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# Local coordinates for $SL(n, \mathbf{C})$ -character varieties of finite-volume hyperbolic 3-manifolds

PERE MENAL-FERRER  
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## Abstract

Given a finite-volume hyperbolic 3-manifold, we compose a lift of the holonomy in  $SL(2, \mathbf{C})$  with the  $n$ -dimensional irreducible representation of  $SL(2, \mathbf{C})$  in  $SL(n, \mathbf{C})$ . In this paper we give local coordinates of the  $SL(n, \mathbf{C})$ -character variety around the character of this representation. As a corollary, this representation is isolated among all representations that are unipotent at the cusps.

*Coordonnées locales pour la variété des  $SL(n, \mathbf{C})$ -caractères des 3-variétés hyperboliques à volume fini*

## Résumé

Étant donnée une 3-variété hyperbolique à volume fini, on compose un relevé dans  $SL(2, \mathbf{C})$  de son holonomie avec la représentation irréductible et  $n$ -dimensionnelle de  $SL(2, \mathbf{C})$  dans  $SL(n, \mathbf{C})$ . Dans cet article on donne des coordonnées locales autour du caractère de cette représentation. Comme corollaire, cette représentation est isolée parmi toutes les représentations qui sont unipotentes aux bouts.

## 1. Introduction

Let  $M^3$  be an oriented, complete, finite-volume, hyperbolic 3-manifold with  $l > 0$  cusps. This manifold is homeomorphic to the interior of a compact manifold  $\overline{M}^3$  with boundary a union of  $l$  tori. Let  $\widetilde{\text{Hol}}: \pi_1(M^3) \rightarrow SL(2, \mathbf{C})$  be a lift of the holonomy of  $M^3$  and compose it with the irreducible  $n$ -dimensional representation

$$\varsigma_n: SL(2, \mathbf{C}) \rightarrow SL(n, \mathbf{C}).$$

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The composition is denoted by

$$\rho_n = \varsigma_n \circ \widetilde{\text{Hol}}: \pi_1(M^3) \rightarrow \text{SL}(n, \mathbf{C}).$$

The variety of characters  $X(M^3, \text{SL}(n, \mathbf{C}))$  is the algebraic quotient of the variety of representations  $\text{hom}(\pi_1(M^3), \text{SL}(n, \mathbf{C}))$  by the action of  $\text{SL}(n, \mathbf{C})$  by conjugation [4]. The character of  $\rho_n$  will be denoted by  $\chi_n$ . In [6] it is proved that  $\chi_n$  is a smooth point of  $X(M^3, \text{SL}(n, \mathbf{C}))$ , with local dimension  $(n-1)l$ . The goal of this paper is to find coordinates for a neighborhood of  $\chi_n$ .

For  $i = 1, \dots, n-1$ , let

$$\sigma_i: \text{SL}(n, \mathbf{C}) \rightarrow \mathbf{C}$$

denote the  $i$ -th elementary symmetric polynomial on the eigenvalues, so that the characteristic polynomial of  $A \in \text{SL}(n, \mathbf{C})$  is

$$P_A(\lambda) = \lambda^n - \sigma_1(A)\lambda^{n-1} + \dots + (-1)^{n-1}\sigma_{n-1}(A)\lambda + (-1)^n.$$

So  $\sigma_1(A)$  is the trace of  $A$ ,  $\sigma_2(A)$  is obtained from  $2 \times 2$  principal minors of  $A$ , and so on. The  $\sigma_i: \text{SL}(n, \mathbf{C}) \rightarrow \mathbf{C}$  are polynomial functions, invariant by conjugation. Thus, for any  $\gamma \in \pi_1(M^3)$ , the map

$$\begin{aligned} \text{hom}(\pi_1(M^3), \text{SL}(n, \mathbf{C})) &\rightarrow \mathbf{C} \\ \rho &\mapsto \sigma_i(\rho(\gamma)) \end{aligned}$$

induces a polynomial map on the character variety

$$\sigma_i^\gamma: X(M^3, \text{SL}(n, \mathbf{C})) \rightarrow \mathbf{C}.$$

The main result of this paper is the following theorem.

**Theorem 1.1.** *Let  $M^3$  be an orientable, complete, finite-volume, hyperbolic 3-manifold with  $l > 0$  cusps. Let  $\gamma_1, \dots, \gamma_l \in \pi_1(M^3)$  be nontrivial peripheral elements, one for each cusp of  $M^3$  (or each boundary component of  $\overline{M^3}$ ). Then*

$$(\sigma_1^{\gamma_1}, \dots, \sigma_{n-1}^{\gamma_1}, \dots, \sigma_1^{\gamma_l}, \dots, \sigma_{n-1}^{\gamma_l}): X(M^3, \text{SL}(n, \mathbf{C})) \rightarrow \mathbf{C}^{l(n-1)}$$

*is a local biholomorphism at  $\chi_n$ .*

**Corollary 1.2.** *The character  $\chi_n$  is isolated among all characters of representations in  $\text{SL}(n, \mathbf{C})$  that are unipotent at the peripheral subgroups.*

When  $n = 2$ , we obtain the following result of Kapovich [3] (see also Bromberg [2]).

**Corollary 1.3.** *The deformation space  $X(M^3, \mathrm{SL}(2, \mathbf{C}))$  is locally parameterized by the trace of  $\gamma_1, \dots, \gamma_l$  around a lift of the holonomy representation.*

The proof relies on a vanishing theorem of [7, 5], that asserts that infinitesimal  $L^2$  deformations are trivial. In this way we determine explicit differential forms on the cusp that describe the infinitesimal deformations and prove Theorem 1.1.

The paper is organized as follows. In Section 2 we state some basic facts about the  $n$ -dimensional irreducible representation  $\varsigma_n: \mathrm{SL}(2, \mathbf{C}) \rightarrow \mathrm{SL}(n, \mathbf{C})$ , which will be required later. Section 3 is devoted to compute the explicit differential forms that give the infinitesimal deformations. Finally, in Section 4 we compute the derivative of  $\sigma_i^{\gamma_j}$  with respect to these infinitesimal deformations.

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## 2. The $n$ -dimensional representation

The irreducible  $n$ -dimensional complex representation

$$\varsigma_n: \mathrm{SL}(2, \mathbf{C}) \rightarrow \mathrm{SL}(n, \mathbf{C})$$

is the symmetric power  $\mathrm{Sym}^{n-1}(\mathbf{C}^2) \cong \mathbf{C}^n$ . More precisely, elements in  $\mathrm{Sym}^{n-1}(\mathbf{C}^2)$  can be seen as homogeneous polynomials of degree  $n-1$  on elements of  $\mathbf{C}^2$  and, for  $A \in \mathrm{SL}(2, \mathbf{C})$ ,  $\varsigma_n(A)$  maps a monomial  $v_1 \cdots v_{n-1}$  to  $A(v_1) \cdots A(v_{n-1})$ , where  $v_1, \dots, v_{n-1} \in \mathbf{C}^2$ .

The induced representation of Lie algebras is denoted by  $\theta_n: \mathfrak{sl}(2, \mathbf{C}) \rightarrow \mathfrak{sl}(n, \mathbf{C})$ . We shall work with the basis for  $\mathfrak{sl}(2, \mathbf{C})$  given by

$$e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad g = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

A straightforward computation shows that:

$$h_+ := \theta_n(f) = \begin{pmatrix} 0 & 1 & 0 & & & \\ 0 & 0 & 2 & & & \\ 0 & 0 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & n-1 \\ & & & & 0 & 0 \end{pmatrix}, \quad (2.1)$$

namely the  $(i, j)$ -entry of  $h_+$  is  $i$  when  $j = i + 1$  and 0 otherwise. Similarly

$$h_- := \theta_n(g) = \begin{pmatrix} 0 & 0 & 0 & & & \\ n-1 & 0 & 0 & & & \\ 0 & n-2 & 0 & & & \\ & & & \ddots & & \\ & & & & 0 & 0 \\ & & & & 1 & 0 \end{pmatrix}. \quad (2.2)$$

This allows to describe  $\varsigma_n$  for  $\pm \exp(\beta f)$  and  $\pm \exp(\beta g)$ ,  $\beta \in \mathbf{C}$ :

$$\varsigma_n\left(\pm \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}\right) = (\pm 1)^{n-1} \varsigma_n(e^{\beta f}) = (\pm 1)^{n-1} e^{\beta h_+}, \quad (2.3)$$

$$\varsigma_n\left(\pm \begin{pmatrix} 1 & 0 \\ \beta & 1 \end{pmatrix}\right) = (\pm 1)^{n-1} \varsigma_n(e^{\beta g}) = (\pm 1)^{n-1} e^{\beta h_-}. \quad (2.4)$$

Notice that both matrices are triangular with 1 in the diagonal, in particular unipotent.

*Notation 2.1.* The group  $\mathrm{SL}(2, \mathbf{C})$  acts on the Lie algebra  $\mathfrak{sl}(n, \mathbf{C})$  by composing the adjoint representation with  $\varsigma_n$ . For  $A \in \mathrm{SL}(2, \mathbf{C})$  and  $a \in \mathfrak{sl}(n, \mathbf{C})$ , this action will be simply denoted by  $Aa$ . Namely,

$$Aa = \mathrm{Ad}_{\varsigma_n(A)}(a) = \varsigma_n(A) a \varsigma_n(A^{-1}).$$

**Lemma 2.2.** *For  $\beta \neq 0$ , the subspace of matrices in  $\mathfrak{sl}(n, \mathbf{C})$  that are invariant by  $\pm \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$  is*

$$\langle h_+, h_+^2, \dots, h_+^{n-1} \rangle.$$

*Proof.* We have  $\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} = \exp(\beta f)$ . Thus the lemma is equivalent to saying that  $\langle h_+, h_+^2, \dots, h_+^{n-1} \rangle$  is the subspace of invariants by the one-parameter group generated by  $f$ . This latter space is exactly

$$\mathrm{Ker}(\mathrm{ad}_{\varsigma_n(f)}) = \{a \in \mathfrak{sl}(n, \mathbf{C}) \mid [\theta_n(f), a] = 0\}.$$

Since  $h_+ = \theta_n(f)$ ,

$$\langle h_+, h_+^2, \dots, h_+^{n-1} \rangle \subseteq \text{Ker}(\text{ad}_{\theta_n(f)}).$$

To prove the equality, we show that  $\text{Ker}(\text{ad}_{\theta_n(f)})$  has dimension  $n - 1$ . To see this, we decompose  $\mathfrak{sl}(n, \mathbf{C})$ , as  $\text{SL}(2, \mathbf{C})$ -module, into irreducible factors using Clebsch-Gordan:

$$\mathfrak{sl}(n, \mathbf{C}) = \text{Sym}^{2n-2}(\mathbf{C}^2) \oplus \dots \oplus \text{Sym}^4(\mathbf{C}^2) \oplus \text{Sym}^2(\mathbf{C}^2).$$

As an endomorphism of  $\text{Sym}^k(\mathbf{C}^2)$ , the rank of  $f$  is  $k$  (use for instance Equation (2.2)), and hence its kernel has dimension 1. The result then follows immediately.  $\square$

We shall also require the following computations. Since  $[e, f] = 2f$  and  $[e, g] = -2g$ ,

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} h_{\pm} = \lambda^{\pm 2} h_{\pm}. \quad (2.5)$$

Hence, for  $i = 1, \dots, n - 1$ ,

$$\begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} h_{\pm}^i = \lambda^{\pm 2i} h_{\pm}^i \quad \text{and} \quad \begin{pmatrix} \lambda & 0 \\ 0 & 1/\lambda \end{pmatrix} h_{\pm}^i = \lambda^{\pm 2i} h_{\pm}^i. \quad (2.6)$$

Finally, we recall the bilinear product

$$\begin{aligned} \mathfrak{sl}(n, \mathbf{C}) \times \mathfrak{sl}(n, \mathbf{C}) &\rightarrow \mathbf{C} \\ (v_1, v_2) &\mapsto \text{trace}(v_1 v_2), \end{aligned} \quad (2.7)$$

which is a multiple of the Killing form. This pairing is nondegenerate, symmetric, bilinear and  $\text{Ad} \circ \varsigma_n$ -invariant (hence  $\text{Ad} \circ \rho_n$ -invariant). In addition  $\text{trace}(h_-^i h_+^j) = 0$  iff  $i \neq j$ . For  $i = 1, \dots, n - 1$ , we denote  $c_i = \text{trace}(h_-^i h_+^i) \neq 0$ . Such a pairing is not unique among nondegenerate, symmetric, bilinear and  $\text{Ad} \circ \varsigma_n$ -invariant pairings, as  $\mathfrak{sl}(n, \mathbf{C})$  is not an irreducible  $\text{SL}(2, \mathbf{C})$ -module.

### 3. Infinitesimal deformations

To describe infinitesimal deformations explicitly, we shall work in cohomology with twisted coefficients. The representation  $\rho_n$  is semisimple, hence by [8, 4] the Zariski tangent space of  $X(M^3, \text{SL}(n, \mathbf{C}))$  at  $\chi_n$  is isomorphic to  $H^1(\pi_1(M^3); \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n})$ , where  $\mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n}$  denotes the Lie algebra with the action obtained by composing  $\rho_n$  and the adjoint representation (this is described in more detail in Section 4, cf. (4.2)).

Since  $M^3$  is aspherical, we shall work with the cohomology of  $M^3$  with coefficients in the flat bundle

$$E_{\text{Ad} \circ \rho_n} = \widetilde{M^3} \times_{\pi_1 M^3} \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n}.$$

Let  $\Omega^p(M^3, E_{\text{Ad} \circ \rho_n})$  denote the space of smooth p-forms on  $M^3$  valued on  $E_{\text{Ad} \circ \rho_n}$ , i.e. the space of smooth sections of the bundle  $\wedge^p T^*M^3 \otimes E_{\text{Ad} \circ \rho_n}$ . The de Rham cohomology of  $\Omega^*(M^3, E_{\text{Ad} \circ \rho_n})$  is denoted by

$$H^*(M^3; E_{\text{Ad} \circ \rho_n}),$$

and it is naturally isomorphic to the group cohomology

$$H^*(\pi_1(M^3); \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n}).$$

The explicit constructions of these identifications will be given in Section 4.

Next we need to recall the inner product on  $\Omega^p(M^3, E_{\text{Ad} \circ \rho_n})$ . In order to do that, start with the homogeneous structure of hyperbolic space,

$$\mathbf{H}^3 \cong \widetilde{M^3} = \text{SL}(2, \mathbf{C}) / \text{SU}(2),$$

that is  $\text{SU}(2)$  is the stabilizer of a base point  $p \in \mathbf{H}^3$ . Fix an  $\text{SU}(2)$ -invariant inner product  $\langle \cdot, \cdot \rangle$  on  $\mathfrak{sl}(n, \mathbf{C})$ . Then define an inner product on the trivial vector bundle  $\mathbf{H}^3 \times \mathfrak{sl}(n, \mathbf{C})$  by using the following rule:

$$\langle v_1, v_2 \rangle_{\gamma p} := \langle \gamma^{-1} v_1, \gamma^{-1} v_2 \rangle, \quad \forall \gamma \in \text{SL}(2, \mathbf{C}), v_1, v_2 \in \{\gamma p\} \times \mathfrak{sl}(n, \mathbf{C}).$$

This descends to a well-defined inner product on  $E_{\text{Ad} \circ \rho_n}$ . By using an orthonormal basis, it induces an inner product on the fibers of  $\wedge^* T^*M^3 \otimes E_{\text{Ad} \circ \rho_n}$ , and on  $\Omega^*(M^3; E_{\text{Ad} \circ \rho_n})$  by integration: if  $\alpha, \beta \in \Omega^*(M^3; E_{\text{Ad} \circ \rho_n})$  then

$$(\alpha, \beta) = \int_M \langle \alpha(x), \beta(x) \rangle_x d \text{vol}.$$

A form  $\alpha \in \Omega^*(M^3; E_{\text{Ad} \circ \rho_n})$  is  $L^2$  if  $|\alpha|^2 = (\alpha, \alpha) < \infty$ .

Recall that  $M^3$  has  $l$  cusps and that it is homeomorphic to the interior of a compact manifold  $\overline{M^3}$  with boundary a union of  $l$  tori. We shall use the following result from [6], based on rigidity results of Raghunathan [7] and Mathshushima-Murakami [5].

**Theorem 3.1** ([6]). *If  $n \geq 2$ , then*

$$\dim_{\mathbf{C}} H^1(M^3; E_{\text{Ad} \circ \rho_n}) = l(n - 1).$$

*In addition, all nontrivial elements in  $H^1(M^3; E_{\text{Ad} \circ \rho_n})$  are nontrivial in  $H^1(\partial \overline{M^3}; E_{\text{Ad} \circ \rho_n})$  and have no  $L^2$ -representative.*

*Remark 3.2.* To simplify, from now on we will assume that  $l = 1$ , i.e.  $M^3$  has a single cusp. The proof below applies to any  $l \in \mathbf{N}$ , because most of the argument is localized at the cusp.

We need to describe the metric on a cusp  $U \subset M^3$ , namely  $U$  is the quotient of a horoball in  $\mathbf{H}^3$  by a rank two parabolic group of isometries. Notice that  $M \setminus \text{int}(U)$  is compact, thus a form  $\Omega^*(M^3, E_{\text{Ad} \circ \rho_n})$  is  $L^2$  (has finite  $L^2$  norm) if and only if its restriction to  $\Omega^*(U, E_{\text{Ad} \circ \rho_n})$  is  $L^2$ .

The cusp  $U$  is diffeomorphic to  $T^2 \times [0, \infty)$ , and it is isometric to the warped product

$$dt^2 + e^{-2t} ds_{T^2}^2,$$

where  $ds_{T^2}^2$  denotes a flat metric on the 2-torus, cf. [1, Ex. 5.2].

Consider  $\vartheta$  any 1-form on the 2-torus  $T^2$ , and view it as a form on  $U$  by pullback from the projection to the first factor  $U = T^2 \times [0, \infty) \rightarrow T^2$ .

Assume that the holonomy of the cusp lies in the group

$$\left\{ \pm \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \mid a \in \mathbf{C} \right\}.$$

Recall that  $h_+ \in \mathfrak{sl}(n, \mathbf{C})$  is defined in (2.1), and that  $h_+^j$  is invariant by the holonomy of the cusp, for  $j = 1, \dots, n-1$ , by Lemma 2.2. In particular, the form  $\vartheta \otimes h_+^j$  is well defined, and it is closed iff  $\vartheta$  is closed.

**Lemma 3.3.** *The 1-form  $\vartheta \otimes h_+^j$  is  $L^2$ , for  $j = 1, \dots, n-1$ .*

*Proof.* Given  $p \in T^2$  and  $t \in [0, \infty)$ , we first compute the norm of  $\vartheta \otimes h_+^j$  at  $(p, t) \in T^2 \times [0, \infty) = U$ , and then we shall show that

$$\int_U |\vartheta \otimes h_+^j|_{(p,t)}^2 d\text{vol}_U < \infty.$$

By compactness, there exists a constant  $C > 0$  such that  $|\vartheta|_{(p,0)} \leq C$  for every point  $p \in T^2$  when  $t = 0$ . Since the metric is the warped product  $dt^2 + e^{-2t} ds_{T^2}^2$ :

$$|\vartheta|_{(p,t)} \leq e^t C.$$

On the other hand, if we work in the half space model for  $\mathbf{H}^3$  and we assume that the horoball is centered at  $\infty$ , the image of  $\pi_1(U)$  is contained in  $\pm \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ . Then the isometry that brings a base point to a lift of  $(p, t)$  in  $\mathbf{H}^3$  is

$$\begin{pmatrix} e^{t/2} & ze^{-t/2} \\ 0 & e^{-t/2} \end{pmatrix},$$



for some  $z \in \mathbf{C}$ . By (2.6) and Lemma 2.2, we have

$$\begin{pmatrix} e^{t/2} & ze^{-t/2} \\ 0 & e^{-t/2} \end{pmatrix}^{-1} h_+^j = \begin{pmatrix} e^{-t/2} & 0 \\ 0 & e^{t/2} \end{pmatrix} h_+^j = e^{-jt} h_+^j.$$

By definition of the metric on the bundle  $E_{\text{Ad} \circ \rho_n}$ :

$$|h_+^j|_{(p,t)} = e^{-jt} |h_+^j|_{(p,0)}.$$

Thus

$$|\vartheta \otimes h_+^j|_{(p,t)} \leq C' e^{(1-j)t},$$

for some constant  $C' > 0$ . In addition, using  $d \text{vol}_U = e^{-2t} d \text{vol}_{T^2} \wedge dt$ , we compute:

$$\int_U |\vartheta \otimes h_+^j|_{(p,t)}^2 d \text{vol}_U \leq C'' \int_0^{+\infty} e^{2(1-j)t-2t} dt = C'' \int_0^{+\infty} e^{-2jt} dt < +\infty.$$

□

We next look for a basis for  $H^1(U; E_{\text{Ad} \circ \rho_n})$  (Lemma 3.4 below). Choose coordinates  $(x, y) \in \mathbf{R}^2$  and view the torus as the quotient  $\mathbf{R}^2/\mathbf{Z}^2$ . Let  $\gamma_1$  and  $\gamma_2$  be two generators of  $\pi_1(T^2)$ , and assume that they act on the universal covering as:

$$\gamma_1(x, y) = (x + 1, y), \quad \gamma_2(x, y) = (x, y + 1), \quad \forall x, y \in \mathbf{R}^2.$$

Assume that their holonomy is defined by

$$\gamma_1 \rightarrow \pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \gamma_2 \rightarrow \pm \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix},$$

for some  $\tau \in \mathbf{C} \setminus \mathbf{R}$ .

For  $i = 1, \dots, n - 1$ , consider the form

$$d(x + \tau y) \otimes \begin{pmatrix} 1 & x + \tau y \\ 0 & 1 \end{pmatrix} h_-^i \in \Omega^1(\mathbf{R}^2; \mathfrak{sl}(n, \mathbf{C})).$$

The projection  $U = T^2 \times [0, \infty) \rightarrow T^2$  lifts to a projection of universal coverings  $\tilde{U} \rightarrow \mathbf{R}^2$ , and this differential form is pulled back to a form on  $\tilde{U}$ . Since it is equivariant, it induces

$$\omega_i \in \Omega^1(U; E_{\text{Ad} \circ \rho_n}),$$

**Lemma 3.4.** *For  $i = 1, \dots, n-1$ ,  $\omega_i \in \Omega^1(U; E_{\text{Ad} \circ \rho_n})$  is closed. Moreover, for any  $a, b \in \mathbf{C}$  such that  $b \neq a\tau$ ,*

$$\left\{ \omega_1, \dots, \omega_{n-1}, (a dx + b dy) \otimes h_+, \dots, (a dx + b dy) \otimes h_+^{n-1} \right\}$$

*is a basis for  $H^1(U; E_{\text{Ad} \circ \rho_n})$ .*

*Proof.* We prove first that the form  $\omega_i$  is closed, by writing

$$\begin{pmatrix} 1 & x + \tau y \\ 0 & 1 \end{pmatrix} h_-^i = \sum_j p_{ij}(x + \tau y) m_{ij},$$

where  $p_{ij}(x + \tau y)$  are polynomials with complex coefficients on the variable  $(x + \tau y)$  and  $m_{ij} \in \mathfrak{sl}(n, \mathbf{C})$ . Therefore

$$\begin{aligned} d \left( d(x + \tau y) \otimes \begin{pmatrix} 1 & x + \tau y \\ 0 & 1 \end{pmatrix} h_-^i \right) = \\ \sum_j d(x + \tau y) \wedge d(p_{ij}(x + \tau y)) \otimes m_{ij} = 0. \end{aligned}$$

Next we want to describe the basis for  $H^1(U; E_{\text{Ad} \circ \rho_n})$ . Knowing that  $\dim H^1(U; E_{\text{Ad} \circ \rho_n}) = 2(n-1)$  [6], we will show that the  $2(n-1)$  differential forms are linearly independent cohomology classes by using a bilinear pairing.

This pairing is induced from the exterior product  $\wedge: \Omega^i(U; E_{\text{Ad} \circ \rho_n}) \times \Omega^j(U; E_{\text{Ad} \circ \rho_n}) \rightarrow \Omega^{i+j}(U; \mathbf{C})$

$$(\vartheta_1 \otimes v_1) \wedge (\vartheta_2 \otimes v_2) = \text{trace}(v_1 v_2) \vartheta_1 \wedge \vartheta_2,$$

where  $\vartheta_1 \in \Omega^i(U)$ ,  $\vartheta_2 \in \Omega^j(U)$  are forms without coefficients (or coefficients in the trivial bundle),  $v_1, v_2 \in \mathfrak{sl}(n, \mathbf{C})$ . Recall that the pairing  $(v_1, v_2) \mapsto \text{trace}(v_1 v_2)$  was described in (2.7) and that  $\text{trace}(h_-^i h_+^j) = \delta_i^j c_i$ , where  $\delta_i^i = 1$ ,  $\delta_i^j = 0$  for  $i \neq j$ , and  $c_i \neq 0$ .

This exterior product induces a cup product in cohomology. Since the pairing and  $h_+^i$  are both invariant by the action of  $(\begin{smallmatrix} 1 & * \\ 0 & 1 \end{smallmatrix})$ , we have:

$$\begin{aligned} ((a dx + b dy) \otimes h_+^i) \wedge \omega_j = \\ \text{trace}(h_+^i \begin{pmatrix} 1 & x + \tau y \\ 0 & 1 \end{pmatrix} h_-^j) (a dx + b dy) \wedge (dx + \tau dy) = \\ \text{trace}(h_+^i h_-^j) (a dx + b dy) \wedge (dx + \tau dy) = c_i \delta_i^j (a\tau - b) dx \wedge dy. \end{aligned}$$

In addition:

$$\omega_i \wedge \omega_j = \text{trace}(h_-^i h_-^j)(dx + \tau dy) \wedge (dx + \tau dy) = 0,$$

and

$$((a dx + b dy) \otimes h_+^i) \wedge ((a dx + b dy) \otimes h_+^j) = 0.$$

Since  $dx \wedge dy$  is the volume form of the torus, the lemma follows.  $\square$

*Remark 3.5.* From now on we apply the previous lemma for  $a = 1$  and  $b = 0$ , namely we take  $a dx + b dy = dx$ .

**Proposition 3.6.** *The image of the following map*

$$H^1(M^3; E_{\text{Ad} \circ \rho_n}) \rightarrow H^1(U; E_{\text{Ad} \circ \rho_n})$$

*is the  $(n - 1)$ -dimensional linear span of*

$$\{\omega_1 + \sum_j a_{1,j} dx \otimes h_+^j, \dots, \omega_{n-1} + \sum_j a_{n-1,j} dx \otimes h_+^j\},$$

*for some  $a_{i,j} \in \mathbf{C}$ .*

*Proof.* By contradiction: if the lemma was not true, then there would be a nontrivial element  $\sum_j a_j dx \otimes h_+^j$  in the image, because it is  $n - 1$  dimensional. But this form is  $L^2$  (Lemma 3.3) contradicting Theorem 3.1.  $\square$

#### 4. Derivating the elementary symmetric polynomials

We want to compute the derivatives of the elementary symmetric polynomials of a peripheral element  $\gamma$  with respect to the infinitesimal deformations of Proposition 3.6.

We first describe the map between closed 1-forms in  $\Omega^1(M^3; E_{\text{Ad} \circ \rho_n})$  and group cocycles in

$$\begin{aligned} Z^1(M^3; \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n}) &= \{d: \pi_1(M^3) \rightarrow \mathfrak{sl}(n, \mathbf{C}) \mid \\ & d(\gamma_1 \gamma_2) = d(\gamma_1) + \text{Ad}_{\rho_n(\gamma_1)}(d(\gamma_2)), \forall \gamma_1, \gamma_2 \in \pi_1(M^3)\} \end{aligned}$$

that induces the isomorphism between de Rham and group cohomology. For this purpose we fix a point  $p \in M^3$ , that will be the base point for  $\pi_1(M^3, p)$ . Let  $\vartheta \in \Omega^1(M^3; E_{\text{Ad} \circ \rho_n})$  be a closed 1-form. In particular it represents an element in de Rham cohomology  $H^1(M^3; E_{\text{Ad} \circ \rho_n})$ . This form is mapped to the cocycle

$$\begin{aligned} d_{\vartheta}: \pi_1(M^3, p) &\rightarrow \mathfrak{sl}(n, \mathbf{C}) \\ [\gamma] &\mapsto \int_{\gamma} \vartheta, \end{aligned} \quad (4.1)$$

where  $\gamma$  is a loop based at  $p$  representing  $[\gamma] \in \pi_1(M^3, p)$ . See [9, §6.3] for details. The map  $d_{\vartheta}: \pi_1(M^3, p) \rightarrow \mathfrak{sl}(n, \mathbf{C})$  is a cocycle, and its group cohomology class only depends on the de Rham cohomology class of the form  $\vartheta$ .

We next describe *Weil's construction* [8, 4], that maps a group cocycle in  $Z^1(\pi_1(M^3); \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n})$  to an infinitesimal deformation of  $\rho_n$ . This construction induces an isomorphism between  $H^1(\pi_1(M^3); \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n})$  and the Zariski tangent space of  $X(\pi_1(M^3), \text{SL}(n, \mathbf{C}))$  at  $\chi_n$ . Weil's construction maps the cocycle  $d \in Z^1(\pi_1(M^3); \mathfrak{sl}(n, \mathbf{C})_{\text{Ad} \circ \rho_n})$  to the first order deformation of the representation  $\rho_n$

$$\rho_{n,\varepsilon}(\gamma) = (\text{Id} + \varepsilon d(\gamma))\rho_n(\gamma), \quad \forall \gamma \in \pi_1(M^3, p). \quad (4.2)$$

Since  $d(\gamma_1\gamma_2) = d(\gamma_1) + \text{Ad}_{\rho_n(\gamma_1)} d(\gamma_2)$ ,  $\rho_{n,\varepsilon}$  is a first order deformation, namely:

$$\rho_{n,\varepsilon}(\gamma_1\gamma_2) = \rho_{n,\varepsilon}(\gamma_1)\rho_{n,\varepsilon}(\gamma_2) + O(\varepsilon^2), \quad \forall \gamma_1, \gamma_2 \in \pi_1(M^3, p).$$

For an elementary symmetric polynomial  $\sigma_i$  and an element  $\gamma \in \pi_1(M^3)$ , recall that we have a function

$$\begin{aligned} \text{hom}(\pi_1(M^3), \text{SL}(n, \mathbf{C})) &\rightarrow \mathbf{C} \\ \rho &\mapsto \sigma_i(\rho(\gamma)) \end{aligned}$$

that induces a polynomial map on the character variety

$$\sigma_i^{\gamma}: X(M^3, \text{SL}(n, \mathbf{C})) \rightarrow \mathbf{C}.$$

For  $\vartheta \in \Omega^1(M^3; E_{\text{Ad} \circ \rho_n})$ , the derivative of  $\sigma_i^{\gamma}$  with respect to the direction of the cohomology class of  $\vartheta$  is

$$\lim_{\varepsilon \rightarrow 0} \frac{\sigma_i((\text{Id} + \varepsilon d_{\vartheta}(\gamma))\rho_n(\gamma)) - \sigma_i(\rho_n(\gamma))}{\varepsilon}, \quad (4.3)$$

where  $d_{\vartheta}$  is as in (4.1).

*Fix  $\gamma \in \pi_1(U)$  a nontrivial peripheral element.* We may assume that the lift of its holonomy is

$$\pm \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

*Remark 4.1.* In order to simplify notation, we shall only deal with the case when the lift is  $+\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ . When it is  $-\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , the argument is completely similar.

We want to compute the derivatives of  $\sigma_i^\gamma$  with respect to the forms of Lemma 3.4.

**Lemma 4.2.** *For a peripheral element  $\gamma \in \pi_1(U)$ , the derivative of  $\sigma_i^\gamma$  with respect to  $dx \otimes h_+^j$  is zero.*

*Proof.* When  $\vartheta = dx \otimes h_+^j$ ,  $d_\vartheta(\gamma) = h_+^j$ , by (4.1). Therefore  $\rho_{n,\varepsilon}(\gamma) = (\text{Id} + \varepsilon h_+^j)\rho_n(\gamma)$  is upper triangular with 1 on the diagonal. In particular  $\sigma_i(\rho_{n,\varepsilon}(\gamma))$  is independent of  $\varepsilon$ , and we get zero when computing the limit (4.3).  $\square$

Now, to analyze the derivative of the  $\sigma_j^\gamma$  with respect to  $\omega_i$ , the differential forms of Lemma 3.4, we shall look at characteristic polynomials. Let  $P_{i,\varepsilon}^\gamma(\lambda)$  denote the characteristic polynomial of  $(\text{Id} + \varepsilon d_{\omega_i}(\gamma))\rho_n(\gamma)$ :

$$P_{i,\varepsilon}^\gamma(\lambda) = \det(\lambda \text{Id} - [\text{Id} + \varepsilon d_{\omega_i}(\gamma)]\rho_n(\gamma)).$$

We write

$$P_{i,\varepsilon}^\gamma(\lambda) = (\lambda - 1)^n + \varepsilon Q_i^\gamma(\lambda) + O(\varepsilon^2),$$

for some polynomial  $Q_i^\gamma(\lambda) \in \mathbf{C}[\lambda]$ . The role of the  $Q_i^\gamma(\lambda)$  comes from the following lemma, whose proof is a consequence of (4.3).

**Lemma 4.3.** *For  $i, j = 1, \dots, n - 1$ , the  $\lambda^{n-j}$ -coefficient of  $Q_i^\gamma(\lambda)$  is the derivative of  $(-1)^j \sigma_j^\gamma$  with respect to  $\omega_i$ .*

To compute  $Q_i^\gamma(\lambda)$  we set the following notation:

$$A = \lambda \text{Id} - \rho_n(\gamma) \quad \text{and} \quad X_i = d_{\omega_i}(\gamma)\rho_n(\gamma),$$

so that

$$P_{i,\varepsilon}^\gamma(\lambda) = \det(A + \varepsilon X_i) = \det(A) \det(\text{Id} + \varepsilon A^{-1} X_i).$$

As the derivative of the determinant at the identity is the trace:

$$Q_i^\gamma(\lambda) = \det(A) \text{trace}(A^{-1} X_i) = (\lambda - 1)^n \text{trace}(A^{-1} X_i). \quad (4.4)$$

With this formula we may prove:

**Proposition 4.4.** *For  $\gamma \in \pi_1(U)$  nontrivial and for  $i = 1, \dots, n - 1$  the following assertions hold:*

$$(1) Q_i^\gamma(0) = 0.$$

$$(2) Q_i^\gamma(\lambda) \text{ is a multiple of } (\lambda - 1)^{n-i-1} \text{ but not of } (\lambda - 1)^{n-i}.$$

*Proof.* At  $\lambda = 0$  we have,  $P_{i,\varepsilon}^\gamma(0) = (-1)^n + O(\varepsilon^2)$ . Indeed the trace of the matrix  $d_{\omega_i}(\gamma)$  is zero, and hence

$$\det(\text{Id} + \varepsilon d_{\omega_i}(\gamma)) = 1 + O(\varepsilon^2).$$

This proves the first assertion. In order to prove the second assertion we use (4.4). To compute  $A^{-1}$ , as we assume that the holonomy of  $\gamma$  is  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ , using (2.3) we write

$$N = \rho_n(\gamma) - \text{Id} = \sum_{j=1}^{n-1} \frac{1}{j!} h_+^j, \quad (4.5)$$

so that

$$A = (\lambda - 1) \text{Id} - N = (\lambda - 1) \left( \text{Id} - (\lambda - 1)^{-1} N \right).$$

As  $N^n = 0$ , the inverse of  $A$  is

$$A^{-1} = (\lambda - 1)^{-1} \sum_{k=0}^{n-1} (\lambda - 1)^{-k} N^k,$$

and (4.4) becomes:

$$Q_i^\gamma(\lambda) = \sum_{k=0}^{n-1} (\lambda - 1)^{n-k-1} \text{trace}(N^k X_i). \quad (4.6)$$

On the other hand, by construction of  $h_-$  (2.2),

$$h_-^i = \begin{pmatrix} 0 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & & \vdots \\ a_{i+1,1} & 0 & & & & 0 \\ 0 & a_{i+2,2} & & & & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & a_{n,n-i} & \cdots & 0 \end{pmatrix},$$

with  $a_{i+1,1}, a_{i+2,2}, \dots, a_{n,n-i} > 0$ . In addition, since  $X^i$  is obtained from  $h_-^i$  by multiplication by upper triangular matrices that have 1 in the

diagonal (see (4.1) and Lemma 3.4),  $X_i$  has the same bottom left  $(n - i)$ -triangular corner as  $h_-^i$ :

$$X_i = \begin{pmatrix} * & * & \cdots & * & \cdots & * \\ \vdots & & & & & \vdots \\ a_{i+1,1} & * & & & & * \\ 0 & a_{i+2,2} & & & & * \\ \vdots & & \ddots & & & \vdots \\ 0 & 0 & \cdots & a_{n,n-i} & \cdots & * \end{pmatrix}, \quad (4.7)$$

with  $a_{i+1,1}, a_{i+2,2}, \dots, a_{n,n-i} > 0$ . In addition, by (4.5)

$$N^k = \begin{pmatrix} 0 & \cdots & b_{1,k+1} & * & \cdots & * \\ 0 & & 0 & b_{2,k+1} & & * \\ \vdots & & & & \ddots & \vdots \\ 0 & & & & & b_{n-k,n} \\ \vdots & & & & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad (4.8)$$

with  $b_{1,k+1} = a_{n-k,n} > 0, \dots, b_{n-k,n} = a_{k+1,1} > 0$ . Using the description of  $N^k$  and  $X_i$ :

$$\text{trace}(N^k X_i) = \begin{cases} 0, & \text{for } k \geq i + 1; \\ \text{trace}(h_+^i h_-^i) > 0, & \text{for } k = i. \end{cases}$$

The second assertion follows from this computation and (4.6).  $\square$

**Corollary 4.5.** *The polynomials  $Q_1^\gamma(\lambda), \dots, Q_{n-1}^\gamma(\lambda)$  form a  $\mathbf{C}$ -basis for the subspace of polynomials in  $\mathbf{C}[\lambda]$  of degree  $\leq n - 1$  that are multiples of  $\lambda$ .*

*Proof.* By the first assertion of Proposition 4.4, it is enough to prove that the polynomials  $Q_i^\gamma(\lambda)$  are linearly independent. Assume that for some  $\alpha_1, \dots, \alpha_{n-1} \in \mathbf{C}$

$$\sum_{i=1}^{n-1} \alpha_i Q_i^\gamma(\lambda) = 0.$$

The second assertion of Proposition 4.4 implies that reduction modulo  $\lambda - 1$  yields  $\alpha_{n-1} = 0$ , reduction modulo  $(\lambda - 1)^2$  yields  $\alpha_{n-2} = 0$ , and so on. Thus the above linear combination must be trivial, as we wanted to prove.  $\square$

*Proof of Theorem 1.1.* By Corollary 4.5 and Lemma 4.3, the  $(n-1) \times (n-1)$  matrix whose  $(i, j)$ -entry is the derivative of  $(-1)^j \sigma_j^\gamma$  with respect to  $\omega_i$  has nonzero determinant. Combining this with Lemma 4.2 and Proposition 3.6, it follows that the differential forms  $d\sigma_1^\gamma, \dots, d\sigma_{n-1}^\gamma$  form a basis for the cotangent space of  $X(M^3, SL(n, \mathbf{C}))$  at  $\chi_n$ . Hence Theorem 1.1 follows from the holomorphic implicit function theorem.  $\square$

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