ANNALES MATHÉMATIQUES

BLAISE PASCAL

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Volume 17, nº 1 (2010), p. 165-181.

<http://ambp.cedram.org/item?id=AMBP_2010__17_1_165_0>

Publication éditée par le laboratoire de mathématiques de l'université Blaise-Pascal, UMR 6620 du CNRS Clermont-Ferrand — France

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http://www.cedram.org/

Annales mathématiques Blaise Pascal **17**, 165-181 (2010)

Asymptotic behavior of weighted quadratic variation of bi-fractional Brownian motion

Rachid Belfadli

Abstract

We prove, by means of Malliavin calculus, the convergence in L^2 of some properly renormalized weighted quadratic variations of bi-fractional Brownian motion (biFBM) with parameters H and K, when $H < 1/4$ and $K \in (0, 1]$.

Comportement asymptotique de la variation quadratique à poids du mouvement brownien bifractionnaire

Résumé

Nous utilisons le calcul de Malliavin pour montrer la convergence dans L^2 de la variation quadratique à poids du mouvement brownien bifractionnaire (biFBM) d'indices H et K lorsque $H < 1/4$ et $K \in (0, 1]$.

1. Introduction

There has been recently a lot of interests in the literature to the study of weighted power variations. More precisely, for a given integer $p > 1$, a smooth enough function $h : \mathbb{R} \to \mathbb{R}$ and a process X, the analysis of the asymptotic behavior, as n tends to infinity, of quantities such as

$$\sum_{l=0}^{n-1} h(X_{l/n})(\Delta X_{l/n})^p \tag{1.1}$$

(or some appropriate renormalized version of them) have been considered in [6, 5, 7]. Here $\Delta X_{l/n}$ stands for the increment $X_{l+1/n} - X_{l/n}$. Notice that (1.1) is called *weighted power variations* because of the presence of the factor $h(X_{l/n})$.

Keywords: Bi-fractional Brownian motion; Weighted quadratic variations; Malliavan calculus.
Math. classification: 60H20, 34F05, 34G20.

This study origins in the work [5] by Nourdin, in the case where X is a fractional Brownian motion (f.B.m, in short). Then, the results of [5] have been improved in [6] by Nourdin, Nualart and Tudor. Let us also stress that the study in [6, 5] has been used in [2, 4] to deduce exact rate of convergence of some approximation schemes of scalar stochastic differential equations driven by a f.B.m. Moreover, for another motivation of this study, we can also mention that the analysis of the asymptotic behavior of (1.1), in the particular case $p = 2$ and X the fractional Brownian motion of Hurst parameter $1/4$, allowed the authors of [7] to derive a new type of change of variable formula for X, with a correction term that is an ordinary Itô integral with respect to a Wiener process that is independent of X.

As we said, a complete description of the nature of the convergence of weighted p-power variation of the form (1.1) in the case where X is the fractional Brownian motion with Hurst parameter $H \in (0, 1)$ has been given in [6, 5, 7]. More precisely, after adequate renormalization, central and non-central limit theorems have been derived there, depending on the value of p and H. In particular, it is shown in [5] that, for weighted quadratic variations ($p = 2$), the following convergence holds for h regular enough and H strictly between 0 and $1/4$:

$$n^{2H-1} \sum_{l=0}^{n-1} h(X_{l/n})(n^{2H}(\Delta X_{l/n})^2 - 1) \quad \xrightarrow[n\to\infty]{L^2} \quad \frac{1}{4} \int_0^1 h''(X_u)\, du. \tag{1.2}$$

As pointed out by Nourdin in [5], (1.2) is somewhat surprising when it is compared to the situation where $h \equiv 1$. Indeed, since the seminal work of Breuer and Major [1], we know that, for any $0 < H < 3/4$:

$$\frac{1}{\sqrt{n}} \sum_{l=0}^{n-1} [n^{2H}(\Delta X_{l/n})^2 - 1] \quad \xrightarrow[n\to\infty]{Law} \quad \mathscr{N}(0, C_H^2) \tag{1.3}$$

where C_H denotes an explicit constant depending only on H. So, instead of an L^2 convergence, we only have a convergence in law in (1.3). Observe that, since $2H - 1 < 1/2$ if and only if $H < 1/4$, convergence (1.2) and (1.3) are, of course, not contradictory.

Motivated by this result, we shall show in the present note that the convergence (1.2) still holds in the case of a more general process, namely the bi-fractional Brownian motion (see below for a precise definition). As in [5], our main tool for the proof is based on the integration by parts formula of Malliavin calculus.

The note is organized as follows. In Section 2 we recall the definition of the bi-f.B.m and present some preliminary results about its Malliavin calculus. In Section 3 we state and prove our result concerning the convergence similar to (1.2), but in the case where X is a bi-f.B.m.

2. Preliminaries and notation

Here we recall the definition of the bi-fractional Brownian motion and present the elements of Malliavin calculus that will be needed in the sequel.

Definition 2.1. Let $H \in (0,1)$ and $K \in (0,1]$. A bi-fractional Brownian motion $(B_t^{H,K})_{t\geq 0}$ of indices H and K is a centered Gaussian process, starting from zero, with covariance function given by

$$R^{H,K}(s,t) \quad := \quad \frac{1}{2^K}\left((t^{2H}+s^{2H})^K - |t-s|^{2HK}\right). \tag{2.1}$$

In particular, by choosing $K=1$ and $H \in (0,1)$ in (2.1), observe that we recover the covariance function of the fractional Brownian motion with Hurst parameter H.

The bi-fractional Brownian motion was introduced by Houdré and Villa in [3], and then further studied by Russo and Tudor in [9], and by Tudor and Xiao in [11]. It enjoys the self-similarity property, that is, for any constant $c > 0$, the processes $\{c^{-HK}B_{ct}^{H,K},\, t \geq 0\}$ and $\{B_t^{H,K},\, t \geq 0\}$ have the same distribution. Moreover, if $K \neq 1$, $B^{H,K}$ does not have stationary increments (see e.g. [10]). It is precisely the main difference with respect to f.B.m.

Let us introduce some basic facts on the Malliavin calculus with respect to $B^{H,K}$ on the time interval $[0,1]$. For a more complete exposition, we refer to [8]. Let $\mathcal{H}$ be the Hilbert space defined as the closure of the linear space $\mathcal{E}$ generated by the indicator functions $(1_{[0,t]},\, t \in [0,1])$ with respect to the following inner product

$$\langle 1_{[0,s]}, 1_{[0,t]}\rangle_{\mathcal{H}} = R^{H,K}(s,t).$$

The mapping $1_{[0,t]} \mapsto B_t^{H,K}$ can be extended to an isometry between $\mathcal{H}$ and the Gaussian space generated by $B^{H,K}$. We denote this isometry by $\varphi \mapsto B^{H,K}(\varphi)$.

Let $\mathscr{S}$ be the set of all smooth cylindrical random variables of the form

$$F = f(B^{H,K}(\varphi_1), B^{H,K}(\varphi_2), \ldots, B^{H,K}(\varphi_n))$$

where $n \geq 0$, $f \in C^\infty$ has a compact support and $\varphi_i \in \mathcal{H}$. The Malliavin derivative of F with respect to $B^{H,K}$ is the element belonging to $L^2(\Omega, \mathcal{H})$ defined by

$$D_s F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(B^{H,K}(\varphi_1), B^{H,K}(\varphi_2), \ldots, B^{H,K}(\varphi_n))\varphi_i(s), \quad s \in [0,1].$$

This operator can be extended to the closure $\mathbb{D}^{1,2}$ of $\mathscr{S}$ with respect to the norm

$$\|F\|_{1,2}^2 := \mathbb{E}[F^2] + \mathbb{E}[|D.F|_{\mathcal{H}}^2].$$

The Malliavin derivative satisfies the following chain rule. For every random vector $F = (F_1, \ldots, F_n)$ with components in $\mathbb{D}^{1,2}$ and for every continuously differentiable function $\varphi : \mathbb{R}^n \to \mathbb{R}$ with bounded partial derivatives, we obtain $\varphi(F_1, ..., F_1) \in \mathbb{D}^{1,2}$ and we have, for any $s \in [0,1]$:

$$D_s \varphi(F_1, ..., F_n) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(F_1, \ldots, F_n) D_s F_i.$$

The divergence operator I is the adjoint of D in the following sense. A random process $u \in L^2(\Omega, \mathcal{H})$ belongs to the domain of I if and only if

$$\mathbb{E}|\langle DF, u\rangle_{\mathcal{H}}| \leq C_u \|F\|_{L^2(\Omega)}, \quad \text{for every } F \in \mathbb{D}^{1,2},$$

where C_u is a constant depending only on u. In that case, $I(u)$ verifies the integration by part formula:

$$\mathbb{E}(FI(u)) = \mathbb{E}(\langle DF, u\rangle_{\mathcal{H}})$$

for any $F \in \mathbb{D}^{1,2}$.

3. Asymptotic behavior of weighted quadratic variations of bifractional Brownian motion.

We will make use of the following assumption on the weight function h.
Assumption ($\mathbf{H_m}$):

$h : \mathbb{R} \to \mathbb{R}$ belongs to C^m and, for any $p > 0$ and any $i = 1, \ldots, m$,

$$\sup_{s\in[0,1]} \mathbb{E}\left[|h^{(i)}(B_s^{H,K})|^p\right] < \infty.$$

The main result of this section is the following:

Theorem 3.1. *Let $B^{H,K}$ be a bifractional Brownian motion with parameters H, K such that $0 < 4H < 1$, $K \in (0,1]$, and let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying ($\boldsymbol{H_4}$). Then we have, as $n \to \infty$:*

$$n^{2HK-1} \sum_{l=0}^{n-1} h(B_{l/n}^{H,K})[n^{2HK}(\Delta B_{l/n}^{H,K})^2 - 2^{1-K}] \xrightarrow{L^2} \frac{1}{2^{2K}} \int_0^1 h''(B_u^{H,K})\, du. \quad (3.1)$$

Remark 3.2. When $K = 1$ (that is when $B^{H,K}$ is a fractional Brownian motion) we recover Theorem 1.1 in [5]. Our proof in the general case follows the same lines.

Proof of the theorem. Throughout the proof, we will denote for simplicity

$$\delta_{k/n} = \mathbf{1}_{[k/n,(k+1)/n]} \quad \text{and} \quad \varepsilon_{k/n} = \mathbf{1}_{[0,k/n]}$$

and we let C stand for a positive generic constant independent of k, l, n that can be different from line to line.

We will need several lemmas. The first one is immediate to check, so its proof is left to the reader.

Lemma 3.3. (1) *If $2HK < 1$, then the sequence φ defined by*

$$\varphi(l) \ := \ \left((l+1)^{2H} + l^{2H}\right)^K - 2^K l^{2HK} \quad (3.2)$$

satisfies $\varphi(l) \sim \frac{2^K HK}{l^{1-2HK}}$ as l goes to infinity. In particular, φ is bounded.

(2) *If $2HK < 1$, then the sequence defined by*

$$\phi(l) \ := \ l^{2HK} + (l+1)^{2HK} - 2^{1-K}\left(l^{2H} + (l+1)^{2H}\right)^K \quad (3.3)$$

satisfies $\phi(l) \sim C/l^{2-2HK}$ as l goes to infinity. In particular, $\sum_{l\geq 0} \phi(l) < \infty$.

Lemma 3.4. (1) *Assume that $2HK<1$. Then, as $n\to\infty$,*

$$\sum_{k,l=0}^{n-1}|\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}| \quad = \quad o(n^{2-2HK}). \tag{3.4}$$

(2) *Assume that $H<1/4$. For $k,l=0,1,\dots,n-1$, set*

$$\begin{aligned} D_{k,l} :=& \Big((k+1)^{2H}+(l+1)^{2H}\Big)^K - \Big((k+1)^{2H}+l^{2H}\Big)^K + \Big(k^{2H}+l^{2H}\Big)^K \\ &- \Big(k^{2H}+(l+1)^{2H}\Big)^K. \end{aligned} \tag{3.5}$$

Then, as $n\to\infty$,

$$\sum_{k,l=0}^{n-1}|D_{k,l}| = o(n^{2-2HK}). \tag{3.6}$$

(3) *Assume that $H<1/4$. Then, as $n\to\infty$,*

$$\sum_{k,l=0}^{n-1}|\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}| \quad = \quad o(n^{2-4HK}). \tag{3.7}$$

Proof of Lemma 3.4. We prove the first point. For $0\le k,l\le n-1$, we have

$$\begin{aligned} \langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}} =& \frac{1}{2^K}n^{-2HK}\Big(\big((k+1)^{2H}+l^{2H}\big)^K \\ & - \big(k^{2H}+l^{2H}\big)^K + |l-k|^{2HK} - |l-k-1|^{2HK}\Big) \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{k,l=0}^{n-1}|\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}| \quad &\le \quad \frac{1}{2^K}n^{-2HK}\sum_{l=0}^{n-1}\Big((n^{2H}+l^{2H})^K - l^{2HK}\Big) \\ & \quad + \frac{1}{2^K}n^{-2HK}\sum_{k,l=0}^{n-1}\Big||l-k|^{2HK}-|l-k-1|^{2HK}\Big| \\ &\sim \quad 2^{-K}n\cdot\left(\int_0^1((1+x^{2H})^K - x^{2HK})\,dx\right) + 2^{-K}n \\ &= \quad Cn = o(n^{2-2HK}), \quad \text{since} \quad 2HK<1. \end{aligned}$$

Concerning the second point, we use the elementary inequality $||x|^K - |y|^K| \leq |x-y|^K$, valid for any $x,y \in \mathbb{R}$ because $K \leq 1$, to see that

$$|D_{k,l}| \leq 2\left((l+1)^{2H} - l^{2H}\right)^K.$$

Consequently, since $\left((l+1)^{2H} - l^{2H}\right)^K$ behaves as l^{2HK-K} for large l, we get

$$\sum_{k,l=0}^{n-1} |D_{k,l}| \quad \leq \quad Cn\sum_{l=1}^{n} \frac{1}{l^{K-2HK}}, \tag{3.8}$$

which is $o(n^{2-2HK})$. Indeed, let γ such that $2HK+1-K < \gamma < -2HK+1$. Then, the series $\sum_{l=1}^{\infty} 1/l^{K-2HK+\gamma}$ is convergent and

$$\begin{aligned} n^{-1+2HK}\sum_{l=1}^{n}\frac{1}{l^{K-2HK}} &= n^{\gamma-1+2HK}\cdot\frac{1}{n^{\gamma}}\sum_{l=1}^{n}\frac{1}{l^{K-2HK}} \\ &\leq n^{\gamma-1+2HK}\sum_{l\geq 1}\frac{1}{l^{K-2HK+\gamma}} \to 0. \end{aligned}$$

For the third point, we have

$$\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathcal{H}} = \frac{1}{2^K}n^{-2HK}\left(D_{k,l} + |k-l-1|^{2HK} + |k-l+1|^{2HK} - 2|k-l|^{2HK}\right),$$

with $D_{k,l}$ defined by (3.5). Then, we obtain as previously

$$\sum_{k,l=0}^{n-1} |\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathcal{H}}| \quad \leq \quad \frac{1}{2^K}n^{-2HK}\sum_{k,l=0}^{n-1}|D_{k,l}| + 2^{1-K}n.$$

Thus, using (3.6) of Lemma 3.4 and the fact that $H < 1/4$, equality (3.7) follows since $n = o(n^{2-4HK})$, which completes the proof.

□

Lemma 3.5. *If $2H < 1$, $0 < K \leq 1$ and g, h are two functions satisfying the condition ($\boldsymbol{H_2}$), then*

$$\begin{aligned} & \sum_{k,l=0}^{n-1} \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\} \\ = \;& \frac{1}{2^{2K}} \frac{1}{n^{2HK}} \sum_{k,l=0}^{n-1} \mathbb{E}[h''(B^{H,K}_{k/n})g(B^{H,K}_{l/n})] + o(n^{2-2HK}) \end{aligned}$$

Proof of Lemma 3.5. For $k, l = 0, 1, \ldots, n-1$, we use the integration by parts formula to write

$$\begin{aligned} & \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})n^{2HK}(\Delta B^{H,K}_{k/n})^2\} \\ = \;& \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})n^{2HK}\Delta B^{H,K}_{k/n} I(\delta_{k/n})\} \\ = \;& \mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})n^{2HK}\Delta B^{H,K}_{k/n}\}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})n^{2HK}\Delta B^{H,K}_{k/n}\}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}n^{2HK}\langle \delta_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}. \end{aligned}$$

But,

$$n^{2HK}\langle \delta_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}} = 2^{1-K} + \phi(k) \qquad (3.9)$$

with ϕ defined as in (3.3). Thus,

$$\begin{aligned} & \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\} \\ = \;& \mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})n^{2HK} I(\delta_{k/n})\}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})n^{2HK} I(\delta_{k/n})\}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}\phi(k) \\ = \;& \mathbb{E}\{h''(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}n^{2HK}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle^2_{\mathcal{H}} \\ + \;& 2\mathbb{E}\{h'(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})\}n^{2HK}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})\}n^{2HK}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle^2_{\mathcal{H}} \qquad (3.10) \\ + \;& \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}\phi(k). \end{aligned}$$

Now, we have

$$\langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathcal{H}}^2 = \frac{1}{2^{2K}} \frac{1}{n^{4HK}} (\varphi(k) - 1)^2 \qquad (3.11)$$

with φ defined by (3.2).
Therefore, using Lemma 3.3, we get

$$\left| \langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathcal{H}}^2 - \frac{1}{2^{2K}} \frac{1}{n^{4HK}} \right| = \frac{1}{2^{2K}} \frac{1}{n^{4HK}} \left| \varphi(k)(\varphi(k) - 2) \right| \leq C \frac{1}{n^{4HK}} \varphi(k).$$

Since $2HK < 1$, we can choose $\beta > 0$ such that $2HK < \beta < 1$ and we set $\gamma = 1 - \beta$. Then $\sum_{l \geq 1} \varphi(l)/l^\beta < \infty$ and consequently,

$$\sum_{l=0}^{n-1} |\langle \varepsilon_{l/n}, \delta_{l/n} \rangle_{\mathcal{H}}^2 - \frac{1}{2^{2K}} \frac{1}{n^{4HK}}| \leq C n^{1-4HK-\gamma}.$$

This implies that, under condition $(\mathbf{H_2})$

$$\begin{aligned} & n^{2HK} \sum_{k,l=0}^{n-1} \left| \mathbb{E}\{h''(B_{k/n}^{H,K}) g(B_{l/n}^{H,K})\} (\langle \varepsilon_{k/n}, \delta_{k/n} \rangle_{\mathcal{H}}^2 - \frac{1}{2^{2K}} \frac{1}{n^{4HK}}) \right| \\ \leq \; & C n^{2-2HK-\gamma} = o(n^{2-2HK}). \end{aligned}$$

Furthermore, using the fact that $2HK \leq 2H \leq 1$, we see that

$$\begin{aligned} n^{2HK} |\langle \varepsilon_{l/n}, \delta_{k/n} \rangle_{\mathcal{H}}| :=& 2^{-K} \Big| \left((k+1)^{2H} + l^{2H} \right)^K \\ & - \left(l^{2H} + k^{2H} \right)^K + |l-k|^{2HK} - |l-1-k|^{2HK} \Big| \\ \leq & 2^{-K} \Big\{ \left| ((k+1)^{2H} - k^{2H})^K \right| \\ & + \left| |l-k|^{2HK} - |l-1-k|^{2HK} \right| \Big\} \leq 2^{1-K} \qquad (3.12) \end{aligned}$$

is bounded independently of k and l. Now, since

$$\begin{aligned} \sum_{k,l=0}^{n-1} \Big\{ & \left((k+1)^{2H} + l^{2H} \right)^K - \left(k^{2H} + l^{2H} \right)^K \\ & + \left| |l-k|^{2HK} - |l-1-k|^{2HK} \right| \Big\} \leq C n^{1+2HK} \end{aligned}$$

by telescoping sum, we deduce that

$$\begin{aligned}
&n^{2HK}\sum_{k,l=0}^{n-1}\{|\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathcal{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}|+|\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}|^2\}\\
\le& C\sum_{k,l=0}^{n-1}|\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}|\\
\le& Cn^{-2HK}\sum_{k,l=0}^{n-1}\Big\{\Big((k+1)^{2H}+l^{2H}\Big)^K\\
&\quad-\Big(k^{2H}+l^{2H}\Big)^K+\Big|(l-k)^{2HK}-(l-1-k)^{2HK}\Big|\Big\}\\
\le& Cn^{-2HK}n^{1+2HK}=Cn=o(n^{2-2HK})\qquad(\text{since } 2HK<1).
\end{aligned}$$

Thus, under condition ($\mathbf{H_2}$), we obtain

$$\begin{aligned}
\sum_{k,l=0}^{n-1}&\Big|2\mathbb{E}\{h'(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})\}n^{2HK}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathcal{H}}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}\Big|\\
&+\Big|\mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})\}n^{2HK}\langle\varepsilon_{l/n},\delta_{k/n}\rangle^2_{\mathcal{H}}\Big|=o(n^{2-2HK}).
\end{aligned}$$

On the other hand, by Lemma 3.3 and once again using condition ($\mathbf{H_2}$)

$$\sum_{k,l=0}^{n-1}\mathbb{E}\{|h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\phi(k)|\}\le C\left(\sum_{k=0}^{\infty}|\phi(k)|\right)\cdot n=o(n^{2-2HK})$$

(since $2HK<1$).

Finally, by combining all the previous estimates with (3.10), the proof of Lemma 3.5 is done. □

Lemma 3.6. *If $H<1/4$, $0<K\le 1$ and g, h are two functions satisfying the condition ($\boldsymbol{H_4}$), then*

$$\begin{aligned}
\sum_{k,l=0}^{n-1}&\mathbb{E}\Big\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\\
&[n^{2HK}(\Delta B^{H,K}_{k/n})^2-2^{1-K}][n^{2HK}(\Delta B^{H,K}_{l/n})^2-2^{1-K}]\Big\}\\
&=\frac{1}{2^{4K}}\frac{1}{n^{4HK}}\sum_{k,l=0}^{n-1}\mathbb{E}[h''(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})]+o(n^{2-4HK}).
\end{aligned}$$

Proof of Lemma 3.6. Using the integration by part formula we have

$$\begin{aligned}
&\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}](\Delta B^{H,K}_{l/n})^2\} \\
&\quad= \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\Delta B^{H,K}_{l/n} I(\delta_{l/n})\} \\
&\quad= \mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\Delta B^{H,K}_{l/n}\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ \mathbb{E}\{h(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\Delta B^{H,K}_{l/n}\}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ 2\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\Delta B^{H,K}_{k/n}\Delta B^{H,K}_{l/n}\}n^{2HK}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\langle\delta_{l/n},\delta_{l/n}\rangle_{\mathcal{H}}.
\end{aligned}$$

It follows from (3.9), that

$$\begin{aligned}
&\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}][n^{2HK}(\Delta B^{H,K}_{l/n})^2 - 2^{1-K}]\} \\
&= n^{2HK}\mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]I(\delta_{l/n})\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&+ n^{2HK}\mathbb{E}\{h(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]I(\delta_{l/n})\}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ 2n^{4HK}\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\Delta B^{H,K}_{k/n} I(\delta_{l/n})\}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\phi(l)
\end{aligned}$$

and, once again by an integration by part formula, it leads to

$$\begin{aligned}
&\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}][n^{2HK}(\Delta B^{H,K}_{l/n})^2 - 2^{1-K}]\} \\
&\quad= n^{2HK}\mathbb{E}\{h''(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle^2_{\mathcal{H}} \\
&+2n^{2HK}\mathbb{E}\{h'(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ 4n^{4HK}\mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\Delta B^{H,K}_{k/n}\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ 4n^{4HK}\mathbb{E}\{h(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})\Delta B^{H,K}_{k/n}\}\langle\varepsilon_{l/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}} \\
&\quad+ 2n^{4HK}\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\langle\delta_{k/n},\delta_{l/n}\rangle^2_{\mathcal{H}} \\
&\quad+ \mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}n^{2HK}\langle\varepsilon_{l/n},\delta_{l/n}\rangle^2_{\mathcal{H}} \\
&\quad+ \mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\phi(l) \\
&\quad=: \sum_{i=1}^{7} A^i_{k,l,n}.
\end{aligned}$$

Consequently, the proof of the lemma will be deduced after the study of the asymptotic behavior of $\sum_{k,l=0}^{n} A^i_{k,l,n}$, as $n \to \infty$, for each $i \in \{1, \dots, 7\}$.

Claim 3.1. *We have, as n goes to infinity,*

$$(1)\quad \sum_{k,l=0}^{n-1} |A^i_{k,l,n}| = o(n^{2-4HK}) \quad \textit{for every } i \neq 6.$$

$$(2)\quad \sum_{k,l=0}^{n-1} A^6_{k,l,n} = \frac{1}{2^{4K} n^{4HK}} \sum_{k,l=0}^{n-1} \mathbb{E}\{h''(B^{H,K}_{k/n}) g''(B^{H,K}_{l/n})\} + o(n^{2-4HK}).$$

Proof of Claim 3.1. We first consider the term $A^1_{k,l,n}$, the study of $A^2_{k,l,n}$ being similar. Using h'' instead of h in (3.10), we can write

$$\begin{aligned} A^1_{k,l,n} &:= \\ &n^{2HK}\mathbb{E}\{h''(B^{H,K}_{k/n}) g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\}\langle \varepsilon_{k/n}, \delta_{l/n}\rangle^2_{\mathcal{H}} \\ &= n^{4HK}\mathbb{E}\{h^{(4)}(B^{H,K}_{k/n}) g(B^{H,K}_{l/n})\}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle^2_{\mathcal{H}}\langle \varepsilon_{k/n}, \delta_{l/n}\rangle^2_{\mathcal{H}} \\ &+ 2n^{4HK}\mathbb{E}\{h^{(3)}(B^{H,K}_{k/n}) g'(B^{H,K}_{l/n})\}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathcal{H}}\langle \varepsilon_{k/n}, \delta_{l/n}\rangle^2_{\mathcal{H}} \\ &+ n^{4HK}\mathbb{E}\{h''(B^{H,K}_{k/n}) g''(B^{H,K}_{l/n})\}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle^2_{\mathcal{H}}\langle \varepsilon_{k/n}, \delta_{l/n}\rangle^2_{\mathcal{H}} \\ &+ n^{2HK}\mathbb{E}\{h''(B^{H,K}_{k/n}) g(B^{H,K}_{l/n})\}\langle \varepsilon_{k/n}, \delta_{l/n}\rangle^2_{\mathcal{H}}\phi(k). \end{aligned}$$

Using that $n^{2HK}|\langle \varepsilon_{k/n}, \delta_{l/n}\rangle_{\mathcal{H}}|$ and $\phi(k)$ are bounded with respect to k, l, n, see (3.12), and using condition ($\mathbf{H_4}$), we have

$$|A^1_{k,l,n}| \leq C n^{-2HK} |\langle \varepsilon_{k/n}, \delta_{l/n}\rangle_{\mathcal{H}}|.$$

According to Lemma 3.4, we deduce

$$\sum_{k,l=0}^{n-1} |A^1_{k,l,n}| = o(n^{2-4HK}).$$

Now, let us consider the term $A^3_{k,l,n}$, the study of the cases $A^i_{k,l,n}$ where $i = 4, 5$ being similar, since each of this terms contains the factor $\langle \delta_{k/n}, \delta_{l/n}\rangle_{\mathcal{H}}$.

As previously, by the Malliavin integration by parts formula, we can write

$$\begin{aligned}
A^3_{k,l,n} &:= 4n^{4HK}\mathbb{E}\{h'(B^{H,K}_{k/n})g(B^{H,K}_{l/n})I(k/n)\}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\\
&= 4n^{4HK}\mathbb{E}\{h''(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}\langle\varepsilon_{k/n},\delta_{k/n}\rangle_{\mathcal{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\\
&+ 4n^{4HK}\mathbb{E}\{h'(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})\}\langle\varepsilon_{l/n},\delta_{k/n}\rangle_{\mathcal{H}}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}.
\end{aligned}$$

Hence, using again that $n^{2HK}\langle\varepsilon_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}$ is bounded and the condition $(\mathbf{H_2})$, we obtain

$$\sum_{k,l=0}^{n-1}|A^3_{k,l,n}| \leq C\sum_{k,l=0}^{n-1}|\langle\delta_{k/n},\delta_{l/n}\rangle_{\mathcal{H}}|$$

which is $o(n^{2-4HK})$ by using point (3) of Lemma 3.4.
For the term $A^6_{k,l,n}$, we use (3.11) and point (1) of Lemma 3.3 to write

$$\langle\varepsilon_{k/n},\delta_{k/n}\rangle^2_{\mathcal{H}} = \frac{1}{2^{2K}}\frac{1}{n^{4HK}} + O(\frac{1}{n^{4HK}}). \tag{3.13}$$

Substituting (3.13) into the expression of $A^6_{k,l,n}$, yields

$$\begin{aligned}
&A^6_{k,l,n} :=\\
&\quad\mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2-2^{1-K}]\}n^{2HK}\langle\varepsilon_{l/n},\delta_{l/n}\rangle^2_{\mathcal{H}}\\
&= \frac{1}{2^{2K}}\frac{1}{n^{2HK}}\mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2-2^{1-K}]\}+O(\frac{1}{n^{2HK}}).
\end{aligned}$$

Therefore, using Lemma 3.5, with g'' instead of g, we obtain

$$\sum_{k,l=0}^{n-1}A^6_{k,l,n} = \frac{1}{2^{4K}}\frac{1}{n^{4HK}}\sum_{k,l=0}^{n-1}\mathbb{E}\{h''(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})\} + o(n^{2-4HK}).$$

Finally, we consider the term $A^7_{k,l,n}$. Still using Malliavin integration by parts formula, we write

$$\begin{aligned} A^7_{k,l,n} &:= \phi(l)\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\} \\ &= \mathbb{E}\{h''(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\}n^{2HK}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle^2_{\mathcal{H}}\phi(l) \\ &+ 2n^{2HK}\phi(l)\mathbb{E}\{h'(B^{H,K}_{k/n})g'(B^{H,K}_{l/n})\}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}\langle \varepsilon_{l/n}, \delta_{k/n}\rangle_{\mathcal{H}} \\ &+ n^{2HK}\phi(l)\mathbb{E}\{h(B^{H,K}_{k/n})g''(B^{H,K}_{l/n})\}\langle \varepsilon_{l/n}, \delta_{l/n}\rangle^2_{\mathcal{H}} \\ &+ \phi(k)\phi(l)\mathbb{E}\{h(B^{H,K}_{k/n})g(B^{H,K}_{l/n})\} \\ &=: (a)_{k,l,n} + (b)_{k,l,n} + (c)_{k,l,n} + (d)_{k,l,n}. \end{aligned}$$

We claim that $\sum_{k,l=0}^{n-1} |A^7_{k,l,n}| = o(n^{2-4HK})$. Indeed, we have by condition $(\mathbf{H_2})$

$$|(a)_{k,l,n}| + |(b)_{k,l,n}| \leq C|\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}||\phi(l)|.$$

But,

$$\begin{aligned} n^{2HK}\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}} &= 2^{-K}\{((k+1)^{2H} + k^{2H})^K - 2^K k^{2HK} - 1\} \\ &=: 2^{-K}\{\varphi(k) - 1\}. \end{aligned}$$

Then, thanks to the first point of Lemma 3.3, we can write

$$\begin{aligned} \sum_{k,l=0}^{n-1} |(a)_{k,l,n}| + |(b)_{k,l,n}| &\leq Cn^{-2HK}\left(\sum_{l=0}^{n-1}|\phi(l)|\right)\left(\sum_{k=0}^{n-1}|\varphi(k) - 1|\right) \\ &\leq Cn^{-2HK}\left(\sum_{l=0}^{\infty}|\phi(l)|\right)\left(n + \sum_{k=1}^{n}|\varphi(k)|\right). \end{aligned}$$

Since $\varphi(k) \sim 1/k^{K-2HK}$ as k goes to infinity and $4H < 1$, we have

$$n^{-2HK}\sum_{k=0}^{n-1}|\varphi(k)| = o(n^{2-4HK}).$$

Combining with $n^{1-2HK} = o(n^{2-4HK})$, since $2HK < 1$, it follows that

$$\sum_{k,l=0}^{n-1} |(a)_{k,l,n}| + |(b)_{k,l,n}| = o(n^{2-4HK}).$$

For the term $\sum_{k,l=0}^{n} |(c)_{k,l,n}|$, we have similarly

$$\begin{aligned}\sum_{k,l=0}^{n-1} |(c)_{k,l,n}| &\leq Cn \sum_{k=0}^{n-1} |\phi(k)||\langle \varepsilon_{k/n}, \delta_{k/n}\rangle_{\mathcal{H}}| \\ &\leq Cn^{1-2HK} \sum_{k=0}^{n-1} |\phi(k)||((k+1)^{2H} - k^{2H})^{K} + 1|\end{aligned}$$

so that (recall that $H < 1/4 < 1/2$)

$$\sum_{k,l=0}^{n-1} |(c)_{k,l,n}| \leq Cn^{1-2HK} \sum_{k=0}^{\infty} |\phi(k)| = o(n^{2-4HK}), \quad \text{since} \quad 2HK < 1.$$

For the last term $\sum_{k,l=0}^{n} |(d)_{k,l,n}|$, we have

$$\sum_{k,l=0}^{n} |(d)_{k,l,n}| \leq \left(\sum_{k=0}^{\infty} |\phi(k)|\right)^2 = o(n^{2-4HK}), \quad \text{since} \quad 2HK < 1.$$

This finishes the proof of Claim 3.1, and thus the proof of Lemma 3.6. □

Combining these two lemmas, the proof of the theorem can be completed along the same lines as in [5]. Indeed, by Lemma 3.6, we have

$$\begin{aligned}&\mathbb{E}\left\{n^{2HK-1} \sum_{k=0}^{n-1} h(B^{H,K}_{k/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\right\}^2 \\ &\qquad = n^{4HK-2} \sum_{k,l=0}^{n-1} \mathbb{E}\Big\{h(B^{H,K}_{k/n})h(B^{H,K}_{l/n}) \\ &\qquad [n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}][n^{2HK}(\Delta B^{H,K}_{l/n})^2 - 2^{1-K}]\Big\} \\ &\qquad = \frac{1}{2^{2K}} \frac{1}{n^2} \sum_{k,l=0}^{n-1} \mathbb{E}[h''(B^{H,K}_{k/n})h''(B^{H,K}_{l/n})] + o(1)\end{aligned}$$

and, using Lemma 3.5, we have

$$\mathbb{E}\Big\{n^{2HK-1}\sum_{k=0}^{n-1} h(B^{H,K}_{k/n})$$

$$[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}] \times \frac{1}{2^{2K}}\frac{1}{n}\sum_{k=0}^{n-1} h''(B^{H,K}_{k/n})\Big\}$$

$$= \sum_{k,l=0}^{n-1} \mathbb{E}\Big\{h(B^{H,K}_{k/n})h''(B^{H,K}_{l/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]\Big\}$$

$$= \frac{1}{2^{4K}}\frac{1}{n^2}\sum_{k,l=0}^{n-1} \mathbb{E}[h''(B^{H,K}_{k/n})h''(B^{H,K}_{l/n})] + o(1).$$

As a consequence, we obtain the convergence

$$\mathbb{E}\Big\{n^{2HK-1}\sum_{k=0}^{n-1} h(B^{H,K}_{k/n})[n^{2HK}(\Delta B^{H,K}_{k/n})^2 - 2^{1-K}]$$

$$- \frac{1}{2^{2K}}\frac{1}{n}\sum_{k=0}^{n-1} h''(B^{H,K}_{k/n})\Big\}^2 \xrightarrow[n\to\infty]{} 0$$

which implies (3.1), since

$$\frac{1}{n}\sum_{k=0}^{n-1} h''(B^{H,K}_{k/n}) \xrightarrow[n\to\infty]{L^2(\Omega)} \int_0^1 h''(B^{H,K}_u)\, du.$$

□

Acknowledgment: This research is supported by the A.I. n° : MA 06/142. The author wishes to express his gratitude to I. Nourdin for the invitation, hospitality and for several helpful comments on the subject of this note. Thanks to the anonymous referee for the useful remarks and valuable suggestions that lead to many improvements in the manuscript.

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