $\mathcal{S}$-normal form of gf when that of f is known and g belongs to $\mathcal{S}$.

Proposition 3.9. *Assume that $\mathcal{C}$ is a left-preGarside category, $\mathcal{S}$ is a seed for $\mathcal{C}$, and $(f_1, ..., f_d)$ is the $\mathcal{S}$-normal form of f. Then, for each g in $\mathcal{S}$, the $\mathcal{S}$-normal form of gf is $(f'_1, ..., f'_d, g_d)$, where $g_0 = g$ and (f'_i, g_i) is the $\mathcal{S}$-normal form of $g_{i-1}f_i$ for i increasing from 1 to d—see Figure 2.*

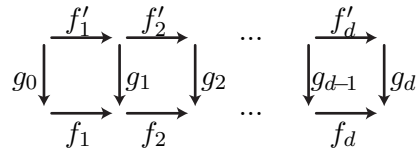
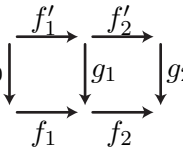


Figure 2. Adding one $\mathcal{S}$-factor g_0 on the left of an $\mathcal{S}$-normal sequence $(f_1, ..., f_d)$: compute the $\mathcal{S}$-normal form (f'_1, g_1) of g_0f_1, then the $\mathcal{S}$-normal form (f'_2, g_2) of g_1f_2, and so on from left to right; the sequence $(f'_1, ..., f'_d, g_d)$ is $\mathcal{S}$-normal.

Proof. For an induction, it is enough to consider the case $d = 2$, hence to prove

Claim. *Assume that the diagram*  *is commutative and* (f_1, f_2) *and* (f_1', g_1) *are* $\mathcal{S}$*-normal. Then* (f_1', f_2') *is* $\mathcal{S}$*-normal.*

So assume that h belongs to $\mathcal{S}$ and satisfies $h \preccurlyeq f_1' f_2'$. Then, a fortiori, we have $h \preccurlyeq f_1' f_2' g_2 = g_0 f_1 f_2$, hence $h \preccurlyeq g_0 f_1$ by Lemma 3.8 since (f_1, f_2) is $\mathcal{S}$-normal. Therefore we have $h \preccurlyeq f_1' g_1$, hence $h \preccurlyeq f_1'$ since (f_1', g_1) is $\mathcal{S}$-normal. □

The results of Proposition 3.7 and 3.9 apply in particular when $\mathcal{C}$ is left-Garside and $\mathcal{S}$ is the family of all simple morphisms, in which case they involve the standard normal form.

In the case of lcm's, Corollary 2.12 shows how a result established for general left-Garside categories induces a similar result for locally left-Garside monoids. The situation is similar with the normal form, provided some additional assumption is satisfied.

Definition 3.10. A left-Garside sequence $(\Delta_x)_{x \in X}$ witnessing that a certain monoid is locally left-Garside is said to be *coherent* if $a \preccurlyeq \Delta_x$ implies $a \preccurlyeq \Delta_y$ for each y such that $a \bullet y$ is defined.

For instance, the family $(\Delta_n)_{n \in \mathbb{N}}$ witnessing for the locally left-Garside structure of the monoid B_∞^+ is coherent. Indeed, a positive n-strand braid a is a left-divisor of Δ_n if and only if it is a left-divisor of $\Delta_{n'}$ for every $n' \geqslant n$. The reason is that being simple is an intrinsic property of positive braids: a positive braid is simple if and only if it can be represented by a braid diagram in which any two strands cross at most once [28].

Proposition 3.11. *Assume that* M *is a locally left-Garside monoid associated with a coherent left-Garside sequence* $(\Delta_x)_{x \in X}$*. Let* $\Sigma = \{a \in M \mid \exists x \in X (a \preccurlyeq \Delta_x)\}$*. Then* Σ *is a seed for* M*, every element of* M *admits a unique* Σ*-normal form, and the counterpart of Propositions 3.7 and 3.9 hold for the* Σ*-normal form in* M*.*

Proof. Axiom ($\mathsf{LG}_3^{\ell oc}$) guarantees that every nontrivial element of M is left-divisible by some nontrivial element of Σ. Then, by hypothesis, $1_x \preccurlyeq \Delta_x$ holds for each object x. Then, assume $a, b \in \Sigma$. There exists x such that $x \bullet a$

and $x \bullet b$ are defined. By definition of Σ, there exists x' satisfying $a \preccurlyeq \Delta_{x'}$, hence, by definition of coherence, we have $a \preccurlyeq \Delta_x$. A similar argument gives $b \preccurlyeq \Delta_x$, whence $\mathrm{lcm}(a,b) \preccurlyeq \Delta_x$, and $\mathrm{lcm}(a,b) \in \Sigma$. So there exists c satisfying $a\,(a\backslash b)\,c = \Delta_x$. By $(\mathsf{LG}_3^{\ell oc})$, we deduce $(a\backslash b)\,c \preccurlyeq \Delta_{x \bullet a}$, whence $a\backslash b \preccurlyeq \Delta_{x \bullet a}$, and we conclude that $a\backslash b$ belongs to Σ. Finally, it directly results from its definition that Σ is closed under left-divisor. Hence Σ is a seed for M in the sense of Definition 3.2.

As, by definition, M is a left-preGarside monoid, Proposition 3.5 applies, guaranteeing the existence and uniqueness of the Σ-normal form on M, and so do Propositions 3.7 and 3.9. □

Thus, the good properties of the greedy normal form are preserved when the assumption that a global Garside element Δ exists is replaced by the weaker assumption that local Garside elements Δ_x exist, provided they make a coherent sequence.

3.4. **Regular left-Garside categories**

It is natural to look for a counterpart of Proposition 3.9 involving right-multiplication by an element of the seed instead of left-multiplication. Such a counterpart exists but, interestingly, the situation is not symmetric, and we need a new argument. The latter demands that the considered category satisfies an additional condition, which is automatically satisfied in a two-sided Garside category, but not in a left-Garside category.

In this section, we only consider the case of a left-Garside category and its simple morphisms, and not the case of a general left-preGarside category with an arbitrary seed—see Remark 3.15. So, we only refer to the standard normal form.

Definition 3.12. We say that a left-Garside category $\mathcal{C}$ is *regular* if the functor ϕ preserves normality of length two sequences: for f_1, f_2 simple with $\partial_1 f_1 = \partial_0 f_2$,

$$(f_1, f_2) \text{ normal} \quad \text{implies} \quad (\phi(f_1), \phi(f_2)) \text{ normal}. \tag{3.4}$$

Proposition 3.13. *Assume that $\mathcal{C}$ is a regular left-Garside category, that $(f_1, ..., f_d)$ is the normal form of a morphism f, and that g is simple. Then the normal form of fg is $(g_0, f_1', ..., f_d')$, where $g_d = g$ and (g_{i-1}, f_i') is the normal form of $f_i g_i$ for i decreasing from d to 1—see Figure 3.*

We begin with an auxiliary observation.

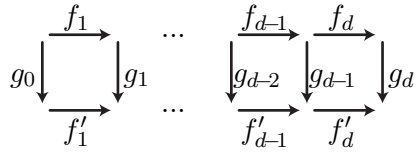


FIGURE 3. Adding one simple factor g_d on the right of a simple sequence $(f_1, ..., f_d)$: compute the normal form $(g_{d-1}f'_d)$ of $f_d g_d$, then the normal form $(g_{d-2}f'_{d-2})$ of $f_{d-1}g_{d-1}$, and so on from right to left; the sequence $(g_0, f'_1, ..., f'_d)$ is normal.

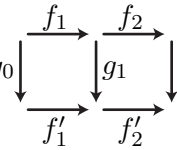
Lemma 3.14. *Assume that $\mathcal{C}$ is a left-Garside category and f_1, f_2 are simple morphisms satisfying $\partial_1 f_1 = \partial_0 f_2$. Then (f_1, f_2) is normal if and only if f_1^* and f_2 are left-coprime,* i.e., $\gcd(f_1^*, f_2)$ *is trivial.*

Proof. The following equalities always hold:

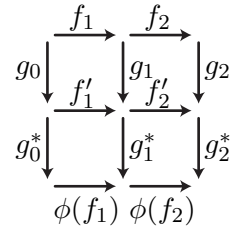
$$H(f_1 f_2) = \gcd(f_1 f_2, \Delta(\partial_0 f_1)) = \gcd(f_1 f_2, f_1 f_1^*) = f_1 \gcd(f_2, f_1^*).$$

Hence (f_1, f_2) is normal, *i.e.*, $f_1 = H(f_1 f_2)$ holds, if and only if $f_1 = f_1 \gcd(f_2, f_1^*)$ does, which is $\gcd(f_2, f_1^*) = 1_{\partial_1 f_1}$ as left-cancelling f_1 is allowed. □

Proof of Proposition 3.13. As in the case of Proposition 3.9 it is enough to consider the case $d = 2$, and therefore it is enough to prove

Claim. *Assume that the diagram* 

is commutative and (f_1, f_2) and (g_1, f'_2) are normal. Then (f'_1, f'_2) is normal.

To prove the claim, we introduce the morphisms g_0^*, g_1^*, g_2^* defined by $g_i g_i^* = \Delta(\partial_0 g_i)$ (Definition 2.1). Then the diagram 

is

commutative. Indeed, applying (2.1), we find

$$\begin{aligned} g_0 f_1' g_1^* &= f_1 g_1 g_1^* = f_1 \Delta(\partial_0 g_0) = f_1 \Delta(\partial_1 f_1) \\ &= \Delta(\partial_0 f_1)\phi(f_1) = \Delta(\partial_0 g_0)\phi(f_1) = g_0 g_0^* \phi(f_1), \end{aligned}$$

hence $f_1' g_1^* = g_0^* \phi(f_1)$ by left-cancelling g_0. A similar argument gives $f_2' g_2^* = g_1^* \phi(f_2)$.

Assume that h is simple and satisfies $h \preccurlyeq f_1' f_2'$. We deduce

$$h \preccurlyeq f_1' f_2' g_2^* = g_0^* \phi(f_1)\phi(f_2).$$

By hypothesis, (f_1, f_2) is normal. Hence the hypothesis that $\mathcal{C}$ is regular implies that $(\phi(f_1), \phi(f_2))$ is normal as well. By Lemma 3.8, $h \preccurlyeq g_0^* \phi(f_1)\phi(f_2)$ implies $h \preccurlyeq g_0^* \phi(f_1) = f_1' g_1^*$. We deduce $h \preccurlyeq \gcd(f_1' f_2', f_1' g_1^*) = f_1' \gcd(f_2', g_1^*)$. By Lemma 3.14, the hypothesis that (g_1, f_2') is normal implies $\gcd(g_1^*, f_2') = 1$, and, finally, we deduce $h \preccurlyeq f_1'$, *i.e.*, (f_1', f_2') is normal. □

Remark 3.15. It might be tempting to mimick the arguments of this section in the general framework of a left-preGarside category $\mathcal{C}$ and a seed $\mathcal{S}$, provided some additional conditions are satisfied. However, it is unclear that the extension can be a genuine one. For instance, if we require that, for each f in $\mathcal{S}$, there exists f^* in $\mathcal{S}$ such that ff^* exists and depends on $\partial_0 f$ only, then the map $\partial_0 f \mapsto ff^*$ is a left-Garside map and we are back to left-Garside categories.

3.5. **Regularity criteria**

We conclude with some sufficient conditions implying regularity. In particular, we observe that, in the two-sided case, regularity is automatically satisfied.

Lemma 3.16. *Assume that $\mathcal{C}$ is a left-Garside category. Then a sufficient condition for $\mathcal{C}$ to be regular is that the functor ϕ be bijective on $\mathcal{H}om(\mathcal{C})$.*

Proof. Assume that $\mathcal{C}$ is a left-Garside category and ϕ is bijective onto $\mathcal{H}om(\mathcal{C})$. First we claim that $\phi(f) \preccurlyeq \phi(g)$ implies $f \preccurlyeq g$ in $\mathcal{C}$. Indeed, assume $\phi(g) = \phi(f)h$. As ϕ is surjective, we have $\phi(g) = \phi(f)\phi(h')$ for some h', hence $\phi(g) = \phi(fh')$ since ϕ is a functor, hence $g = fh'$ since ϕ is injective.

Next, we claim that $\phi(f)$ is simple if and only if f is simple. That the condition is sufficient directly follows from Definition 2.1. Conversely,

assume that $\phi(f)$ is simple. This means that there exists g satisfying $\phi(f)g = \Delta(\partial_0\phi(f))$. As ϕ is surjective, there exists g' satisfying $g = \phi(g')$. Applying (2.2), we obtain $\phi(fg') = \Delta(\partial_0\phi(f)) = \phi(\Delta(\partial_0 f)$, hence $fg' = \Delta(\partial_0 f)$ by injectivity of ϕ.

Finally, assume that (f_1, f_2) is normal, and g is a simple morphism left-dividing $\phi(f_1)\phi(f_2)$, hence satisfying $gh = \phi(f_1)\phi(f_2)$ for some h. As ϕ is surjective, we have $g = \phi(g')$ and $h = \phi(h')$ for some g', h'. Moreover, by the claim above, the hypothesis that g is simple implies that g' is simple as well. Then we have $\phi(g')\phi(h') = \phi(f_1)\phi(f_2)$, hence $g'h' = f_1 f_2$ since ϕ is a functor and it is injective. The hypothesis that (f_1, f_2) is normal implies $g' \preccurlyeq f_1$, hence $g = \phi(g') \preccurlyeq \phi(f_1)$. So $(\phi(f_1), \phi(f_2))$ is normal, and $\mathcal{C}$ is regular. □

Proposition 3.17. *Every Garside category is regular.*

Proof. Assume that $\mathcal{C}$ is a left-Garside category with respect to Δ and a right-Garside category with respect to ∇ satisfying $\Delta(x) = \nabla(x')$ for $x' = \partial_1\Delta(x)$. Put $\psi(x') = \partial_0\nabla(x')$ for x' in $\mathcal{O}bj(\mathcal{C})$ and, for g simple in $\mathcal{H}om(\mathcal{C})$, hence a right-divisor of $\nabla(\partial_1 g)$, denote by ${}^*g$ the unique simple morphism satisfying ${}^*g g = \nabla(\partial_0 g)$, and put $\psi(g) = {}^{**}g$. Then arguments similar to those of Lemma 2.4 give the equality

$$\nabla(\partial_0 g)\, g = \psi(g)\, \nabla(\partial_1 g) \tag{3.5}$$

which is an exact counterpart of (2.1). Let $f : x \to y$ be any morphism in $\mathcal{C}$. Put $x' = \phi(x)$ and $y' = \phi(y)$. By construction, we also have $x = \psi(x')$ and $y = \psi(y')$. Applying (2.1) to $f : x \to y$, we obtain $\Delta(x)\, \phi(f) = f\, \Delta(y)$, which is also

$$\nabla(x')\, \phi(f) = f\, \nabla(y'). \tag{3.6}$$

On the other hand, applying (3.5) to $\phi(f) : x' \to y'$ yields

$$\nabla(x')\, \phi(f) = \psi(\phi(f))\, \nabla(y'). \tag{3.7}$$

Comparing (3.6) and (3.7) and right-cancelling $\nabla(y')$, we deduce $\psi(\phi(f)) = f$. A symmetric argument gives $\phi(\psi(g)) = g$ for each g, and we conclude that ψ is the inverse of ϕ, which is therefore bijective. Then we apply Lemma 3.16. □

Remark. The above proof shows that, if $\mathcal{C}$ is a left-Garside category that is Garside, then the associated functor ϕ is bijective both on $\mathcal{O}bj(\mathcal{C})$ and on $\mathcal{H}om(\mathcal{C})$. Let us mention without proof that this necessary condition is actually also sufficient.

Apart from the previous very special case, we can state several weaker regularity criteria that are close to the definition and will be useful in Section 6. We recall that $H(f)$ denotes the maximal simple morphism left-dividing f.

Proposition 3.18. *A left-Garside category $\mathcal{C}$ is regular if and only if ϕ preserves the head function on product of two simples: for f_1, f_2 simple with $\partial_1 f_1 = \partial_0 f_2$,*

$$H(\phi(f_1 f_2)) = \phi(H(f_1 f_2)); \tag{3.8}$$

Proof. Assume that $\mathcal{C}$ is regular, and that f_1, f_2 satisfy $\partial_1 f_1 = \partial_0 f_2$. Let (f_1', f_2') be the formal form of $f_1 f_2$—which has length 2 at most by Proposition 3.9. Then, $(\phi(f_1'), \phi(f_2'))$ is normal and satisfies $\phi(f_1')\phi(f_2') = \phi(f_1 f_2)$, so $(\phi(f_1'), \phi(f_2'))$ is the normal form of $f(f_1 f_2)$. Hence we have $H(f_1 f_2) = f_1'$ and $H(\phi(f_1 f_2)) = \phi(f_1')$, which is (3.8).

Conversely, assume (3.8) and let (f_1, f_2) be normal. By construction, we have $f_1 = H(f_1 f_2)$, hence $\phi(f_1) = H(\phi(f_1 f_2))$ by hypothesis. This means that the normal form of $\phi(f_1 f_2)$ is $(\phi(f_1), g)$ for some g satisfying $\phi(f_1 f_2) = \phi(f_1)g$. Now $\phi(f_2)$ is such a morphism g, and, by (LG_1), it is the only one. So the normal form of $\phi(f_1 f_2)$ is $(\phi(f_1), \phi(f_2))$, and $\mathcal{C}$ is regular. □

Proposition 3.19. *Assume that $\mathcal{C}$ is a left-Garside category $\mathcal{C}$. Then two sufficient conditions for $\mathcal{C}$ to be regular are*

(i) The functor ϕ preserves left-coprimeness of simple morphisms: for f, g simple with $\partial_0 f = \partial_0 g$,

$$\gcd(f, g) = 1 \quad \textit{implies} \quad \gcd(\phi(f), \phi(g)) = 1. \tag{3.9}$$

(ii) The functor ϕ preserves the gcd operation on simple morphisms: for f, g simple with $\partial_0 f = \partial_0 g$,

$$\gcd(\phi(f), \phi(g)) = \phi(\gcd(f, g)), \tag{3.10}$$

and, moreover, $\phi(f)$ is nontrivial whenever f is nontrivial.

Proof. Assume (i). Let (f, g) be normal. By Lemma 3.14, $\gcd(f^*, g)$ is 1. By (3.9), we deduce $\gcd(\phi(f^*), \phi(g)) = 1$. By Lemma 2.5, this equality is also $\gcd(\phi(f)^*, \phi(g)) = 1$, which, by Lemma 3.14 again, means that $(\phi(f), \phi(g))$ is normal. Hence $\mathcal{C}$ is regular.

On the other hand, it is clear that (ii) implies (i). □

4. **Self-distributivity**

We quit general left-Garside categories, and turn to one particular example, namely a certain category (two categories actually) associated with the left self-distributive law. The latter is the algebraic law

$$x(yz) = (xy)(xz) \tag{LD}$$

extensively investigated in [18].

We first review some basic results about this law and the associated free LD-systems, *i.e.*, the binary systems that obey the LD-law. The key notion is the notion of an LD-expansion, with two derived categories $\mathcal{LD}_0^+$ and $\mathcal{LD}^+$ that will be our main subject of investigation from now on.

4.1. **Free LD-systems**

For each algebraic law (or family of algebraic laws), there exist universal objects in the category of structures that satisfy this law, namely the free systems. Such structures can be uniformly described as quotients of absolutely free structures under convenient congruences.

Definition 4.1. We let T_n be the set of all bracketed expressions involving variables $x_1, ..., x_n$, *i.e.*, the closure of $\{x_1, ..., x_n\}$ under $t_1 \star t_2 = (t_1)(t_2)$. We use T for the union of all sets T_n. Elements of T are called *terms*.

Typical terms are x_1, $x_2 \star x_1$, $x_3 \star (x_3 \star x_1)$, etc. It is convenient to think of terms as rooted binary trees with leaves indexed by the variables: the trees associated with the previous terms are $\overset{\bullet}{x_1}$, $\overset{\wedge}{x_2 x_1}$, and $\overset{\diagup\!\wedge}{x_3 \; x_3 x_1}$, respectively. The system $(T_n, \star)$ is the absolutely free system (or algebra) generated by $x_1, ..., x_n$, and every binary system generated by n elements is a quotient of this system. So is in particular the free LD-system of rank n.

Definition 4.2. We denote by $=_{\mathrm{LD}}$ the least congruence (*i.e.*, equivalence relation compatible with the product) on $(T_n, \star)$ that contains all pairs of the form

$$(t_1 \star (t_2 \star t_3), (t_1 \star t_2) \star (t_1 \star t_3)).$$

Two terms t, t' satisfying $t =_{\mathrm{LD}} t'$ are called *LD-equivalent*.

The following result is then standard.

Proposition 4.3. *For each $n \leqslant \infty$, the binary system $(T_n/{=_{\mathrm{LD}}}, \star)$ is a free LD-system based on $\{x_1, ..., x_n\}$.*

4.2. **LD-expansions**

The relation $=_{\mathrm{LD}}$ is a complicated object, about which many questions remain open. In order to investigate it, it proved useful to introduce the subrelation of $=_{\mathrm{LD}}$ that corresponds to applying the LD-law in the expanding direction only.

Definition 4.4. Let t, t' be terms. We say that t' is an *atomic LD-expansion* of t, denoted $t \to^1_{\mathrm{LD}} t'$, if t' is obtained from t by replacing some subterm of the form $t_1 \star (t_2 \star t_3)$ with the corresponding term $(t_1 \star t_2) \star (t_1 \star t_3)$. We say that t' is an *LD-expansion* of t, denoted $t \to_{\mathrm{LD}} t'$, if there exists a finite sequence of terms $t_0, ..., t_p$ satisfying $t_0 = t$, $t_p = t'$, and $t_{i-1} \to^1_{\mathrm{LD}} t_i$ for $1 \leqslant i \leqslant p$.

By definition, being an LD-expansion implies being LD-equivalent, but the converse is not true. For instance, the term $(x \star x) \star (x \star x)$ is an (atomic) LD-expansion of $x \star (x \star x)$, but the latter is not an LD-expansion of the former. However, it should be clear that $=_{\mathrm{LD}}$ is generated by $\to_{\mathrm{LD}}$, so that two terms t, t' are LD-equivalent if and only if there exists a finite zigzag $t_0, t_1, ..., t_{2p}$ satisfying $t_0 = t$, $t_{2p} = t'$, and $t_{i-1} \to_{\mathrm{LD}} t_i \ {}_{\mathrm{LD}}\!\leftarrow t_{i+1}$ for each odd i.

The first nontrivial result about LD-equivalence is that the above zigzags may always be assumed to have length two.

Proposition 4.5. [17] *Two terms are LD-equivalent if and only if they admit a common LD-expansion.*

This result is similar to the property that, if a monoid M satisfies Ore's conditions—as the braid monoid B_n^+ does for instance—then every element in the universal group of M can be expressed as a fraction of the form ab^{-1} with a, b in M. Proposition 4.5 plays a fundamental role in the sequel, and we need to recall some elements of its proof.

Definition 4.6. [17] A binary operation $\circledast$ on terms is recursively defined by

$$t \circledast x_i = t \star x_i, \quad t \circledast (t_0 \star t_1) = (t \circledast t_0) \star (t \circledast t_1). \tag{4.1}$$

Next, for each term t, the term $\phi(t)$ is recursively defined by[2]

$$\phi(x_i) = x_i \quad \phi(t_0 \star t_1) = \phi(t_0) \circledast \phi(t_1). \tag{4.2}$$

The idea is that $t \circledast t'$ is obtained by distributing t everywhere in t' once. Then $\phi(t)$ is the image of t when $\star$ is replaced with $\circledast$ everywhere in the unique expression of t in terms of variables. Examples are given in Figure 4. A straightforward induction shows that $t \circledast t'$ is always an LD-expansion of $t \star t'$ and, therefore, that $\phi(t)$ is an LD-expansion of t.

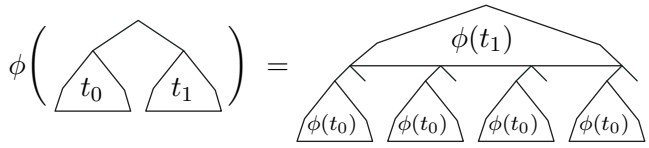


FIGURE 4. The fundamental LD-expansion $\phi(t)$ of a term t, recursive definition: $\phi(t_0 \star t_1)$ is obtained by distributing $\phi(t_0)$ everywhere in $\phi(t_1)$.

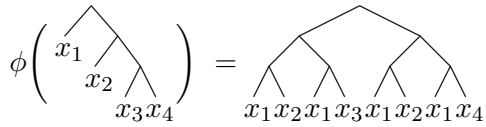


FIGURE 5. The fundamental LD-expansion $\phi(t)$ of a term t, a concrete example: $t = x_1 \star (x_2 \star (x_3 \star x_4))$ gives $\phi(t) = x_1 \circledast (x_2 \circledast (x_3 \circledast x_4))$.

The main step for establishing Proposition 4.5 consists in proving that $\phi(t)$ plays with respect to atomic LD-expansions the same role as the one played by Garside's fundamental braid Δ_n with respect to Artin's generators σ_i—which makes it natural to call $\phi(t)$ the *fundamental LD-expansion* of t.

Lemma 4.7. [17] [18, Lemmas V.3.11 and V.3.12] (*i*) *The term $\phi(t)$ is an LD-expansion of each atomic LD-expansion of t.*

(*ii*) *If t' is an LD-expansion of t, then $\phi(t')$ is an LD-expansion of $\phi(t)$.*

[2]In [17] and [18], ∂ is used instead of ϕ, an inappropriate notation in the current context.

Sketch of proof. One uses induction on the size of the involved terms. Once Lemma 4.7 is established, an easy induction on d shows that, if there exists a length d sequence of atomic LD-expansions connecting t to t', then $\phi^d(t)$ is an LD-expansion of t'. Then a final induction on the length of a zigzag connecting t to t' shows that, if t and t' are LD-equivalent, then $\phi^d(t)$ is an LD-expansion of t' for sufficiently large d (namely for d at least the number of "zag"s in the zigzag). □

4.3. The category $\mathcal{LD}_0^+$

A category (and a quiver) is naturally associated with every graph, and the previous results invite to introduce the category associated with the LD-expansion relation $\rightarrow_{\mathrm{LD}}$.

Definition 4.8. We denote by $\mathcal{LD}_0^+$ the category whose objects are terms, and whose morphisms are all pairs of terms (t, t') satisfying $t \rightarrow_{\mathrm{LD}} t'$.

By construction, the category $\mathcal{LD}_0^+$ is left- and right-cancellative, and Proposiion 4.5 means that any two morphisms of $\mathcal{LD}_0^+$ with the same source admit a common right-multiple. Moreover, a natural candidate for being a left-Garside map is obtained by defining $\Delta(t) = (t, \phi(t))$ for each term t.

Question 1. Is $\mathcal{LD}_0^+$ a left-Garside category?

Question 1 is currently open. We shall see in Section 6.3 that it is one of the many forms of the so-called Embedding Conjecture. The missing part is that we do not know that *least* common multiples exist in $\mathcal{LD}_0^+$, the problem being that we have no method for proving that a common LD-expansion of two terms is possibly a least common LD-expansion.

5. The monoid LD^+ and the category $\mathcal{LD}^+$

The solution for overcoming the above difficulty consists in developing a more precise study of LD-expansions that takes into account the position where the LD-law is applied. This leads to introducing a certain monoid LD^+ whose elements provide natural labels for LD-expansions, and, from there, a new category $\mathcal{LD}^+$, of which $\mathcal{LD}_0^+$ is a projection. This category $\mathcal{LD}^+$ is the one on which a left-Garside structure will be proved to exist.

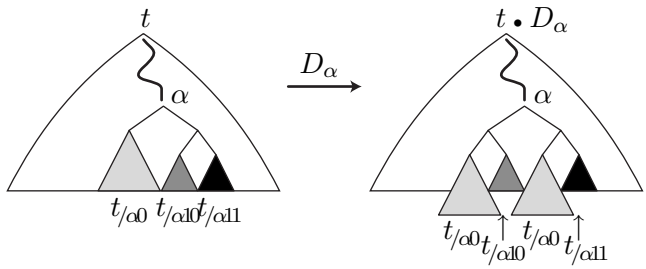


FIGURE 6. Action of D_α to a term t: the LD-law is applied to expand t at position α, *i.e.*, to replace the subterm $t_{/\alpha}$, which is $t_{/\alpha 0} \star (t_{/\alpha 10} \star t_{/\alpha 11})$, with $(t_{/\alpha 0} \star t_{/\alpha 10}) \star (t_{/\alpha 0} \star t_{/\alpha 11})$; in other words, the light grey subtree is duplicated and distributed to the left of the dark grey and black subtrees.

5.1. **Labelling LD-expansions**

By definition, applying the LD-law to a term t means selecting some subterm of t and replacing it with a new LD-equivalent term. When terms are viewed as binary rooted trees, the position of a subterm can be specified by describing the path that connects the root of the tree to the root of the considered subtree, hence typically by a binary address, *i.e.*, a finite sequence of 0's and 1's, according to the convention that 0 means "forking to the left" and 1 means "forking to the right". Hereafter, we use $\mathbb{A}$ for the set of all such addresses, and ϵ for the empty address, which corresponds to the position of the root in a tree.

Notation 5.1. For t a term and α an address, we denote by $t_{/\alpha}$ the subterm of t whose root (when viewed as a subtree) has address α, if it exists, *i.e.*, if α is short enough.

So, for instance, if t is the term $x_1 \star (x_2 \star x_3)$, we have $t_{/0} = x_1$, $t_{/10} = x_2$, whereas $t_{/00}$ is not defined, and $t_{/\epsilon} = t$ holds, as it holds for every term.

Definition 5.2. (See Figure 6.) We say that t' is a D_α-*expansion* of t, denoted $t' = t \bullet D_\alpha$, if t' is the atomic LD-expansion of t obtained by applying LD at the position α, *i.e.*, replacing the subterm $t_{/\alpha}$, which is $t_{/\alpha 0} \star (t_{/\alpha 10} \star t_{/\alpha 11})$, with the term $(t_{/\alpha 0} \star t_{/\alpha 10}) \star (t_{/\alpha 0} \star t_{/\alpha 11})$.

By construction, every atomic LD-expansion is a D_α-expansion for a unique α. The idea is to use the letters D_α as labels for LD-expansions.

As arbitrary LD-expansions are compositions of finitely many atomic LD-expansions, hence of D_α-expansions, it is natural to use finite sequences of D_α to label LD-expansions. In other words, we extend the (partial) action of D_α on terms into a (partial) action of finite sequences of D_α's. Thus, for instance, we write

$$t' = t \bullet D_\alpha D_\beta D_\gamma$$

to indicate that t' is the LD-expansion of t obtained by successively applying the LD-law (in the expanding direction) at the positions α, then β, then γ.

If S is a nonempty set, we denote by S^* the free monoid generated by S, *i.e.*, the family of all words on the alphabet S (finite sequences of elements of S) equiped with concatenation.

Lemma 5.3. *In the sense of Definition 1.8, Definition 5.2 gives a partial action of the free monoid $\{D_\alpha \mid \alpha \in \mathbb{A}\}^*$ on T (the set of terms).*

Proof. Conditions (i) and (ii) of Definition 1.8 follow from the construction. The point is to prove (iii), *i.e.*, to prove that, if $w_1, ..., w_n$ are arbitrary finite sequences of letters D_α, then there exists at least one term t such that $t \bullet w_i$ is defined for each i. This is what [18, Proposition VII.1.21] states. □

5.2. **The monoid** LD^+

There exist clear connections between the action of various D_α's: different sequences may lead to the same transformation of terms. We shall now identify a natural family of such relations and introduce the monoid presented by these relations.

Lemma 5.4. *For all α, β, γ, the following pairs have the same action on terms:*

(i) $D_{\alpha 0 \beta} D_{\alpha 1 \gamma}$ *and* $D_{\alpha 1 \gamma} D_{\alpha 0 \beta}$;	*("parallel case")*
(ii) $D_{\alpha 0 \beta} D_\alpha$ *and* $D_\alpha D_{\alpha 00 \beta} D_{\alpha 10 \beta}$;	*("nested case 1")*
(iii) $D_{\alpha 10 \beta} D_\alpha$ *and* $D_\alpha D_{\alpha 01 \beta}$;	*("nested case 2")*
(iv) $D_{\alpha 11 \beta} D_\alpha$ *and* $D_\alpha D_{\alpha 11 \beta}$;	*("nested case 3")*
(v) $D_\alpha D_{\alpha 1} D_\alpha$ *and* $D_{\alpha 1} D_\alpha D_{\alpha 1} D_{\alpha 0}$.	*("critical case")*

Sketch of proof. The commutation relation of the parallel case is clear, as the transformations involve disjoint subterms. The nested cases are

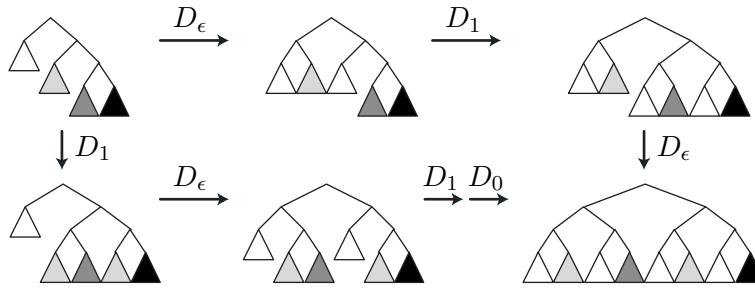


Figure 7. Relations between D_α-expansions: the critical case. We read that the action of $D_\epsilon D_1 D_\epsilon$ and $D_1 D_\epsilon D_1 D_0$ coincide.

commutation relations as well, but, because one of the involved subterms is nested in the other, it may be moved, and even possibly duplicated, when the main expansion is performed, so that the nested expansion(s) have different names before and after the main expansion. Finally, the critical case is specific to the LD-law, and there is no way to predict it except a direct verification, as shown in Figure 7. □

Definition 5.5. Let R_{LD} be the family of all relations of Lemma 5.4. We define LD^+ to be the monoid $\langle \{D_\alpha \mid \alpha \in \mathbb{A}\} \mid R_{\mathrm{LD}} \rangle^+$.

Lemma 5.4 immediately implies

Proposition 5.6. *The partial action of the free monoid* $\{D_\alpha \mid \alpha \in \mathbb{A}\}^*$ *on terms induces a well defined partial action of the monoid* LD^+.

For t a term and a in LD^+, we shall naturally denote by $t \bullet a$ the common value of $t \bullet w$ for all sequences w of letters D_α that represent a.

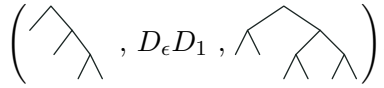
Remark. In this way, each LD-expansion receives a label that is an element of LD^+, thus becoming a labelled LD-expansion. However, we do not claim that a labelled LD-expansion is the same as an LD-expansion. Indeed, we do not claim that the relations of Lemma 5.4 exhaust all possible relations between the action of the D_α's on terms. A priori, it might be that different elements of LD^+ induce the same action on terms, so that one pair (t, t') might correspond to several labelled expansions with different labels. As we shall see below, the uniqueness of the labelling is another form of the above mentioned Embedding Conjecture.

5.3. The category $\mathcal{LD}^+$

We are now ready to introduce our main subject of interest, namely the category $\mathcal{LD}^+$ of labelled LD-expansions. The starting point is the same as for $\mathcal{LD}_0^+$, but the difference is that, now, we explicitly take into account the way the source is expanded into the target.

Definition 5.7. We denote by $\mathcal{LD}^+$ the category whose objects are terms, and whose morphisms are triples (t, a, t') with a in LD^+ and $t \bullet a = t'$.

In other words, $\mathcal{LD}^+$ is the category associated with the partial action of LD^+ on terms, in the sense of Section 1.9. We recall our convention that, when the morphisms of a category are triples, the source is the first entry, and the target is the last entry. So, for instance, a typical morphism in $\mathcal{LD}^+$ is the triple



whose source is the term $x \star (x \star (x \star x))$ (we adopt the default convention that specifying no variable means using some fixed variable x), and whose target is the term $(x \star x) \star ((x \star x) \star (x \star x))$.

5.4. The element Δ_t

We aim at proving that the category $\mathcal{LD}^+$ is a left-Garside category. To this end, we need to define the Δ-morphisms. As planned in Section 4.3, the latter will be constructed using the LD-expansions $(t, \phi(t))$. Defining a labelled version of this expansion means fixing some canonical way of expanding a term t into the corresponding term $\phi(t)$. A natural solution then exists, namely following the recursive definition of the operations $\circledast$ and ϕ.

For w a word in the letters D_α, we denote by $\mathrm{sh}_0(w)$ the word obtained by replacing each letter D_α of w with the corresponding letter $D_{0\alpha}$, *i.e.*, by *shifting* all indices by 0. Similarly, we denote by $\mathrm{sh}_\gamma(w)$ the word obtained by appending γ on the left of each address in w. The LD-relations of Lemma 5.4 are invariant under shifting: if w and w' represent the same element a of LD^+, then, for each γ, the words $\mathrm{sh}_\gamma(w)$ and $\mathrm{sh}_\gamma(w')$ represent the same element, naturally denoted $\mathrm{sh}_\gamma(a)$, of LD^+. For each a in LD^+,

the action of $\mathrm{sh}_\gamma(a)$ on a term t corresponds to the action of a to the γ-subterm of t: so, for instance, if $t' = t \bullet a$ holds, then $t' \star t_1 = (t \star t_1) \bullet \mathrm{sh}_0(a)$ holds as well, since the 0-subterm of $t \star t_1$ is t, whereas that of $t' \star t_1$ is t'.

Definition 5.8. For each term t, the elements δ_t and Δ_t of LD^+ are defined by the recursive rules

$$\delta_t = \begin{cases} 1 & \text{for } t \text{ of size 0, } \textit{i.e.}\text{, when } t \text{ is a variable } x_i, \\ D_\epsilon \cdot \mathrm{sh}_0(\delta_{t_0}) \cdot \mathrm{sh}_1(\delta_{t_1}) & \text{for } t = t_0 \star t_1. \end{cases} \tag{5.1}$$

$$\Delta_t = \begin{cases} 1 & \text{for } t \text{ of size 0,} \\ \mathrm{sh}_0(\Delta_{t_0}) \cdot \mathrm{sh}_1(\Delta_{t_1}) \cdot \delta_{\phi(t_1)} & \text{for } t = t_0 \star t_1. \end{cases} \tag{5.2}$$

Example 5.9. Let t be $x \star (x \star (x \star x))$. Then $t_{/0}$ is x, and, therefore, $\Delta_{t_{/0}}$ is 1. Next, $t_{/1}$ is $x \star (x \star x)$, so (5.2) reads $\Delta_t = \mathrm{sh}_1(\Delta_{t_{/1}}) \cdot \delta_{\phi(t_{/1})}$. Then $\phi(t_{/1})$ is $(x \star x) \star (x \star x)$. Applying (5.1), we obtain

$$\delta_{\phi(t_{/1})} = D_\epsilon \cdot \mathrm{sh}_0(\delta_{x \star x}) \cdot \mathrm{sh}_1(\delta_{x \star x}) = D_\epsilon D_0 D_1.$$

On the other hand, using (5.2) again, we find

$$\Delta_{t_{/1}} = \mathrm{sh}_0(\Delta_x) \cdot \mathrm{sh}_1(\Delta_{x \star x}) \cdot \delta_{x \star x} = 1 \cdot 1 \cdot D_\epsilon = D_\epsilon,$$

and, finally, we obtain $\Delta_t = D_1 D_\epsilon D_0 D_1$. According to the defining relations of the monoid LD^+, this element is also $D_\epsilon D_1 D_\epsilon$. Note the compatibility of the result with the examples of Figures 5 and 7.

Lemma 5.10. *For all terms t_0, t, we have*

$$(t_0 \star t) \bullet \delta_t = t_0 \circledast t, \tag{5.3}$$

$$t \bullet \Delta_t = \phi(t). \tag{5.4}$$

The proof is an easy inductive verification.

5.5. **Connection with braids**

Before investigating the category $\mathcal{LD}^+$ more precisely, we describe the simple connection existing between $\mathcal{LD}^+$ and the positive braid category $\mathcal{B}^+$ of Example 1.10.

Lemma 5.11. *Define* $\pi : \{D_\alpha \mid \alpha \in \mathbb{A}\} \to \{\sigma_i \mid i \geqslant 1\} \cup \{1\}$ *by*

$$\pi(D_\alpha) = \begin{cases} \sigma_{i+1} & \text{if } \alpha \text{ is the address } 1^i, \text{ i.e., } 11...1, \text{ } i \text{ times } 1, \\ 1 & \text{otherwise.} \end{cases} \tag{5.5}$$

Then π *induces a surjective monoid homomorphism of* LD^+ *onto* B_∞^+.

Proof. The point is that each LD-relation of Lemma 5.4 projects under π onto a braid relation. All relations involving addresses that contain at least one 0 collapse to mere equalities. The remaining relations are

$$D_{1^i} D_{1^j} = D_{1^j} D_{1^i} \quad \text{with } j \geqslant i+2,$$

which projects to the valid braid relation $\sigma_{i-1}\sigma_{j-1} = \sigma_{j-1}\sigma_{i-1}$, and

$$D_{1^i} D_{1^j} D_{1^i} = D_{1^j} D_{1^i} D_{1^j} D_{1^i 0}, \quad \text{with } j = i+1,$$

which projects to the not less valid braid relation

$$\sigma_{i-1}\sigma_{j-1}\sigma_{i-1} = \sigma_{j-1}\sigma_{i-1}\sigma_{j-1}.$$

□

We introduced a category $\mathcal{C}(M, X)$ for each monoid M partially acting on X in Definition 1.9. The braid category $\mathcal{B}^+$ and our current category $\mathcal{LD}^+$ are of these type. For such categories, natural functors arise from morphisms between the involved monoids, and we fix the following notation.

Definition 5.12. Assume that M, M' are monoids acting on sets X and X', respectively. A morphism $\varphi\colon M \to M'$ and a map $\psi\colon X \to X'$ are called *compatible* if

$$\psi(x \bullet a) = \psi(x) \bullet \varphi(a) \tag{5.6}$$

holds whenever $x \bullet a$ is defined. Then, we denote by $[\varphi, \psi]$ the functor of $\mathcal{C}(M, X)$ to $\mathcal{C}(M', X')$ that coincides with ψ on objects and maps (x, a, y) to $(\psi(x), \varphi(a), \psi(y))$.

Proposition 5.13. *Define the* right-height $\mathrm{ht}(t)$ *of a term* t *by* $\mathrm{ht}(x_i) = 0$ *and* $\mathrm{ht}(t_0 \star t_1) = \mathrm{ht}(t_1) + 1$. *Then the morphism* π *of* (5.5) *is compatible with* ht, *and* $[\pi, \mathrm{ht}]$ *is a surjective functor of* $\mathcal{LD}^+$ *onto* $\mathcal{B}^+$.

The parameter $\mathrm{ht}(t)$ is the length of the rightmost branch in t viewed as a tree or, equivalently, the number of final)'s in t viewed as a bracketed expression.

Proof. Assume that (t, a, t') belongs to $\mathcal{H}om(\mathcal{LD}^+)$. Put $n = \mathrm{ht}(t)$. The LD-law preserves the right-height of terms, so we have $\mathrm{ht}(t') = n$ as well. The hypothesis that $t \bullet a$ exists implies that the factors D_{1^i} that occur in some (hence in every) expression of a satisfy $i < n - 1$. Hence $\pi(a)$ is a braid of B_n^+, and $n \bullet \pi(a)$ is defined. Then the compatibility condition (5.6) is clear, and $[\pi, \mathrm{ht}]$ is a functor of $\mathcal{LD}^+$ to $\mathcal{B}^+$.

Surjectivity is clear, as each braid σ_i belongs to the image of π. □

Moreover, a simple relation connects the elements Δ_t of LD^+ and the braids Δ_n.

Proposition 5.14. *If t is a term with non-zero right-height n, we have $\pi(\Delta_t) = \Delta_n$.*

Proof. We first prove that $\mathrm{ht}(t) = n$ implies

$$\pi(\delta_t) = \sigma_1\sigma_2...\sigma_n \tag{5.7}$$

using induction on the size of t. If t is a variable, we have $\mathrm{ht}(t) = 0$ and $\delta_t = 1$, so the equality is clear. Otherwise, write $t = t_0 \star t_1$. By definition, we have

$$\delta_t = D_\epsilon \cdot \mathrm{sh}_0(\delta_{t_0}) \cdot \mathrm{sh}_1(\delta_{t_1}).$$

Let sh denote the endomorphism of B_∞^+ that maps σ_i to σ_{i+1} for each i. Then π collapses every term in the image of sh_0, and $\pi(\mathrm{sh}_1(a)) = \mathrm{sh}(\pi(a))$ holds for each a in LD^+. So, using the induction hypothesis $\pi(\delta_{t_1}) = \sigma_1...\sigma_{n-1}$, we deduce

$$\pi(\delta_t) = \sigma_1 \cdot 1 \cdot \mathrm{sh}(\sigma_1...\sigma_{n-1}) = \sigma_1...\sigma_n,$$

which is (5.7). Put $\Delta_0 = 1$ $(= \Delta_1)$. We prove that $\mathrm{ht}(t) = n$ implies $\pi(\Delta_t) = \Delta_n$ for $n \geqslant 0$, using induction on the size of t again. If t is a variable, we have $n = 0$ and $\Delta_t = 1$, as expected. Otherwise, write $t = t_0 \star t_1$. The definition gives

$$\Delta_t = \mathrm{sh}_0(\Delta_{t_0}) \cdot \mathrm{sh}_1(\Delta_{t_1}) \cdot \delta_{\phi(t_1)}.$$

As above, π collapses the term in the image of sh_0, and it transforms sh_1 into sh. Hence, using the induction hypothesis $\pi(\Delta_{t_1}) = \Delta_{n-1}$ and (5.7) for $\phi(t_1)$, whose right-height is that of t_1, we obtain

$$\pi(\Delta_t) = 1 \cdot \mathrm{sh}(\Delta_{n-1}) \cdot \sigma_1\sigma_2 \dots \sigma_{n-1} = \Delta_n.$$ □

6. **Main results**

We can now state the two main results of this paper.

Theorem 6.1. *For each term t, put $\Delta(t) = (t, \Delta_t, \phi(t))$. Then $\mathcal{LD}^+$ is a left-Garside category with left-Garside map Δ, and $[\pi, \mathrm{ht}]$ is a surjective right-lcm preserving functor of $\mathcal{LD}^+$ onto the positive braid category $\mathcal{B}^+$.*

Theorem 6.2. *Unless the category $\mathcal{LD}^+$ is not regular, the Embedding Conjecture of* [18, Chapter IX] *is true.*

6.1. **Recognizing left-preGarside monoids**

Owing to Proposition 1.12 and to the construction of $\mathcal{LD}^+$ from the partial action of the monoid LD^+ on terms, the first part of Theorem 6.1 is a direct consequence of

Proposition 6.3. *The monoid LD^+ equipped with its partial action on terms via self-distributivity is a locally left-Garside monoid with associated left-Garside sequence $(\Delta_t)_{t\in T}$.*

This is the result we shall prove now. The first step is to prove that LD^+ is left-preGarside. To do it, we appeal to general tools that we now describe. As for (LG_0), we have an easy sufficient condition when the action turns out to be monotonous in the following sense.

Proposition 6.4. *Assume that M has a partial action on X and there exists a map $\mu : X \to \mathbb{N}$ such that $a \neq 1$ implies $\mu(x \bullet a) > \mu(x)$. Then M satisfies (LG_0).*

Proof. Assume that $(a_1, ..., a_\ell)$ is a $\prec$-increasing sequence in $\mathrm{Div}(a)$. By definition of a partial action, there exists x in X such that $x \bullet a$ is defined, and this implies that $x \bullet a_i$ is defined for each i. Next, the hypothesis that $(a_1, ..., a_\ell)$ is $\prec$-increasing implies that there exist $b_2, ..., b_\ell \neq 1$ satisfying $a_i = a_{i-1}b_i$ for each i. We find

$$\mu(x \bullet a_i) = \mu((x \bullet a_{i-1}) \bullet b_i) > \mu(x \bullet a_{i-1}),$$

and the sequence $(\mu(x \bullet a_1), ..., \mu(x \bullet a_\ell))$ is increasing. As $\mu(x \bullet a_1) \geqslant \mu(x)$ holds, we deduce $\ell \leqslant \mu(x \bullet a) - \mu(x) + 1$ and, therefore, M satisfies (LG_0). □

As for conditions (LG_1) and (LG_2), we appeal to the subword reversing method of [21]. We recall that S^* denotes the free monoid generated by S. We use ϵ for the empty word.

Definition 6.5. Let S be any set. A map $C : S \times S \to S^*$ is called a *complement* on S. Then, we denote by R_C the family of all relations $aC(a,b) = bC(b,a)$ with $a \neq b$ in S, and by $\widehat{C}$ the unique (possibly partial) map of $S^* \times S^*$ to S^* that extends C and obeys the recursive rules

$$\widehat{C}(u, v_1v_2) = \widehat{C}(u,v_1)\widehat{C}(\widehat{C}(v_1,u),v_2),\ \widehat{C}(v_1v_2,u) = \widehat{C}(v_2,\widehat{C}(u,v_1)). \quad (6.1)$$

Proposition 6.6 ([21] or [18, Prop. II.2.5.])**.** *Assume that M is a monoid satisfying* (LG_0) *and admitting the presentation $\langle S, R_C\rangle^+$, where C is a complement on S. Then the following are equivalent:*

(*i*) *The monoid M is left-preGarside;*

(*ii*) *For all a, b, c in S, we have*

$$\widehat{C}(\widehat{C}(\widehat{C}(a,b),\widehat{C}(a,c)),\widehat{C}(\widehat{C}(b,a),\widehat{C}(b,c))) = \epsilon. \quad (6.2)$$

6.2. **Proof of Therem 6.1**

We shall now prove that the monoid LD^+ equipped with its partial action on terms via left self-distributivity satisfies the criteria of Section 6.1. Here, and in most subsequent developments, we heavily appeal to the results of [18], some of which have quite intricate proofs.

Proof of Theorem 6.1. First, each term t has a natural size $\mu(t)$, namely the number of inner nodes in the associated binary tree. Then the hypothesis of Proposition 6.4 clearly holds: if t' is a nontrivial LD-expansion of t, then the size of t' is larger than that of t. Then, by Proposition 6.4, LD^+ satisfies (LG_0).

Next, we observe that the presentation of LD^+ in Definition 5.5 is associated with a complement on the set $\{D_\alpha \mid \alpha \in \mathbb{A}\}$. Indeed, for each pair of addresses α, β, there exists in the list R_{LD} exactly one relation of the type $D_\alpha ... = D_\beta ....$ Hence, in view of Proposition 6.6, and because we know that LD^+ satisfies (LG_0), it suffices to check that (6.2) holds in LD^+ for each triple $D_\alpha, D_\beta, D_\gamma$. This is Proposition VIII.1.9 of [18]. Hence LD^+ satisfies (LG_1) and (LG_2), and it is a left-preGarside monoid.

Let us now consider the elements Δ_t of Definition 5.8. First, Lemma 5.10 implies that $t \bullet \Delta_t$ is defined for each term t, and it is equal to $\phi(t)$. Next, assume that $t \bullet D_\alpha$ is defined. Then Lemma VII.3.16 of [18] states that D_α is a left-divisor of Δ_t in LD^+, whereas Lemma VII.3.17 of [18] states that Δ_t is a left-divisor of $D_\alpha \Delta_{t \bullet D_\alpha}$. Hence Condition ($\mathsf{LG}_3^{loc}$) of Definition 1.11

is satisfied, and the sequence $(\Delta_t)_{t\in T}$ is a left-Garside sequence in LD^+. Thus LD^+ is a locally left-Garside monoid, which completes the proof of Proposition 6.3.

By Proposition 1.12, we deduce that $\mathcal{LD}^+$, which is $\mathcal{C}(\mathrm{LD}^+, T)$ by definition, is a left-Garside category with left-Garside map Δ as defined in Theorem 6.1.

As for the connection with the braid category $\mathcal{B}^+$, we saw in Proposition 5.13 that $[\pi, \mathrm{ht}]$ is a surjective functor of $\mathcal{LD}^+$ onto $\mathcal{B}^+$, and it just remains to prove that it preserves right-lcm's. This follows from the fact that the homomorphism π of LD^+ to B_∞^+ preserves right-lcm's, which in turn follows from the fact that LD^+ and B_∞^+ are associated with complements C and $\overline{C}$ satisfying, for each pair of addresses α, β,

$$\pi(C(D_\alpha, D_\beta)) = \overline{C}(\pi(D_\alpha), \pi(D_\beta)). \tag{6.3}$$

Indeed, let a, b be any two elements of LD^+. Let u, v be words on the alphabet $\{D_\alpha \mid \alpha \in \mathbb{A}\}$ that represent a and b, respectively. By [18, Proposition II.2.16], the word $\widehat{C}(u, v)$ exists, and $u\widehat{C}(u, v)$ represents $\mathrm{lcm}(a, b)$. Then $\pi(u\widehat{C}(u, v))$ represents a common right-multiple of the braids $\pi(a)$ and $\pi(b)$, and, by (6.3), we have

$$\pi(u\widehat{C}(u, v)) = \pi(u)\widehat{\overline{C}}(\pi(u), \pi(v)).$$

This shows that the braid represented by $\pi(u\widehat{C}(u, v))$, which is $\pi(\mathrm{lcm}(a, b))$ by definition, is the right-lcm of the braids $\pi(a)$ and $\pi(b)$. So the morphism π preserves right-lcm's, and the proof of Theorem 6.1 is complete. □

6.3. **The Embedding Conjecture**

From the viewpoint of self-distributive algebra, the main benefit of the current approach might be that it leads to a natural program for possibly establishing the so-called Embedding Conjecture. This conjecture, at the moment the most puzzling open question involving free LD-systems, can be stated in several equivalent forms.

Proposition 6.7. [18, Section IX.6] *The following are equivalent:*
(*i*) *The monoid* LD^+ *embeds in a group;*
(*ii*) *The monoid* LD^+ *admits right-cancellation;*
(*iii*) *The categories* $\mathcal{LD}_0^+$ *and* $\mathcal{LD}^+$ *are isomorphic;*

(iv) The functor ϕ associated with the category $\mathcal{LD}^+$ is injective;
(v) For each term t, the LD-expansions of t make an upper-semilattice;
(vi) The relations of Lemma 5.4 generate all relations that connect the action of D_α's by self-distributivity.

Each of the above properties is conjectured to be true: this is the *Embedding Conjecture*.

We turn to the proof of Theorem 6.2. So our aim is to show that the Embedding Conjecture is true whenever the category $\mathcal{LD}^+$ is regular. To this end, we shall use some technical results from [18], plus the following criterion, which enables one to prove right-cancellability by only using simple morphisms.

Proposition 6.8. *Assume that $\mathcal{C}$ is a left-Garside category and the associated functor ϕ is injective on $\mathcal{O}bj(\mathcal{C})$. Then the following are equivalent:*
(i) $\mathcal{H}om(\mathcal{C})$ admits right-cancellation;
(ii) The functor ϕ is injective on $\mathcal{H}om(\mathcal{C})$.
Moreover, if $\mathcal{C}$ is regular, (i) and (ii) are equivalent to
(iii) The functor ϕ is injective on simple morphisms of $\mathcal{C}$.

Proof. Assume that f, g are morphisms of $\mathcal{C}$ that satisfy $\phi(f) = \phi(g)$. As ϕ is a functor, we first deduce

$$\phi(\partial_0 f) = \partial_0(\phi(f)) = \partial_0(\phi(g)) = \phi(\partial_0 g),$$

hence $\partial_0 f = \partial_0 g$ as ϕ is injective on objects. A similar argument gives $\partial_1 f = \partial_1 g$. Then, (2.1) gives

$$f\Delta(\partial_1 f) = \Delta(\partial_0 f)\phi(f) = \Delta(\partial_0 g)\phi(g) = g\Delta(\partial_1 g) = g\Delta(\partial_1 f).$$

If we can cancel $\Delta(\partial_1 f)$ on the right, we deduce $f = g$ and, therefore, (i) implies (ii).

Conversely, assume that h is simple and $fh = gh$ holds. By multiplying by h^*, we deduce $fhh^* = ghh^*$, *i.e.*, $f\Delta(\partial_0 h) = g\Delta(\partial_0 h)$. As we have $\partial_1 f = \partial_0 h = \partial_1 g$ by hypothesis, applying (2.1) gives

$$\Delta(\partial_0 f)\phi(f) = f\Delta(\partial_1 f) = g\Delta(\partial_1 g) = \Delta(\partial_0 g)\phi(g) = \Delta(\partial_0 f)\phi(g),$$

hence $\phi(f) = \phi(g)$ by left-cancelling $\Delta(\partial_0 f)$. If (ii) holds, we deduce $f = g$, *i.e.*, h is right-cancellable. As simple morphisms generate $\mathcal{H}om(\mathcal{C})$, we deduce that every morphism is right-cancellable and, therefore, (ii) implies (i).

It is clear that (ii) implies (iii). So assume that $\mathcal{C}$ is regular and (iii) holds. Let f, g satisfy $\phi(f) = \phi(g)$. Let $(f_1, ..., f_d)$ and $(g_1, ..., g_e)$ be the normal forms of f and g, respectively. The regularity assumption implies that every length two subsequence of $(\phi(f_1), ..., \phi(f_d))$ and $(\phi(g_1), ..., \phi(g_e))$ is normal. Moreover, (iii) guarantees that $\phi(f_i)$ and $\phi(g_j)$ are nontrivial. Hence $(\phi(f_1), ..., \phi(f_d))$ and $(\phi(g_1), ..., \phi(g_e))$ are normal. As ϕ is a functor, we have $\phi(f_1)...\phi(f_d) = \phi(f) = \phi(g) = \phi(g_1)...\phi(g_e)$, and the uniqueness of the normal form implies $d = e$, and $\phi(f_i) = \phi(g_i)$ for each i. Then (iii) implies $f_i = g_i$ for each i, hence $f = g$. □

So, in order to prove Theorem 6.2, it suffices to show that the category $\mathcal{LD}^+$ satisfies the hypotheses of Proposition 6.8, and this is what we do now.

Lemma 6.9. *The functor ϕ of $\mathcal{LD}^+$ is injective on objects,* i.e.*, on terms.*

Proof. We show using induction on the size of t that $\phi(t)$ determines t. The result is obvious if t has size 0, *i.e.*, when t is a single variable x_i. Assume $t = t_0 \star t_1$. By construction, the term $\phi(t)$ is obtained by substituting every variable x_i occurring in the term $\phi(t_1)$ with the term $\phi(t_0) \star x_i$. Hence $\phi(t_0)$ is the $1^{n-1}0$-subterm of $\phi(t)$, where n is the common right-height of t and $\phi(t)$. From there, $\phi(t_1)$ can be recovered by replacing the subterms $\phi(t_0) \star x_i$ of $\phi(t)$ by x_i. Then, by induction hypothesis, t_0 and t_1, hence t, can be recovered from $\phi(t_1)$ and $\phi(t_0)$. □

Lemma 6.10. *The functor ϕ of $\mathcal{LD}^+$ is injective on simple morphisms.*

Proof. Assume that f, f' are morphisms of $\mathcal{LD}^+$ satisfying $\phi(f) = \phi(f')$, say $f = (t, a, s)$ and $f' = (t', a', s')$. The explicit description of Lemma 2.7 implies $\phi(t) = \phi(t')$, hence $t = t'$ by Lemma 6.9. Similarly, we have $\phi(s) = \phi(s')$, hence $s' = s$. Therefore, we have $t \bullet a = t \bullet a' = s$. By Proposition VII.126 of [18], we deduce that $t \bullet a = t \bullet a'$ holds for every term t for which both $t \bullet a$ and $t \bullet a'$ are defined. Then Proposition IX.6.6 of [18] implies $a = a'$ provided a or a' is simple. □

We can now complete the argument.

Proof of Theorem 6.2. The category $\mathcal{LD}^+$ is left-Garside, with an associated functor ϕ that is injective both on objects and on simple morphisms. By Proposition 6.8, if $\mathcal{LD}^+$ is regular, then $\mathcal{H}om(\mathcal{LD}^+)$ admits right-cancellation, which is one of the forms of the Embedding Conjecture, namely (ii) in Proposition 6.7. □

6.4. **A program for proving the regularity of $\mathcal{LD}^+$**

At this point, we are left with the question of proving (or disproving)

Conjecture 1. The left-Garside category $\mathcal{LD}^+$ is regular.

The regularity criteria of Section 3.5 lead to a natural program for possibly proving Conjecture 1 and, therefore, the Embedding Conjecture.

We begin with a preliminary observation.

Lemma 6.11. *The left-Garside sequence $(\Delta_t)_{t\in T}$ on LD^+ is coherent (in the sense of Definition 3.10).*

Proof. The question is to prove that, if t is a term and $t \bullet a$ is defined and $a \preccurlyeq \Delta_{t'}$ holds for some t', then we necessarily have $a \preccurlyeq \Delta_t$. This is a direct consequence of Proposition VIII.5.1 of [18]. Indeed, the latter states that an element a is a left-divisor of some element Δ_t if and only if a can be represented by a word in the letters D_α that has a certain special form. This property does not involve the term t, and it implies that, if a left-divides Δ_t, then it automatically left-divides every element $\Delta_{t'}$ such that $t' \bullet a$ is defined. □

So, according to Proposition 3.11, we obtain a well defined notion of a simple element in LD^+: an element a of LD^+ is called *simple* if it left-divides at least one element of the form Δ_t. Then simple elements form a seed in LD^+, and are eligible for a normal form satisfying the general properties described in Section 3. In this context, applying Proposition 3.19(ii) leads to the following criterion.

Proposition 6.12. *Assume that, for each term t and for all simple elements a, b of LD^+ such that $t \bullet a$ and $t \bullet b$ are defined, we have*

$$\gcd(\phi_t(a), \phi_t(b)) = \phi_t(\gcd(a, b)). \tag{6.4}$$

Then Conjecture 1 is true.

Proof. Let f, g be two simple morphisms in $\mathcal{LD}^+$ that satisfy $\partial_0 f = \partial_0 g = t$. By definition, f has the form $(t, a, t \bullet a)$ for some a satisfying $a \preccurlyeq \Delta_t$, hence simple in LD^+. Similarly, f has the form $(t, b, t \bullet b)$ for some simple element b, and we have $\gcd(f, g) = (t, \gcd(a, b), t \bullet \gcd(a, b))$. On the other hand, Lemma 2.7 gives $\phi(f) = (\phi(t), \phi_t(a), \phi(t \bullet a))$ and $\phi(g) = (\phi(t), \phi_t(b), \phi(t \bullet b))$, whence

$$\gcd(\phi(f), \phi(g)) = (\phi(t), \gcd(\phi_t(a), \phi_t(b)), \phi(t) \bullet \gcd(\phi_t(a), \phi_t(b))).$$

If (6.4) holds, we deduce

$$\gcd(\phi(f),\phi(g)) = \phi(\gcd(f,g)).$$

Moreover, if $t\bullet a$ is defined, then $a \neq 1$ implies $\mu(\phi(t\bullet a)) > \mu(\phi(t))$, whence $\phi_t(a) \neq 1$. Then, Proposition 3.19(*ii*) implies that $\mathcal{LD}^+$ is regular. □

Example 6.13. Assume $a = D_\epsilon$, $b = D_1$, and $t = x \star (x \star (x \star x))$. Then $t \bullet a$ and $t \bullet b$ are defined. On the other hand, we have $\phi(t) = ((x\star x)\star(x\star x)) \star ((x \star x) \star (x \star x))$. An easy computation gives $\phi_t(D_\epsilon) = D_0D_1$ and $\phi_t(D_1) = D_\epsilon$, see Figure 8. We find $\gcd(\phi_t(a),\phi_t(b)) = 1 = \gcd(a,b)$, and (6.4) is true in this case.

Note that the couterpart of (6.4) involving right-lcm's fails. In the current case, we have

$$\operatorname{lcm}(\phi_t(a),\phi_t(b)) = \phi_t(\operatorname{lcm}(a,b)) \cdot D_0D_1 :$$

the terms $\phi(t \bullet D_\epsilon)$ and $\phi(t \bullet D_1)$ admit a common LD-expansion that is smaller than $\phi_t(t \bullet \operatorname{lcm}(D_\epsilon, D_1))$, which turns out to be $\phi^2(t)$, see Figure 8 again.

The reader may similarly check that (6.4) holds for $t = (x \star (x \star x)) \star (x \star (x \star x))$ with $a = D_0$ and $b = D_1$; the values are $\phi_t(D_0) = D_{000}D_{010}D_{100}D_{110}$ and $\phi_t(D_1) = D_\epsilon$.

Proposition 6.12 leads to a realistic program that would reduce the proof of the Embedding Conjecture to a (long) sequence of verifications. Indeed, it is shown in Proposition VIII.5.15 of [18] that every simple element a of LD^+ admits a unique expression of the form

$$a = \prod_{\alpha\in\mathbb{A}}^{>} D_\alpha^{(e_\alpha)},$$

where $D_\alpha^{(e)}$ denotes $D_{\alpha 1^{e-1}}...D_{\alpha 1}D_\alpha$ and $>$ refers to the unique linear ordering of $\mathbb{A}$ satisfying $\alpha > \alpha 0\beta > \alpha 1\gamma$ for all α,β,γ. In this way, we associate with every simple element a of LD^+ a sequence of nonnegative integers $(e_\alpha)_{\alpha\in\mathbb{A}}$ that plays the role of a sequence of coordinates for a. Then it *should* be possible to

- express the coordinates of $\phi_t(a)$ in terms of those of a,
- express the coordinates of $\gcd(a,b)$ in terms of those of a and b.

If this were done, proving (or disproving) the equalities (6.4) should be easy.

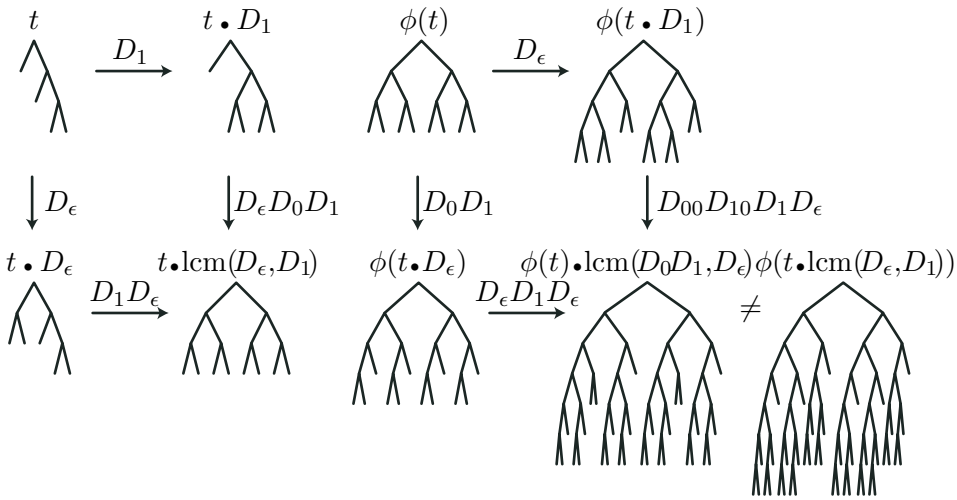


FIGURE 8. The left diagram shows an instance of Relation (6.4): for the considered choice of t, we find $\Delta_t = D_\epsilon D_1 D_\epsilon$, $\Delta_{t \bullet D_1} = D_1 D_\epsilon D_0 D_{00} D_1 D_{10}$, leading to $\phi_t(D_1) = D_\epsilon$ and $\phi_t(D_\epsilon) = D_0 D_1$. Here $\phi_t(D_\epsilon)$ and $\phi_t(D_1)$ are left-coprime, so (6.4) is true. The right diagram shows that the counterpart involving lcm's fails.

Remark. Contrary to the braid relations, the LD-relations of Lemma 5.4 are not symmetric. However, it turns out that the presentation of LD^+ is also associated with what can naturally be called a left-complement, namely a counterpart of a (right)-complement involving left-multiples. But the latter fails to satisfy the counterpart of (6.2), and it is extremely unlikely that one can prove that the monoid LD^+ is possibly right-cancellative (which would imply the Embedding Conjecture) using some version of Proposition 6.6.

7. **Reproving braids properties**

Proposition 5.13 and Theorem 6.1 connect the Garside structures associated with self-distributivity and with braids, both being previously known to exist. In this section, we show how the existence of the Garside structure of braids can be (re)-proved to exist assuming the existencce of the Garside structure of $\mathcal{LD}^+$ only. So, for a while, we pretend that we do

not know that the braid monoid B_n^+ has a Garside structure, and we only know about the Garside structure of $\mathcal{LD}^+$.

7.1. **Projections**

We begin with a general criterion guaranteeing that the projection of a locally left-Garside monoid is again a locally left-Garside monoid.

If $S, \overline{S}$ are two alphabets and π is a map of S to $\overline{S}^*$ (the free monoid on $\overline{S}$), we still denote by π the alphabetical homomorphism of S^* to $\overline{S}^*$ that extends π, defined by $\pi(s_1...s_\ell) = \pi(s_1)...\pi(s_\ell)$.

Lemma 7.1. *Assume that*

- *M is a locally left-Garside monoid associated with a complement C on S;*
- *$\overline{M}$ is a monoid associated with a complement $\overline{C}$ on $\overline{S}$ and satisfying (LG_0);*
- *$\pi : S \to \overline{S} \cup \{\epsilon\}$ satisfies $\pi(S) \supseteq \overline{S}$ and*

$$\text{For all } a, b \text{ in } S, \text{ we have } \overline{C}(\pi(a), \pi(b)) = \pi(C(a, b)). \tag{7.1}$$

Then $\overline{M}$ is left-preGarside, and π induces a surjective right-lcm preserving homomorphism of M onto $\overline{M}$.

Proof. An easy induction shows that, if u, v are words on S and $\widehat{C}(u, v)$ exists, then $\widehat{\overline{C}}(\pi(u), \pi(v))$ exists as well and we have

$$\widehat{\overline{C}}(\pi(u), \pi(v)) = \pi(\widehat{C}(u, v)). \tag{7.2}$$

Let $\overline{a}, \overline{b}, \overline{c}$ be elements of $\overline{S}$. By hypothesis, there exist a, b, c in S satisfying $\pi(a) = \overline{a}$, $\pi(b) = \overline{b}$, $\pi(c) = \overline{c}$. As M is left-preGarside, by the direct implication of Proposition 6.6, the relation (6.2) involving a, b, c is true in S^*. Applying π and using (7.2), we deduce that the relation (6.2) involving $\overline{a}, \overline{b}, \overline{c}$ is true in $\overline{S}^*$. Then, as $\overline{M}$ satisfies (LG_0) by hypothesis, the converse implication of Proposition 6.6 implies that $\overline{M}$ is left-preGarside.

Then, by definition, the relations $aC(a, b) = bC(b, a)$ with $a, b \in S$ make a presentation of M. Now, for a, b in S, we find

$$\pi(a)\overline{C}(\pi(a), \pi(b)) = \pi(aC(a, b)) = \pi(bC(b, a)) = \pi(b)C(\pi(b), \pi(a))$$

in $\overline{M}$, which shows that the homomorphism of S^* to $\overline{M}$ that extends π induces a well defined homomorphism of M to $\overline{M}$. This homomorphism, still denoted π, is surjective since, by hypothesis, its image includes $\overline{S}$.

Finally, we claim that π preserves right-lcm's. The argument is almost the same as in the proof of Theorem 6.1, with the difference that, here, we do not assume that common multiples necessarily exist. Let a, b be two elements of M that admit a common right-multiple. Let u, v be words on S^* that represent a and b, respectively. By Proposition II.2.16 of [18], the word $\widehat{C}(u, v)$ exists, and $u\widehat{C}(u, v)$ represents $\mathrm{lcm}(a, b)$. Then the word $\pi(u\widehat{C}(u, v))$ represents a common right-multiple of $\pi(a)$ and $\pi(b)$ in $\overline{M}$, and, by (7.2), we have

$$\pi(u\widehat{C}(u, v)) = \pi(u)\, \widehat{\overline{C}}(\pi(u), \pi(v)),$$

which shows that the element represented by $\pi(u\widehat{C}(u, v))$, which, by definition, is $\pi(\mathrm{lcm}(a, b))$, is the right-lcm of $\pi(a)$ and $\pi(b)$ in $\overline{M}$. □

We turn to locally left-Garside monoids, *i.e.*, we add partial actions in the picture. Although lengthy, the following result is easy. It just says that, if M is a locally left-Garside monoid, then its image under a projection that is compatible with the various ingredients of the Garside structure is again locally left-Garside.

Proposition 7.2. *Assume that*

• *M is a locally left-Garside monoid associated with a complement C on S and $(\Delta_x)_{x\in X}$ is a left-Garside sequence for the involved action of M on X;*

• *$\overline{M}$ is a monoid associated with a complement $\overline{C}$ on $\overline{S}$ that has a partial action on $\overline{X}$ and satisfies* (LG_0)*;*

• *$\pi : S \to \overline{S} \cup \{\epsilon\}$ satisfies* (7.2)*, $\theta : \overline{S} \to S$ is a section for π, $\varpi : X \to \overline{X}$ is a surjection, and*

For x in X and a in M, if $x \bullet a$ is defined, then so is $\varpi(x) \bullet \pi(a)$ and we have $\varpi(x) \bullet \pi(a) = \varpi(x \bullet a)$; (7.3)

For $\overline{x}$ in $\overline{X}$ and $\overline{a}$ in $\overline{S}$, if $\overline{x} \bullet \overline{a}$ is defined, then so is $x \bullet \theta(\overline{a})$ for each x satisfying $\varpi(x) = x$; (7.4)

For x in X, the value of $\pi(\Delta_x)$ depends on $\varpi(x)$ only. (7.5)

For $\overline{x}$ in $\overline{X}$, let $\Delta_{\overline{x}}$ be the common value of $\pi(\Delta_x)$ for $\varpi(x) = \overline{x}$. Then $\overline{M}$ is locally left-Garside, with associated left-Garside sequence $(\Delta_{\overline{x}})_{\overline{x}\in\overline{X}}$, and π induces a surjective right-lcm preserving homomorphism of M onto $\overline{M}$.

Proof. First, the hypotheses of Lemma 7.1 are satisfied, hence $\overline{M}$ is left-preGarside and π induces a surjective lcm-preserving homomorphism of M onto $\overline{M}$.

Next, by (7.5), the definition of the elements $\Delta_{\overline{x}}$ for $\overline{x}$ in $\overline{X}$ is unambiguous. It remains to check that $(\Delta_{\overline{x}})_{\overline{x}\in\overline{X}}$ is a left-Garside sequence with respect to the action of $\overline{M}$ on $\overline{X}$. So, assume $\overline{x} \in \overline{M}$, and let x be any element of M satisfying $\varpi(x) = \overline{x}$.

First, $x \bullet \Delta_x$ is defined, hence, by (7.3), so is $\varpi(x) \bullet \pi(\Delta_x)$, which is $\overline{x} \bullet \Delta_{\overline{x}}$.

Assume now $\overline{a} \neq 1$ and $\overline{x}\bullet\overline{a}$ is defined. As $\overline{S}$ generates $\overline{M}$, we can assume $a \in \overline{S}$ without loss of generality. By (7.4), the existence of $\overline{x} \bullet \overline{a}$ implies that of $x \bullet \theta(\overline{a})$. As $(\Delta_x)_{x\in X}$ is a left-Garside sequence for the action of M on X, we have $a' \preccurlyeq \Delta_x$ for some $a' \neq 1$ left-dividing $\theta(\overline{a})$. By construction, $\theta(\overline{a})$ lies in $\overline{S}$, and it is an atom in M. So the only possiblity is $a' = \theta(\overline{a})$, *i.e.*, we have $\theta(\overline{a}) \preccurlyeq \Delta_x$. Applying π, we deduce $\overline{a} \preccurlyeq \Delta_{\overline{x}}$ in $\overline{M}$.

Finally, under the same hypotheses, we have $\Delta_x \preccurlyeq \theta(\overline{a})\, \Delta_{x\bullet\theta(\overline{a})}$ in M. Using

$$\pi(\Delta_{x\bullet\theta(\overline{a})}) = \Delta_{\varpi(x\bullet\theta(\overline{a}))} = \Delta_{\varpi(x)\bullet\overline{a}} = \Delta_{\overline{x}\bullet\overline{a}},$$

we deduce $\Delta_{\overline{x}} \preccurlyeq \overline{a}\, \Delta_{\overline{x}\bullet\overline{a}}$ in $\overline{M}$, always under the hypothesis $\overline{a} \in \overline{S}$. The case of an arbitrary element $\overline{a}$ for which $\overline{x} \bullet \overline{a}$ exists then follows from an easy induction on the length of an expression of $\overline{a}$ as a product of elements of $\overline{S}$. □

It should then be clear that, under the hypotheses of Proposition 7.2, $[\pi, \varpi]$ is a surjective, right-lcm preserving functor of $\mathcal{C}(M, X)$ to $\mathcal{C}(\overline{M}, \overline{X})$.

7.2. The case of $\mathcal{LD}^+$ and $\mathcal{B}^+$

Applying the criterion of Section 7.1 to the categories $\mathcal{LD}^+$ and $\mathcal{B}^+$ is easy.

Proposition 7.3. *The monoid B^+_∞ is a locally left-Garside monoid with respect to its action on $\mathbb{N}$, and $(\Delta_n)_{n\in\mathbb{N}}$ is a left-Garside sequence in B^+_∞.*

Proof. Hereafter, we denote by C the complement on $\{D_\alpha \mid \alpha \in \mathbb{A}\}$ associated with the LD-relations of Lemma 5.4, and by $\overline{C}$ the complement on $\{\sigma_i \mid i \geqslant 1\}$ associated with the braid relations of (1.2). We consider the maps π of Lemma 5.11, and the right-height ht from terms to nonnegative integers. Finally, we define θ by $\theta(\sigma_i) = D_{1^{i-i}}$. We claim that these data satisfy all hypotheses of Proposition 7.2. The verifications are easy. That

the complements C and $\overline{C}$ satisfy (7.1) follows from a direct inspection. For instance, we find

$$\pi(C(D_1, D_\epsilon)) = \pi(D_\epsilon D_1 D_0) = \sigma_1\sigma_2 = \overline{C}(\sigma_2, \sigma_1) = \overline{C}(\pi(D_1), \pi(D_\epsilon)),$$

and similar relations hold for all pairs of generators D_α, D_β.

Then, the action of LD^+ on terms preserve the right-height, whereas the action of braids on $\mathbb{N}$ is trivial, so (7.3) is clear. Next, θ is a section for π, and we observe that $t \bullet \theta(\sigma_i)$ is defined if and only if the right-height of t is at least $i+1$, hence if and only if $\mathrm{ht}(t) \bullet \sigma_i$ is defined, so (7.4) is satisfied. Finally, we observed in Proposition 5.13 that $\pi(\Delta_t)$ is equal to $\Delta_{\mathrm{ht}(t)}$, hence it depends on $\mathrm{ht}(t)$ only. So (7.5) is satisfied.

Therefore, Proposition 7.2 applies, and it gives the expected result. □

Corollary 7.4. (*i*) *The braid category $\mathcal{B}^+$ is a left-Garside category.*
(*ii*) *For each n, the braid monoid B_n^+ is a Garside monoid.*

Proof. Point (*i*) follows from Proposition 1.12 once we know that B_∞^+ is locally left-Garside. Point (*ii*) follows from Proposition 1.13 since, for each n, the submonoid B_n^+ of B_∞^+ is $(B_\infty^+)_n$ in the sense of Definition 1.8. □

Thus, as announced, the Garside structure of braids can be recovered from the left-Garside structure of $\mathcal{LD}^+$.

8. **Intermediate categories**

We conclude with a different topic. The projection of the self-distributivity category $\mathcal{LD}^+$ to the braid category $\mathcal{B}^+$ described above is rather trivial in that terms are involved through their right-height only and the corresponding action of braids on integers is just constant. Actually, one can consider alternative projections corresponding to less trivial braid actions and leading to two-step projections

$$\mathcal{LD}^+ \longrightarrow \mathcal{C}(B_\infty^+, X) \longrightarrow \mathcal{B}^+.$$

We shall describe two such examples.

8.1. **Action of braids on sequences of integers**

Braids act on sequences of integers via their permutations. Indeed, the rule

$$(x_1, ..., x_n) \bullet \sigma_i = (x_1, ..., x_{i-1}, x_{i+1}, x_i, x_{i+2}, ..., x_n) \tag{8.1}$$

defines an action of B_n^+ on $\mathbb{N}^n$, whence a partial action of B_∞^+ on $\mathbb{N}^*$, where $\mathbb{N}^*$ denotes the set of all finite sequences in $\mathbb{N}$. In this way, we obtain a new category $\mathcal{C}(B_\infty^+, \mathbb{N}^*)$, which clearly projects to $\mathcal{B}^+$.

We shall now describe an explicit projection of $\mathcal{LD}^+$ onto this category. We recall that terms have been defined to be bracketed expressions constructed from a fixed sequence of variables $x_1, x_2, ...$ (or as binary trees with leaves labelled with variables x_p), and that, for t a term and α a binary address, $t_{/\alpha}$ denotes the subterm of t whose root, when t is viewed as a binary tree, has address α.

Proposition 8.1. *Let $\widehat{\mathcal{B}}^+$ be the category associated with the partial action* (8.1) *of B_∞^+ on $\mathbb{N}^*$. Then $\widehat{\mathcal{B}}^+$ is a Garside category, and the projection $[\pi, \mathrm{ht}]$ of $\mathcal{LD}^+$ onto $\mathcal{B}^+$ factors through $\widehat{\mathcal{B}}^+$ into*

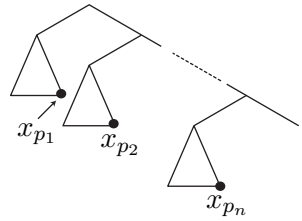
$$\mathcal{LD}^+ \xrightarrow{[\pi, \widehat{\pi}]} \widehat{\mathcal{B}}^+ \xrightarrow{[\mathrm{id}, \mathrm{lg}]} \mathcal{B}^+,$$

where $\widehat{\pi}$ is defined for $\mathrm{ht}(t) = n$ by

$$\widehat{\pi}(t) = (\mathrm{var}_R(t_{/0}), \mathrm{var}_R(t_{/10}), ..., \mathrm{var}_R(t_{/1^{n-1}0})),$$

$\mathrm{var}_R(t)$ denoting the index of the rightmost variable occurring in t.

So, a typical morphism of $\widehat{\mathcal{B}}^+$ is $((1,2,2), \sigma_1, (2,1,2))$, and the projection of terms to sequences of integers is given by



$$\overset{\widehat{\pi}}{\mapsto} (p_1, p_2, ..., p_n).$$

Sketch of proof. The point is to check that the action of the LD-law on the indices of the right variables of the subterms with addresses 1^i0 is compatible with the action of braids on sequences of integers. It suffices to consider the basic case of $D_{1^{i-1}}$, and the expected relation is shown

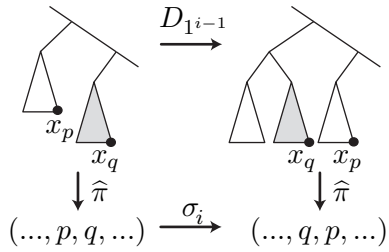


FIGURE 9. Compatibility of the action of $D_{1^{i-1}}$ on sequences of "subright" variables and of the action of σ_i on sequences of integers.

in Figure 9. Details are easy. Note that, for symmetry reasons, the category $\widehat{\mathcal{B}}^+$ is not only left-Garside, but even Garside. □

8.2. **Action of braids on LD-systems**

The action of positive braids on sequences of integers defined in (8.1) is just one example of a much more general situation, namely the action of positive braids on sequences of elements of any LD-system. It is well known—see, for instance, [18, Chapter I]—that, if $(S, \star)$ is an LD-system, *i.e.*, $\star$ is a binary operation on S that obeys the LD-law, then

$$(x_1, ..., x_n) \bullet \sigma_i = (x_1, ..., x_{i-1}, x_i \star x_{i+1}, x_i, x_{i+2}, ..., x_n) \qquad (8.2)$$

induces a well defined action of the monoid B_n^+ on the set S^n, and, from there, a partial action of B_∞^+ on the set S^* of all finite sequences of elements of S.

Proposition 8.2. *Assume that $(S, \star)$ is an LD-system, and let $\mathcal{B}_S^+$ be the category associated with the partial action* (8.2) *of B_∞^+ on S^*. Then $\mathcal{B}_S^+$ is a left-Garside category, and, for each sequence $\boldsymbol{s} = (s_1, s_2, ...)$ of elements of S, the projection $[\pi, \mathrm{ht}]$ of $\mathcal{LD}^+$ onto $\mathcal{B}^+$ factors through $\mathcal{B}_S^+$ into*

$$\mathcal{LD}^+ \xrightarrow{[\pi, \pi_{\boldsymbol{s}}]} \mathcal{B}_S^+ \xrightarrow{[\mathrm{id}, \mathrm{lg}]} \mathcal{B}^+,$$

where $\pi_{\boldsymbol{s}}$ is defined for $\mathrm{ht}(t) = n$ by

$$\pi_{\boldsymbol{s}}(t) = (\mathrm{eval}_{\boldsymbol{s}}(t_{/0}), ..., \mathrm{eval}_{\boldsymbol{s}}(t_{/1^{n-1}0})),$$

$\mathrm{eval}_{\boldsymbol{s}}(t)$ *being the evaluation of* t *in* $(S,\star)$ *when* x_p *is given the value* s_p *for each* p.

We skip the proof, which is an easy verification similar to that of Proposition 8.1.

When $(S,\star)$ is $\mathbb{N}$ equipped with $x\star y = y$ and we map x_p to p for each p, we obtain the category $\widehat{\mathcal{B}}^+$ of Proposition 8.1. In this case, the (partial) action of braids is not constant as in the case of $\mathcal{B}^+$, but it factors through an action of the associated permutations, and it is therefore far from being free. By contrast, if we take for S the braid group B_∞ with the operation $\star$ defined by $x\star y = x\,\mathrm{sh}(y)\,\sigma_1\,\mathrm{sh}(x)^{-1}$, where we recall sh is the shift endomorphism of B_∞ that maps σ_i to σ_{i+1} for each i, and if we send x_p to 1 (or to any other fixed braid) for each p, then the corresponding action (8.2) of B_∞^+ on $(B_\infty)^*$ is *free*, in the sense that $a = a'$ holds whenever $s\bullet a = s\bullet a'$ holds for at least one sequence s in $(B_\infty)^*$: this follows from Lemma III.1.10 of [24]. This suggests that the associated category $\mathcal{C}(B_\infty^+,(B_\infty)^*))$ has a very rich structure.

Appendix: Other algebraic laws

The above approach of self-distributivity can be developed for other algebraic laws as well. However, at least from the viewpoint of Garside structures, the case of self-distributivity seems quite particular.

The case of associativity

Associativity is the law $x(yz) = (xy)z$. It is syntactically close to self-distributivity, the only difference being that the variable x is not duplicated in the right hand side. Let us say that a term t' is an A-expansion of another term t if t' can be obtained from t by applying the associativity law in the left-to-right direction only, *i.e.*, by iteratively replacing subterms of the form $t_1\star(t_2\star t_3)$ by the corresponding term $(t_1\star t_2)\star t_3$. Then the counterpart of Proposition 4.5 is true, *i.e.*, two terms t,t' are equivalent up to associativity if and only if they admit a common A-expansion, a trivial result since every size n term t admits as an A-expansion the term $\phi(t)$ obtained from t by pushing all brackets to the left.

As in Sections 4.3 and 5.2, we can introduce the category $\mathcal{A}_0^+$ whose objects are terms, and whose morphisms are pairs (t, t') with t' an A-expansion of t. As in Section 5.1, we can take positions into account, using A_α when associativity is applied at address α, and introduce a monoid A^+ that describes the connections between the generators A_α [22]. Here the relations of Lemma 5.4 are to be replaced by analogous new relations, among which the MacLane–Stasheff Pentagon relations $A_\alpha^2 = A_{\alpha 1} A_\alpha A_{\alpha 0}$. The monoid A^+ turns out to be a well known object: indeed, it is (isomorphic to) the submonoid F^+ of R. Thompson's group F generated by the standard generators $x_1, x_2, \ldots$ [11]. Also, the orbits of the partial action of the monoid A^+ on terms are well known: these are the (type A) associahedra, equipped with the structure known as Tamari lattice.

Now, as in Section 5, we can introduce the category $\mathcal{A}^+$, whose objects are terms, and whose morphisms are triples (t, a, t') with a in A^+ and $t \bullet a = t'$. Using $\psi(t)$ for the term obtained from t by pushing all brackets to the right, we have

Proposition. *The categories $\mathcal{A}_0^+$ and $\mathcal{A}^+$ are isomorphic; $\mathcal{A}_0^+$ is left-Garside with Garside map $t \mapsto (t, \phi(t))$, and right-Garside with Garside map $t \mapsto (\psi(t), t)$.*

This result might appear promising. It is not! Indeed, the involved Garside structure(s) is trivial: the maps ϕ and ψ are constant on each orbit of the action of A^+ on terms, and it easily follows that every morphism in $\mathcal{A}_0^+$ and $\mathcal{A}^+$ is left-simple and right-simple so that, for instance, the greedy normal form of any morphism always has length one[3]. The only observation worth noting is that $\mathcal{A}^+$ provides an example where the left- and the right-Garside structures are not compatible, and, therefore, we have no Garside structure in the sense of Definition 1.7.

Central duplication

We conclude with still another example, namely the exotic *central duplication* law $x(yz) = (xy)(yz)$ of [20]. The situation there turns out to be similar to that of self-distributivity, and a nontrivial left-Garside structure appears. As there is no known connection between this law and other

[3]We do not claim that the monoid A^+ is not an interesting object in itself: actually it is, with rich nontrivial algebraic and geometric properties, see [22]; we only say that the current Garside category approach is not relevant here.

widely investigated objects like braids, it is probably not necessary to go into details.

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