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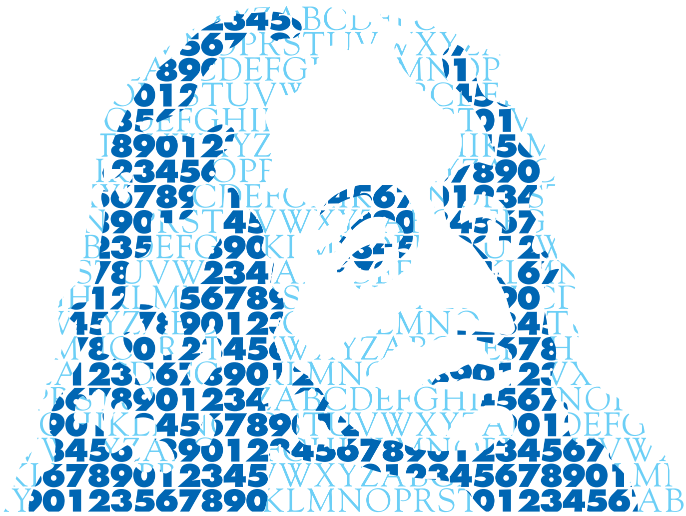

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Optimal boundedness of central oscillating multipliers on compact Lie groups

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Abstract

Fefferman-Stein, Wainger and Sjölin proved optimal H^p boundedness for certain oscillating multipliers on $\mathbf{R}^d$. In this article, we prove an analogue of their result on a compact Lie group.

1. Introduction

Let G be a connected, simply connected, compact semisimple Lie group of dimension n. In this paper, we will study the $H^p(G)$ boundedness for the oscillating multiplier operator

$$T_{\gamma,\beta}(f)(x) = K_{\gamma,\beta} * f(x),\ \gamma \geq 0 \text{ and } 0 < \beta < 1.$$

Here $K_{\gamma,\beta}$ is a central kernel defined by

$$K_{\Omega,\gamma,\beta}(y) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|^{\beta}}}{\| \lambda+\delta \|^{\gamma}} d_\lambda \chi_\lambda(\xi),$$

where y is conjugate to the element $\exp\xi$ in a fixed maximal torus of G (the detailed definition can be found in the second section). Thus the operator $T_{\gamma,\beta}$ has the Fourier expansion

$$T_{\gamma,\beta}(f)(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|^{\beta}}}{\| \lambda+\delta \|^{\gamma}} d_\lambda \chi_\lambda * f(x)$$

for any $f \in C^\infty(G)$.

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The formulation of $T_{\gamma,\beta}$ is an analogue of the oscillating multiplier operator $S_{\gamma,\beta}(f)$ on $\mathbf{R}^d$:

$$S_{\alpha,\beta}(f)(x) = \int_{\mathbf{R}^d} \widehat{f}(\xi)\Psi(|\xi|)m(\xi)e^{i\langle x,\xi\rangle}d\xi,$$

where

$$m(\xi) = \frac{e^{i|\xi|^\beta}}{|\xi|^\gamma}$$

and $\Psi(|\xi|) \in C^\infty(\mathbf{R}^d)$, satisfying $\Psi(|\xi|) = 0$ for $|\xi| < 1/2$ and $\Psi(|\xi|) = 1$ for $|\xi| > 1$.

It is well known that the operator $S_{\gamma,\beta}$ is bounded on $H^p(\mathbf{R}^d)$ if and only if $|1/2 - 1/p| \leq \gamma/(d\beta)$ for all $0 < p < \infty$ (see [14, 19, 20, 22]). We notice that when $1 < p < \infty$, the boundedness of $S_{\gamma,\beta}$ has been generalized to many different settings of Lie groups and manifolds (see [1, 5, 15, 18]). In a recent paper [5], we established the following optimal $L^p(G)$ boundedness of $T_{\gamma,\beta}$ on a compact Lie group.

Theorem 1.1. *Let G be a connected, simply connected, compact semisimple Lie group of dimension n. For $0 < \beta < 1$, the operator $T_{\gamma,\beta}$ is bounded on $L^p(G)$ if and only if $|\frac{1}{2} - \frac{1}{p}| \leq \frac{\gamma}{n\beta}$ for all $1 < p < \infty$.*

In [5], we are able to extend Theorem 1.1 to $H^p(G)$ for $0 < p_0 < p \leq 1$. However, the method in [5] only allows us to obtain the result when p_0 is close to 1. Thus, the aim of this paper is to give a complete solution of Theorem 1.1 by establishing the following optimal H^p boundedness for all $p > 0$.

Theorem 1.2. *Let G be a connected, simply connected, compact semisimple Lie group of dimension n and $0 < \beta < 1$. The operator $T_{\gamma,\beta}$ is bounded on $H^p(G)$ if and only if $|\frac{1}{2} - \frac{1}{p}| \leq \frac{\gamma}{n\beta}$ for all $0 < p < \infty$.*

We want to point out that the extension of Theorem 1.1 to all $0 < p < \infty$ is not trivial and actually it is quite involved, due to the structure of a semi-simple Lie group. Our proof will use powerful results of Clerc [8], in which the Weyl denominator and its derivatives are carefully estimated based on the classification of the root system. This allows us to obtain sharp estimates on the kernel and its derivatives. This same method was recently also used in [6] to study the wave problem ($\beta = 1$).

In order to prove the main theorem, we use a standard analytic interpolation argument (see [2]). Define an analytic family of operators

$$T_{z,\beta}(f)(x) = \sum_{\lambda+\delta\in\Lambda\setminus\{0\}} \frac{e^{i\|\lambda+\delta\|^{\beta}}}{\| \lambda+\delta \|^{z}} d_\lambda \chi_\lambda * f(x), \quad z \in \mathbf{C}.$$

By the Plancherel theorem, it is easy to see that

$$\|T_{z,\beta}(f)\|_{L^2(G)} \preceq \|f\|_{L^2(G)}$$

if $\Re(z) = 0$. Thus, to complete the proof of the sufficiency part of Theorem 1.2, by the analytic interpolation theorem it suffices to show the following endpoint estimate.

Proposition 1.3. *Let* $p < \frac{2\beta}{\beta+2}$ *and* $\gamma_p = n\beta(\frac{1}{p} - \frac{1}{2})$*. If* $\Re(z) = \gamma_p$*, then*

$$\|T_{z,\beta}(f)\|_{H^p(G)} \preceq \|f\|_{H^p(G)}\,.$$

The plan of this paper is as follows: in Section 2, we will recall some necessary notation and known results on a compact Lie group; the kernel $K_{z,\beta}$ and its derivatives will be carefully estimated in Section 3; we will prove Proposition 1.3 in Section 4.

In this paper, we use the notation $A \preceq B$ to mean that there is a positive constant C independent of all essential variables such that $A \leq CB$. We use the notation $A \simeq B$ to mean that there are two positive constants c_1 and c_2 independent of all essential variables such that $c_1 A \leq B \leq c_2 A$.

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2. Notation and known results

2.1. Some definitions

Let G be a connected, simply connected, compact semisimple Lie group of dimension$n$. Let $\mathbf{g}$ be the Lie algebra of G and τ the Lie algebra of a fixed maximal torus T in G of dimension m. Let A be a system of positive roots for $(\mathbf{g}, \tau)$ so $Card(A) = \frac{n-m}{2}$, and let $\delta = \frac{1}{2}\sum_{a\in A} a$.

Let $|\ .\ |$ be the norm of $\mathbf{g}$ induced by the negative of the Killing form B on $\mathbf{g}^{\mathbf{C}}$, the complexification of $\mathbf{g}$. The norm $|\ .\ |$ induces a bi-invariant metric d on G. Furthermore, since $B|_{\tau^{\mathbf{C}}\times\tau^{\mathbf{C}}}$ is nondegenerate, given $\lambda \in$

$\hom_{\mathbf{C}}(\tau^{\mathbf{C}}, \mathbf{C})$, there is a unique H_λ in $\tau^{\mathbf{C}}$ such that $\lambda(H) = B(H, H_\lambda)$ for each $H \in \tau^{\mathbf{C}}$. We let $\langle . , . \rangle$ and $\| \, . \, \|$ denote the inner product and norm transferred from τ to $\hom_{\mathbf{C}}(\tau, i\mathbf{R})$ by means of this canonical isomorphism.

Let $\mathbf{N} = \{H \in \tau : \exp H = I\}$, where I is the identity in G. The weight lattice P is defined by $P = \{\lambda \in \tau : \langle \lambda, n \rangle \in 2\pi\mathbf{Z}$ for any $n \in \mathbf{N}\}$ with dominant weights defined by $\Lambda = \{\lambda \in P : \langle \lambda, a \rangle \geq 0$ for any $a \in A\}$. Λ provides a full set of parameters for the equivalence classes of unitary irreducible representation of G: for $\lambda \in \Lambda$, the representation U_λ has dimension

$$d_\lambda = \prod_{a \in A} \frac{\langle \lambda + \delta, a \rangle}{\langle \delta, a \rangle}$$

and its associated character is

$$\chi_\lambda(\xi) = \frac{\sum_{w \in W} \epsilon(w) \; e^{i \langle w(\lambda + \delta), \xi \rangle}}{\sum_{w \in W} e^{i \langle w\delta, \xi \rangle}}$$

where $\xi \in \tau$, W is the Weyl group, and $\epsilon(w)$ is the signature of $w \in W$. Any function $f \in L^1(G)$ has the Fourier series

$$\sum_{\lambda \in \Lambda} d_\lambda \chi_\lambda * f(x).$$

The oscillating multiplier

$$T_{z,\beta}(f)(x) = \sum_{\lambda + \delta \in \Lambda \setminus \{0\}} \frac{e^{i \| \lambda + \delta \|^\beta}}{\| \lambda + \delta \|^z} d_\lambda \chi_\lambda * f(x)$$

is initially defined on all $f \in C^\infty$. Thus $T_{z,\beta}$ is a convolution operator

$$T_{z,\beta}(f)(x) = K_{z,\beta} * f(x),$$

where $K_{z,\beta}$ is a central kernel defined by

$$K_{z,\beta}(y) = \sum_{\lambda + \delta \in \Lambda \setminus \{0\}} \frac{e^{i \| \lambda + \delta \|^\beta}}{\| \lambda + \delta \|^z} d_\lambda \chi_\lambda(\xi),$$

and $\exp \xi \in T$ is conjugate to y. Let Q be a fixed fundamental domain of T and

$$Q_\nu = \{\xi + \nu : \; \xi \in Q\}, \;\; \nu \in \mathbf{N}.$$

Up to sets of measure zero, $\{Q_\nu\}$ is a sequence of mutually disjoint subsets in the Lie algebra τ.

Any element $y \in G$ is conjugate to exactly one element in $\exp(\overline{Q})$. We denote $y \sim \exp \xi$ if y is conjugate to $\exp \xi$.

2.2. **Hardy spaces** $H^p(G)$

There are many equivalent definitions of the Hardy space H^p. The reader can see [3, 4, 7, 8, 9, 10, 11, 13, 16] and the references therein for more details and background on the Hardy space. Below, we briefly review the definition of H^p by using the heat kernel, and the atomic characterization of Hardy spaces.

For the heat kernel (see [21])

$$W_t = \sum_{\lambda+\delta\in\Lambda} e^{-t\{\|\lambda+\delta\|^2-\|\delta\|^2\}} d_\lambda \chi_\lambda,$$

the Hardy space $H^p(G)$, $0 < p < \infty$, is the collection of all distributions $f \in S'(G)$ such that

$$\|f\|_{H^p(G)} = \|\sup_{t>0}\{|W_t * f|\}\|_{L^p(G)} < \infty.$$

An exceptional atom is an L^∞ function bounded by 1. In order to define a regular atom, one considers a faithful unitary representation Π of G such that $\Pi(G) \approx U(L, \mathbf{C})$. Then G can be identified as a submanifold in a real vector space V underlying $\mathrm{End}(\mathbf{C}^L)$. A regular p-atom for $0 < p \leq 1$ is a function $a(x)$ supported in a ball $B(x_0, \rho)$ such that

$$\|a\|_{L^2(G)} \leq \rho^{-n(\frac{1}{p}-\frac{1}{2})}, \quad \int_G a(x)\wp(\Pi(x))dx = 0,$$

where $\wp$ is any polynomial on V of degree less than or equal to ℓ for any integer $\ell \leq [n(\frac{1}{p} - 1)]$, the integer part of $n(\frac{1}{p} - 1)$.

The space $H^p_a(G)$, $0 < p \leq 1$, is the space of all $f \in S'(G)$ having the form

$$f = \sum c_k a_k \text{ with } \sum |c_k|^p < \infty,$$

where each $a(x)$ is either a regular p-atom, or an exceptional atom. The "norm" $\|f\|_{H^p_a}$ is the infimum of all expressions $(\sum |c_k|^p)^{1/p}$ for which we have a representation $f = \sum c_k a_k$. As we discussed in [5] (see also [3], [4]), to show that the operator $T_{z,\beta}$ is bounded on $H^p(G)$, it suffices to prove

$$\|T_{z,\beta}(a)\|_{L^p(G)} \preceq 1$$

uniformly for any regular p-atom $a(x)$ with support in $B(I,\rho)$, where $0<\rho<r$ and r is a fixed sufficiently small number.

2.3. **Decomposing the Weyl denominator**

Denote

$$D(\theta)=\sum_{w\in W} e^{i<w\delta,\theta>}.$$

$D(\theta)$ is called the Weyl denominator and it satisfies

$$D(\theta)=\prod_{\alpha\in A}\sin\frac{\alpha(\theta)}{2},$$

where

$$\alpha(\theta)=\langle\alpha,\theta\rangle.$$

In this section, we introduce some notation in [8]. Let B_s be the set of all simple roots in A and let B_L be the set of all largest roots. For $\theta\in Q$, introduce the following sets

$$I=I_\theta=\{\alpha\in B_s:\alpha(\theta)\le\frac{1}{R}\}$$

$$J=J_\theta=\{\beta\in B_L:\beta(\theta)\ge 2\pi-\frac{1}{R}\}.$$

When R is a large number, elements in I_θ and J_θ are independent. We define two facets

$$F_{I,J}=\{\xi\in\overline{Q}:\alpha(\xi)=0 \text{ for } \alpha\in I,\ \beta(\xi)=2\pi \text{ for } \beta\in J,$$

$$0<\alpha(\xi)<2\pi \text{ for } \alpha\in B_s\backslash I,\ 0<\beta(\xi)<2\pi \text{ for } \beta\in B_L\backslash J\}$$

and let $\mathcal{F}_{I,J}$ be the affine subspace generated by $F_{I,J}$ so that

$$\mathcal{F}_{I,J}=\{\xi\in\tau:\alpha(\xi)=0 \text{ for } \alpha\in I \text{ and } \beta(\xi)=2\pi \text{ for } \beta\in J\}.$$

A positive root γ is $R-$singular of type I at θ, if the following equivalent conditions are satisfied:

(i) $\gamma\{\mathcal{F}_{I,J}\}=\{0\}$,

(ii) $\gamma\{F_{I,J}\}=\{0\}$,

(iii) γ can be written as

$$\gamma=\sum_{\alpha\in I} n_\alpha\alpha,\ n_\alpha\in\mathbf{N}.$$

A positive root γ is R-singular of type II at θ, if the following equivalent conditions are satisfied:
(i) $\gamma\{\mathcal{F}_{I,J}\} = \{2\pi\}$,
(ii) $\gamma\{F_{I,J}\} = \{2\pi\}$,
(iii) γ can be written as, for some $\beta \in J$

$$\gamma = \beta - \sum_{\alpha\in I} n_\alpha \alpha,\ n_\alpha \in \mathbf{N}.$$

Both $R-$singular roots of types I and II are called singular roots. By this definition, it is easy to see that if γ is a positive non-singular root, then

$$\frac{1}{R} < \gamma(\theta) < 2\pi - \frac{1}{R}.$$

For a singular root α of type I, let S_α be the orthogonal symmetry with respect to the hyperplane $\alpha = 0$. For a singular root β of type II, let $\widetilde{S}_\beta$ be the orthogonal symmetry with respect to the hyperplane $\alpha = 2\pi$. Let $W_{I,J}$ be the group generated by

$$\{S_\alpha\}_{\alpha\in I} \cup \{\widetilde{S}_\beta\}_{\beta\in J}.$$

This group is a finite subgroup of the affine Weyl group. Now we define

$$\Gamma_\theta = \Gamma_\theta^{(R)} = \text{convex hull of } \{w\theta\}_{w\in W_{I,J}}.$$

Also, we write the root system

$$A = A_s \cup A_{ns}$$

where A_s is the set of all singular ($R-$singular) roots and A_{ns} is the set of all non-singular roots. Denote by μ^R the number of singular roots and denote

$$D^R(\theta) = \prod_{a\in A_{ns}} \sin(\frac{\langle\alpha,\theta\rangle}{2}).$$

Thus

$$D(\theta) = D^R(\theta) \prod_{a\in A_s} \sin(\frac{\langle\alpha,\theta\rangle}{2}).$$

The above definition of $D(\theta)$ and $\Gamma_\theta^{(R)}$ can be defined on the torus T itself. In fact, if $x \in T$, then $x = \exp\theta$ for some θ in $\overline{Q}$. We define

$$d(\exp\theta) = D(\theta),\ \text{and } \Gamma_x^{(R)} = \exp\Gamma_\theta^{(R)}.$$

Some properties of the domain $\Gamma_x^{(R)}$ can be found in [8]. In particular, it is known from Lemma 2.9 in [8] that there exists a constant c such that

$$\left|D^R(\xi)\right| \geq c\left|D^R(\theta)\right| \tag{2.1}$$

for all ξ in $\Gamma_\theta^{(R)}$.

2.4. **Derivatives on central functions.**

Fixing a vector basis of $\mathbf{g^C}$, say $Y_1, Y_2, \ldots, Y_n$, we denote the element $Y_1^{j_1} Y_2^{j_2} \cdots Y_n^{j_n}$ by Y^J, where $J = (j_1, \ldots, j_n)$. As J varies over all possible $n-$tuples, the $\{Y^J\}$ forms a basis of the complex universal enveloping algebra $U(\mathbf{g})$ of $\mathbf{g}$. Similarly, we fix a basis $\Theta_1, \ldots, \Theta_m$ of $\tau^{\mathbf{C}}$ and use the notation Θ^I for $\Theta_1^{i_1}\Theta_2^{i_2}\cdots\Theta_m^{i_m}$, with $I = (i_1, \ldots, i_m)$. We can find the following two theorems in [8].

Theorem 2.1. *Let p, q be positive integers. Assume that f is a C^∞ central function. Then for each I with $|I| \leq p$ and J with $|J| \leq q$, there exists a constant C such that*

$$\left|\Theta^I Y^J f(y)\right| \preceq C \sum_{j=0}^{p+q} R^j \sum_{|K| \leq p+q-j} \sup_{v \in \Gamma_y^{(R)}} \left|\Theta^K f(v)\right|.$$

Theorem 2.2. *Let p be a positive integer. Assume $f(y) = d(y)^{-1} g(y)$ and that g is a C^∞ central function which is skew-invariant by the Weyl group. For each I with $|I| \leq p$, there exists a constant C such that*

$$\left|\Theta^I f(y)\right| \preceq C \left|d^R(y)\right|^{-1} \sum_{j=0}^{p} R^j \sum_{|K| \leq p+\mu^R-j} \sup_{v \in \Gamma_y^{(R)}} \left|\Theta^K g(v)\right|.$$

3. **Estimates on the kernel** $K_{z,\beta}$**,** $\Re(z) > n$

In this section, we denote $\gamma = \Re(z)$ and assume $\gamma > n$. First, we recall the following lemma, which is an easy modification of results in [12] or [17].

Lemma 3.1. *Let $A \subseteq \mathbf{R}^m$ denote an open set and $\Phi \in C_0^\infty(A)$. If $\Psi \in C^\infty(A)$ satisfies*

$$|\det(\partial^2/\partial x_i \partial x_j \Psi(x))| \geq C > 0$$

for all $x \in \operatorname{supp}(\Phi)$, *then for large* $|\lambda|$,

$$\Big|\int_A e^{i(\lambda\Psi(x)+\langle\xi,x\rangle)}\Phi(x)dx\Big| \preceq |\lambda|^{-m/2}\|\Phi\|_{C^{2m}}.$$

By the definition, it is easy to see

$$K_{z,\beta}(y) = \frac{\sum\limits_{\lambda+\delta\in\Lambda\backslash\{0\}} \frac{e^{i\|\lambda+\delta\|^\beta}}{\|\lambda+\delta\|^z}(\prod\limits_{a\in A}\langle\delta+\lambda,a\rangle)\sum_{w\in W}\epsilon(w)\ e^{i\langle w(\lambda+\delta),\xi\rangle}}{D(\xi)\prod\limits_{\alpha\in A}\langle\alpha,\delta\rangle}$$

$$= \frac{(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\sum\limits_{\lambda\in P\backslash\{0\}}\frac{e^{i\|\lambda\|^\beta}}{\|\lambda\|^z}\ e^{i\langle\lambda,\xi\rangle}}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle},$$

where y is conjugate to $\exp\xi \in T$.

Choose a C^∞ radial function ψ on $\mathbf{R}^m$ such that

$$\psi(t) = 0 \text{ if } t\in(0,c_1) \text{ and } \psi(t) = 1 \text{ if } t \in (c_2,\infty),$$

where c_1 and c_2 are two fixed positive numbers such that

$$K_{z,\beta}(y) = \frac{(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\sum\limits_{\mu\in P}\frac{e^{i\|\mu\|^\beta}}{\|\mu\|^z}\psi(\|\mu\|)\,e^{i\langle\mu,\xi\rangle}}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle}.$$

Since $\Re(z)=\gamma>n$, the above summation is absolutely convergent. By the Poisson summation formula (see [11]) we obtain

$$K_{z,\beta}(y) \simeq \frac{\sum\limits_{\nu\in\mathbf{N}}(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\Re_\nu(\xi)}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle},$$

where

$$\Re_0(\xi) = \int_{\mathbf{R}^m}\frac{e^{i\|H\|^\beta}}{\|H\|^z}\psi(\|H\|)e^{-i\langle\xi,H\rangle}dH,$$

and

$$\Re_\nu(\xi) = \Re_0(\xi+\nu).$$

For simplicity, we write

$$\Re(\xi) = \Re_0(\xi) \text{ and } \Theta(H') = (\prod_{a\in A}\langle\alpha,\frac{H}{\|H\|}\rangle).$$

Also, without loss of generality, we may use the Euclidean norm $|\ .\ |$ instead of $\|\ .\ \|$. Thus, for $\nu \in \mathbf{N}$,

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})\Re_\nu(\xi) = (\prod_{a\in A}\frac{\partial}{\partial\alpha})\Re(\xi+\nu) \simeq \int_{\mathbf{R}^m}\frac{e^{i\Phi(H,\xi+\nu)}}{|H|^{z-\frac{n-m}{2}}}\Theta(H')\psi(\,|H|\,)dH,$$

where

$$\Phi(H,\xi+\nu) = |H|^\beta - \langle\xi+\nu, H\rangle$$

is the phase function.

Fix a small $\rho > 0$. Let $\varphi(t)$ be a C^∞ function on $(0,\infty)$ that satisfies

$$\varphi(t) = 0 \text{ if } t < 0.05\ ,\ \varphi(t) = 1 \text{ if } t > 0.1,$$

and let

$$\varphi_\infty(t) = \varphi(\rho t),\ \ \varphi_0(t) = 1 - \varphi(\rho t).$$

For each $\nu \in \mathbf{N}$, we write

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})\Re_\nu(\xi) = (\prod_{a\in A}\frac{\partial}{\partial\alpha})\ \Re_{\nu,0}(\xi) + (\prod_{a\in A}\frac{\partial}{\partial\alpha})\ \Re_{\nu,\infty}(\xi),$$

where

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})\ \Re_{\nu,0}\,(\xi) = \int_{\mathbf{R}^m}\frac{e^{i\Phi(H,\xi+\nu)}}{|H|^{z-\frac{n-m}{2}}}\Theta(H')\psi(\,|H|\,)\varphi_0(|H|)dH$$

and

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})\ \Re_{\nu,\infty}(\xi) = \int_{\mathbf{R}^m}\frac{e^{i\Phi(H,\xi+\nu)}}{|H|^{z-\frac{n-m}{2}}}\Theta(y')\psi(|H|)\varphi_\infty(|H|)dH.$$

Furthermore, for $y \sim \exp\xi$, we define four central kernels:

$$\Im_{\infty,0}(y) = \frac{\sum\limits_{\nu\in\mathbf{N}\backslash\{0\}}(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\Re_{\nu,0}(\xi)}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle},\ \ \Im_{\infty,\infty}(y) = \frac{\sum\limits_{\nu\in\mathbf{N}\backslash\{0\}}(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\Re_{\nu,\infty}(\xi)}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle},$$

and

$$\Im_{0,0}(y) = \frac{(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\Re_{0,0}(\xi)}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle},\ \ \Im_{0,\infty}(y) = \frac{(\prod\limits_{a\in A}\frac{\partial}{\partial\alpha})\Re_{0,\infty}(\xi)}{D(\xi)\prod\limits_{a\in A}\langle a,\delta\rangle}.$$

Then, we decompose the kernel $K_{z,\beta}$ by

$$K_{z,\beta}(y) \simeq \Im_{0,\infty}(\xi) + \Im_{\infty,\infty}(\xi) + \Im_{0,0}(\xi) + \Im_{\infty,0}(\xi).$$

We will give different estimates on these kernels.

Lemma 3.2. *Fix a small positive ρ such that $\rho \leq 1/c_1$. Let $\xi \in Q_\nu \cap \{\xi : |\xi| > 1000n\rho^{1-\beta}\}$. For any positive integer L and $N = \gamma - \frac{n+m}{2} + L$, we have*

$$|\Im_{0,\infty}(\xi)| \preceq \frac{|\xi|^{-L}\rho^N}{| D(\xi) \prod_{a\in A} \langle a, \delta\rangle |}, \tag{3.1}$$

and

$$\|\Im_{\infty,\infty}\|_{L^1(G)} \preceq \rho^N.$$

Proof. For $|\xi| \geq 1000n\rho^{1-\beta}$, there is at least one coordinate ξ_j satisfying

$$|\xi_j| \geq 1000\rho^{1-\beta}.$$

On the other hand, if H lies on the support of φ_∞ we have

$$|H|^{\beta-1} \leq 100\rho^{1-\beta}.$$

Thus we have

$$\left|\frac{\partial}{\partial y_j}\Phi(H,\xi)\right| \geq |\xi_j| - \beta \mid H \mid^{\beta-1} \succeq |\xi_j| \simeq |\xi|\,.$$

By this observation, we perform integration by parts on H_j-variable to obtain

$$\left|\int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{\mid H\mid^{z-\frac{n-m}{2}}}\Theta(H')\varphi_\infty(\mid H \mid)dH\right| \preceq |\xi|^{-L}\rho^{\gamma-\frac{n+m}{2}+L}.$$

This shows (3.1). As a consequence, we obtain

$$\left|\int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi+\nu)}}{\mid H\mid^{z-\frac{n-m}{2}}}\Theta(H')\varphi_\infty(\mid H \mid)dH\right| \preceq |\xi+\nu|^{-L}\rho^{\gamma-\frac{n+m}{2}+L}.$$

Thus, by the Weyl integral formula we have

$$\|\Im_{\infty,\infty}\|_{L^1(G)} \preceq \sum_{\nu\in\mathbf{N}\setminus\{0\}} \rho^{\gamma-\frac{n+m}{2}+L}\int_Q |\xi+\nu|^{-L}|D(\xi)|\,d\xi \preceq \rho^N$$

after choosing a suitably large L and letting $\gamma - \frac{n+m}{2} + L = N$. □

Lemma 3.3. *Assume $|\xi| > 1000nc_1^{\beta-1}$. We have, for any multi-index J and any positive integer L,*

$$\left|(\frac{\partial}{\partial\xi})^J\int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{\mid H\mid^{z-\frac{n-m}{2}}}\Theta(H')\varphi_0(\mid H \mid)\psi(\,|H|\,)dH\right| \preceq (1+|\xi|)^{-L}.$$

Proof. It is easy to see that

$$(\frac{\partial}{\partial\xi})^J \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|H|^{z-\frac{n-m}{2}}} \Theta(H')\varphi_0(\|H\|)\psi(\,|H|\,)dH$$

$$= \sum_{k=1}^{|J|} \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|H|^{z-\frac{n-m}{2}}} P_k(H)\Theta(H')\varphi_0(|H|)\psi(\,|H|\,)dH = \sum_{k=1}^{|J|} \mathcal{L}_k(\xi),$$

where P_k is a homogeneous polynomial of degree k. Without loss of generality, we may write, for each k,

$$\mathcal{L}_k(\xi) = \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|\,y\,|^{z-\frac{n-m}{2}-k}} \Theta(H')\psi(\,|H|\,)\varphi_0(|H|)\,dH.$$

The support condition of ψ implies that the phase function satisfies

$$\sum_{j=1}^{n} \left| \frac{\partial}{\partial H_j}\Phi(H,\xi) \right| \succeq |\xi|\,.$$

Thus the lemma follows by integration by parts N times for a suitably large N. □

Lemma 3.4. *Assume* $1000nc_1^{\beta-1} \geq |\xi| > 1000\rho^{1-\beta}$. *For any multi-index* J, *if*

$$\frac{(\alpha - \frac{n}{2} - |J|)}{1-\beta} - \frac{m}{2} < 0,$$

then we have

$$\left| (\frac{\partial}{\partial\xi})^J \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|\,H\,|^{z-\frac{n-m}{2}}} \Theta(H')\varphi_0(|H|)\psi(|H|)\,dH \right| \preceq |\xi|^{\frac{(\gamma-\frac{n}{2}-|J|)}{1-\beta}-\frac{m}{2}}.$$

Proof. As in the previous lemma, we need to show for each k,

$$|\mathcal{L}_k(\xi)| = \left| \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|H|^{z-\frac{n-m}{2}-k}} \Theta(H')\psi(\,|H|\,)\varphi_0(|H|)dH \right|$$

$$\preceq |\,\xi\,|^{\frac{(\gamma-\frac{n}{2}-k)}{1-\beta}-\frac{m}{2}} + 1.$$

Choose C^∞ functions $\Gamma_j(t)$, $j = 1, 2, 3$ such that

$$\Gamma_1(t) = 1 \text{ if } \beta t^{\beta-1} > 3 \text{ and } \Gamma_1(t) = 0 \text{ if } \beta t^{\beta-1} \leq 2,$$

$$\Gamma_2(t) = 1 \text{ if } \beta t^{\beta-1} < \frac{0.1}{n} \text{ and } \Gamma_2(t) = 0 \text{ if } \beta t^{\beta-1} \geq 0.2.$$

Let

$$\Gamma_3(t) = 1 - \Gamma_1(t) - \Gamma_2(t),$$

and

$$\Gamma_j(t,\xi) = \Gamma_j(|\xi|^{\frac{1}{1-\beta}}\, t),\ j = 1,2,3.$$

Therefore,

$$\mathcal{L}_k(\xi) = \sum_{j=1}^{3} \int_{\mathbf{R}^m} \frac{e^{i\Phi(H,\xi)}}{|H|^{z-\frac{n-m}{2}-k}} \Gamma_j(|\xi|^{\frac{1}{1-\beta}}\,|H|)\Theta(H')\psi(|H|)\varphi_0(|H|)\,dH$$

$$= \sum_{j=1}^{3} \mathcal{L}_{k,j}(\xi).$$

By polar coordinates,

$$\mathcal{L}_{k,1}(\xi) \simeq \int_{S^{m-1}} \Theta(H') \times \Big\{ \int_0^\infty \frac{e^{ig(t)}}{t^{z-\frac{n-m}{2}-k-m+1}} \Gamma_1(t\,|\xi|^{\frac{1}{1-\beta}})\psi(t)\varphi_0(t)dt \Big\} d\sigma(H^{'}),$$

where S^{m-1} is the unit sphere in $\mathbf{R}^m$ with the induced Lebesgue measure $d\sigma(H')$, and the phase function

$$g(t) = t^\beta - it\langle \xi, H'\rangle$$

satisfies

$$\left|\frac{d}{dt}g(t)\right| \succeq t^{\beta-1},\ \text{and}\ \frac{d}{dt}\frac{1}{g'(t)} = \frac{\beta(1-\beta)t^{\beta-2}}{g'(t)^2} = O\Big(\frac{1}{t^\beta}\Big),$$

if t lies in the support of $\Gamma_1(t\,|\xi|^{\frac{1}{1-\beta}})$. By this observation, it is easy to see that after using integration by parts N times for a sufficiently large N, we have

$$|\mathcal{L}_{k,1}(\xi)| \preceq 1\,. \tag{3.2}$$

For the term $\mathcal{L}_{k,2}(\xi)$, it is easy to check that, in some direction H_j,

$$\left|\frac{\partial}{\partial H_j}\Phi(H,\xi)\right| \geq |\xi| \succeq |H|$$

if H lies in the support of $\Gamma_2(H\,|\xi|^{\frac{1}{1-\beta}})$. Again, performing integration by parts in the H_j direction for sufficiently many times, we obtain

$$|\mathcal{L}_{k,2}(\xi)| \preceq 1. \tag{3.3}$$

For the term $\mathcal{L}_{k,3}(\xi)$, changing variables we have

$$\mathcal{L}_{k,3}(\xi) \simeq$$

$$\frac{|\xi|^{-i\Im(z)/(\beta-1)}}{|\xi|^{\frac{(\gamma-\frac{n+m}{2}-k)}{\beta-1}}}\int_{\mathbf{R}^m}\frac{e^{i\Psi(H,\xi)}}{\mid H\mid^{z-\frac{n-m}{2}-k}}\Gamma_3(|H|)\Theta(H')(\psi\varphi_0)(|H|\,|\xi|^{\frac{1}{\beta-1}})\,dH,$$

where

$$\Psi(H,\xi) = |\xi|^{\frac{\beta}{\beta-1}}\,|H|^{\beta} - |\xi|^{\frac{1}{\beta-1}}\,\langle \xi', H\rangle.$$

Recalling that the support of Γ_3 lies in the set

$$S = \Big\{ y : (\frac{0.1}{\beta n})^{1-\beta} \le |H| \le (\frac{3}{\beta})^{1-\beta}\Big\},$$

without loss of generality, we may also assume that $|\xi|$ is small such that

$$|H|\,|\xi|^{\frac{1}{\beta-1}} > c_2 \text{ for all } H \in S.$$

We now have

$$\mathcal{L}_{k,3}(\xi) \simeq \frac{|\xi|^{-i\Im(z)/(\beta-1)}}{|\xi|^{\frac{(\gamma-\frac{n+m}{2}-k)}{\beta-1}}}\int_{\mathbf{R}^m} e^{i\Psi(H,\xi)}\widetilde{\Gamma_3}(|H|)\Theta(H')\varphi_0(\rho|H|\,|\xi|^{\frac{1}{\beta-1}})dH,$$

where

$$\widetilde{\Gamma_3}(|H|) = \frac{\Gamma_3(|H|)}{|H|^{z-\frac{n-m}{2}-k}}$$

is a C^∞function supported in the set S. By Lemma 3.1 we have

$$|\mathcal{L}_{k,3}(\xi)| \preceq |\xi|^{\frac{(\gamma-\frac{n+m}{2}-k)}{1-\beta}}|\xi|^{\frac{m\beta}{2(1-\beta)}} = |\xi|^{\frac{(\gamma-\frac{n}{2}-k)}{1-\beta}-\frac{m}{2}}. \tag{3.4}$$

We now obtain the lemma by combining (3.2),(3.3) (3.4). □

Next, we fix a number η such that $d(u,I) \le 2\eta$ implies $\alpha(\varpi) \le \pi$ for all $\alpha \in A$, where $u \sim \exp\varpi$.

Lemma 3.5. *Assume $d(u,I) \le \eta$. For any non-negative integer L and any multi-index M with $|M| = q$, we have*

$$\left|Y^M \Im_{\infty,0}(u)\right| \preceq R^q \mid d^R(u)\mid^{-1}\sum_{\nu\neq 0}|\nu|^{-L}.$$

Proof. By [8], $d(u)\Im_{\infty,0}(u)$ is skew-invariant by the Weyl group. Thus we invoke Theorems 2.1 and 2.2 and (2.1) to obtain

$$\left|Y^M\Im_{\infty,0}(u)\right| \preceq \sum_{|J|\le q}\sup_{x\in\Gamma_u^{(R)}}\left|\Theta^J\ \Im_{\infty,0}(x)\right| \preceq R^q|d^R(u)|^{-1}$$

$$\times\sum_{|J|\le q}\ \sum_{|I|\le|J|+\mu^R}\ \sup_{x\in\Gamma_u^{(R)}}\ \sup_{y\in\Gamma_x^{(R)}}\left|(\frac{\partial}{\partial\xi})^I\sum_{\nu\in\mathbf{N}\setminus\{0\}}(\prod_{a\in A}\frac{\partial}{\partial\alpha})\Re_{\nu,0}(\xi)\right|,$$

where $\exp\xi\sim y$ and

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})\Re_{\nu,0}(\xi)\simeq(\frac{\partial}{\partial\xi})^I\int_{\mathbf{R}^m}\frac{e^{i\Phi(H,\xi+\nu)}}{|\ H\ |^{z-\frac{n-m}{2}}}\Theta(H')\varphi_0(|\ H\ |)\psi(\,|H|\,)dH\,.$$

Observing that there is a $\sigma>0$ such that for all $\xi\in Q$,

$$|\xi+\nu|\ge\sigma \text{ if } \nu\ne 0,$$

and

$$|\xi+\nu|\simeq|\nu| \text{ if } |\nu| \text{ is large},$$

we obtain Lemma 3.5 from Lemma 3.3 and 3.4. □

Now we turn to estimate $Y^I\Im_{0,0}(u)$. We have the following estimate.

Lemma 3.6. *Let the number η be the same as in Lemma 3.5. For any multi-index I with $|I|=q$, if*

$$d(u,I)\le\eta \text{ and } N>-\gamma+\frac{n-1}{2}+q,$$

then we have

$$\left|Y^I\Im_{0,0}(u)\right|\preceq\sup_{y\in\Gamma_u^{(R)}}(|\xi|^{-\frac{(\gamma-\frac{n}{2})-q}{\beta-1}}\ |\ \xi\ |^{-\frac{n}{2}})+|\xi|^{-N-\frac{n+1}{2}}),$$

where $y\sim\exp\xi$.

Proof. By Theorem 2.1, without loss of generality, we may write

$$\left|Y^I\Im_{0,0}(u)\right|\preceq\sum_{|J|=q}\ \sup_{y\in\Gamma_u^{(R)}}\ \left|(\frac{\partial}{\partial\xi})^J\Im_{0,0}(y)\right|$$

By [22, Ch.4] (or see [8]), we have

$$\Re_{0,0}(\xi)\simeq\int_0^\infty e^{it^\beta}\ \psi(t)\varphi_0(t)V_{\frac{m-2}{2}}(t\ |\ \xi\ |)t^{-z+m-1}dt,$$

where

$$V_{\frac{m-2}{2}}(t)=\frac{J_{\frac{m-2}{2}}(t)}{t^{\frac{m-2}{2}}},$$

and $J_{\frac{m-2}{2}}(t)$ is the Bessel function of order $\frac{m-2}{2}$. An easy computation shows (see [5]),

$$(\prod_{a\in A}\frac{\partial}{\partial\alpha})V_{\frac{m-2}{2}}(t\mid\xi\mid)\simeq(\prod_{a\in A}\langle\alpha,\xi\rangle)V_{\frac{n-2}{2}}(t\mid\xi\mid)t^{n-m}.$$

Thus, we obtain

$$\Im_{0,0}(y)\simeq\frac{(\prod_{a\in A}\langle\alpha,\xi\rangle)}{(\prod_{a\in A}\sin\frac{\langle\alpha,\xi\rangle}{2})}\int_0^\infty e^{it^\beta}\psi(t)\ \varphi_0(t)V_{\frac{n-2}{2}}(t\mid\xi\mid)t^{-z+n-1}dt.$$

By choosing a sufficiently large R, without loss of generality, we may assume $\alpha(\xi)\leq\pi$ for all $\alpha\in A$. Thus, taking the advantage that

$$\frac{(\prod_{a\in A}\langle\alpha,\xi\rangle)}{(\prod_{a\in A}\sin\frac{\langle\alpha,\xi\rangle}{2})}$$

is an analytic function, we may assume

$$\left|(\frac{\partial}{\partial\xi})^J\Im_{0,0}(\exp\xi)\right|$$

$$\preceq\left|(\frac{\partial}{\partial\xi})^J\int_0^\infty e^{it^\beta}\psi(t)\ \varphi_0(t)V_{\frac{n-2}{2}}(t\mid\xi\mid)t^{-z+n-1}dt\right|.$$

Using a derivative formula for the Bessel function (see [23]) we have

$$\left|(\frac{\partial}{\partial\xi})^J\Im_{0,0}(\exp\xi)\right|\leq\sum_{k=0}^q|\wp_k(\xi)|$$

where

$$\wp_k(\xi)=\sum_{k=0}^q\frac{P_k(\xi)}{\mid\xi\mid^{\frac{n-2}{2}+k}}\int_0^\infty e^{it^\beta}\varphi_0(t)\psi(t)J_{\frac{n-2}{2}+k}(t\mid\xi\mid)t^{-z+\frac{n}{2}+k}dt,$$

and P_k is a homogeneous polynomial of degree k.

For each k, using the asymptotic expansion of the Bessel function (see [23]), for any positive integer N we have

$$\wp_k(\xi) = \sum_{\mu=0}^{N} a_\mu \frac{P_k(\xi)}{|\ \xi\ |^{\frac{n-1}{2}+\mu+k}} \int_0^\infty e^{it^\beta} \quad e^{-it|\xi|}\ \varphi_0(t)\psi(t)t^{-z+\frac{n-1}{2}-\mu+k}dt$$

$$+\sum_{\mu=0}^{N} b_\mu \frac{P_k(\xi)}{|\ \xi\ |^{\frac{n-1}{2}+k+\mu}} \int_0^\infty e^{it^\beta} \quad e^{it|\xi|}\varphi_0(t)\psi(t)t^{-z+\frac{n-1}{2}-\mu+k}dt$$

$$+O\Big(\frac{P_k(\xi)}{|\ \xi\ |^{\frac{n+1}{2}+N+k}} \int_0^\infty \varphi_0(t)\psi(t)\frac{dt}{t^{z-\frac{n-1}{2}+N+1-k}}\Big)$$

$$\preceq \sum_{\mu=0}^{N} |\ \mathcal{F}_{k,\mu}(\xi)| + O\Big(\frac{1}{|\xi|^{\frac{m+1}{2}+N}}\Big),$$

where

$$\mathcal{F}_{k,\mu}(\xi) = \frac{1}{|\ \xi\ |^{\frac{n-1}{2}+\mu}} \int_0^\infty e^{it^\beta} \quad e^{\pm it|\xi|}\ \varphi_0(t)\psi(t)t^{-z+\frac{n-1}{2}-\mu+k}dt.$$

Let Γ_j, $j = 1, 2, 3$ be defined in Lemma 3.4. We write

$$\mathcal{F}_{k,\mu}(\xi) = \sum_{j=1}^{3} \mathcal{F}_{k,\mu,j}(\xi),$$

where, for simplicity of notation, we write

$$\mathcal{F}_{k,\mu,j}(\xi) = \frac{1}{|\ \xi\ |^{\frac{n-1}{2}+\mu}} \int_0^\infty e^{it^\beta} \quad e^{\pm it|\xi|}\ \varphi_0(t)\Gamma_j(t\,|\xi|^{\frac{1}{1-\beta}})\psi(t)t^{-z+\frac{n-1}{2}-\mu+k}dt$$

for $j = 1, 2, 3$.

By the same argument as we estimate $\mathcal{L}_{k,1}$and $\mathcal{L}_{k,2}$ in Lemma 3.4, we have

$$|\mathcal{F}_{k,\mu,1}(\xi)| + |\mathcal{F}_{k,\mu,2}(\xi)| \preceq 1.$$

By changing variables we have that

$$\mathcal{F}_{k,\mu,3}(\xi) = \frac{|\xi|^{\frac{-\gamma+\frac{n-1}{2}-\mu+k+1}{\beta-1}}\ |\xi|^{-i\Im(z)/(\beta-1)}}{|\ \xi\ |^{\frac{n-1}{2}+\mu}}$$

$$\times \int_0^\infty e^{i\phi_\pm(t,\xi)}\ (\varphi_0\psi)(|\xi|^{\frac{1}{\beta-1}}\ t)\Gamma_3(t)t^{-z+\frac{n-1}{2}-\mu+k}dt,$$

where

$$\phi_\pm(t,\xi) = |\xi|^{\frac{\beta}{\beta-1}}\ t^\beta \pm t.$$

Thus, by Lemma 3.1 and an easy computation we have

$$|\mathcal{F}_{k,\mu,3}(\xi)| \preceq \frac{|\xi|^{\frac{-\gamma+\frac{n-1}{2}+k+1}{\beta-1}}}{|\ \xi\ |^{\frac{n-1}{2}}} |\xi|^{\frac{\beta}{2(1-\beta)}} \frac{|\xi|^{\frac{\mu}{1-\beta}}}{|\ \xi\ |^{\mu}}.$$

Therefore,

$$\sum_{k=0}^{q} |\wp_k(\xi)| \preceq \sum_{k=0}^{q}\sum_{\mu=0}^{N}\sum_{j=1}^{3} |\mathcal{F}_{k,\mu,j}(\xi)| + O(\frac{1}{|\ \xi\ |^{\frac{m+1}{2}+N}})$$

$$\preceq \sum_{k=0}^{q}\sum_{\mu=0}^{N} \frac{|\xi|^{\frac{-\gamma+\frac{n-1}{2}+k+1}{\beta-1}}}{|\ \xi\ |^{\frac{n-1}{2}}} |\xi|^{\frac{\beta}{2(1-\beta)}} \frac{|\xi|^{\frac{\mu}{1-\beta}}}{|\ \xi\ |^{\mu}} + O(\frac{1}{|\ \xi\ |^{\frac{m+1}{2}+N}})$$

$$\preceq |\xi|^{\frac{-\gamma+\frac{n}{2}}{\beta-1}} |\xi|^{\frac{q}{\beta-1}} |\ \xi\ |^{-\frac{n}{2}} + O(\frac{1}{|\ \xi\ |^{\frac{m+1}{2}+N}}).$$

This completes the proof of the lemma. □

Lemma 3.7. *Let the number η be the same as in Lemma 3.5. If $d(u,I) > \frac{\eta}{100}$ then for any multi-index M with $|M| = q$ and any $L > 0$,*

$$|\ Y^M(\Im_{0,0}(u) + \Im_{\infty,0}(u))\ | \preceq R^q\ |\ d^R(u)\ |^{-1} \left\{1 + \sum_{\nu\neq 0} |\nu|^{-L}\right\}.$$

Proof. Observe that $(\Im_{0,0}(u)+\Im_{\infty,0}(u))d(u)$ is skew-invariant by the Weyl group. The proof is the same as that of Lemma 3.5. □

4. Proofs of the theorem

4.1. H^p Boundedness Of $T_{z,\beta}$

We will prove Proposition 1.3 in this section. Precisely, we will prove the $H^p(G)$ boundedness of the operator $T_{z,\beta}$ with

$$\Re(z) = \gamma_p = n\beta(\frac{1}{p} - \frac{1}{2}).$$

Observing that the assumption

$$p < \frac{2\beta}{\beta+2}$$

implies $\gamma_p > n$, we can use the estimates of $K_{z,\beta}$ obtained in Section 3. Moreover, as mentioned in Section 2, to prove the H^p boundedness of $T_{z,\beta}$ it suffices to prove

$$\| T_{z,\beta}(a) \|_{L^p(G)} \preceq 1,$$

for any p-atom $a(x)$ of support in $B(I,\rho)$ with sufficiently small ρ.

Let $\varsigma = 1-\beta$. We have

$$\| T_{z,\beta}(a) \|_{L^p}^p = \int_{d(x,I)\le 10^4 n\rho^\varsigma} | T_{z,\beta}(a)(x) |^p \, dx$$

$$+ \int_{d(x,I)>10^4 n\rho^\varsigma} | T_{z,\beta}(a)(x) |^p \, dx = I_1 + I_2.$$

By Hölder's inequality, we have

$$I_1 \preceq \| T_{z,\beta}(a) \|_{L^2(G)}^p \, \rho^{\frac{2-p}{2} n\varsigma} \preceq \| a \|_{L^2_{-a_p}(G)}^p \, \rho^{\frac{2-p}{2} n\varsigma}$$

$$\preceq \| a \|_{L^r(G)}^p \, \rho^{(1-\frac{2}{p})n\varsigma}, \; 1/2 = 1/r - \alpha_p/n.$$

Note $\frac{1}{p} - \frac{1}{2} = \frac{\gamma_p}{n\beta}$ and $\varsigma = 1-\beta$. An easy computation shows

$$I_1 \preceq \rho^{p\gamma_p - n(1-p/2)} \rho^{\frac{2-p}{2}\varsigma n} \preceq 1.$$

By Hölder's inequality and Lemma 3.2, we have

$$\int_{d(x,I)>10^4 n\rho^\varsigma} \left| \int_G a(y) \Im_{\infty,\infty}(xy^{-1}) dy \right|^p dx$$

$$\preceq \|a\|_{L^1(G)} \, \| \Im_{\infty,\infty} \|_{L^1(G)} \preceq 1.$$

Using Hölder's inequality and Lemma 3.2 again , we have

$$\int_{d(x,I)>10^4 n\rho^\varsigma} \left| \int_G a(y) \Im_{0,\infty}(xy^{-1}) dy \right|^p dx$$

$$\preceq \|a\|_{L^1(G)} \int_{d(x,I)>8^4 n\rho^\varsigma} | \Im_{0,\infty}(x) | \, dx$$

$$\preceq \rho^{(\gamma_p - \frac{n+m}{2} + L)} \rho^{-\frac{n}{p}+n} \int_{\{|\theta|\ge 10 n\rho^\varsigma\}\cap Q} | D(\theta) | \, |\theta|^{-L} \, d\theta \preceq 1$$

if we choose a large positive integer L.

It remains to show

$$\int_{d(x,I)>10^4 n\rho^\varsigma} | \int_G a(y)(\Im_{\infty,0}(xy^{-1}) + \Im_{0,0}(xy^{-1})) dy \, |^p \, dx \preceq 1.$$

Let η be the number defined in Lemma 3.6 and let $n_0 = 2[\frac{n}{p} - n] + 2$. Using the cancellation condition of a, we have

$$\int_G a(y)\Im_{\infty,0}(xy^{-1})dy$$

$$= \int_G a(y)\{\Im_{\infty,0}(xy^{-1}) - T^x_{n_0}(\Im_{\infty,0}(y))\}dy$$

where $T^x_{n_0}(\Im_{\infty,0})$ is the Taylor polynomial of $\Im_{\infty,0}$ at x. Hence

$$\left|\int_G a(y)\Im_{\infty,0}(xy^{-1})dy\right|$$
$$\preceq \rho^{n_0+1}\int_{B(I,\rho)} |a(y)|\, dy \sup_{z\in B(x,\rho),|J|\le n_0+1} \left|Y^J\Im_{\infty,0}(z)\right|$$
$$\preceq \rho^{-\frac{n}{p}+n+1+n_0} \sup_{z\in B(I,\rho),|J|\le n_0+1} \left|L_x Y^J\Im_{\infty,0}(z)\right|.$$

where L_x is the left shift operator defined by $L_x f(y) = f(xy)$ for any function f. From Lemma 3.4.4 in [8], we know that $d^R(x) \simeq d^R(z)$ if the distance $d(x,z)$ of x and z is sufficiently small. Observe that we may fix any small $r > 0$ and assume $\rho \le r/100$ in our proof. Choosing $R = \frac{1}{\sqrt{\rho}}$, by Lemma 3.5 we have that

$$\sup_{z\in B(I,\rho),|J|\le n_0+1} \{\left|L_x Y^J\Im_{\infty,0}(z)\right| \preceq \rho^{-\frac{n_0}{2}} \mid d^R(x)\mid^{-1} \sum_{\nu\neq 0} |\nu|^{-L},$$

for any positive integer L. Thus

$$\int_{10^4 n\rho^{\varsigma}<d(x,I)\le\frac{\eta}{2}} \mid \int_{B(I,\rho)} a(y)\Im_{\infty,0}(xy^{-1})dy \mid^p dx \preceq 1.$$

Replacing Lemma 3.5 by Lemma 3.7, we use a similar argument to obtain

$$\int_{d(x,I)>\frac{\eta}{2}} \mid \int_{B(I,\rho)} a(y)(\Im_{\infty,0}(xy^{-1}) + \Im_{0,0}(xy^{-1}))dy \mid^p dx \preceq 1.$$

Finally, by the cancellation of a we obtain

$$\int_{10^4 n\rho^{\varsigma}<d(x,I)\le\frac{\eta}{2}} \mid \int_{B(I,\rho)} a(y)\Im_{0,0}(xy^{-1})dy \mid^p dx$$

$$= \int_{10^4 n\rho^{\varsigma}<d(x,I)\le\frac{\eta}{2}} \mid \int_{B(I,\rho)} a(y)\{\Im_{0,0}(xy^{-1}) - T^x_q(\Im_{0,0}(y))\}dy \mid^p dx$$

$$\preceq \rho^{-n+pn+pq} \int_{10^4 n\rho^{\varsigma}<d(x,I)\leq \frac{\eta}{2}} | \sup_{z\in B(I,\rho),|J|\leq q+1} \{ \left| L_x Y^J \Im_{0,0}(z) \right|^p dx,$$

where q is a suitably large number such that

$$q + \frac{n-1}{2} - \gamma_p > 1.$$

Choosing $N = q + \frac{n-1}{2} - \gamma_p + 1$, by Lemma 3.6 we have

$$\int_{10^4 n\rho^{\varsigma}<d(x,I)\leq \frac{\eta}{2}} | \int_{B(I,\rho)} a(y)\Im_{0,0}(xy^{-1})dy |^p \, dx$$

$$\preceq \rho^{-n+pn+pq} \int_{|\theta|>9^4 n\rho^{\varsigma}} | \theta |^{-p(\frac{\gamma_p-\frac{n}{2}}{\beta-1})-p\frac{m}{2}} | \theta |^{\frac{pq}{\beta-1}} | D(\theta) |^{2-p} \, d\theta$$

$$+O\Big(\rho^{-n+pn+pq} \int_{|\theta|>10\rho^{\varsigma}} |\theta|^{-pN-\frac{n+1}{2}p} \mid D(\theta) \mid^2 d\theta\Big)$$

$$\preceq \rho^{-n+pn+pq} \rho^{p(\gamma_p-\frac{n}{2})+(1-\beta)(-\frac{n}{2}p-\frac{pq}{1-\beta}+n)} + 1$$

$$\preceq \rho^{p\gamma_p-\beta n+\frac{np\beta}{2}} + 1 = 2.$$

This proves the H^p boundedness of $T_{z,\beta}$, which completes the proof of Proposition 1.3 and the sufficiency part of Theorem 1.2.

4.2. **Proof Of The Necessity Part In Theorem 1.2**

Recall that we define the analytic family

$$T_{z,\beta}(f)(x) = K_{z,\beta}(f) * f(x) = \sum_{\lambda+\delta\in\Lambda\backslash\{0\}} \frac{e^{i\|\lambda+\delta\|^\beta}}{\| \lambda+\delta \|^z} d_\lambda \chi_\lambda * f(x), \ z \in \mathbf{C}.$$

In [5], using the analytic interpolation, we proved

$$\|T_{\gamma,\beta}(f)\|_{H^p(G)} \preceq \|f\|_{H^p(G)} \tag{4.1}$$

if and only if $\left|\frac{1}{p} - \frac{1}{2}\right| \leq \frac{\gamma}{n\beta}$ for all $1 \leq p < \infty$.

Clearly, (4.1) is also a necessity condition for the H^p boundedness of $T_{\gamma,\beta}$ for $0 < p \leq 1$. Otherwise, an analytic interpolation on the family $\{T_{z,\beta}\}$ would yield a contradiction to the proven case for $p > 1$.

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