

# ANNALES MATHÉMATIQUES



## BLAISE PASCAL

BECHIR AMRI & MOHAMED SIFI

**Riesz transforms for Dunkl transform**

Volume 19, n° 1 (2012), p. 247-262.

<[http://ambp.cedram.org/item?id=AMBP\\_2012\\_\\_19\\_1\\_247\\_0](http://ambp.cedram.org/item?id=AMBP_2012__19_1_247_0)>

© Annales mathématiques Blaise Pascal, 2012, tous droits réservés.

L'accès aux articles de la revue « Annales mathématiques Blaise Pascal » (<http://ambp.cedram.org/>), implique l'accord avec les conditions générales d'utilisation (<http://ambp.cedram.org/legal/>). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

*Publication éditée par le laboratoire de mathématiques  
de l'université Blaise-Pascal, UMR 6620 du CNRS  
Clermont-Ferrand — France*

**cedram**

*Article mis en ligne dans le cadre du  
Centre de diffusion des revues académiques de mathématiques  
<http://www.cedram.org/>*

# Riesz transforms for Dunkl transform

BECHIR AMRI  
MOHAMED SIFI

## Abstract

In this paper we obtain the  $L^p$ -boundedness of Riesz transforms for the Dunkl transform for all  $1 < p < \infty$ .

*Transformées de Riesz associées à la transformée de Dunkl*

## Résumé

Dans cet article, nous étudions la bornitude des transformées de Riesz associées à la transformée de Dunkl sur les espaces  $L^p$ ,  $1 < p < \infty$ .

## 1. Introduction

On the Euclidean space  $\mathbb{R}^N$ ,  $N \geq 1$ , the ordinary Riesz transform  $R_j$ ,  $j = 1, \dots, N$  is defined as the multiplier operator

$$\widehat{R_j(f)}(\xi) = -i \frac{\xi_j}{\|\xi\|} \widehat{f}(\xi). \quad (1.1)$$

It can also be defined by the principal value of the singular integral

$$R_j(f)(x) = d_0 \lim_{\varepsilon \rightarrow 0} \int_{\|x-y\| > \varepsilon} \frac{x_j - y_j}{\|x - y\|} f(y) dy$$

where  $d_0 = 2^{\frac{N}{2}} \frac{\Gamma(\frac{N+1}{2})}{\sqrt{\pi}}$ . It follows from the general theory of singular integrals that Riesz transforms are bounded on  $L^p(\mathbb{R}^N, dx)$  for all  $1 < p < \infty$ . What is done in this paper is to extend this result to the context of Dunkl theory where a similar operator is already defined.

Dunkl theory generalizes classical Fourier analysis on  $\mathbb{R}^N$ . It started twenty years ago with Dunkl's seminal work [3] and was further developed by several mathematicians. See for instance the surveys [5, 6, 7, 9] and the

---

Keywords: Dunkl transforms, Riesz Transforms, Singular integrals.

Math. classification: 17B22, 32A55, 43A32, 42A45.

references cited therein. The study of the  $L^p$ -boundedness of Riesz transforms for Dunkl transform on  $\mathbb{R}^N$  goes back to the work of S. Thangavelu and Y. Xu [10] where they established boundedness result only in a very special case of  $N = 1$ . It has been noted in [10] that the difficulty arises in the application of the classical  $L^p$ - theory of Caldéron-Zygmund, since Riesz transforms are singular integral operators. In this paper we describe how this theory can be adapted in Dunkl setting and gives an  $L^p$ -result for Riesz transforms for all  $1 < p < \infty$ . More precisely, through the fundamental result of M. Rösler [6] for the Dunkl translation of radial functions, we reformulate a Hörmander type condition for singular integral operators. The Riesz kernel is given by acting Dunkl operator on Dunkl translation of radial function.

This paper is organized as follows. In Section 2 we present some definitions and fundamental results from Dunkl’s analysis. The Section 3 is devoted to proving  $L^p$ -boundedness of Riesz transforms. As applications, we will prove a generalized Riesz and Sobolev inequalities. Throughout this paper  $C$  denotes a constant which can vary from line to line.

**Acknowledgments.** The authors are partially supported by the project DGRST 04/UR/15-02 and the cooperation programs PHC Utique / CMCU 07G 1501 and 10G 1503.

## 2. Preliminaries

In this section we collect notations and definitions and recall some basic facts. We refer to [5, 3, 6, 7, 9].

Let  $G \subset O(\mathbb{R}^N)$  be a finite reflection group associated to a reduced root system  $R$  and  $k : R \rightarrow [0, +\infty)$  be a  $G$ -invariant function (called multiplicity function). Let  $R_+$  be a positive root subsystem. We shall assume that  $R$  is normalized in the sense that  $\|\alpha\|^2 = \langle \alpha, \alpha \rangle = 2$  for all  $\alpha \in R$ , where  $\langle \cdot, \cdot \rangle$  is the standard Euclidean scalar product on  $\mathbb{R}^N$ .

The Dunkl operators  $T_\xi$ ,  $\xi \in \mathbb{R}^N$  are the following  $k$ -deformations of directional derivatives  $\partial_\xi$  by difference operators :

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\alpha \in R_+} k(\alpha) \langle \alpha, \xi \rangle \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}, \quad x \in \mathbb{R}^N$$

RIESZ TRANSFORMS FOR DUNKL TRANSFORM

where  $\sigma_\alpha$  denotes the reflection with respect to the hyperplane orthogonal to  $\alpha$ . For the standard basis vectors of  $\mathbb{R}^N$ , we simply write  $T_j = T_{e_j}$ .

The operators  $\partial_\xi$  and  $T_\xi$  are intertwined by a Laplace-type operator

$$V_k f(x) = \int_{\mathbb{R}^N} f(y) d\mu_x(y),$$

associated to a family of compactly supported probability measures

$$\{\mu_x \mid x \in \mathbb{R}^N\}.$$

Specifically,  $\mu_x$  is supported in the the convex hull  $co(G.x)$ .

For every  $\lambda \in \mathbb{C}^N$ , the simultaneous eigenfunction problem,

$$T_\xi f = \langle \lambda, \xi \rangle f, \quad \xi \in \mathbb{R}^N$$

has a unique solution  $f(x) = E_k(\lambda, x)$  such that  $E_k(\lambda, 0) = 1$ , which is given by

$$E_k(\lambda, x) = V_k(e^{\langle \lambda, \cdot \rangle})(x) = \int_{\mathbb{R}^N} e^{\langle \lambda, y \rangle} d\mu_x(y), \quad x \in \mathbb{R}^N.$$

Furthermore  $\lambda \mapsto E_k(\lambda, x)$  extends to a holomorphic function on  $\mathbb{C}^N$ .

Let  $m_k$  be the measure on  $\mathbb{R}^N$ , given by

$$dm_k(x) = \prod_{\alpha \in R_+} |\langle \alpha, x \rangle|^{2k(\alpha)} dx.$$

For  $f \in L^1(m_k)$  (the Lebesgue space with respect to the measure  $m_k$ ) the Dunkl transform is defined by

$$\mathcal{F}_k(f)(\xi) = \frac{1}{c_k} \int_{\mathbb{R}^N} f(x) E_k(-i\xi, x) dm_k(x), \quad c_k = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} dm_k(x).$$

This new transform shares many analogous properties of the Fourier transform.

- (i) The Dunkl transform is a topological automorphism of  $\mathcal{S}(\mathbb{R}^N)$  (*Schwartz space*).
- (ii) (*Plancherel Theorem*) The Dunkl transform extends to an isometric automorphism of  $L^2(m_k)$ .
- (iii) (*Inversion formula*) For every  $f \in L^1(m_k)$  such that  $\mathcal{F}_k f \in L^1(m_k)$ , we have

$$f(x) = \mathcal{F}_k^2 f(-x), \quad x \in \mathbb{R}^N.$$

(iv) For all  $\xi \in \mathbb{R}^N$  and  $f \in \mathcal{S}(\mathbb{R}^N)$

$$\mathcal{F}_k(T_\xi(f))(x) = \langle i\xi, x \rangle \mathcal{F}_k(f)(x), \quad x \in \mathbb{R}^N. \quad (2.1)$$

Let  $x \in \mathbb{R}^N$ , the Dunkl translation operator  $\tau_x$  is defined on  $L^2(m_k)$  by,

$$\mathcal{F}_k(\tau_x(f))(y) = E_k(ix, y)\mathcal{F}_k f(y), \quad y \in \mathbb{R}^N. \quad (2.2)$$

If  $f$  is a continuous radial function in  $L^2(m_k)$  with  $f(y) = \tilde{f}(\|y\|)$ , then

$$\tau_x(f)(y) = \int_{\mathbb{R}^N} \tilde{f}\left(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle y, \eta \rangle}\right) d\mu_x(\eta). \quad (2.3)$$

This formula is first proved by M. Rösler [6] for  $f \in \mathcal{S}(\mathbb{R}^N)$  and recently is extended to continuous functions by F. and H. Dai Wang [2].

We collect below some useful facts :

(i) For all  $x, y \in \mathbb{R}^N$ ,

$$\tau_x(f)(y) = \tau_y(f)(x). \quad (2.4)$$

(ii) For all  $x, \xi \in \mathbb{R}^N$  and  $f \in \mathcal{S}(\mathbb{R}^N)$ ,

$$T_\xi \tau_x(f) = \tau_x T_\xi(f). \quad (2.5)$$

(iii) For all  $x \in \mathbb{R}^N$  and  $f, g \in L^2(m_k)$ ,

$$\int_{\mathbb{R}^N} \tau_x(f)(-y)g(y)dm_k(y) = \int_{\mathbb{R}^N} f(y)\tau_x g(-y)dm_k(y). \quad (2.6)$$

(iv) For all  $x \in \mathbb{R}^N$  and  $1 \leq p \leq 2$ , the operator  $\tau_x$  can be extended to all radial functions  $f$  in  $L^p(m_k)$  and the following holds

$$\|\tau_x(f)\|_{p,k} \leq \|f\|_{p,k}. \quad (2.7)$$

$\|\cdot\|_{p,k}$  is the usual norm of  $L^p(m_k)$ .

### 3. Riesz transforms for the Dunkl transform.

In Dunkl setting the Riesz transforms (see [10]) are the operators  $\mathcal{R}_j$ ,  $j = 1 \dots N$  defined on  $L^2(m_k)$  by

$$\mathcal{R}_j(f)(x) = d_k \lim_{\varepsilon \rightarrow 0} \int_{|y| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} dm_k(y), \quad x \in \mathbb{R}^N$$

RIESZ TRANSFORMS FOR DUNKL TRANSFORM

where

$$d_k = 2^{\frac{p_k-1}{2}} \frac{\Gamma(\frac{p_k}{2})}{\sqrt{\pi}}; \quad p_k = 2\gamma_k + N + 1 \quad \text{and} \quad \gamma_k = \sum_{\alpha \in R^+} k(\alpha).$$

It has been proved by S. Thangavelu and Y. Xu [10], that  $\mathcal{R}_j$  is a multiplier operator given by

$$\mathcal{F}_k(\mathcal{R}_j(f))(\xi) = -i \frac{\xi_j}{\|\xi\|} \mathcal{F}_k(f)(\xi), \quad f \in \mathcal{S}(\mathbb{R}^N), \quad \xi \in \mathbb{R}^N, \quad (3.1)$$

The authors state that if  $N = 1$  and  $2\gamma_k \in \mathbb{N}$  the operator  $\mathcal{R}_j$  is bounded on  $L^p(m_k)$ ,  $1 < p < \infty$ . In [1] this result is improved by removing  $2\gamma_k \in \mathbb{N}$ , where Riesz transform is called Hilbert transform. If  $\gamma_k = 0$  ( $k = 0$ ), this operator coincides with the usual Riesz transform  $R_j$  given by (1.1). Our interest is to prove the boundedness of this operator for  $N \geq 2$  and  $k \geq 0$ . To do this, we invoke the theory of singular integrals. Our basic is the following,

**Theorem 3.1.** *Let  $\mathcal{K}$  be a measurable function on  $\mathbb{R}^N \times \mathbb{R}^N \setminus \{(x, g.x); x \in \mathbb{R}^N, g \in G\}$  and  $S$  be a bounded operator from  $L^2(m_k)$  into itself, associated with a kernel  $\mathcal{K}$  in the sense that*

$$S(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}(x, y) f(y) dm_k(y), \quad (3.2)$$

for all compactly supported function  $f$  in  $L^2(m_k)$  and for a.e  $x \in \mathbb{R}^N$  satisfying  $g.x \notin \text{supp}(f)$ , for all  $g \in G$ . If  $\mathcal{K}$  satisfies

$$\int_{\min_{g \in G} \|g.x-y\| > 2\|y-y_0\|} |\mathcal{K}(x, y) - \mathcal{K}(x, y_0)| dm_k(x) \leq C, \quad y, y_0 \in \mathbb{R}^N, \quad (3.3)$$

then  $S$  extends to a bounded operator from  $L^p(m_k)$  into itself for all  $1 < p \leq 2$ .

*Proof.* We first note that  $(\mathbb{R}^N, m_k)$  is a space of homogenous type, that is, there is a fixed constant  $C > 0$  such that

$$m_k(B(x, 2r)) \leq C m_k(B(x, r)), \quad \forall x \in \mathbb{R}^N, \quad r > 0 \quad (3.4)$$

where  $B(x, r)$  is the closed ball of radius  $r$  centered at  $x$  (see [8], Ch 1). Then we can adapt to our context the classical technic which consist to show that  $S$  is weak type (1,1) and conclude by Marcinkiewicz interpolation theorem.

In fact, the Calderón-Zygmund decomposition says that for all  $f \in L^1(m_k) \cap L^2(m_k)$  and  $\lambda > 0$ , there exist a decomposition of  $f$ ,  $f = h + b$  with  $b = \sum_j b_j$  and a sequence of balls  $(B(y_j, r_j))_j = (B_j)_j$  such that for some constant  $C$ , depending only on the multiplicity function  $k$

- (i)  $\|h\|_\infty \leq C\lambda$ ;
- (ii)  $\text{supp}(b_j) \subset B_j$ ;
- (iii)  $\int_{B_j} b_j(x) dm_k(x) = 0$ ;
- (iv)  $\|b_j\|_{1,k} \leq C \lambda m_k(B_j)$ ;
- (v)  $\sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda}$ .

The proof consists in showing the following inequality hold for  $w = h$  and  $w = b$  :

$$\rho_\lambda(S(w)) = m_k\left(\left\{x \in \mathbb{R}^N; |S(w)(x)| > \frac{\lambda}{2}\right\}\right) \leq C \frac{\|f\|_{1,k}}{\lambda}. \quad (3.5)$$

By using the  $L^2$ -boundedness of  $S$  we get

$$\rho_\lambda(S(h)) \leq \frac{4}{\lambda^2} \int_{\mathbb{R}^N} |S(h)(x)|^2 dm_k(x) \leq \frac{C}{\lambda^2} \int_{\mathbb{R}^N} |h(x)|^2 dm_k(x). \quad (3.6)$$

From (i) and (v),

$$\int_{\cup B_j} |h(x)|^2 dm_k(x) \leq C\lambda^2 \mu_k(\cup B_j) \leq C\lambda \|f\|_{1,k}. \quad (3.7)$$

Since on  $(\cup B_j)^c$ ,  $f(x) = h(x)$ , then

$$\int_{(\cup B_j)^c} |h(x)|^2 dm_k(x) \leq C\lambda \|f\|_{1,k}. \quad (3.8)$$

From (3.6), (3.7) and (3.8), the inequality (3.5) is satisfied for  $h$ .

Next we turn to the inequality (3.5) for the function  $b$ . Consider

$$B_j^* = B(y_j, 2r_j); \quad \text{and} \quad Q_j^* = \bigcup_{g \in G} g.B_j^*.$$

Then

$$\rho_\lambda(S(b)) \leq m_k\left(\bigcup_j Q_j^*\right) + m_k\left\{x \in \left(\bigcup_j Q_j^*\right)^c; |S(b)(x)| > \frac{\lambda}{2}\right\}.$$

Now by (3.4) and (v)

$$m_k\left(\bigcup_j Q_j^*\right) \leq |G| \sum_j m_k(B_j^*) \leq C \sum_j m_k(B_j) \leq C \frac{\|f\|_{1,k}}{\lambda}.$$

Furthermore if  $x \notin Q_j^*$ , we have

$$\min_{g \in G} \|g \cdot x - y_j\| > 2\|y - y_j\|, \quad y \in B_j.$$

Thus, from (3.2), (iii), (ii), (3.3), (iv) and (v)

$$\begin{aligned} & \int_{(\cup Q_j^*)^c} |S(b)(x)| dm_k(x) \\ & \leq \sum_j \int_{(Q_j^*)^c} |S(b_j)(x)| dm_k(x) \\ & = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} \mathcal{K}(x, y) b_j(y) dm_k(y) \right| dm_k(x) \\ & = \sum_j \int_{(Q_j^*)^c} \left| \int_{\mathbb{R}^N} b_j(y) (\mathcal{K}(x, y) - \mathcal{K}(x, y_j)) dm_k(y) \right| dm_k(x) \\ & \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{(Q_j^*)^c} |\mathcal{K}(x, y) - \mathcal{K}(x, y_j)| dm_k(x) dm_k(y) \\ & \leq \sum_j \int_{\mathbb{R}^N} |b_j(y)| \int_{\min_{g \in G} \|g \cdot x - y_j\| > 2\|y - y_j\|} |\mathcal{K}(x, y) - \mathcal{K}(x, y_j)| dm_k(x) dm_k(y) \\ & \leq C \sum_j \|b_j\|_{1,k} \\ & \leq C \|f\|_{1,k}. \end{aligned}$$

Therefore,

$$\begin{aligned} m_k\left\{x \in \left(\bigcup_j Q_j^*\right)^c; |S(b)(x)| > \frac{\lambda}{2}\right\} \\ \leq \frac{2}{\lambda} \int_{(\cup Q_j^*)^c} |S(b)(x)| dm_k(x) \leq C \frac{\|f\|_{1,k}}{\lambda}. \end{aligned}$$



This achieves the proof of (3.5) for  $b$ . □

Now, we will give an integral representation for the Riesz transform  $\mathcal{R}_j$ . For this end, we put for  $x, y \in \mathbb{R}^N$  and  $\eta \in co(G.x)$

$$A(x, y, \eta) = \sqrt{\|x\|^2 + \|y\|^2 - 2 \langle y, \eta \rangle} = \sqrt{\|y - \eta\|^2 + \|x\|^2 - \|\eta\|^2}.$$

It is easy to check that

$$\min_{g \in G} \|g.x - y\| \leq A(x, y, \eta) \leq \max_{g \in G} \|g.x - y\|. \quad (3.9)$$

The following inequalities are clear

$$\begin{aligned} \left| \frac{\partial A^\ell}{\partial y_r}(x, y, \eta) \right| &\leq CA^{\ell-1}(x, y, \eta), \\ \left| \frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x, y, \eta) \right| &\leq CA^{\ell-2}(x, y, \eta) \end{aligned} \quad (3.10)$$

and for  $\alpha \in R_+$ ,

$$\begin{aligned} \left| \frac{\partial A^\ell}{\partial y_r}(x, \sigma_\alpha.y, \eta) \right| &\leq CA^{\ell-1}(x, \sigma_\alpha.y, \eta), \\ \left| \frac{\partial^2 A^\ell}{\partial y_r \partial y_s}(x, \sigma_\alpha.y, \eta) \right| &\leq CA^{\ell-2}(x, \sigma_\alpha.y, \eta), \end{aligned} \quad (3.11)$$

where  $r, s = 1, \dots, N$  and  $\ell \in \mathbb{R}$ .

Let us set

$$\begin{aligned} \mathcal{K}_j^{(1)}(x, y) &= \int_{\mathbb{R}^N} \frac{\eta_j - y_j}{A^{p_k}(x, y, \eta)} d\mu_x(\eta) \\ \mathcal{K}_j^{(\alpha)}(x, y) &= \frac{1}{\langle y, \alpha \rangle} \int_{\mathbb{R}^N} \left[ \frac{1}{A^{p_k-2}(x, y, \eta)} - \frac{1}{A^{p_k-2}(x, \sigma_\alpha.y, \eta)} \right] d\mu_x(\eta), \\ \mathcal{K}_j(x, y) &= d_k \left\{ \mathcal{K}_j^{(1)}(x, y) + \sum_{\alpha \in R_+} \frac{k(\alpha)\alpha_j}{p_k - 2} \mathcal{K}_j^{(\alpha)}(x, y) \right\}, \end{aligned}$$

where  $\alpha \in R_+$ .

**Proposition 3.2.** *If  $f \in L^2(m_k)$  with compact support, then for all  $x \in \mathbb{R}^N$  satisfying  $g.x \notin \text{supp}(f)$  for all  $g \in G$ , we have*

$$\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y).$$

RIESZ TRANSFORMS FOR DUNKL TRANSFORM

*Proof.* Let  $f \in L^2(m_k)$  be a compact supported function and  $x \in \mathbb{R}^N$ , such that  $g.x \notin \text{supp}(f)$  for all  $g \in G$ . For  $0 < \varepsilon < \min_{g \in G} \min_{y \in \text{supp}(f)} |g.x - y|$  and  $n \in \mathbb{N}$ , we consider  $\tilde{\varphi}_{n,\varepsilon}$  a  $C^\infty$ -function on  $\mathbb{R}$ , such that:

- $\tilde{\varphi}_{n,\varepsilon}$  is odd .
- $\tilde{\varphi}_{n,\varepsilon}$  is supported in  $\{t \in \mathbb{R}; \varepsilon \leq |t| \leq n + 1\}$ .
- $\tilde{\varphi}_{n,\varepsilon} = 1$  in  $\{t \in \mathbb{R}; \varepsilon + \frac{1}{n} \leq t \leq n\}$ .
- $|\tilde{\varphi}_{n,\varepsilon}| \leq 1$ .

Let

$$\tilde{\phi}_{n,\varepsilon}(t) = \int_{-\infty}^t \frac{\tilde{\varphi}_{n,\varepsilon}(u)}{|u|^{p_k-1}} du \quad \text{and} \quad \phi_{n,\varepsilon}(y) = \tilde{\phi}_{n,\varepsilon}(\|y\|),$$

for  $t \in \mathbb{R}$  and  $y \in \mathbb{R}^N$ . Clearly,  $\phi_{n,\varepsilon}$  is a  $C^\infty$  radial function supported in the ball  $B(0, n + 1)$  and

$$\lim_{n \rightarrow +\infty} \tilde{\varphi}_{n,\varepsilon}(\|y\|) = 1, \quad \forall y \in \mathbb{R}^N, \|y\| > \varepsilon.$$

The dominated convergence theorem, (2.5) and (2.6) yield

$$\begin{aligned} \int_{\|y\| > \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} dm_k(y) &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} \tilde{\varphi}_{n,\varepsilon}(\|y\|) dm_k(y) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} \tau_x(f)(-y) T_j(\phi_{n,\varepsilon})(y) dm_k(y) \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} f(y) T_j \tau_x(\phi_{n,\varepsilon})(-y) dm_k(y). \end{aligned}$$

Now we have

$$\begin{aligned} T_j \tau_x(\phi_{n,\varepsilon})(-y) &= \int_{\mathbb{R}^N} \frac{(\eta_j - y_j) \tilde{\varphi}_{n,\varepsilon}(A(x, y, \eta))}{A^{p_k}(x, y, \eta)} d\mu_x(\eta) \\ &+ \sum_{\alpha \in R_+} k(\alpha) \alpha_j \int_{\mathbb{R}^N} \frac{\tilde{\varphi}_{n,\varepsilon}(A(x, \sigma_\alpha \cdot y, \eta)) - \tilde{\varphi}_{n,\varepsilon}(A(x, y, \eta))}{\langle y, \alpha \rangle} d\mu_x(\eta), \end{aligned}$$

where from (3.9)

$$\varepsilon < A(x, y, \eta) ; \quad \varepsilon < A(x, \sigma_\alpha \cdot y, \eta), \quad y \in \text{supp}(f), \eta \in \text{co}(G \cdot x).$$

Then with the aid of dominated convergence theorem

$$\lim_{n \rightarrow \infty} T_j \tau_x(\phi_{n, \varepsilon})(-y) = \frac{1}{d_k} \mathcal{K}_j(x, y),$$

and

$$d_k \int_{\|y\| \geq \varepsilon} \tau_x(f)(-y) \frac{y_j}{\|y\|^{p_k}} dm_k(y) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y).$$

Letting  $\varepsilon \rightarrow 0$ , it follows that

$$\mathcal{R}_j(f)(x) = \int_{\mathbb{R}^N} \mathcal{K}_j(x, y) f(y) dm_k(y),$$

which proves the result. □

Now, we are able to state our main result.

**Theorem 3.3.** *The Riesz transform  $\mathcal{R}_j$ ,  $j = 1 \dots N$ , is a bounded operator from  $L^p(m_k)$  into itself, for all  $1 < p < \infty$ .*

*Proof.* Clearly, from (3.1) and Plancherel's theorem  $\mathcal{R}_j$  is bounded from  $L^2(m_k)$  into itself, with adjoint operator  $\mathcal{R}_j^* = -\mathcal{R}_j$ . Thus, via duality it's enough to consider the range  $1 < p \leq 2$  and apply Theorem 3.1. In view of Proposition 3.2 it only remains to show that  $\mathcal{K}_j$  satisfies condition (3.3).

Let  $y, y_0 \in \mathbb{R}^N$ ,  $y \neq y_0$  and  $x \in \mathbb{R}^N$ , such that

$$\min_{g \in G} \|g \cdot x - y\| > 2\|y - y_0\|. \tag{3.12}$$

By mean value theorem,

RIESZ TRANSFORMS FOR DUNKL TRANSFORM

$$\begin{aligned}
 |\mathcal{K}_j^{(1)}(x, y) - \mathcal{K}_j^{(1)}(x, y_0)| &= \left| \sum_{i=0}^N (y_i - (y_0)_i) \int_0^1 \frac{\partial \mathcal{K}_j^{(1)}}{\partial y_i}(x, y_t) dt \right| \\
 &= \left| \sum_{i=0}^N (y_i - (y_0)_i) \int_0^1 \int_{\mathbb{R}^N} \left( \frac{\delta_{i,j}}{A^{p_k}(x, y_t, \eta)} \right. \right. \\
 &\quad \left. \left. + \frac{p_k((y_t)_i - \eta_i)(\eta_j - (y_t)_j)}{A^{p_k+2}(x, y_t, \eta)} \right) d\mu_x(\eta) \right| \\
 &\leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \frac{1}{A^{p_k}(x, y_t, \eta)} d\mu_x(\eta) dt.
 \end{aligned}$$

where  $y_t = y_0 + t(y - y_0)$  and  $\delta_{i,j}$  is the Kronecker symbol. In view of (3.9) and (3.12), we obtain

$$\|y - y_0\| < A(x, y_t, \eta), \quad \eta \in co(G.x).$$

Therefore,

$$\begin{aligned}
 & \left| \mathcal{K}_j^{(1)}(x, y) - \mathcal{K}_j^{(1)}(x, y_0) \right| \\
 & \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \frac{1}{\left( \|y - y_0\|^2 + A^2(x, y_t, \eta) \right)^{\frac{p_k}{2}}} d\mu_x(\eta) dt. \\
 & \leq C \|y - y_0\| \int_0^1 \tau_x(\psi)(y_t) dt
 \end{aligned}$$

where  $\psi$  is the function defined by

$$\psi(z) = \frac{1}{\left( \|y - y_0\|^2 + \|z\|^2 \right)^{\frac{p_k}{2}}}, \quad z \in \mathbb{R}^N.$$

Using Fubini's theorem, (2.4) and (2.7), we get

$$\begin{aligned}
 & \int_{\min_{g \in G} \|g.x - y\| > 2\|y - y_0\|} |\mathcal{K}_j^{(1)}(x, y) - \mathcal{K}_j^{(1)}(x, y_0)| dm_k(x) \\
 & \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \tau_{-y_t}(\psi)(x) dm_k(x) dt \\
 & \leq C \|y - y_0\| \int_{\mathbb{R}^N} \psi(z) dm_k(z) = C \int_{\mathbb{R}^N} \frac{du}{(1 + u^2)^{\frac{p_k}{2}}} = C'.
 \end{aligned}$$

This established the condition (3.3) for  $\mathcal{K}_j^{(1)}$ .

To deal with  $\mathcal{K}_j^{(\alpha)}$ ,  $\alpha \in R_+$ , we put for  $x, y \in \mathbb{R}^N$ ,  $\eta \in co(G.x)$  and  $t \in [0, 1]$

$$\begin{aligned} U(x, y, \eta) &= A^{2p_k-4}(x, y, \eta), \\ V_\alpha(x, y, \eta) &= A^{p_k-2}A_\alpha^{p_k-2}(A^{p_k-2} + A_\alpha^{p_k-2}), \\ h_{\alpha,t}(y) &= y + t(\sigma_\alpha \cdot y - y) = y - t \langle y, \alpha \rangle, \end{aligned}$$

By mean value theorem we have

$$\begin{aligned} \mathcal{K}_j^{(\alpha)}(x, y) &= \int_{\mathbb{R}^N} \frac{1}{\langle y, \alpha \rangle} \frac{U(x, \sigma_\alpha \cdot y, \eta) - U(x, y, \eta)}{V_\alpha(x, y, \eta)} d\mu_x(\eta) \\ &= - \int_{\mathbb{R}^N} \int_0^1 \frac{\partial_\alpha U(x, h_{\alpha,t}(y), \eta)}{V_\alpha(x, y, \eta)} dt d\mu_x(\eta) \end{aligned}$$

and

$$\begin{aligned} \mathcal{K}_j^{(\alpha)}(x, y) - \mathcal{K}_j^{(\alpha)}(x, y_0) &= \int_{\mathbb{R}^N} \int_0^1 \int_0^1 \partial_{y-y_0} \left( \frac{\partial_\alpha U(x, h_{\alpha,t}(\cdot), \eta)}{V_\alpha(x, \cdot, \eta)} \right) (y_\theta) d\theta dt d\mu_x(\eta). \end{aligned} \quad (3.13)$$

Here the derivations are taken with respect to the variable  $y$ .

To simplify, let us denote by

$$A = A(x, y_\theta, \eta); \quad A_\alpha = A(x, \sigma_\alpha \cdot y_\theta, \eta)$$

Then using (3.10) and the fact

$$\|\eta - h_{\alpha,t}(y_\theta)\| \leq \max(\|\eta - y_\theta\|, \|\eta - \sigma_\alpha(y_\theta)\|),$$

we obtain

$$\begin{aligned} \left| \frac{\partial U}{\partial y_r}(x, h_{\alpha,t}(y_\theta), \eta) \right| &\leq C \left( A^{2p_k-5} + A_\alpha^{2p_k-5} \right) \\ \left| \frac{\partial^2 U}{\partial y_r \partial y_s}(x, h_{\alpha,t}(y_\theta), \eta) \right| &\leq C \left( A^{2p_k-6} + A_\alpha^{2p_k-6} \right), \quad r, s = 1, \dots, N. \end{aligned}$$

This gives us the following estimates

$$\left| \partial_\alpha U(x, h_{\alpha,t}(y_\theta), \eta) \right| \leq C \left( A^{2p_k-5} + A_\alpha^{2p_k-5} \right), \quad (3.14)$$

$$\left| \partial_{y-y_0} \left( \partial_\alpha U(x, h_{\alpha,t}(\cdot), \eta) \right) (y_\theta) \right| \leq C \|y - y_0\| \left( A^{2p_k-6} + A_\alpha^{2p_k-6} \right). \quad (3.15)$$

By (3.10) and (3.11), we also have

$$\left| \frac{\partial V_\alpha}{\partial y_r} ((x, y_\theta, \eta)) \right| \leq C A^{p_k-3} A_\alpha^{p_k-3} (A^{p_k-2} + A_\alpha^{p_k-2}) (A + A_\alpha).$$

The elementary inequality  $\frac{u+v}{u^\ell+v^\ell} \leq \frac{3}{u^{\ell-1}+v^{\ell-1}}$ ,  $u, v > 0$ ,  $\ell \geq 1$ , leads to

$$\begin{aligned} & \frac{\left| \partial_{y-y_0} V_\alpha(x, y_\theta, \eta) \right|}{V_\alpha^2(x, y_\theta, \eta)} \\ & \leq C \|y - y_0\| \frac{A_\alpha + A}{A^{p_k-1} A_\alpha^{p_k-1} (A^{p_k-2} + A_\alpha^{p_k-2})} \\ & \leq C \|y - y_0\| \frac{1}{A^{p_k-1} A_\alpha^{p_k-1} (A^{p_k-3} + A_\alpha^{p_k-3})}. \end{aligned} \quad (3.16)$$

Now (3.14), (3.15) and (3.16) yield

$$\begin{aligned} & \left| \partial_{y-y_0} \left( \frac{\partial_\alpha U(x, h_{\alpha,t}(\cdot), \eta)}{V_\alpha(x, \cdot, \eta)} \right) (y_\theta) \right| \\ & \leq C \|y - y_0\| \frac{A^{2p_k-6} + A_\alpha^{2p_k-6}}{A^{p_k-2} A_\alpha^{p_k-2} (A^{p_k-2} + A_\alpha^{p_k-2})} \\ & + C \|y - y_0\| \frac{A^{2p_k-5} + A_\alpha^{2p_k-5}}{A^{p_k-1} A_\alpha^{p_k-1} (A^{p_k-3} + A_\alpha^{p_k-3})} \\ & \leq C \|y - y_0\| \left( \frac{1}{A^2 A_\alpha^{p_k-2}} + \frac{1}{A^{p_k-2} A_\alpha^2} \right) \\ & + C \|y - y_0\| \left( \frac{1}{A A_\alpha^{p_k-1}} + \frac{1}{A^{p_k-1} A_\alpha} \right) \\ & \leq C \|y - y_0\| \left( \frac{1}{A^{p_k}} + \frac{1}{A_\alpha^{p_k}} \right) \end{aligned}$$

where in the last equality we have used the fact that  $\frac{1}{uv^{\ell-1}} \leq \frac{1}{u^\ell} + \frac{1}{v^\ell}$ ,  $u, v > 0$  and  $\ell \geq 1$ .

Thus, in view of (3.13),

$$\begin{aligned} & \left| \mathcal{K}_j^{(\alpha)}(x, y) - \mathcal{K}_j^{(\alpha)}(x, y_0) \right| \\ & \leq C \|y - y_0\| \int_0^1 \int_{\mathbb{R}^N} \left[ \frac{1}{A^{pk}(x, y_\theta, \eta)} + \frac{1}{A^{pk}(x, \sigma_\alpha y_\theta, \eta)} \right] d\mu_x(\eta) d\theta. \end{aligned}$$

Then by same argument as for  $\mathcal{K}_j^{(1)}$  we obtain

$$\int_{\min_{g \in G} \|g \cdot x - y\| > 2\|y - y_0\|} |\mathcal{K}_j^{(2)}(x, y) - \mathcal{K}_j^{(2)}(x, y_0)| dm_k(x) \leq C,$$

which established the condition (3.3) for the kernel  $\mathcal{K}_j^{(\alpha)}$  and furnishes the proof.  $\square$

As applications, we will prove a generalized Riesz and Sobolev inequalities

**Corollary 3.4** (Generalized Riesz inequalities). *For all  $1 < p < \infty$  there exists a constant  $C_p$  such that*

$$\|T_r T_s(f)\|_{k,p} \leq C_p \|\Delta_k f\|_{k,p}, \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^N), \quad (3.17)$$

where  $\Delta_k$  is the Dunkl laplacian:  $\Delta_k f = \sum_{r=1}^N T_r^2(f)$

*Proof.* From (2.1) and (3.1) one can see that

$$T_r T_s(f) = \mathcal{R}_r \mathcal{R}_s(-\Delta_k)(f), \quad r, s = 1 \dots N, \quad f \in \mathcal{S}(\mathbb{R}^N).$$

Then (3.17) is concluded by Theorem 3.3.  $\square$

**Corollary 3.5** (Generalized Sobolev inequality). *For all  $1 < p \leq q < 2\gamma(k) + N$  with  $\frac{1}{q} = \frac{1}{p} - \frac{1}{2\gamma(k) + N}$ , we have*

$$\|f\|_{q,k} \leq C_{p,q} \|\nabla_k f\|_{p,k} \quad (3.18)$$

for all  $f \in \mathcal{S}(\mathbb{R}^N)$ . Here  $\nabla_k f = (T_1 f, \dots, T_N f)$  and  $|\nabla_k f| = \left( \sum_{r=1}^N |T_r f|^2 \right)^{\frac{1}{2}}$ .

*Proof.* For all  $f \in \mathcal{S}(\mathbb{R}^N)$ , we write

$$\begin{aligned} \mathcal{F}_k(f)(\xi) &= \frac{1}{\|\xi\|} \sum_{r=1}^N \frac{-i\xi_r}{\|\xi\|} \left( i\xi_r \mathcal{F}_k(f)(\xi) \right) \\ &= \frac{1}{\|\xi\|} \sum_{r=1}^d \frac{-i\xi_r}{\|\xi\|} \left( \mathcal{F}_k(T_r f)(\xi) \right). \end{aligned}$$

This yields to the following identity

$$f = I_k^1 \left( \sum_{j=1}^N \mathcal{R}_j(T_j f) \right),$$

where

$$I_k^\beta(f)(x) = (d_k^\beta)^{-1} \int_{\mathbb{R}^N} \frac{\tau_y f(x)}{\|y\|^{2\gamma(k)+N-\beta}} dm_k(y),$$

here

$$d_k^\beta = 2^{-\gamma(k)-N/2+\beta} \frac{\Gamma(\frac{\beta}{2})}{\Gamma(\gamma(k) + \frac{N-\beta}{2})}.$$

Theorem 1.1 of [4] asserts that  $I_k^\beta$  a bounded operator from  $L^p(m_k)$  to  $L^q(m_k)$ . Then (3.18) follows from Theorem 3.3.  $\square$

## References

- [1] B. AMRI, A. GASMI & M. SIFI – “Linear and bilinear multiplier operators for the dunkl transform”, *Mediterranean Journal of Mathematics* **7** (2010), p. 503–521.
- [2] F. DAI & H. WANG – “A transference theorem for the dunkl transform and its applications”, *Journal of Functional Analysis* **258** (2010), p. 4052–4074.
- [3] C. F. DUNKL – “Differential–difference operators associated to reflection groups”, *Trans. Amer. Math.* **311** (1989), p. 167–183.
- [4] S. HASSANI, S. MUSTAPHA & M. SIFI – “Riesz potentials and fractional maximal function for the dunkl transform”, *J. Lie Theory* **19** (2009, no. 4), p. 725–734.
- [5] M. DE JEU – “The dunkl transform”, *Invent. Math.* **113** (1993), p. 147–162.



- [6] M. ROSLER – “Dunkl operators: theory and applications, in orthogonal polynomials and special functions (leuven, 2002),  $\mathbb{R}^n$ ”, *Lect. Notes Math.* **1817** (2003), p. 93–135.
- [7] ———, “A positive radial product formula for the dunkl kernel”, *Trans. Amer. Math. Soc.* **355** (2003), p. 2413–2438.
- [8] E. M. STEIN – *Harmonic analysis: Reals-variable methods, orthogonality and oscillatory integrals*, PrincetonS, New Jersey, 1993.
- [9] S. THANGAVELU & Y. XU – “Convolution operator and maximal function for dunkl transform”, *J. Anal. Math.* **97** (2005), p. 25–55.
- [10] ———, “Riesz transforms and riesz potentials for the dunkl transform”, *J. Comp. and Appl. Math.* **199** (2007), p. 181–195.

BECHIR AMRI  
Department of Mathematics  
University of Tunis  
Preparatory Institute of Engineer Studies  
of Tunis  
1089 Montfleury, Tunis, Tunisia  
bechir.amri@ipeit.rnu.tn

MOHAMED SIFI  
Department of Mathematics  
University of Tunis El Manar  
Faculty of Sciences of Tunis  
2092 Tunis El Manar, Tunis, Tunisia  
mohamed.sifi@fst.run.tn