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Abstract

In this work, we describe the historic links between the study of 3-dimensional manifolds (specially knot theory) and the study of the topology of complex plane curves with a particular attention to the role of braid groups and Alexander-like invariants (torsions, different instances of Alexander polynomials). We finish with detailed computations in an example.

Tresses à Pau – une introduction

Résumé

Dans ce travail, nous décrivons les liaisons historiques entre l'étude de variétés de dimension 3 (notamment, la théorie de nœuds) et l'étude de la topologie des courbes planes complexes, dont l'accent est posé sur le rôle des groupes de tresses et des invariantes du type Alexander (torsions, différents incarnations des polynômes d'Alexander). Nous finissons en présentant un example avec des calculs détaillés.

1. Historic

The conference *Tresses in Pau*, held in Pau from the 5th to the 8th of October 2009, was devoted to low dimensional topology and interactions with algebraic geometry. It was organized around three mini-courses and this volume contains their notes. We would like to thank the authors for their effort to produce high-quality notes of these courses.

These two branches of mathematics, low dimensional topology and algebraic geometry, have a long common history, as it can be seen in the work of Klein or Poincaré. In particular, the study of 3 or 4-manifolds, and knot theory were developed in parallel to the study of topological properties of singular plane algebraic curves since the late 30's. Principal

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questions concern topological or smooth classifications of manifolds, or isotopies of embeddings and stratifications of discriminants.

As a natural invariant, the fundamental group of a manifold or a knot complement contains much topological information. It was introduced by Dyck [13], following Cayley [6], and mainly developed by Wirtinger [41, 42] in the case of knots (the aim of Wirtinger was to compute this group for algebraic knots or links). From the successive results of Dehn [12, 11], Fox [15] and later Papakyriakopoulos [35, 36] on the peripheral system, Waldhausen [40] showed that it is strong enough to determine the knot. Fundamental groups of knots and links are also helpful to compute groups of manifolds constructed by Dehn surgery or ramified coverings. One interesting feature of Wirtinger's presentation is the following: the 2-complex associated to the presentation has the homotopy type of the complement of the link.

A parallel work in order to study the topology of complex surfaces was initiated by Enriques and Zariski. Their main idea was to generalize Riemann's classical work for Riemann surfaces of a multivalued function. In order to understand the topology of a complex projective manifold, the best way is to project this surface onto the complex projective space (of the same dimension) and, then, interpreting the manifold as a ramified covering along the discriminant locus of the projection (a hypersurface). A main step in this process is to compute the fundamental group of the complement of the hypersurface. In fact, as a consequence of Zariski-Lefschetz theory, by taking generic sections it is enough to study this fundamental group in the case of surfaces, i.e., for complements of curves. Zariski [43] and van Kampen [20] developed their well-known method which has been extensively used since then.

Much more common objects are studied and applied in both settings, but there is one specially important: the braid groups. Indeed, this conference focuses on several aspects of them. Braid groups appeared historically with the works of Hurwitz [19] on ramified covering of surfaces and Magnus [30]. Their beauty and wealth comes from the several ways to define them, as trajectories of particles, fundamental group of configuration spaces (Fox-Neuwirth [14]), algebraic objects (Artin [3]) or mapping class groups. Juan González-Meneses' lecture [18] is devoted to these different approaches. Several classical results on braid groups are proved, with many original proofs.

The braids were used in knot theory via a famous theorem of Alexander [1] that expresses any link as the *closure* of a braid. From that braid it is possible to obtain invariants of the knot (or link), as the Alexander and Jones polynomials, among others. There is also a presentation of the fundamental group of the complement in terms of the braid, with the same feature as Wirtinger presentation.

Similarly, braids appear as a main tool in the understanding of the topology of complex plane curves via its *braid monodromy*. This invariant, relatively to a given pencil of lines, provides complete information about the embedding of the curve (as shown by Kulikov-Teicher [24] and Carmona [4]). It can be understood as a formalization of the Zariski-van Kampen method to give a presentation of the fundamental group of a singular plane curve complement, as Chisini [7] realized in the thirties. Much later, in the eighties, it was extensively used by Moishezon (e.g., [34]) in order to get information about complex surfaces. Later on, Libgober [26] proved that the homotopy type of the complement of an affine curve can be retreived from the presentation of the fundamental group issued from braid monodromy, as in the case of links. The lecture given by José Ignacio Cogolludo [8] provides a detailed exposition of braid monodromy for plane singular algebraic curves.

The lecture given by Gwénaël Massuyeau [31] deals with Redemeister torsion, and a famous related invariant, the Alexander polynomial. It was introduced by Alexander [2] for knot and links, by combinatorial constructions on the diagram. The torsion invariants were constructed by Reidemeister and Franz [16], for lens spaces first, from a triangulation and the action of the fundamental group on the universal covering of the considered space. Then Milnor [33, 32] showed (see also Turaev [37, 38]) in the case of link complements that, if the coefficients are taken over a field of rational functions, the Reidemeister torsion specializes to the Alexander polynomial. This result was extended to non-abelian specializations of the torsion, and twisted Alexander polynomials (see Lin [29], Wada [39], and Kitano [23]). It is worth saying that, in the non-Abelian case, the invariant depends essentially on the fundamental group and a representation. The point of view of torsion on them gives nice tools of computation and properties. For 3-manifolds and knot theory, it was used for concordance, estimation of the genus of to give obstruction to the fiberedness (see Kirk-Livingston [21, 22], Friedl-Vidussi [17]), in relation with Thurston norm.

In the case of plane singular curves, the Alexander polynomial (introduced in this case by Libgober [25]) appears also to be a more manageable invariant than the fundamental group, and also sensitive to the position of singularities (except in the case of ordinary double points). It was extensively studied by Libgober and many other authors. As proof of their close relationship with knot theory, Libgober also showed that the Alexander polynomial of an affine curve divides both the product of Alexander polynomial of the algebraic links of its singular points and the Alexander polynomial of the link at infinity. There are several generalizations of Alexander polynomials. Twisted polynomials (associated to a representation) were introduced by Cogolludo-Florens [9] and they proved that these polynomials may detect the presence of nodal points. The point of view of torsion appears here to offer new perspectives on the invariant. Libgober [28] also introduced characteristic varieties for algebraic (reducible) curves which may be seen as a generalization of multivariate Alexander polynomials for links.

For knot theory, by the results of Alexander and Markov, braid group representations can be used to construct invariants. As an example, the reduced Burau representation allows to recover the Alexander polynomial. Libgober [27], inspired by this construction showed that by composing the braid monodromy of a curve with such a representation, one obtains isotopy invariants of the curve. The particular case of Burau returns the Alexander polynomial.

As it can be seen from this historical introduction, the role of Anatoly Libgober in these subjects is very important. Four months before this workshop, the conference LIB60BER (Topology of Algebraic Varieties: A Conference in Honor of the 60th Birthday of Anatoly Libgober) was held in Jaca (Aragón), only one hundred kilometers south of Pau.

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2. **Alexander invariants**

Consider a group G and a surjective homomorphism $\chi : G \to \mathbb{Z}^\mu$. Let K be the kernel of χ, and $K' = [K, K]$ its commutator. We get

$$0 \to K/K' \to G/K' \to \mathbb{Z}^\mu \to 0.$$

From this exact sequence, K/K' can be viewed as a $\mathbb{Z}[\mathbb{Z}^\mu]$-module, where $\mathbb{Z}[\mathbb{Z}^\mu]$ is the group ring which can be identified with the (multivariable) Laurent polynomial ring. This module is an invariant of (G, χ) and the *Alexander polynomial* of (G, χ) is its order. If G is the fundamental group of a CW-complex X, then χ induces a normal covering with group of deck transformation $\mathbb{Z}^\mu$ and the module is a topological invariant of (X, χ). From this, the invariant can be described by homological considerations on the chain complex, where the coefficients -and boundary maps- are *twisted* by the homomorphism χ. This can be recasted in terms of Reidemeister torsion, a combinatorial invariant of the chain complex. In particular, if the complex is 2-dimensional (up to homotopy), then the Alexander polynomial coincides exactly with the torsion associated to the coefficients $\mathbb{Q}(t)$, see [31]. This point of view provides nice computation tools, in particular.

The Alexander invariants of links in S^3 are defined from their complement. This complement has the homotopy type of a finite CW-complex, and the abelianization of its fundamental group is generated by the meridians. In this historical setting, it carries only metabelian information on the fundamental group. A natural generalization, introduced by Lin [29], associates a polynomial invariant to a complex together with a choice of a (linear) representation of its fundamental group. It was shown that these *twisted polynomials* can also be interpreted as specializations of the Reidemeister torsion.

The case of plane curves is very similar. The abelianization of the fundamental group of a plane curve complement is generated by the meridians of the irreducible components. Similarly to the case of links, Libgober [26] showed that the complement of a place curve has the homotopy type of a 2-complex. Cogolludo-Florens [9] used this to show the relation with Reidemeister torsion and extend this to twisted Alexander polynomials. This allows to extend the classical divisibility properties (by Libgober and Degtyarev [10]) to the twisted case, and to provide a better geometrical understanding of them.

We now briefly present a construction due to Libgober of a family of invariants of continuous equisingular families of plane curves, using representation of braid groups. As stated in the introduction, in the case of the reduced Burau representation, this gives a direct way to obtain the Alexander polynomial from the braid monodromy. This result illustrates the strong relationships between the three subjects of lectures given in the conference.

Let $\mathcal{C}$ be a curve in $\mathbb{P}^2$ transversal to the line at infinity L_∞, $\mathbb{C}^2 = \mathbb{P}^2 \setminus L_\infty$. Consider a linear projection $\pi : \mathbb{C}^2 \to \mathcal{C}$ of the affine part of $\mathbb{P}^2$ from a point at infinity (not belonging to $\mathcal{C}$) such that:

- The fibers of π are all transversal to $\mathcal{C}$ except for a finite set of singular fibers $F_1, \ldots F_n$, over the points $p_1, \ldots, p_n$ in $\mathcal{C}$.

- The singular fibers either have a simple tangency to $\mathcal{C}$ or pass through a singular point, transversally to their tangent cone.

Such a generic projection induces a locally trivial fibration of the pair $(\mathbb{C}^2 \setminus \cup F_i, \mathcal{C} \setminus \cup F_i)$ over $\mathbb{C} \setminus \{p_1, \ldots, p_n\}$. Consider a disk D in $\mathbb{C}$ containing the points $p_1, \ldots, p_n$. Let p be a base point in the boundary ∂D. Let $\gamma_1, \ldots, \gamma_n$ be a system of generators of the free group $\pi_1(\mathbb{C} \setminus \{p_1, \ldots, p_n\})$, where each γ_i is the class of a small closed loop around p_i. Recall that a generic fiber of π is homeomorphic to the 2-disk D, with d puncture points corresponding to the intersection with the curve. Let $\mathbb{B}_d$ be the mapping class group of D_d, that is the group of braids with d strings. The braid monodromy is the following homomorphism of groups:

$$\varphi : \pi_1(\mathbb{C} \setminus \{p_1, \ldots, p_n\}) \to \mathbb{B}_d, \quad \gamma_i \mapsto \varphi_i = \varphi(\gamma_i).$$

Let $\rho : \mathbb{B}_d \longrightarrow \mathrm{GL}_{d-1}(\mathbb{Z}[t, t^{-1}])$ be the reduced Burau representation. Let us consider the $\mathbb{Z}[t, t^{-1}]$-module

$$M = H_0(\pi_1(\mathbb{C} \setminus \{p_1, \ldots, p_n\}), \rho \circ \varphi).$$

Theorem 2.1. *The order of the $\mathbb{Z}[t, t^{-1}]$-module M is equal to the Alexander polynomial of $\mathcal{C}$, multiplied by $(1 + t + \cdots + t^{d-1})$.*

The proof of this theorem uses Fox calculus. An alternative geometrical proof should be obtained via the Reidemeister torsion, using multiplicativity and Mayer–Vietoris arguments.

3. A detailed example

The following example illustrates some aspects appearing in this volume, see [5].

We consider a classical curve $\mathcal{C} \subset \mathbb{C}^2$ (Figure 3.1), called the deltoid, given by the equation:

$$f(x, y) := (x^2 + y^2)^2 - 48x(x^2 + y^2) + 72(x^2 + y^2) + 64x^3 - 432 = 0. \quad (3.1)$$

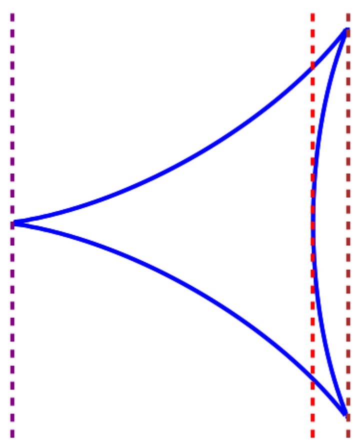

Figure 3.1. Real deltoid with non-transversal vertical lines

Its compactification in $\mathbb{P}^2$ is the tricuspidal quartic and the line at infinity L_∞ is the unique bitangent. Though this does not follow the genericity hypothesis of [8, §3], it is not hard to see that all the statements in that section apply to any affine curve as long as the equation of $\mathcal{C}$ is monic in y (which is the case), i.e., there are no vertical asymptotes.

In order to compute the braid monodromy of this curve with respect to the projection on the x-axis, we need to compute the discriminant of f with respect to y:

$$\Delta(x) := 1048576(x - 2)(x + 6)^3(x - 3)^6. \quad (3.2)$$

Following the notation of [8, Def. 3.3], we consider the set $Z_3 := \{z_1 := 3, z_2 := 2, z_3 := -6\}$. We choose suitable closed disks $\mathbb{D}_x, \mathbb{D}_y \subset \mathbb{C}$ and $z_0 \in \partial\mathbb{D}_x$, $z_0 \in \mathbb{R}_{\gg 3}$. The group $\pi_1(\mathbb{D}_x \setminus Z_3; z_0)$ has a geometric basis $(\gamma_1, \gamma_2, \gamma_3)$ constructed as follows. Let $z_i^+, z_i^- \in \mathbb{R}$ be such that $z_i^\pm := z_i \pm \varepsilon$, $0 < \varepsilon \ll 1$. We define the following paths:

- α_0 is the segment $[z_0, z_1^+]$ and α_i, $i = 1, 2$, is the segment $[z_i^-, z_{i+1}^+]$;
- δ_i is the path running counterclockwise along the circle of center z_i and radius ε starting and ending at z_i^+;
- δ_i^+ is the path running counterclockwise along the upper semicircle of center z_i and radius ε starting at z_i^+ and ending at z_i^-.

Then,

$$\begin{aligned}
\gamma_1 :=&\alpha_0 \cdot \delta_1 \cdot \alpha_0^{-1},\\
\gamma_2 :=&(\alpha_0 \cdot \delta_1^+ \cdot \alpha_1) \cdot \delta_2 \cdot (\alpha_0 \cdot \delta_1^+ \cdot \alpha_1)^{-1}\\
\gamma_3 :=&(\alpha_0 \cdot \delta_1^+ \cdot \alpha_1 \cdot \delta_2^+ \cdot \alpha_2) \cdot \delta_3 \cdot (\alpha_0 \cdot \delta_1^+ \cdot \alpha_1 \cdot \delta_2^+ \cdot \alpha_2)^{-1}.
\end{aligned}$$

We want to describe the braid monodromy $\mu : \pi_1(\mathbb{D}_x \setminus Z_3; z_0) \to \mathbb{B}_4$ in terms of classical Artin generators. Considering the basis $(\gamma_1, \gamma_2, \gamma_3)$, the map μ is defined by the images of the basis by μ, producing an element in $B := (\mu(\gamma_1), \mu(\gamma_2), \mu(\gamma_3)) \in (\mathbb{B}_4)^3$, which is usually called a *Braid Monodromy Factorization* BMF [8, Def. 3.3]. For generic curves at infinity of degree d, i.e. its link at infinity is the Hopf link with d components, the product of the coordinates of a BMF (in the reversed order) is the positive generator of the center of the braid group. This will not be the case for our deltoid, since the link at infinity is not a Hopf link.

We consider the generators of $\mathbb{B}_4$ as in [18, §1.5], but in order to be coherent with notations in [8], we number the strings from right to left. In order to *draw the braids*, we would like to use the standard projection $\mathbb{C} \to \mathbb{R}$ by taking the real part. This projection is not generic for our braids since we may have pairs of conjugate complex numbers. In order to avoid this problem, we perturb the projection near the conjugate complex numbers such that the image of the number with negative imaginary part is (slightly) smaller than the image of the number with positive imaginary part.

Using this convention and the computations in [8, Ex. 1.53], we obtain:

$$\begin{array}{cccccccc} \alpha_0\mapsto & 1 & \delta_1\mapsto & \sigma_1^3\sigma_3^3 & \delta_1^+\mapsto & \sigma_1^2\sigma_3^2 & \alpha_1\mapsto & 1\\ \delta_2\mapsto & \sigma_2 & \delta_2^+\mapsto & 1 & \alpha_2\mapsto & \sigma_1^{-1}\sigma_3^{-1} & \delta_3\mapsto & \sigma_2^3. \end{array}$$

The calculation of α_2 and δ is slightly more complicated and follows the ideas in [8, Example 2.20]. Since any conjugation of a BDM is also a BDM, one can replace B by $B_1 := B^{\sigma_1^2\sigma_3^2}$ to obtain a simpler one:

$$B_1 = (\sigma_1^3\sigma_3^3, \sigma_2, \sigma_1^{-1}\sigma_3^{-1}\sigma_2^3\sigma_3\sigma_1).$$

From B_1 it is easy to compute the fundamental group of $\mathbb{C}^2\setminus\mathcal{C}$.

Proposition 3.1. *The presentation*

$$\langle x,y,z : xyx=yxy, xzx=zxz, yzy=zyz\rangle$$

is a Zariski presentation for $\pi_1(\mathbb{C}^2\setminus\mathcal{C})$; in particular, $\mathbb{C}^2\setminus\mathcal{C}$ has the homotopy type of the 2-complex associated to the presentation, see [8, Theorem 3.3]*.*

Proof. We fix a basis $a_1,\dots,a_4$ for the fundamental group of a vertical line (minus the points in $\mathcal{C}$). Since the first braid $\sigma_1^3\sigma_3^3$ corresponds to two singular points (of multiplicity two) we obtain two relations:

$$a_1 = a_1^{\sigma_1^3\sigma_3^3} = a_1^{\sigma_1^3} = (a_2a_1)a_2(a_2a_1)^{-1},$$
$$a_3 = a_3^{\sigma_1^3\sigma_3^3} = a_3^{\sigma_3^3} = (a_4a_3)a_4(a_4a_3)^{-1}.$$

For the braid σ_2 we obtain the relation $a_2=a_3$. In order to obtain the relation for $\sigma_1^{-1}\sigma_3^{-1}\sigma_2^3\sigma_3\sigma_1)$ one needs to work some more. One obtains $b_2=(b_3b_2)^{-1}b_3(b_3b_2)$, where $b_i=a_i^{\sigma_3\sigma_1}$:

$$(a_2a_1a_2^{-1}) = (a_3a_2a_1a_2^{-1})a_3(a_3a_2a_1a_2^{-1})^{-1}.$$

To obtain the presentation of the statement we perform two Tietze transformations which do not change the homotopy type of the associated 2-complexes: we eliminate the generator a_3 and we rename the generators as $x=a_2a_1a_2^{-1}$, $y=a_2$ and $z=a_4$. □

Remark 3.2. This proposition works even if the braid monodromy is not generic, as it was stated by Libgober [26]. Following carefully his proof, it is not hard to see that neither the genericity of the braid nor the genericity at infinity of the affine line are needed in the statement. The only needed condition is the non-existence of vertical asymptotes.

We finish this section by computing the Alexander polynomial from braid monodromy using Libgober's method as stated in the previous section. As in the above remark, this method also works for non-generic braid monodromies (with no vertical asymptotes). Let us consider the reduced Burau representation

$$\sigma_1 \mapsto \begin{pmatrix} -t & 0 & 0 \\ t & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \sigma_2 \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & -t & 0 \\ 0 & t & 1 \end{pmatrix}, \sigma_3 \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & -t \end{pmatrix}.$$

We obtain the following matrix list:

$$S_1 := \begin{pmatrix} -t^3 & 0 & 0 \\ t^3 - t^2 + t & 1 & t^2 - t + 1 \\ 0 & 0 & -t^3 \end{pmatrix}, \quad S_2 := \begin{pmatrix} 1 & 1 & 0 \\ 0 & -t & 0 \\ 0 & t & 1 \end{pmatrix},$$

$$S_3 := \begin{pmatrix} t - t^2 & -\frac{t^2-t+1}{t} & -\frac{t^2-t+1}{t} \\ (t-t^2)(t^2-t+1) & -t^3 + 2\,t^2 - 2\,t + 2 & (1-t)(t^2-t+1) \\ -t^3 + t^2 - t & -t^2 + t - 1 & -(t-1)t \end{pmatrix}.$$

To apply Libgober's method we must compute the greatest common divisor of the three minors of the matrix $(S_1 - I_3|S_2 - I_3|S_3 - I_3)$. It is clear we can forget the second, fourth, sixth, and last two columns. We can perform row operations in order to get $(1, 0, 0)$ in the new third column. We eliminate the first row and the third column and we obtain:

$$\begin{pmatrix} -(t^2-t+1)(t^2+t+1) & t^2-t+1 & -(t^2+1)(t^2-t+1) \\ (t+1)(t^2-t+1)t & -(t+1)(t^2-t+1) & 0 \end{pmatrix}$$

If we add the first row (multiplied by $t+1$) to the second one, then we can perform column operations to obtain zeroes in the first row outside the second column, obtaining:

$$\begin{pmatrix} 0 & t^2-t+1 & 0 \\ -(t^3+t^2+t+1)(t^2-t+1) & 0 & -(t^3+t^2+t+1)(t^2-t+1) \end{pmatrix}.$$

We know that the gcd of the minors is the product of the Alexander polynomial and $t^3 + t^2 + t + 1$. We obtain (as it was already known) that the Alexander polynomial equals $(t^2 - t + 1)^2$.

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