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# $L_{p,q}$-cohomology of warped cylinders

Yaroslav Kopylov

**Abstract**

We extend some results by Gol′dshtein, Kuz′minov, and Shvedov about the $L_p$-cohomology of warped cylinders to $L_{p,q}$-cohomology for $p \neq q$. As an application, we establish some sufficient conditions for the nontriviality of the $L_{p,q}$-torsion of a surface of revolution.

## *Cohomologie $L_{p,q}$ des cylindres tordus*

**Résumé**

On généralise quelques résultats par Gol′dshtein, Kuz′minov et Shvedov sur la cohomologie $L_p$ des cylindres tordus à cohomologie $L_{p,q}$ pour $p \neq q$. Comme application, on établit des conditions suffisantes pour la non-nullité de la torsion $L_{p,q}$ d'une surface de révolution.

## 1. Introduction

Let $M$ be a Riemannian manifold. For $1 \leq p \leq \infty$ and a positive continuous function $\sigma : M \to \mathbb{R}$, denote by $L^j_p(M, \sigma)$ the Banach space of measurable forms of degree $j$ on $M$ with the finite norm

$$\|\omega\|_{L^j_p(M,\sigma)} = \begin{cases} \left\{ \int_M |\omega(x)|^p \sigma^p(x)\, dx \right\}^{1/p} & \text{if } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x\in M} |\omega(x)|\sigma(x) & \text{if } p = \infty. \end{cases}$$

Here $dx$ stands for the volume element of $M$ and $|\omega(x)|$ is the modulus of the exterior form $\omega(x)$. In the usual way, we also define the spaces $L_{p,\mathrm{loc}}(M)$.

Denote by $D^j(M) = C_0^{\infty,j}(M)$ the space of smooth forms of degree $j$ on $M$ having compact support included in $\operatorname{Int} M$. A form $\psi \in L^{j+1}_{1,\mathrm{loc}}(M)$

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is called the (*weak*) *differential* $d\omega$ of $\omega \in L^j_{1,\mathrm{loc}}(M)$ if

$$\int\limits_U \omega \wedge du = (-1)^{j+1} \int\limits_U \psi \wedge u$$

for every orientable domain $U \subset \operatorname{Int} M$ and every form $u \in D^{\dim M - j - 1}(M)$ having support in $U$.

For two weights $\sigma_j$, $\sigma_{j+1}$ on $M$, put

$$W^j_{p,q}(M, \sigma_j, \sigma_{j+1}) = \{\omega \in L^j_p(M, \sigma_j) \mid d\omega \in L^{j+1}_q(M, \sigma_{j+1})\}.$$

The space $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ is endowed with the norm

$$\|\omega\|_{W^j_{p,q}(M,\sigma_j,\sigma_{j+1})} = \|\omega\|_{L^j_p(M,\sigma_j)} + \|d\omega\|_{L^{j+1}_q(M,\sigma_{j+1})}.$$

If $p = q$ then it is often more convenient to consider the equivalent norm

$$\|\omega\|_{W^j_p(M,\sigma_j,\sigma_{j+1})} = \left(\|\omega\|^p_{L^j_p(M,\sigma_j)} + \|d\omega\|^p_{L^{j+1}_p(M,\sigma_{j+1})}\right)^{1/p}.$$

In the sequel we let $V^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ denote the closure of $D^j(M)$ in the norm of $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$.

Given an arbitrary subset $A \subset M$, let $W^j_{p,q}(M, A, \sigma_j, \sigma_{j+1})$ be the closure in $W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ of the subspace spanned by all forms $\omega \in W^j_{p,q}(M, \sigma_j, \sigma_{j+1})$ which vanish on some neighborhood of $A$ (depending on $\omega$).

Let $Z^j_q(M, \sigma_j)$ be the subspace in $W^j_{q,q}(M, \sigma_j, \sigma_j)$ that consists of all forms $\omega$ such that $d\omega = 0$ and let

$$\begin{aligned} B^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = \{\theta \in W^j_{q,q}(M, \sigma_j, \sigma_j) \\ \mid \theta = d\psi \quad \text{for some} \quad \psi \in W^{j-1}_{p,q}(M, \sigma_{j-1}, \sigma_j)\}. \end{aligned}$$

The spaces

$$H^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$$

and

$$\overline{H}^j_{p,q}(M, \sigma_{j-1}, \sigma_j) = Z^j_q(M, \sigma_j)/\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j),$$

where $\overline{B}^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ is the closure of $B^j_{p,q}(M, \sigma_{j-1}, \sigma_j)$ in $L^j_q(M, \sigma_j)$ (equivalently, in $W^j_{q,q}(M, \sigma_j, \sigma_j)$) are called the $j$th $L_{p,q}$-*cohomology* and the $j$th *reduced $L_{p,q}$-cohomology* of the Riemannian manifold $M$ *with weights $\sigma_{j-1}$ and $\sigma_j$*. The quotient space

$$T^j_{p,q}(M,\sigma_{j-1},\sigma_j) = \overline{B}^j_{p,q}(M,\sigma_{j-1},\sigma_j)/B^j_{p,q}(M,\sigma_{j-1},\sigma_j)$$

will be referred to as the $L_{p,q}$-*torsion* of $M$ with the given weights. Clearly, the space $T^j_{p,q}(M,\sigma_{j-1},\sigma_j)$ is isomorphic to the closure of the zero in $H^j_{p,q}(M,\sigma_{j-1},\sigma_j)$.

Given a subset $A \subset M$, the *relative nonreduced and reduced $L_{p,q}$-cohomology spaces* $H^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ and $\overline{H}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ are defined as

$$H^j_{p,q}(M,A,\sigma_{j-1},\sigma_j) = Z^j_q(M,A,\sigma_j)/B^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$$

and

$$\overline{H}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j) = Z^j_q(M,A,\sigma_j)/\overline{B}^j_{p,q}(M,A,\sigma_{j-1},\sigma_j),$$

where the relative spaces $Z^j_q(M,A,\sigma_j)$ and $B^j_{p,q}(M,A,\sigma_{j-1},\sigma_j)$ are defined as their absolute analogs above with the spaces $W^j_{p,q}(M,\sigma_j,\sigma_j)$ and $W^{j-1}_{p,q}(M,\sigma_{j-1},\sigma_j)$ replaced by the spaces

$$W^j_{p,q}(M,A,\sigma_j,\sigma_j) \text{ and } W^{j-1}_{p,q}(M,A,\sigma_{j-1},\sigma_j).$$

For $p = q$, we write the subscript $p$ instead of $p,p$ throughout. If the weights involved in the definition of the corresponding space are equal to 1 then they will be omitted.

The spaces $W_{p,q}$ and $L_{p,q}$-cohomology were introduced at the beginning of the 1980's by Gol′dshtein, Kuz′minov, and Shvedov [3, 4, 5, 6, 7, 8], who obtained many results concerning $W_{p,q}$-forms and especially $L_p$-cohomology. Later $L_{p,q}$-cohomology was considered in [11, 12, 13, 14, 15, 17, 22].

In this paper, we, following [9, 10], look for conditions of the nontriviality of the $L_{p,q}$-cohomology and $L_{p,q}$-torsion on warped cylinders, a class of warped products of Riemannian manifolds. By the *warped product* $X \times_f Y$ of two Riemannian manifolds $(X, g_X)$ and $(Y, g_Y)$ *with the warping function* $f : X \to \mathbb{R}_+$ we mean the product manifold $X \times Y$ endowed with the metric $g_X + f^2(x)g_Y$. If $X = [a, b[$ is a half-interval on the real line then $X \times_f Y$ is referred to as the *warped cylinder*. The study of the $L_2$-cohomology of warped cylinders was initiated by Cheeger [2].

The structure of the article is as follows. In Section 2, we adapt the results of [9] about the $L_p$-cohomology of a half-interval to the case $p \neq q$.

After that, using these $L_{p,q}$-results, in Section 3, we prove a partial $L_{p,q}$-generalization of Theorem 1 of [9] about the $L_p$-cohomology of a warped cylinder $[a,b[\times_f Y$ depending on the analytic properties of the function $f$. As an application, we obtain an extension of the necessary condition for the triviality of the $L_{p,q}$-torsion of a surface of revolution in $\mathbb{R}^{n+2}$ [16] from the case $p=q$ to arbitrary $p$,$q$ such that $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$.

## 2. Weighted $L_{p,q}$-cohomology of a half-interval

Consider a half-interval $[a,b[$, $-\infty<a<b\leq\infty$ and positive continuous functions $v_0,v_1:[a,b[\to\mathbb{R}$. For $1<p,q<\infty$, the space $W^0_{p,q}([a,b[,v_0,v_1)$ can be identified with the space of the functions $g\in L_p([a,b[,v_0)$ whose weak derivative $g'\in L_q([a,b[,v_1)$. As above, endow $W^0_{p,q}([a,b[,v_0,v_1)$ with the norm

$$\|g\|_{W^0_{p,q}([a,b[,v_0,v_1)}=\left(\int_a^b|g(t)|^pv_0^pdt\right)^{1/p}+\left(\int_a^b|g'(t)|^qv_1^qdt\right)^{1/q}.$$

From the classical Sobolev Embedding Theorem it follows that the functions of the class $W^0_{p,q}([a,b[,v_0,v_1)$ are continuous on $[a,b[$. Consider also the space

$$W^0_{p,q}([a,b[,\{a\},v_0,v_1)=\{f\in W^0_{p,q}([a,b[,\{a\},v_0,v_1)\mid f(a)=0\}.$$

We have

$$H^1_{p,q}([a,b[,v_0,v_1)=W^1_q([a,b[,v_1,v_1)/dW^0_{p,q}([a,b[,v_0,v_1);$$
$$H^1_{p,q}([a,b[,\{a\},v_0,v_1)=W^1_q([a,b[,\{a\},v_1,v_1)/dW^0_{p,q}([a,b[,\{a\},v_0,v_1).$$

The spaces $\overline{H}^1_{p,q}([a,b[,v_0,v_1)$ and $\overline{H}^1_{p,q}([a,b[,\{a\},v_0,v_1)$ are described similarly.

We call the following assertion the lemma about the Hardy inequality [1, 10, 21]:

**Lemma 2.1.** *Suppose that* $1\leq p,q\leq\infty$, $\frac{1}{q}+\frac{1}{q'}=1$, $\alpha,\beta\in[-\infty,\infty]$, $I_{\alpha,\beta}$ *is the interval with endpoints* $\alpha$ *and* $\beta$, $v_0$ *and* $v_1$ *are continuous positive functions on* $I_{\alpha,\beta}$. *Then for the existence of a global constant* $C$ *such that*

$$\left|\int_\alpha^\beta\left|v_0(t)\int_\alpha^\tau g(t)dt\right|^pd\tau\right|^{1/p}\leq C\left|\int_\alpha^\beta|v_1(t)g(t)|^qdt\right|^{1/q}$$

*for every* $g \in L_q(I_{\alpha,\beta}, v_1)$, *it is necessary and sufficient that*

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) < \infty\,.$$

*Here*

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) = \sup_{\tau \in I_{\alpha,\beta}} \left\{ \left| \int_\tau^\beta |v_0(t)|^p dt \right|^{1/p} \left| \int_\alpha^\tau |v_1(t)|^{-q'} dt \right|^{1/q'} \right\}$$

*if* $p \geq q$;

$$\chi_{p,q}(\alpha, \beta, v_0, v_1) = \left| \int_\alpha^\beta \left( \left| \int_\alpha^\tau |v_1(t)|^{-q'} dt \right|^{p-1} \left| \int_\tau^\beta |v_0(t)|^p dt \right| \right)^{\frac{q}{q-p}} |v_1(\tau)|^{-q'} d\tau \right|^{\frac{q-p}{pq}}$$

*if* $p < q$.

*If* $p = 1$ $(q' = \infty)$ *then the corresponding integral must be replaced by* ess sup.

The constant $\chi_{p,q}(\alpha, \beta, v_0, v_1)$ will be referred to as the *Hardy constant.*

The following lemma was proved in [9] for $p = q$ and $v_0 = v_1$. The proof given in [9] holds for different $p$ and $q$ and different $v_0$ and $v_1$.

**Lemma 2.2.** *Suppose that* $\alpha, \beta \in [-\infty, \infty]$, $v_0, v_1 : I_{\alpha,\beta} \to \mathbb{R}$ *are positive continuous functions, and* $\chi_{p,q}(\alpha, \beta, v_0, v_1) = \infty$. *Then there exists a nonnegative function* $h$ *such that*

$$\left| \int_\alpha^\beta v_1^q(t) h^q(t) dt \right| < \infty, \qquad \left| \int_\alpha^\beta v_0^p(\tau) \left| \int_\alpha^\tau h(t) dt \right|^p d\tau \right| = \infty.$$

As in [9], Lemma 2 yields the following assertion.

**Theorem 2.3.** *If* $v_0$, $v_1$ *are positive continuous functions on* $[a, b[$ *and* $1 < p, q < \infty$ *then*

(1) $H^1_{p,q}([a, b[, \{a\}, v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty$;

(2) $H^1_{p,q}([a, b[, v_0, v_1) = 0 \iff \chi_{p,q}(a, b, v_0, v_1) < \infty$ *or* $\chi_{p,q}(b, a, v_0, v_1) < \infty$.

Let

$$0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0 \tag{2.1}$$

be an exact sequence of Banach complexes, i.e., complexes in the category of Banach spaces and bounded linear operators. Sequence (2.1) yields an exact sequence of the cohomology spaces

$$\dots \to H^{k-1}(C) \xrightarrow{\partial} H^k(A) \xrightarrow{\varphi^*} H^k(B) \xrightarrow{\psi^*} H^k(C) \to \dots$$

with continuous operators $\partial^*$, $\varphi^*$, $\psi^*$ and a semi-exact sequence of the reduced cohomology spaces

$$\dots \to \overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^*} \overline{H}^k(B) \xrightarrow{\overline{\psi}^*} \overline{H}^k(C) \to \dots \tag{2.2}$$

Under certain conditions, sequence (2.2) is exact at some terms (see [10, 18, 20]). In particular, Gol′dshtein, Kuz′minov, and Shvedov proved the following assertion in [10, Theorem 1(1)]:

**Lemma 2.4.** *If $H^k(C)$ is separated and $\dim \partial(H^{k-1}(C)) < \infty$ then the sequence $\overline{H}^{k-1}(C) \xrightarrow{\overline{\partial}} \overline{H}^k(A) \xrightarrow{\overline{\varphi}^*} \overline{H}^k(B) \xrightarrow{\overline{\psi}^*} \overline{H}^k(C)$ is exact.*

As was explained in [12], we can describe the $j$th weighted $L_{p,q}$-cohomology of an $n$-dimensional Riemannian manifold $M$ with given weights $\sigma_{j-1}$ and $\sigma_j$ in terms of Banach complexes. To this end, consider an arbitrary sequence $\pi = \{p_0, p_1, \dots, p_n\} \subset [1, \infty]$ with $p_{j-1} = p$ and $p_j = q$ and a sequence of positive continuous weights $\sigma = \{\sigma_k\}_{k=0}^n$ with the given $\sigma_{j-1}$ and $\sigma_j$. Given a subset $A \subset M$, put

$$W_\pi^k(M, A, \sigma) = W_{p_k, p_{k+1}}(M, A, \sigma_k, \sigma_{k+1}).$$

Here we have assumed that $p_{n+1} = p_n$ and $\sigma_{n+1} = \sigma_n$.

Since the exterior differential is a bounded operator

$$d^{k-1} : W_\pi^{k-1}(M, A, \sigma) \to W_\pi^k(M, A, \sigma),$$

we obtain a Banach complex

$$0 \to W_\pi^0(M, A, \sigma) \xrightarrow{d^0} W_\pi^1(M, A, \sigma) \to \dots \xrightarrow{d^{n-1}} W_\pi^n(M, A, \sigma) \to 0. \tag{2.3}$$

By the $k$-th $L_\pi$*-cohomology* $H_\pi^k(M, A, \sigma)$ (*reduced $k$-th $L_\pi$-cohomology* $\overline{H}_\pi^k(M, A, \sigma)$) *of the Riemannian manifold $M$ with respect to $A$ with weight* $\sigma$ we mean the cohomology (reduced cohomology) of (2.3). Thus,

$$H_\pi^k(M, A, \sigma) = H_{p_{k-1}, p_k}^k(M, A, \sigma_{k-1}, \sigma_k)$$

and

$$\overline{H}_\pi^k(M, A, \sigma) = \overline{H}_{p_{k-1}, p_k}^k(M, A, \sigma_{k-1}, \sigma_k)$$

for all $k$. In particular,

$$H_\pi^j(M, A, \sigma) = H_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j),$$
$$\overline{H}_\pi^j(M, A, \sigma) = \overline{H}_{p,q}^j(M, A, \sigma_{j-1}, \sigma_j).$$

Take $M = [a,b[$, $A = \{a\}$, $1 < p,q < \infty$, $\pi = \{p,q\}$, and a pair of weights $v = \{v_0, v_1\}$. We have the following exact sequence of Banach complexes:

$$0 \to W^*_\pi([a,b[,\{a\},v) \xrightarrow{\mathfrak{j}} W^*_\pi([a,b[,v) \xrightarrow{\mathfrak{i}} H^*(\{a\}) \to 0,$$

where $H^*(\{a\})$ is the complex with the only nontrivial term $H^0(\{a\}) = \mathbb{R}$. Here the mappings $\mathfrak{i}$ and $\mathfrak{j}$ are defined as follows: $\mathfrak{j}$ is the inclusion mapping; if $g \in W^0_\pi([a,b[,v)$ then $\mathfrak{i}g = g(a)$ (recall that $g$ is continuous) and in dimension one $\mathfrak{j}$ is zero. Lemma 2.4 yields the exact sequence

$$\mathbb{R} = H^0(\{a\}) \xrightarrow{\overline{\partial}} \overline{H}^1_{p,q}([a,b[,\{a\},v_0,v_1) \xrightarrow{\overline{\mathfrak{j}}^*} \overline{H}^1_{p,q}([a,b[,v_0,v_1).$$

Thus, we infer the following assertion, proved for $p = q$ in [9]. With what has been said above, the proof of [9] extends to the case of $p \neq q$ without change.

**Theorem 2.5.** *If $v_0$, $v_1$ are positive continuous functions on $[a,b[$, $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$ then*

(1) $\overline{H}^1_{p,q}([a,b[,v_0,v_1) = 0$;

(2) $\overline{H}^1_{p,q}([a,b[,\{a\},v_0,v_1) = 0$ *if and only if* $\int_a^b v_1^{-q'}(t)dt = \infty$ *or* $\int_a^b v_0^p(t)dt < \infty$;

(3) *If* $\overline{H}_{p,q}([a,b[,\{a\},v_0,v_1) \neq 0$ *then*

$$\overline{\partial} : \mathbb{R} = H^0(\{a\}) \to \overline{H}^1_{p,q}([a,b[,\{a\},v_0,v_1)$$

*is an isomorphism.*

## 3. $L_{p,q}$-cohomology of the warped cylinder $C^f_{a,b}$

Let $Y$ be an orientable manifold of dimension $n$, $C^f_{a,b}Y = [a,b[\times_f Y$. Put $Y_a = \{a\} \times Y$. Generally speaking, $C^f_{a,b}$ is a Lipschitz Riemannian manifold in the sense of [3] but we will assume throughout for simplicity that $\partial Y = \varnothing$ to make $C^f_{a,b}$ smooth, which will be enough for our purposes.

Suppose that $1 < p < \infty$ and $1 < q < \infty$.

In [9], Gol′dshtein, Kuz′minov, and Shvedov introduced the bilinear mapping

$$\nu : L_p^{j-1}(Y) \times L_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to L_p^j(C_{a,b}^f Y),$$

$\nu(\varphi, gdt) = gdt \wedge \varphi$. In [9] it was proved that $\nu$ is continuous and if $\varphi \in Z_p^{j-1}(Y)$ then $\nu_\varphi = \nu(\varphi, \cdot) : L_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to L_p^j(C_{a,b}^f Y)$ induces continuous mappings

$$\nu_\varphi^* : H_p^1([a,b[, f^{\frac{n}{p}-j+1}) \to H_p^j(C_{a,b}^f Y);$$

$$\tilde{\nu}_\varphi^* : H_p^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}) \to H_p^j(C_{a,b}^f Y, Y_a).$$

Supposing that $\varphi \in Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$, we similarly become convinced that the mapping $\nu_\varphi = \nu(\varphi, \cdot)$ induces continuous mappings

$$\nu_\varphi^* : H_{p,q}^1([a,b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H_{p,q}^j(C_{a,b}^f Y);$$

$$\tilde{\nu}_\varphi^* : H_{p,q}^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H_{p,q}^j(C_{a,b}^f Y, Y_a).$$

Now, assume that $\psi \in L_{p'}^{n+1-j}(Y)$ ($p' = \frac{p}{p-1}$) and $\omega \in L_p^j(C_{a,b}Y)$. Write $\omega$ in the form $\omega = \omega_A + dt \wedge \omega_B$, where $\omega_A$, $\omega_B$ do not contain $dt$ [10]. Following [9], introduce the continuous operator

$$\mu_\psi\colon\ L_p^j(C_{a,b}^f Y) \to L_p^1([a,b[, f^{\frac{n}{p}-j+1})$$

by the formula

$$\mu_\psi \omega = \left( \int_Y \omega_B(t) \wedge \psi \right) dt.$$

The following lemma was proved in [9] for $p = q$ and $\psi \in V_{p'}^{n-j+1}(Y)$. The proof in [9] easily extends to $p \neq q$:

**Lemma 3.1.** *If $\psi \in D^{n-j+1}(Y)$ and $d\psi = 0$ then $\mu_\psi$ induces continuous mappings*

$$\mu_\psi^* : H_{p,q}^j(C_{a,b}^f Y) \to H_{p,q}^1([a,b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1});$$

$$\tilde{\mu}_\psi^* : H_{p,q}^j(C_{a,b}^f Y, Y_a) \to H_{p,q}^1([a,b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1})$$

We have the following theorem partially generalizing item 7 of Theorem 1 in [9]:

**Theorem 3.2.** *Suppose that $Y$ is an orientable $n$-dimensional Riemannian manifold, $\infty < a < b \le \infty$, $f : [a,b[ \to \mathbb{R}$ is a positive continuous function, $1 < p < \infty$, $1 < q < \infty$. Assume that there exists $\varphi \in$*

*$Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$ such that $\int_Y \varphi \wedge \gamma \neq 0$ for some form $\gamma \in D^{n-j+1}(Y)$, $d\gamma = 0$.*

*The following hold:*

*(1) if $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ then $H^j_{p,q}(C^f_{a,b}Y, Y_a) \neq 0$;*

*(2) if $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ and $\chi_{p,q}(b, a, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$ then $T^j_{p,q}(C^f_{a,b}Y) \neq 0$ and, hence, $\dim H^j_{p,q}(C^f_{a,b}Y) = \infty$.*

*Proof.* Let $\varphi \in Z_p^{j-1}(Y) \cap Z_q^{j-1}(Y)$ be a cocycle having the property mentioned in the theorem and let $\gamma \in D^{n-j+1}(M)$ be a form such that $\int_Y \varphi \wedge \gamma = 1$. Then $\mu^*_\gamma \circ \nu^*_\varphi = \mathrm{id}$, $\tilde{\mu}^*_\gamma \circ \tilde{\nu}^*_\varphi = \mathrm{id}$ [9]. Consequently, the mappings

$$\nu^*_\varphi : H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H^j_{p,q}(C^f_{a,b}Y)$$

and

$$\tilde{\nu}^*_\varphi : H^1_{p,q}([a, b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \to H^j_{p,q}(C^f_{a,b}Y, Y_a)$$

are injective.

Suppose that $\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$. Then, by Theorem 2.3, $H^1_{p,q}([a, b[, \{a\}, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0$. Therefore, $H^j_{p,q}(C^f_{a,b}Y, Y_a) \neq 0$.

Assume now that

$$\chi_{p,q}(a, b, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty$$

and

$$\chi_{p,q}(b, a, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = \infty.$$

Then, by Theorem 2.3, $H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0$. Since, by Theorem 2.5, $\overline{H}_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) = 0$, we have

$$T^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1}) \neq 0.$$

Now, if we had $T^j_{p,q}(C^f_{a,b}Y) = 0$, $\nu^*_\varphi$ would be a continuous injective mapping with values in the Hausdorff space $H^j_{p,q}(C^f_{a,b}Y)$, and so the cohomology space $H^1_{p,q}([a, b[, f^{\frac{n}{p}-j+1}, f^{\frac{n}{q}-j+1})$ would also be Hausdorff, i.e., without torsion. Thus, $T^j_{p,q}(C^f_{a,b}Y) \neq 0$. The theorem is proved. $\square$

## $L_{p,q}$-torsion of a surface of revolution

Let $M$ be a surface of revolution in $\mathbb{R}^{n+2}$, i.e., the $(n+1)$-dimensional surface defined by the equation

$$f^2(x_1) = x_2^2 + \dots + x_{n+2}^2, \quad (x_1, \dots, x_{n+2}) \in \mathbb{R}^{n+2},\ x_1 \ge 0, \tag{3.1}$$

where $f : [0, \infty[ \to \mathbb{R}$ is a positive smooth function. The manifold $M$ is the product $[0, \infty[ \times \mathbb{S}^n$ endowed with the metric

$$g_M = (1 + f'^2(x_1))dx_1^2 + f^2(x_1)dy^2$$

induced from $\mathbb{R}^{n+2}$, where $dx_1^2$ and $dy^2$ are the conventional Riemannian metrics on $[0, \infty[$ and the sphere $\mathbb{S}^n$. In other words, $M$ may be considered as the warped product $[0, \infty[ \times_F \mathbb{S}^n$, where $F = f \circ G^{-1}$, $G(x) = \int_0^x \sqrt{1 + f'^2(t)}dt$.

In [17], we have proved the following fact:

**Theorem 3.3.** *Suppose that $f$ is unbounded, $p, q \in [1, \infty[$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, $1 \le j \le n+1$. Then $T^j_{p,q}(M) \neq 0$.*

Kuz′minov and Shvedov [19] established that when $f$ is bounded from above, $T^j_p(M)$ is zero for all $j$, $2 \le j \le n$ and that, for $j = 1, n+1$, the triviality of $T^j_p(M)$ depends on the finiteness of some Hardy constants. This is due to the connection between the $L_p$-cohomology of the warped product $C^f_{a,b}Y$ and the weighted $L_p$-cohomology of $[a, b[$ given in the mentioned papers [9, 10]. Above we have shown that there is a connection of this type for $L_{p,q}$-cohomology. Namely, by Theorem 3.2, since $\mathbb{S}^n$ is compact and the de Rham cohomology $H^{j-1}(\mathbb{S}^n)$ of $\mathbb{S}^n$ is nontrivial if $j = 1, n+1$, for $T^j_{p,q}(M)$ $(j = 1, n+1)$ to be zero, it is necessary that $\chi_{p,q}(0, \infty, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$ or $\chi_{p,q}(\infty, 0, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$.

The main result of this section is a generalization of Theorems 2 and 2′ of [16] and is formulated as follows:

**Theorem 3.4.** *Let $M$ be the surface of revolution* (3.1). *Suppose that $1 < p < \infty$, $1 < q < \infty$, $\frac{1}{q} - \frac{1}{p} < \frac{1}{n+1}$, $j \in \{1, n+1\}$. If $T^j_{p,q}(M) = 0$ then $\lim\limits_{x \to \infty} f(x) = 0$ and $\operatorname{vol} M < \infty$.*

*Proof.* Put $k = j - 1$, $q' = \frac{q}{q-1}$.

We have the following equalities:

$$\chi^0_{p,q} \equiv \chi_{p,q}(0,\infty,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$
$$= \sup_{\tau>0}\Big\{\Big(\int\limits_{\tau}^{\infty} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int\limits_{0}^{\tau} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\Big\};$$

$$\chi^\infty_{p,q} \equiv \chi_{p,q}(\infty,0,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$
$$= \sup_{\tau>0}\Big\{\Big(\int\limits_{0}^{\tau} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int\limits_{\tau}^{\infty} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\Big\}$$

if $p \geq q$;

$$\chi^0_{p,q} \equiv \chi_{p,q}(0,\infty,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$
$$=\Big(\int\limits_0^\infty\Big[\Big(\int\limits_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{p-1}\int\limits_{H(x)}^{\infty} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\Big)^{\frac{q-p}{qp}}; \quad (3.2)$$

$$\chi^\infty_{p,q} \equiv \chi_{p,q}(\infty,0,F^{\frac{n}{p}-k},F^{\frac{n}{q}-k})$$
$$=\Big(\int\limits_0^\infty\Big[\Big(\int\limits_{H(x)}^{\infty} f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{p-1}\int\limits_{0}^{H(x)} f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\Big)^{\frac{q-p}{qp}}$$

if $p < q$. Here $H(x)$ is the function inverse to the arc length function $G(x) = \int_0^x \sqrt{1+f'^2(t)}dt$.

The main element in the proof of Theorem 3.4 is the following lemma which has some independent interest.

**Lemma 3.5.** *If $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$, $1<p<\infty$, $1<q<\infty$, $0\leq k\leq n$, then the following hold*:

(1) *if $\chi^0_{p,q}<\infty$ or $\chi^\infty_{p,q}<\infty$ then $\lim\limits_{t\to\infty} f(t)=0$*;

(2) *if* $\frac{n}{p} - k \leq 0$ *then* $\chi^0_{p,q} = \infty$;
(3) *if* $\frac{n}{q} - k \geq 0$ *then* $\chi^\infty_{p,q} = \infty$.

*Proof.* Suppose first that $p \geq q$.

Assume that $\chi^0_{p,q} < \infty$. Then

$$\int_0^\infty f^{n-kp}(t)\sqrt{1+f'^2(t)}dt < \infty \tag{3.3}$$

Since

$$f^{n-kp}(t)\sqrt{1+f'^2(t)} \geq f^{n-kp}(t)|f'(t)|,$$

it follows that the integral

$$\int_0^\infty f^{n-kp}(t)f'(t)dt$$
$$= \begin{cases} \frac{1}{n-kp+1} \lim\limits_{t\to\infty}(f^{n-kp+1}(t) - f^{n-kp+1}(0)) & \text{if } n-kp \neq -1, \\ \lim\limits_{t\to\infty} \log\frac{f(t)}{f(0)} & \text{if } n-kp = -1 \end{cases} \tag{3.4}$$

is finite.

There appear several possibilities:

(a) $\frac{n}{p} - k > 0$. The above implies that there exists a finite limit $\lim\limits_{t\to\infty} f(t)$, which is zero by (3.3).

(b) $\frac{n}{p} - k = 0$. This is impossible in view of (3.3).

(c) $-\frac{1}{p} < \frac{n}{p} - k < 0$. Then $n - kp + 1 > 0$ and $f(t)$ has a finite limit as $t \to \infty$, which contradicts (3.3).

(d) $\frac{n}{p} - k = -\frac{1}{p}$. A contradiction to (3.3).

(e) $\frac{n}{p} - k < -\frac{1}{p}$. In this case, $n - kp < -1$. Hence, $\lim\limits_{t\to\infty} f(t) = \infty$.

Note that, since $\frac{1}{q}-\frac{1}{p}<\frac{1}{n+1}$, we have $k+1>\frac{n+1}{p}+1>\frac{n+1}{q}$, whence $-(\frac{n}{q}-k)q'+1>0$. We infer

$$\begin{aligned}
&\Big(\int_\tau^\infty f^{n-kp}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/p}\Big(\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt\Big)^{1/q'}\\
&\quad\geq \Big(\int_\tau^\infty f^{n-kp}(t)|f'(t)|dt\Big)^{1/p}\Big(\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)|f'(t)|dt\Big)^{1/q'}\\
&\quad\geq \Big|\int_\tau^\infty f^{n-kp}(t)f'(t)dt\Big|^{1/p}\Big|\int_0^\tau f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\Big|^{1/q'}\\
&\quad\geq \Big(\frac{f^{n-kp+1}(\tau)}{|n-kp+1|}\Big)^{1/p}\Big|\frac{f^{-\frac{n}{q}-k)q'+1}(\tau)-f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\Big|^{1/q'}\\
&\quad= C\cdot f^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)|1-f^{-(\frac{n}{q}-k)q'+1}(0)f^{(\frac{n}{q}-k)q'-1}(\tau)|^{1/q'}. \qquad (3.5)
\end{aligned}$$

The last quantity in (3.5) is equivalent to $Cf^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)$ as $\tau\to\infty$ and, hence, tends to infinity. Therefore, $\chi^0_{p,q}=\infty$, and we obtain a contradiction.

Thus, if $\chi^0_{p,q}<\infty$ then $\lim\limits_{t\to 0}f(t)=0$ and $\frac{n}{p}-k>0$.

Suppose now that $\chi^\infty_{p,q}<\infty$. Then

$$\int_0^\infty f^{-(\frac{n}{q}-k)q'}(t)\sqrt{1+f'^2(t)}dt<\infty \qquad (3.6)$$

and, hence, there exists a finite integral

$$\begin{aligned}
&\int_0^\infty f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\\
&\quad=\begin{cases}\dfrac{\lim\limits_{t\to\infty} f^{-(\frac{n}{q}-k)q'+1}(t)-f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1} & \text{if } -(\frac{n}{q}-k)q'\neq -1,\\ \lim\limits_{t\to\infty}\log\frac{f(t)}{f(0)} & \text{if } -(\frac{n}{q}-k)q'=-1.\end{cases} \qquad (3.7)
\end{aligned}$$

As in the case $\chi^0_{p,q} < \infty$, we infer that either $\frac{n}{q} - k < 0$ and $\lim\limits_{t\to\infty} f(t) = 0$ or $(\frac{n}{q} - k)q' > 1$ and $\lim\limits_{t\to\infty} f(t) = \infty$. In the latter case we have:

$$\begin{aligned}\chi^\infty_{p,q} &\geq \sup_{\tau>0}\Big\{\Big(\frac{f^{-(\frac{n}{q}-k)q'+1}(\tau)}{1-(\frac{n}{q}-k)q'}\Big)^{1/q'}\Big|\frac{f^{n-kp+1}(\tau)-f^{n-kp+1}(0)}{n-kp+1}\Big|^{1/p}\Big\}\\ &= C\sup_{\tau>0}\{f^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)|1-f^{n-kp+1}(0)f^{-(n-kp+1)}(\tau)|^{1/p}\}, \quad (3.8)\end{aligned}$$

where $C = \text{const} > 0$. Since $k < \frac{n+1}{q} - 1 < \frac{n+1}{p}$, we have $n - kp + 1 > 0$, and, hence, the last quantity in (3.8) behaves like $Cf^{\frac{n+1}{p}-\frac{n+1}{q}+1}(\tau)$ and, consequently, tends to infinity as $\tau \to \infty$. Hence, $\chi^\infty_{p,q} = \infty$; a contradiction.

Thus, if $\chi^\infty_{p,q} < \infty$ then $\lim\limits_{t\to 0} f(t) = 0$ and $\frac{n}{q} - k < 0$.

We now pass to the case $p < q$.

Suppose that $\chi^0_{p,q} < \infty$. Then, as above, we have (3.3) and (3.4) and conclude that either $\frac{n}{p} - k > 0$ and $\lim\limits_{t\to\infty} f(t) = 0$ or $\frac{n}{p} - k < -\frac{1}{p}$ and $\lim\limits_{t\to\infty} f(t) = \infty$. Show that the latter case is impossible. By (3.2), we infer

$$\begin{aligned}(\chi^0_{p,q})^{\frac{pq}{q-p}} &\geq \int_0^\infty\Big[\Big|\int_0^{H(x)} f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\Big|^{p-1}\Big|\int_{H(x)}^\infty f^{n-kp}(t)f'(t)dt\Big|\Big]^{\frac{q}{q-p}}\\ &\qquad\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\\ &= \int_0^\infty\Big[\Big|\frac{f^{-(\frac{n}{q}-k)q'+1}(H(x))-f^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\Big|^{p-1}\Big|\frac{f^{n-kp+1}(H(x))}{n-kp+1}\Big|\Big]^{\frac{q}{q-p}}\\ &\qquad\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx\\ &= \int_0^\infty\Big[\Big|\frac{F^{-(\frac{n}{q}-k)q'+1}(s)-F^{-(\frac{n}{q}-k)q'+1}(0)}{-(\frac{n}{q}-k)q'+1}\Big|^{p-1}\Big|\frac{F^{n-kp+1}(s)}{n-kp+1}\Big|\Big]^{\frac{q}{q-p}}F^{-(\frac{n}{q}-k)q'}(s)ds\\ &= C\int_0^\infty F^N(s)|1-F^{-(\frac{n}{q}-k)q'+1}(0)F^{(\frac{n}{q}-k)q'-1}(s)|ds. \quad (3.9)\end{aligned}$$

Here $C = \text{const} > 0$ and

$$N = \left(\left(-\left(\frac{n}{q}-k\right)q'+1\right)(p-1)+n-kp+1\right)\frac{q}{q-p}-\left(\frac{n}{q}-k\right)q'$$
$$= \left[\left(1-\frac{p-1}{q-1}\right)\frac{q}{q-p}-\frac{1}{q-1}\right]n-\left[\left(\frac{q(p-1)}{q-1}-p\right)\frac{q}{q-p}+\frac{1}{q-1}\right]k+\frac{pq}{q-p}$$
$$= n-k+\frac{pq}{q-p} > 0.$$

Moreover, $(\frac{n}{q}-k)q'-1 < 0$, since $\frac{n}{q} < \frac{n+1}{q} < \frac{n+1}{p} < k$. Consequently, the expression under the last integral in (3.9) is equivalent to $CF^{n-k+\frac{pq}{q-p}}(s)$, i.e., tends to $\infty$ as $s \to \infty$ and, thus, the integral does not exist. A contradiction.

Suppose now that $\chi^{\infty}_{p,q} < \infty$. Then we have (3.6) and (3.7) and infer that, in this case, either $\frac{n}{q}-k < 0$ and $\lim\limits_{t\to\infty} f(t) = 0$ or $q'(\frac{n}{q}-k) > 1$ and $\lim\limits_{t\to\infty} f(t) = \infty$. In the latter case, we infer

$$(\chi^{\infty}_{p,q})^{\frac{pq}{q-p}} \geq \int\limits_0^{\infty}\Bigg[\Bigg|\int\limits_0^{H(x)} f^{n-kp}(t)f'(t)dt\Bigg|\Bigg|\int\limits_{H(x)}^{\infty} f^{-(\frac{n}{q}-k)q'}(t)f'(t)dt\Bigg|^{p-1}\Bigg]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$= \int\limits_0^{\infty}\Bigg[\Bigg|\frac{f^{n-kp+1}(H(x))-f^{n-kp+1}(0)}{n-kp+1}\Bigg|\Bigg|\frac{f^{-(\frac{n}{q}-k)q'+1}(H(x))}{-(\frac{n}{q}-k)q'+1}\Bigg|^{p-1}\Bigg]^{\frac{q}{q-p}}$$
$$\times f^{-(\frac{n}{q}-k)q'}(x)\sqrt{1+f'^2(x)}dx$$
$$= \int\limits_0^{\infty}\Bigg[\Bigg|\frac{F^{n-kp+1}(s)-F^{n-kp+1}(0)}{n-kp+1}\Bigg|\Bigg|\frac{F^{-(\frac{n}{q}-k)q'+1}(s)}{-(\frac{n}{q}-k)q'+1}\Bigg|^{p-1}\Bigg]^{\frac{q}{q-p}} F^{-(\frac{n}{q}-k)q'}(s)ds$$
$$= C\int\limits_0^{\infty} F^N(s)|1-F^{n-kp+1}(0)F^{-(n-kp+1)}(s)|ds. \quad (3.10)$$

Here, as above, $C = \text{const} > 0$, $N = n-k+\frac{pq}{q-p} > 0$, and $-(n-kp+1) < 0$. Thus, the expression under the integral is equivalent to $CF^{n-k+\frac{pq}{q-p}}(s)$, i.e., tends to infinity as $s \to \infty$.

Lemma 3.5 is completely proved. □

Now, return to the proof of Theorem 3.4. Suppose that $T^j_{p,q}(M) = 0$ for $j = 1$ or $j = n+1$. Then, by Theorem 3.2,

$$\chi_{p,q}(0, \infty, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$$

(and, hence, $\int_0^\infty f^{(\frac{n}{p}-j+1)p}(t)\sqrt{1+f'^2(t)}dt < \infty$) or

$$\chi_{p,q}(\infty, 0, F^{\frac{n}{p}-j+1}, F^{\frac{n}{q}-j+1}) < \infty$$

(and, hence, $\int_0^\infty f^{-(\frac{n}{q}-j+1)q'}(t)\sqrt{1+f'^2(t)}dt < \infty$). By Lemma 3.5, this implies that $\lim\limits_{t\to\infty} f(t) = 0$ and, in both cases,

$$\operatorname{vol} M = s_n \int_0^\infty f^n(t)\sqrt{1+f'^2(t)}dt < \infty.$$

Here $s_n$ stands for the volume of the $n$-dimensional unit sphere in $\mathbb{R}^{n+1}$.

The theorem is proved. $\square$

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