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Additivity rates and PPT property forrandom quantum channels

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Abstract

Inspired by Montanaro's work, we introduce the concept of *additivity rates* of a quantum channel L, which give the first order (linear) term of the minimum output p-Rényi entropies of $L^{\otimes r}$ as functions of r. We lower bound the additivity rates of arbitrary quantum channels using the operator norms of several interesting matrices including partially transposed Choi matrices. As a direct consequence, we obtain upper bounds for the classical capacity of the channels. We study these matrices for random quantum channels defined by random subspaces of a bipartite tensor product space. A detailed spectral analysis of the relevant random matrix models is performed, and strong convergence towards free probabilistic limits is shown. As a corollary, we compute the threshold for random quantum channels to have the positive partial transpose (PPT) property. We then show that a class of random PPT channels violate generically additivity of the p-Rényi entropy for all $p \geq 30.95$.

1. Introduction

In this paper, we focus on three questions related to additivity properties of quantum channels. First, we introduce the concept of additivity rates by which we can bound additivity violations for tensor powers of channels. Then, we use these results to upper bound the classical capacity of

quantum channels. Finally, we prove the existence of PPT quantum channels violating the additivity of minimum output p-Rényi entropy. In the following three subsections, we introduce the above questions and present our main results.

1.1. **Additivity rates of quantum channels**

One of the most important conjectures in quantum information theory had been the additivity of p-Rényi entropy: for any (quantum) channels $L_{1,2}$,

$$H_p^{\min}(L_1 \otimes L_2) = H_p^{\min}(L_1) + H_p^{\min}(L_2) \tag{1.1}$$

for $1 \leq p \leq \infty$. Here, $H_p^{\min}(\cdot)$ is defined for a quantum channel L by

$$H_p^{\min}(L) = \min_X H_p(L(X)) \tag{1.2}$$

where X runs over all the quantum states, and the p-Rényi entropy $H_p(\cdot)$ is defined by

$$H_p(X) = \frac{1}{1 - p} \log \left(\text{Tr } X^p \right). \tag{1.3}$$

Note that $H_p(\cdot)$ becomes von Neumann entropy as $p \to 1$. This conjecture was made first for $p = 1$ in [41], and then for $p \in (1, \infty)$ in [1]. More detailed explanations about additivity questions can be found in [39].

These conjectures were disproved by Hayden and Winter for $1 < p < \infty$ [34] and by Hastings for $p = 1$ [33]. The case $p = 0$, and p close to 0, was disproved in [23]. Importantly, the violation in the case $p = 1$ implies that we can increase the classical capacity of some quantum channels by using entangled inputs [48]. Then, an important question comes to our mind: how much one can increase the classical capacity by using entanglement over as many quantum states as possible. Although this question on classical capacity should be most important, it is difficult to treat it directly. On the other hand, approaches via Rényi entropy only involve eigenvalues of matrices and this fact enables us to use random matrix and free probability to investigate the generic behavior of random quantum channels on this issue. In this paper, for natural class of random quantum channels, we bound additivity violation of Rényi entropy.

In fact, Montanaro [44] investigated on the limit of additivity violation for $p = \infty$ and extended the result to $1 \leq p < \infty$ by using the monotonicity

of the Schatten p-norms in p. In our paper, we study this problem first for $p = 2$ and then extend it to $0 \le p < 2$. His paper and ours both depend on estimate of norm of random matrices. However, those two random matrices are different and give different estimates. Detailed discussions on this matter are made in Section 6.4 and Section 8.2.

Informally, our main theorem on limitation of additivity violation can be stated as follows.

Theorem 1.1. *Consider a sequence of random quantum channels $L_n :$ $\mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, defined via random embeddings of $\mathbb{C}^d$ into $\mathbb{C}^n \otimes \mathbb{C}^k$, where k is a fixed parameter and $d \sim tnk$ for a fixed $t \in (0,1)$. Then, almost surely as $n \to \infty$, for all $p \in [0,2]$, there exist constants $\alpha_p \in [0,1]$ such that, for all $r \ge 1$,*

$$\frac{1}{r} H_p^{\min}(L_n^{\otimes r}) \ge \alpha_p H_p^{\min}(L_n). \tag{1.4}$$

The constants α_p satisfy the following relations

(1) When $0 < t < 1/2$ is a constant,

$$\alpha_p = o(1) + \frac{p-1}{2p}\left[1 + \frac{2\log 2 + \log(1-t)}{\log t}\right] \cdot \mathbf{1}_{(1,2]}(p). \tag{1.5}$$

(2) When k is large and $t \asymp k^{-\tau}$ with $\tau > 0$,

$$\alpha_p = o(1) + \begin{cases} \frac{p-1}{2p} & \text{if } 0 < \tau \le 1 - 1/p \\ \tau/2 & \text{if } 1 - 1/p \le \tau \le 2 \\ 1 & \text{if } \tau \ge 2. \end{cases} \tag{1.6}$$

The above statements hold for the complementary channels $L^C : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$, where the roles of $\mathbb{C}^k$ and $\mathbb{C}^n$ are swapped.

In the result above, the larger the constant α_p is, the more restrictive the additivity violation is. A precise definition of additivity rates is given in Definition 3.1 and more detailed estimates on α are made in Theorem 8.4. Also, our model of random quantum channels together with the idea of complementarity is described in detail at the beginning of Section 2.1.

1.2. **Range of capacity**

In [36, 46], the Holevo capacity of quantum channels, denoted by $\chi(\cdot)$, was proved to be the capacity of transmitting classical information without entangled inputs. Here, $\chi(\cdot)$ is defined for quantum channels L:

$$\chi(L) = \max_{\{p_i, X_i\}} \left[H_1\left(\sum_i p_i L(X_i)\right) - \sum_i p_i H_1(L(X_i)) \right], \qquad (1.7)$$

where the (p_i, X_i) are all possible ensembles with (p_i) and (X_i) being a probability distribution and quantum states, respectively. It is not difficult to see that

$$\chi(L) \leq H_1^{\max}(L) - H_1^{\min}(L). \qquad (1.8)$$

Here, $H_1^{\max}(L) = \max_X H_1(L(X))$; note that this quantity is trivially additive, $H_1^{\max}(L^{\otimes r}) = r H_1^{\max}(L)$. The equality is saturated, for example, if the channel L has covariant property [37], and it is also the case in our setting for a similar reason, which will be discussed later. Since the classical capacity, denoted by $\mathcal{C}_{cl}(\cdot)$ is obtained by regularizing Holevo capacity $\chi(\cdot)$ [36, 46], we can show the following estimate.

Theorem 1.2. *Consider a sequence of random quantum channels $L_n :$ $\mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, defined via random embeddings of $\mathbb{C}^d$ into $\mathbb{C}^n \otimes \mathbb{C}^k$, where k is a fixed parameter and $d \sim tnk$ for a fixed $t \in (0,1)$. Then, almost surely in the regime $1 \ll k \ll n$, the classical capacity is asymptotically bounded as follows:*

i) When $0 < t < 1/2$ is a constant, we have

$$\limsup_{n \to \infty} \mathcal{C}_{cl}(L_n) \leq \log k + \log 2 + \frac{1}{2} \log t(1-t) + o(1). \qquad (1.9)$$

ii) When $t \asymp k^{-\tau}$ and $0 < \tau \leq 2$, we have, for some constant $c > 0$

$$\limsup_{n \to \infty} \mathcal{C}_{cl}(L_n) \leq \left(1 - \frac{\tau}{2}\right) \log k + c. \qquad (1.10)$$

iii) When $t \asymp k^{-\tau}$ and $\tau > 2$, the classical capacity is almost surely bounded by a constant.

1.3. **PPT property and additivity violation**

Another topic treated in this current paper is the *positive partial transpose* property (PPT) for quantum channels; a quantum channel L is called *PPT* iff the partial transposition of its Choi matrix is positive semi-definite. The importance of PPT channels stems from their recent use in the proofs of superactivation for the quantum capacity, see [49]. Hence, it is interesting to investigate typical PPT/non-PPT property for random quantum channels. Also, we show that there exist PPT channels which violate additivity of Rényi p entropy. This result is interesting because all entanglement-breaking channels are proved to be additive [42, 47]. Note that the set of entanglement-breaking channels is contained by the set of PPT channels but for qubit channels, these sets are the same. The above two problems are investigated in Section 10, and we obtain the following results.

First, in Section 10.1 we have

Theorem 1.3. *Consider a sequence L_n of random quantum channels of parameters k, t, and let*

$$t_{PPT} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{k^2}}\right).$$ (1.11)

If $t \in (0, t_{PPT})$ then, almost surely as $n \to \infty$, the sequence L_n has the PPT property, whereas if $t \in (t_{PPT}, 1)$, then, almost surely, the sequence L_n does not have the PPT property. We say that the value t_{PPT} is a threshold for the PPT property of random quantum channels.

Second, we have

Theorem 1.4. *Consider a sequence of random quantum channels $L_n : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, defined via random embeddings of $\mathbb{C}^d$ into $\mathbb{C}^n \otimes \mathbb{C}^k$ and let $d \sim \frac{n}{4k}$. Suppose one of the following two procedures are made:*

- *fix $k \geq 76$ and take large enough p and n, or*

- *fix $p \geq 30.95$ and take large enough n and k*

then, typically L_n are PPT and violate additivity:

$$H_p^{\min}(L_n \otimes \bar{L}_n) < 2H_p^{\min}(L_n).$$ (1.12)

This theorem is divided into two theorems: Theorem 10.5 and Theorem 10.6 in Section 10.2.

1.4. **Structure of the paper**

The paper is divided roughly into two parts: Sections 3 and 4 deal with the general theory of additivity rates and their lower bounds, while Sections 5-10 deal with random quantum channels.

More precisely, after recalling some basic definitions and results in Section 2, we introduce in Section 3 one of the main topics of this paper: additivity rates of quantum channels. Then, in Section 4, we introduce lower bounds for minimum output Rényi entropies, which are additive with respect to tensor products. These results can be used to lower bound the additivity rates. In Sections 5 and 6 we study these bounds for random quantum channels. After recalling some known results about the minimum output entropies of random quantum channels in Section 7, we give lower bounds for the additivity rates of random quantum channels in Section 8, limiting the possible violations of the additivity of the minimum output entropies of these channels. Based on previous results, we present in Section 9 upper bounds for the classical capacitiy of (random) quantum channels, and in Section 10 examples of random channels which are PPT and violate the additivity of the minimum Rényi output entropies.

2. **Preliminaries**

In this section, we go through basic definitions and knowledge, which are needed through this current paper. We give definitions on quantum states, channels and entropy in Section 2.1, and then make quick overviews on graphical Weingarten calculus and free probability in Section 2.2 and Sec 2.3, respectively, as much as we need.

Let us start by introducing some notation. In this paper, the operator Tr denotes the usual, unnormalized trace. The reader may choose log to denote the logarithm in basis 2 or e, depending on her/his background. Finally, we use the following asymptotic notations for sequences:

$$x_n \sim y_n \iff \lim_{n\to\infty} x_n/y_n = 1 \tag{2.1}$$

$$x_n \asymp y_n \iff 0 < \liminf_{n\to\infty} x_n/y_n \le \limsup_{n\to\infty} x_n/y_n < \infty. \tag{2.2}$$

2.1. **Quantum states and channels**

In this paper we will consider quantum channels $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, defined via the Stinespring dilation [51]

$$L(X) = [\mathrm{Tr}_{\mathbb{C}^n} \otimes \mathrm{id}_{\mathbb{C}^k}](VXV^*), \qquad (2.3)$$

for an isometry $V : \mathbb{C}^d \to \mathbb{C}^n \otimes \mathbb{C}^k$. In this case, the dimensions of input, output and environment spaces are d, k and n, respectively. If we swap the roles of $\mathbb{C}^k$ and $\mathbb{C}^n$, we get another channel L^C, called the *complementary* channel [38, 40]. Outputs of this channel $L^C : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_n(\mathbb{C})$ share non-zero eigenvalues with those of the channel L as long as inputs are pure states. Hence, our results on entropy bounds are also shared by L and L^C. For such maps L and L^C we know that $L \otimes \mathrm{id}_{\mathbb{C}^m}$ and $L^C \otimes \mathrm{id}_{\mathbb{C}^m}$ are positive for any $m \geq 0$. This property is called *completely positivity*. Later, we shall consider random quantum channels obtained by choosing the isometry V randomly. The probability distribution of the random variable V will be the uniform one on the set of isometries, obtained by truncating a Haar-distributed unitary matrix $U \in \mathcal{U}(kn)$: V will consist of the first d columns of a $kn \times kn$ random Haar unitary matrix. For now, let us introduce a key concept in this paper, the *Choi matrix* of a quantum channel [11]. To a channel L, we associate its Choi matrix $C_L \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$, defined by

$$C_L = [L \otimes \mathrm{id}](E_d) = \sum_{i,j=1}^{d} L(e_i e_j^*) \otimes e_i e_j^*, \qquad (2.4)$$

where $E_d \in \mathcal{M}_{d^2}(\mathbb{C})$ is the (unnormalized) *maximally entangled state*

$$E_d = \sum_{i,j=1}^{d} e_i e_j^* \otimes e_i e_j^*. \qquad (2.5)$$

It is a classical result that a linear map L is completely positive if and only if its Choi matrix C_L is positive semidefinite.

Finally, we shall denote by $\mathcal{M}_d^{1,+}(\mathbb{C})$ the set of d-dimensional *quantum states*

$$\mathcal{M}_d^{1,+}(\mathbb{C}) = \{X \in \mathcal{M}_d(\mathbb{C}) : \mathrm{Tr}X = 1 \text{ and } X \geq 0\}. \qquad (2.6)$$

Let us now introduce the entropic quantities we are interested in. The *Shannon* entropy of a probability vector $x \in \mathbb{R}^k$, $x_i \geq 0$, $\sum_i x_i = 1$ is

defined by

$$H(x) = -\sum_{i=1}^{k} x_i \log x_i, \tag{2.7}$$

where we put $0 \log 0 = 0$. This quantity is extended, via functional calculus, to quantum states, where it is known as the *von Neumann entropy*. The *Rényi entropies* are a one-parameter generalizations of these quantities. They are defined for any $p \in [0, \infty]$, as follows:

$$H_p(x) = \begin{cases} \log \#\{i \,:\, x_i \neq 0\}, & \text{if } p = 0 \\ H(x), & \text{if } p = 1 \\ \dfrac{1}{1-p} \log \sum_{i=1}^{k} x_i^p, & \text{if } p \neq 0, 1, \infty \\ -\log \max\{x_i\}, & \text{if } p = \infty. \end{cases} \tag{2.8}$$

The same quantities are defined for quantum states, and satisfy $H_p(X) \in [0, \log k]$. In what follows, we shall use the following well-known result [7]:

Lemma 2.1. *For a fixed probability vector x (resp. quantum state X), the function*

$$[0, \infty] \ni p \mapsto H_p(x) \tag{2.9}$$

(resp. $p \mapsto H_p(X)$) is non-increasing in p. In a similar manner, for a fixed quantum channel L, the function $p \mapsto H_p^{\min}(L)$ is non-increasing in p.

Recall from the introduction that the *minimum output p-Rényi entropy* of a quantum channel L is defined by

$$H_p^{\min}(L) = \min_{X \in \mathcal{M}_d^{1,+}(\mathbb{C})} H_p(L(X)), \tag{2.10}$$

for an arbitrary entropy parameter $p \in [0, \infty]$. The functionals $H_p^{\min}$ are *sub-additive*, in the sense that for any quantum channels L, K, we have

$$H_p^{\min}(L \otimes K) \leq H_p^{\min}(L) + H_p^{\min}(K). \tag{2.11}$$

For a pair of quantum channels (L, K) such that $L \otimes K$ has no pure outputs, define the *relative violation of minimum output p-entropy* of the pair (L, K) by

$$v_p(L, K) := \frac{H_p^{\min}(L) + H_p^{\min}(K)}{H_p^{\min}(L \otimes K)} \in [1, \infty]. \tag{2.12}$$

With this notation, we call the pair (L, K) p-additive iff. $v_p(L, K) = 1$.

2.2. **The graphical Weingarten integration formula**

The model of random quantum channels we are interested in involves random isometries, which can be seen as truncated random Haar unitary matrices. Since our approach to understanding statistics of such channels is the moment method, we shall compute integrals of polynomials in the entries of unitary matrices. The main result here is the Weingarten formula, which was introduced by Weingarten [53] in the physics literature and rigorously developed by Collins [12], and Collins and Śniady [19].

Theorem 2.2. *Let n be a positive integer and $i = (i_1, \ldots, i_p)$, $i' = (i'_1, \ldots, i'_p)$, $j = (j_1, \ldots, j_p)$, $j' = (j'_1, \ldots, j'_p)$ be p-tuples of positive integers from $\{1, 2, \ldots, n\}$. Then, the following integral over the Haar measure of $\mathcal{U}_n$ can be evaluated as*

$$\int_{\mathcal{U}_n} U_{i_1 j_1} \cdots U_{i_p j_p} \bar{U}_{i'_1 j'_1} \cdots \bar{U}_{i'_p j'_p} \, dU$$
$$= \sum_{\alpha, \beta \in \mathcal{S}_p} \delta_{i_1 i'_{\alpha(1)}} \cdots \delta_{i_p i'_{\alpha(p)}} \delta_{j_1 j'_{\beta(1)}} \cdots \delta_{j_p j'_{\beta(p)}} \operatorname{Wg}(n, \alpha^{-1}\beta), \tag{2.13}$$

where the function Wg *is called the* Weingarten function *(see the next definition). If $p \neq p'$ then*

$$\int_{\mathcal{U}(n)} U_{i_1 j_1} \cdots U_{i_p j_p} \bar{U}_{i'_1 j'_1} \cdots \bar{U}_{i'_{p'} j'_{p'}} \, dU = 0. \tag{2.14}$$

For a permutation $\sigma \in \mathcal{S}_p$, $\#\sigma$ denotes the number of cycles of σ, and $|\sigma|$ is the length of σ, i.e. the minimal number of transpositions that multiply to σ. Note that the length function defines a distance on $\mathcal{S}_p$, via $d(\sigma, \tau) = |\sigma^{-1}\tau|$. Let us recall the definition of the unitary Weingarten function.

Definition 2.3. The unitary Weingarten function $\operatorname{Wg}(n, \sigma)$ is a combinatorial function which depends on a dimension parameter n and on a permutation σ in the symmetric group $\mathcal{S}_p$. It is the inverse of the function $\sigma \mapsto n^{\#\sigma}$ with respect to the following convolution operation:

$$\forall \sigma, \pi \in \mathcal{S}_p, \quad \sum_{\tau \in \mathcal{S}_p} \operatorname{Wg}(n, \sigma^{-1}\tau) n^{\#(\tau^{-1}\pi)} = \delta_{\sigma, \pi}. \tag{2.15}$$

In the large n limit (p is being kept fixed), it has the following asymptotics

$$\operatorname{Wg}(n, \sigma) = n^{-(p+|\sigma|)}(\operatorname{Mob}(\sigma) + O(n^{-2})), \tag{2.16}$$

where the Möbius function Mob is multiplicative on the cycles of σ and its value on a r-cycle is

$$(-1)^{r-1}\mathrm{Cat}_{r-1}, \qquad (2.17)$$

where $\mathrm{Cat_r}$ are the Catalan numbers. Note that we omit the dimension in the Weingarten function when there is no confusion and write $\mathrm{Wg}(\sigma)$ for $\mathrm{Wg}(n, \sigma)$. Also, we use the notation $\mathrm{Mob}(\alpha, \beta) = \mathrm{Mob}(\alpha^{-1}\beta)$.

When applying the above integration formula, especially in the cases where the degree of the polynomial to be integrated is high, one has to deal often with sums indexed by a large set of indices. Computing such sums is a tedious task, so we use here a graphical formulation of the Weingarten formula, introduced in [22]. Here we sketch the main ideas, referring the reader to original work [22] for the technical details. This method has been used recently in relation to channel capacities [16, 13, 20, 30], entanglement theory [4, 18] and condensed matter physics [14].

The Weingarten graphical calculus builds up on the *tensor diagrams* introduced by theoretical physicists and adds to it the ability to perform averages over diagrams containing Haar unitary matrices. In the graphical formalism, tensors (vectors, linear forms, matrices, etc.) are represented by *boxes*, see Figure 2.1, left diagram. To each box, one attaches labels of different shapes, corresponding to vector spaces. The labels can be filled (black) or empty (white) corresponding to spaces or their duals: a (p, q)-tensor will be represented by a box with p black labels and q white labels attached. The example in Figure 2.1, left corresponds to a (square) matrix $A \in \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C})$.

Besides boxes, our diagrams contain *wires*, which connect the labels attached to boxes. Each wire corresponds to a tensor contraction between a vector space V and its dual V^* ($V \times V^* \to \mathbb{C}$). See Figure 2.1 for the example of the partial trace. A *diagram* is simply a collection of such boxes and wires and corresponds to an element in a tensor product space (which might be degenerate, i.e. the scalars $\mathbb{C}$).

Let us now describe briefly how one computes expectation values of such diagrams containing boxes U corresponding to Haar-distributed random unitary matrices. The idea in [22] was to implement in the graphical formalism the Weingarten formula in Theorem 2.2. Each pair of permutations (α, β) in (2.13) will be used to eliminate U and $\bar{U}$ boxes and wires will be added between the black, resp. white, labels of the box U with index i and the black, resp. white, labels of the box $\bar{U}$ with index $\alpha(i)$, resp.

FIGURE 2.1. Diagram for a matrix A acting on a tensor product space and for its partial trace $[\mathrm{id}\otimes\mathrm{Tr}](A)$, obtained by contracting with a wire the labels corresponding to the second tensor factor.

$\beta(i)$. In this way, for each pair of permutations, one obtains a new diagram $\mathcal{D}_{\alpha,\beta}$, called a *removal* of the original diagram corresponding to α, β. The graphical Weingarten formula is described in the following theorem [22].

Theorem 2.4. *If $\mathcal{D}$ is a diagram containing boxes $U, \bar{U}$ corresponding to a Haar-distributed random unitary matrix $U \in \mathcal{U}(n)$, the expectation value of $\mathcal{D}$ with respect to U can be decomposed as a sum of removal diagrams $\mathcal{D}_{\alpha,\beta}$, weighted by Weingarten functions:*

$$\mathbb{E}_U(\mathcal{D}) = \sum_{\alpha,\beta} \mathcal{D}_{\alpha,\beta}\, \mathrm{Wg}(n, \alpha^{-1}\beta). \qquad (2.18)$$

Since the above Weingarten formula is written as a sum over permutations we review next some additional properties of the symmetric group endowed with the distance $d(\alpha, \beta) = |\alpha^{-1}\beta|$. The function $|\cdot|$ has nice properties, for example, $|\alpha^{-1}| = |\alpha|$ and $|\alpha\beta| = |\beta\alpha|$; we refer the readers to [45] for more details. For three permutations α, β, γ the triangle inequality holds:

$$|\alpha^{-1}\gamma| \leq |\alpha^{-1}\beta| + |\beta^{-1}\gamma|. \qquad (2.19)$$

When the bound above is saturated, we say that β is on a *geodesic* between α and γ, and write $\alpha - \beta - \gamma$. When γ is the full cycle permutation, $\gamma = (p \cdots 3\,2\,1)$, permutations lying on the geodesic between id and γ are simply called *geodesics permutations*. In [45, Proposition 23.23], it is shown that geodesic permutations are in bijection with *non-crossing partitions*. Recall that a partition π of $\{1, \ldots, p\}$ is said to be non-crossing if

$$\forall i < j < k < l, \quad i \sim k \text{ and } j \sim l \implies i \sim j \sim k \sim l, \qquad (2.20)$$

where $\sim$ denotes the equivalence relation on $\{1, \ldots, p\}$ induced by π. We denote by $NC(p)$ the set of non-crossing partitions on p elements. Moreover, the isomorphism between geodesic permutations and non-crossing

partitions respects many combinatorial properties of the objects, such as the number of cycles (resp. blocks); see [45, Section 23] for more details.

2.3. Some elements of free probability

An excellent reference for the theory of free probability is [45]; we recall now only some basic facts from this theory needed in the current paper.

A C^* probability space is a unital C^*-algebra $\mathcal{A}$ equipped with a state τ, which gives a norm; $\|a\|_\tau = \lim_{p\to\infty}(\tau(a^p))^{1/p}$. Such a C^* probability space is denoted by $(\mathcal{A}, \tau, \|\cdot\|_\tau)$.

The convergence of the eigenvalues of random matrices can be stated in the language of C^* probability spaces as follows. Note that we define two types of convergence: the convergence in distribution (which is the convergence of all moments) and the strong convergence (which implies, in particular, the convergence of the extreme eigenvalues of the matrices). In this paper, we are interested in the operator norms of random matrices, and the usual convergence in distribution does not guarantee the convergence of the norms in the case when the size of the matrices grows; hence we shall make use of the strong convergence.

Definition 2.5. Suppose we have C^*-probability spaces: $(\mathcal{A}, \tau, \|\cdot\|_\tau)$ and $(\mathcal{A}_N, \tau_N, \|\cdot\|_{\tau_N})$ with $N \in \mathbb{N}$, where τ and τ_N are faithful traces. For l-tuple elements $a = (a_1, \ldots, a_l)$ in $\mathcal{A}$ and $a^{(N)} = (a_1^{(N)}, \ldots, a_l^{(N)})$ in $\mathcal{A}^{(N)}$,

i) we say $a^{(N)}$ converges to a in distribution if

$$\lim_{N\to\infty} \tau_N[P(a^{(N)}, a^{(N)*})] = \tau[P(a, a^*)], \qquad (2.21)$$

ii) we say $a^{(N)}$ converges to a strongly in distribution if in addition

$$\lim_{N\to\infty} \|[P(a^{(N)}, a^{(N)*})]\|_{\tau_N} = \|P(a, a^*)\|_\tau. \qquad (2.22)$$

Here, P is any polynomial in non-commuting $2l$ variables.

The strong asymptotic freeness of random unitary matrices and deterministic matrices has been proved by Collins and Male:

Proposition 2.6 ([15]). *Suppose we have C^*-probability spaces: $(\mathcal{A}, \tau, \|\cdot\|_\tau)$ and $(\mathcal{M}_N(\mathbb{C}), \tau_N, \|\cdot\|_{\tau_N})$ with $N \in \mathbb{N}$. Here, τ is a faithful trace and τ_N is the usual normalized trace on the $N \times N$ matrix space $\mathcal{M}_N(\mathbb{C})$. Take*

- *a p-tuple of free Haar unitary elements $u = (u_1, \ldots, u_p)$ in $\mathcal{A}$, and*

- *a p-tuple of i.i.d. Haar-distributed unitary matrices $U^{(N)} = (U_1^{(N)}, \ldots, U_p^{(N)})$ in $\mathcal{M}_N(\mathbb{C})$.*

Suppose we are given

- *a q-tuple of elements $y = (y_1, \ldots, y_q)$ free from u in $\mathcal{A}$, and*

- *a q-tuple of matrices $Y^{(N)} = (Y_1^{(N)}, \ldots, Y_q^{(N)})$ independent from $U^{(N)}$ in $\mathcal{M}_N(\mathbb{C})$.*

such that $Y^{(N)}$ converges to y strongly in distribution. Then, almost surely $(U^{(N)}, Y^{(N)})$ converges to (u, y) strongly in distribution.

The following useful statement was proved by Male:

Proposition 2.7 (Proposition 7.3 in [43])**.** *Suppose we have C^*-probability spaces: $(\mathcal{A}, \tau, \|\cdot\|_\tau)$ and $(\mathcal{A}_N, \tau_N, \|\cdot\|_{\tau_N})$ with $N \in \mathbb{N}$, where τ and τ_N are faithful traces. Take*

- *a l-tuple of self-adjoint elements $z = (z_1, \ldots, z_l)$ in $\mathcal{A}$, and*

- *a l-tuple of self-adjoint elements $z^{(N)} = (z_1^{(N)}, \ldots, z_l^{(N)})$ in $\mathcal{A}_N$.*

If we assume that $z^{(N)}$ converges to z strongly in distribution, then we have strong convergence in the following sense: for any polynomial P in l non-commuting variables with coefficients in $\mathcal{M}_k(\mathbb{C})$,

$$\lim_{N \to \infty} \|P(z^{(N)})\|_{\tau_k \otimes \tau_N} = \|P(z)\|_{\tau_k \otimes \tau} \tag{2.23}$$

Note that in the above result, one can drop the self-adjointness assumption by considering real and imaginary parts of the operators involved.

We prove next a simple lemma about push-forwards of free additive convolution powers of probability measures and we recall a well-known result about the free multiplicative convolution product of Bernoulli distributions $b_t = (1 - t)\delta_0 + t\delta_1$. Recall that given two free elements a, b having distributions μ, ν, the distributions of $a + b$ and respectively $a^{-1/2}ba^{-1/2}$ are denoted by $\mu \boxplus \nu$, respectively $\mu \boxtimes \nu$, and they are called the free additive (resp. multiplicative) convolutions of μ and ν (for the latter, we require $a \geq 0$). We denote by $f_\# \mu$ the push-forward of a measure μ by a measurable function f: if the random variable X has distribution μ, then $f(X)$ has distribution $f_\# \mu$.

Lemma 2.8. *Let μ be a compactly supported probability measure on $\mathbb{R}$ so that, for any $T \geq 1$, $\mu^{\boxplus T}$ is well-defined. Then, we have, for any $a, b \in \mathbb{R}$*

$$((x \mapsto ax + b)_{\#}\mu)^{\boxplus T} = (x \mapsto ax + Tb)_{\#}(\mu^{\boxplus T}). \tag{2.24}$$

Proof. First, let v_μ be an element in the C^*-algebra which gives the probability measure μ so that $av_\mu + b$ induces the probability measure μ_{ax+b}. Then, first by using multi-linear property of cummulant,

$$\kappa_n(av_\mu) = \kappa_n(av_\mu, \ldots, av_\mu) = a^n \kappa_n(v_\mu). \tag{2.25}$$

Also, shift does not change cummulants κ_n except for the case when $n = 1$: $\kappa_1(av + b) = \kappa_1(av) + b$. Therefore,

$$\begin{aligned}
\kappa_n((av_\mu + b)^{\boxplus T}) &= T[a^n \kappa_n(v_\mu) + b\delta_{1,n}] \\
&= \kappa_n(av_\mu^{\boxplus T}) + Tb\delta_{1,n} = \kappa_n(av_\mu^{\boxplus T} + Tb).
\end{aligned} \tag{2.26}$$

This completes the proof. $\qquad\square$

Proposition 2.9. *The free multiplicative convolution of two Bernoulli distributions b_s, b_t (with $s, t \in [0,1]$) is given by*

$$b_s \boxtimes b_t = (1 - \min(s,t))\delta_0 + \max(s + t - 1, 0)\delta_1$$
$$+ \frac{\sqrt{(\varphi^+(s,t) - x)(x - \varphi^-(s,t))}}{2\pi x(1-x)} \mathbf{1}_{[\varphi^-(s,t),\varphi^+(s,t)]}(x) \, dx, \tag{2.27}$$

where the bounds of the a.c. part of the support are given by

$$\varphi^\pm(s,t) = s + t - 2st \pm 2\sqrt{st(1-s)(1-t)}. \tag{2.28}$$

Equivalently, for any $T \geq 1$,

$$b_s^{\boxplus T} = \max(0, 1 - Ts)\delta_0 + \max(0, 1 - T(1-s))\delta_T$$
$$+ \frac{T\sqrt{(\gamma^+(s,T) - x)(x - \gamma^-(s,T))}}{2\pi x(T-x)} \mathbf{1}_{[\gamma^-(s,T),\gamma^+(s,T)]}(x) \, dx, \tag{2.29}$$

where $\gamma^\pm(s,T) = (T-2)s + 1 \pm 2\sqrt{(T-1)s(1-s)}$. Note that $\varphi^\pm(s,t) = t\gamma^\pm(s, 1/t)$.

Proof. The first claim is taken from [52, Example 3.6.7], while the second one follows from the fact that (see [45, Exercise 14.21])

$$(1 - T^{-1})\delta_0 + T^{-1}b_s^{\boxplus T} = b_s \boxtimes \left((1 - T^{-1})\delta_0 + T^{-1}\delta_T\right). \tag{2.30}$$

$\qquad\square$

3. **Additivity rates for quantum channels**

In this section, we discuss how the functional $H_p^{\min}(\cdot)$ behaves with respect to tensor products. The ideas and results developed here will be applied to random quantum channels later. Our inspiration comes from [44], where a multiplicative version of the additivity rates was established.

3.1. **Definition and basic properties**

First, as is explained in Section 1.1, because of non-additivity properties of quantum channels, we do not know how $H_p^{\min}(L^{\otimes r})$ behaves as r grows. So, we introduce a notion which quantifies $H_p^{\min}(L^{\otimes r})$ in terms of $H_p^{\min}(L)$:

Definition 3.1. For a quantum channel L and an entropy parameter $p \in [0, \infty]$, define the *p-additivity rate* of L by

$$\alpha_p(L) = \sup \left\{ a \in [0,1] \ : \ \liminf_{r \to \infty} \frac{1}{r} H_p^{\min}(L^{\otimes r}) \geq a H_p^{\min}(L) \right\}. \qquad (3.1)$$

Different characterizations as well as some basic properties of p-additivity rates can readily be obtained from basic properties of the entropy functionals.

Proposition 3.2. *The p-additivity rate of a quantum channel L can be characterized in the following equivalent ways:*

$$\alpha_p(L) = \sup \left\{ a \in [0,1] \ : \ \forall r \geq 1, \quad \frac{1}{r} H_p^{\min}(L^{\otimes r}) \geq a H_p^{\min}(L) \right\} \qquad (3.2)$$

$$= \begin{cases} \displaystyle\lim_{r \to \infty} \frac{H_p^{\min}(L^{\otimes r})}{r H_p^{\min}(L)} & , \quad \text{if } H_p^{\min}(L) > 0 \\ 1 & , \quad \text{if } H_p^{\min}(L) = 0. \end{cases} \qquad (3.3)$$

$$= \begin{cases} \displaystyle\inf_{r \geq 1} \frac{H_p^{\min}(L^{\otimes r})}{r H_p^{\min}(L)} & , \quad \text{if } H_p^{\min}(L) > 0 \\ 1 & , \quad \text{if } H_p^{\min}(L) = 0. \end{cases} \qquad (3.4)$$

Proof. The statements follow from Fekete's sub-additive lemma [50, Lemma 1.2.1] and (2.11): the $\liminf$ in (3.1) is actually a limit and it is equal to the infimum of the sequence $H_p^{\min}(L^{\otimes r})/r$. The zero entropy case follows from the fact that if the channel L has zero minimum output entropy for some p, then the same holds for all tensor powers $L^{\otimes r}$ because

$0 \leq H_p^{\min}(L^{\otimes r}) \leq rH_p^{\min}(L) = 0$. Moreover, such a channel is additive with any channel (see [27, Lemma 1]). $\qquad\square$

Proposition 3.3. *The p-additivity rate functionals have the following set of properties:*

(1) *The additivity relation $H_p^{\min}(L^{\otimes r}) = rH_p^{\min}(L)$ holds for all $r \geq 1$ if and only if $\alpha_p(L) = 1$.*

(2) *Monotonicity with respect to tensor powers:*

$$\alpha_p(L^{\otimes s}) \geq \alpha_p(L), \tag{3.5}$$

for all integer tensor powers $s \geq 1$.

(3) *Convex-like behaviour with respect to tensor products:*

$$\begin{aligned}\alpha_p(L \otimes K) &\leq v_p(L, K)\left[t\alpha_p(L) + (1-t)\alpha_p(K)\right]\\ &\leq v_p(L, K)\max\left\{\alpha_p(L), \alpha_p(K)\right\},\end{aligned} \tag{3.6}$$

where $v_p(L, K)$ is the relative violation of the minimum p-output entropy (2.12) and $t = H_p^{\min}(L)/[H_p^{\min}(L) + H_p^{\min}(K)] \in [0, 1]$; if $H_p^{\min}(L) = H_p^{\min}(K) = 0$, just put $t = 0$.

(4) *Additivity violations yield upper bounds:*

$$\alpha_p(L) \leq \frac{1}{v_p(L, L)}, \quad \forall p \in [0, \infty]. \tag{3.7}$$

Proof. The first property follows directly from the definition. For the second one, in the case when $H_p^{\min}(L) > 0$, write

$$\alpha_p(L^{\otimes s}) = \lim_{r \to \infty} \frac{H_p^{\min}((L^{\otimes s})^{\otimes r})}{rH_p^{\min}(L^{\otimes s})} \geq \lim_{r \to \infty} \frac{H_p^{\min}(L^{\otimes sr})}{srH_p^{\min}(L)} = \alpha_p(L). \tag{3.8}$$

The last two statements follow from the definition of the relative violation v_p. $\qquad\square$

Remark 3.4. In Proposition 3.3, we set an upper bound for $\alpha_p(L)$ by using the relative violation $v_p(L, L)$. However, in Section 8.3, we lower bound $\alpha_p(L \otimes \bar{L})$ by using $v_p(L, \bar{L})$, where $\bar{L}$ is the complex conjugate of L.

3.2. **Examples: the Werner-Holevo and the antisymmetric channels**

In Proposition 3.3, we have seen that any p-additive channel L has unit additivity rate $\alpha_p(L) = 1$. We discuss next some examples of non-additive channels. Below, we shall denote with $A^\top$ the transposition of a matrix A.

Example 3.5. The Werner-Holevo channel $W_d : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$,

$$W_d(X) = \frac{1}{d-1}[\text{Tr}(X)I_d - X^\top] \tag{3.9}$$

is the first known example of a quantum channel that violates the additivity of the minimum p-output Rényi entropy. In [54], it has been shown that W_3 violates the additivity for any value $p > 4.79$. From [54] we have explicitly

$$v_p(W_d, W_d) \tag{3.10}$$

$$\geq \frac{2\log(d-1)}{\log\left\{(d^2-1)[(1-2/d)/(d-1)^2]^p + [(2-2/d)/(d-1)^2]^p\right\}/(p-1)},$$

from which we can infer the following upper bounds for additivity rates (see (3.7)):

$$\alpha_5(W_3) \leq \frac{1}{v_5(W_3, W_3)} \leq 0.989$$

$$\alpha_\infty(W_3) \leq \frac{1}{v_\infty(W_3, W_3)} \leq \frac{\log 3}{\log 4}. \tag{3.11}$$

Example 3.6. In [31], the authors construct explicit counterexamples to the additivity relation, for all values $p > 2$, by considering the natural embedding of the anti-symmetric subspace $\Lambda^2(\mathbb{C}^d)$ into $\mathbb{C}^d \otimes \mathbb{C}^d$. This yields a channel $A_d : \mathcal{M}_{d(d-1)/2}(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$. From [31], one has

$$v_p(A_d, A_d) \geq \frac{2\log 2}{\frac{p}{p-1}\log\left[2\frac{d}{d-1}\right]}. \tag{3.12}$$

From the above relation, using (3.7), one gets, for example

$$\forall p > 2, \quad \alpha_p(A_d) \leq \frac{1}{v_p(A_d, A_d)} = \frac{p}{2(p-1)}(1 + \log_2[d/(d-1)])$$

$$\xrightarrow[d\to\infty]{} \frac{p}{2(p-1)} < 1. \tag{3.13}$$

4. **Additive bounds for the Rényi entropies via (partial) traces and transpositions**

In this section we introduce several *additive* bounds for the Rényi entropies of quantum channels (we focus on $p = 2, \infty$) that we obtain by considering the operator norm of the vectorized version of the isometry defining the channel, after applying one or several traces or transpositions. We perform an exhaustive study of this method, concluding that the method yields 5 non-trivial bounds, including the one studied by Montanaro [44]. The key point is that the bounds we are providing are additive with respect to tensor powers of channels, so they can be used to bound the additivity rates defined in the previous sections. Interesting bounds for the classical capacity of quantum channels can be obtained from these bounds.

Recall that the 2, resp. ∞-minimum output Rényi entropies of a quantum channel L are

$$H_2^{\min}(L) = \min_X \frac{\log \mathrm{Tr}\left[L(X)^2\right]}{1 - 2} = -\max_X \log \mathrm{Tr}\left[L(X)^2\right] \qquad (4.1)$$

$$H_\infty^{\min}(L) = -\max_X \log \|L(X)\|, \qquad (4.2)$$

where X runs over all the input quantum states. In what follows, we shall write $\Theta(A) = A^\top$ for the transposition map, which is an involution on matrix algebras. Moreover, for bi-partite matrices $B \in \mathcal{M}_p(\mathbb{C}) \otimes \mathcal{M}_q(\mathbb{C})$, we write B^Γ for the *partial transposition* of B with respect to the second subsystem,

$$B^\Gamma = [\mathrm{id}_p \otimes \Theta_q](B). \qquad (4.3)$$

Equivalently, the partial transposition operation can be defined on simple tensors by $(B_1 \otimes B_2)^\Gamma = B_1 \otimes B_2^\top$.

4.1. **Quantities arising from vectorized isometries**

The starting point of our study is the vectorization of the isometry $V : \mathbb{C}^d \to \mathbb{C}^n \otimes \mathbb{C}^k$ defining the channel L as in (2.3). To this isometry we associate its vectorization $v \in \mathbb{C}^n \otimes \mathbb{C}^k \otimes \mathbb{C}^d$ (which is a tripartite tensor) by the relation

$$v = \sum_{i=1}^n \sum_{j=1}^k \sum_{s=1}^d \langle e_i \otimes f_j, V g_s \rangle e_i \otimes f_j \otimes g_s, \qquad (4.4)$$

where $\{e_i\}$, $\{f_j\}$, $\{g_s\}$ are orthonormal bases of respectively $\mathbb{C}^n$, $\mathbb{C}^k$, $\mathbb{C}^d$. The Choi matrix C_L of the channel L (see (2.4)) is related to the third order tensor v by the partial trace operation:

$$C_L = [\mathrm{Tr}_n \otimes \mathrm{id_k} \otimes \mathrm{id_d}](vv^*). \tag{4.5}$$

For a graphical representation of the vectorization v and its relation to the Choi matrix C_L, see Figure 4.1. Note also that $\|v\|^2 = \mathrm{Tr}(V^*V) = d$.
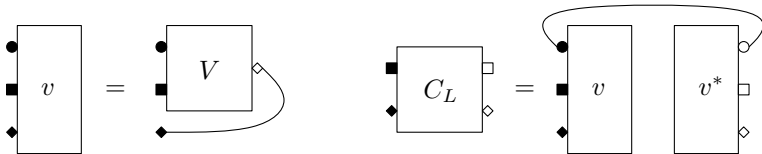

FIGURE 4.1. The vectorization v of the isometry V defining a quantum channel is an order 3 tensor. The partial trace of the orthogonal projection of v with respect to the first tensor factor gives the Choi matrix of the channel. In both figures, round shaped labels correspond to $\mathbb{C}^n$, squares to $\mathbb{C}^k$, and diamonds to $\mathbb{C}^d$.

We shall now apply different operations to the orthogonal projection on v and take the operator norm of the resulting matrix, in order to obtain a scalar quantity $B(v)$; in the next subsection, we show that some of these quantities are (additive) bounds for the 2 or the ∞ minimum output Rényi entropies of quantum channels. We are interested in three operations: the identity id, the trace Tr, and the transposition Θ. These operators will act on the three tensor factors of vv^*, and we shall denote by $B_{QRS} = B_{QRS}(L)$ the quantity

$$B_{QRS} = \|[Q_n \otimes R_k \otimes S_d](vv^*)\|, \tag{4.6}$$

where $Q, R, S \in \{\mathrm{id}, \mathrm{Tr}, \Theta\}$. As an illustration, in the case of the Choi matrix, where we apply the trace map on the first factor and the identity on the other two, we have $B_{\mathrm{Tr,id,id}}(L) = \|C_L\|$.

We gather in Table 4.1 the $27 = 3^3$ possibilities we obtain by applying on each of the tensor factors the maps above. We obtain 5 different bounds (B_I, B_C, $B_{C\Gamma}$, $B_{Cc\Gamma}$, and $B_{M\Gamma}$) which appear several times in the list, as well as some trivial, constant bounds which do not depend on the channel.

Let us first comment on the equalities in Table 4.1. These follow from the following basic facts about the operator norm. First, we note that the

TABLE 4.1. Bounds obtained by applying traces and transpositions to the orthogonal projection on v. The defining occurrences of the bounds appear in red, while the other occurrences appear in blue; trivial, constant bounds are black.

N°	$\mathbb{C}^n$	$\mathbb{C}^k$	$\mathbb{C}^d$	Norm
1.	id	id	id	$= d$
2.	id	id	Tr	$= 1$
3.	id	id	Θ	$= 1$
4.	id	Tr	id	$= B_I$
5.	id	Tr	Tr	$= B_C$
6.	id	Tr	Θ	$=: B_{Cc\Gamma}$
7.	id	Θ	id	$= B_I$
8.	id	Θ	Tr	$=: B_{M\Gamma}$
9.	id	Θ	Θ	$= B_C$
10.	Tr	id	id	$=: B_C$
11.	Tr	id	Tr	$=: B_I$
12.	Tr	id	Θ	$=: B_{C\Gamma}$
13.	Tr	Tr	id	$= 1$
14.	Tr	Tr	Tr	$= d$

N°	$\mathbb{C}^n$	$\mathbb{C}^k$	$\mathbb{C}^d$	Norm
15.	Tr	Tr	Θ	$= 1$
16.	Tr	Θ	id	$= B_{C\Gamma}$
17.	Tr	Θ	Tr	$= B_I$
18.	Tr	Θ	Θ	$= B_C$
19.	Θ	id	id	$= B_C$
20.	Θ	id	Tr	$= B_{M\Gamma}$
21.	Θ	id	Θ	$= B_I$
22.	Θ	Tr	id	$= B_{Cc\Gamma}$
23.	Θ	Tr	Tr	$= B_C$
24.	Θ	Tr	Θ	$= B_I$
25.	Θ	Θ	id	$= 1$
26.	Θ	Θ	Tr	$= 1$
27.	Θ	Θ	Θ	$= d$

operator norm is invariant under *global* transposition. Moreover, consider a rank one projection xx^* acting on a bipartite Hilbert space $\mathbb{C}^p \otimes \mathbb{C}^q$ (here, p is the product of some, possible empty, subset of $\{n, k, d\}$ and $q = nkd/p$). Then, one has

$$\|[\mathrm{id}_p \otimes \mathrm{Tr}_q](xx^*)\| = \|[\mathrm{Tr}_p \otimes \mathrm{id}_q](xx^*)\| = \|[\mathrm{id}_p \otimes \Theta_q](xx^*)\|$$
$$= \|[\Theta_p \otimes \mathrm{id}_q](xx^*)\| = \lambda_1, \tag{4.7}$$

where λ_1 is the largest Schmidt coefficient of the vector $x \in \mathbb{C}^p \otimes \mathbb{C}^q$. The statement above is well known in the case of the partial traces. In the case of the partial transpositions, it is also straightforward, see [35, Lemma III.3]. Let us use these simple facts to prove the equality cases in Table 4.1.

First, the constant value d appears 3 times in the table, and the equality of the three quantities is precisely equation (4.7) with the choice $p = 1$, $q = nkd$. The same relation (4.7) implies the equality of the quantities in the rows 2, 3, 13, and 25 with the choice $p = nk$, $q = d$; the common value is the operator norm of the orthogonal projection on the image of V (see row 2), which is 1. Moreover, applying a global transposition to the quantities on rows 15 and 26, we obtain, respectively, the quantities on rows 13 and 2, which are 1; we have shown thus the equality of all the quantities which give 1 in Table 4.1.

The choice $p = k$, $q = nd$ in (4.7) yields the equality of the rows 4,7,11,21; the common value is the operator norm output of the identity $\|L(I)\|$, as seen from row 11. Applying a global transposition on the rows 17 and 24, we obtain the rows 11 and 4, giving the same value $B_I := \|L(I)\|$.

Similarly, the choice $p = n$, $q = kd$ in (4.7) yields the equality of the rows 5, 9, 10, 19; the common value is the operator norm of the Choi matrix $\|C_L\|$, as seen from row 10. Applying a global transposition on the rows 23 and 18, we obtain the rows 5 and 10, giving the same value $B_C := \|C_L\|$.

The 6 remaining cases are the ones corresponding to the 6 permutations of the operators id, Tr, and Θ acting on the 3 legs of the tensor v. The fact that each of the quantities $B_{C\Gamma}$, $B_{Cc\Gamma}$, and $B_{M\Gamma}$ appears twice is a consequence of the invariance of the norm by global transposition. The quantity $B_{C\Gamma}$ corresponds to the partial transposition of the Choi matrix of L: $B_{C\Gamma} := \|C_L^\Gamma\|$. The quantity $B_{Cc\Gamma}$ corresponds to the partial transposition of the Choi matrix of the complementary channel L^c: $B_{Cc\Gamma} := \|C_{L^c}^\Gamma\|$. Finally, $B_{M\Gamma}$ corresponds to Montanaro's bound [44, Fact 1 and Proposition 4]: it is the norm of the partial transposition of the orthogonal projection on the image of V, $B_{M\Gamma} := \|M_L^\Gamma\|$, where M is the projection on the image of V, $M_L = VV^*$.

Let us note the important fact that the last two bounds we considered, $\|C_{L^c}^\Gamma\|$ and $\|M_L^\Gamma\|$, are not defined in terms of the channel L, but in terms of the isometry V, or its vectorized version v (whereas the first two are defined in terms of the Choi matrix of L). In the next lemma, we show that these two quantities do not depend on the actual choice of the isometry V defining the channel, but only on the channel itself, so the notations $B_{Cc\Gamma}(L)$ and $B_{M\Gamma}(L)$ are justified.

Lemma 4.1. *Consider a fixed quantum channel* $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ *and let* $V : \mathbb{C}^d \to \mathbb{C}^n \otimes \mathbb{C}^k$ *be a Stinespring isometry for* L, *i.e.* $L(X) = [\mathrm{Tr}_n \otimes \mathrm{id}_k](VXV^*)$. *Then, the bounds* $B_{Cc\Gamma} = \|C_{L^c}^{\Gamma}\|$ *and* $B_{M\Gamma} = \|M_L^{\Gamma}\|$, *defined using* V, *do not depend on the choice of the isometry* V.

Proof. Let r be the rank of the Choi matrix C_L of L and consider a *minimal* purification $v_0 \in \mathbb{C}^r \otimes (\mathbb{C}^k \otimes \mathbb{C}^d)$ of C_L. The vectorized version of the Stinespring isometry V is another purification of C_L (not necessarily minimal). Since any purification of a quantum mixed state is related to a minimal one by an isometry, there exist an isometric operator $W : \mathbb{C}^r \to \mathbb{C}^n$ such that $v = (W \otimes I_{kd})v_0$. We now have that

$$vv^* = (W \otimes I_{kd})v_0 v_0^* (W \otimes I_{kd})^*, \tag{4.8}$$

so, after taking a partial trace, a partial transposition, the operator norm, and using $W^*W = I_r$, we can conclude. □

To summarize, we have associated to a quantum channel L the following quantities:

$$B_C(L) = \|C_L\| \tag{4.9}$$

$$B_{C\Gamma}(L) = \|C_L^{\Gamma}\| \tag{4.10}$$

$$B_{Cc\Gamma}(L) = \|C_{L^c}^{\Gamma}\| \tag{4.11}$$

$$B_{M\Gamma}(L) = \|M_L^{\Gamma}\| \tag{4.12}$$

$$B_I(L) = \|L(I)\|, \tag{4.13}$$

where C_L, resp. C_{L^c} are the Choi matrices of the channel L, resp. of the complementary channel V^c and $M_L = VV^*$ is the projection on the image of the Stinespring isometry V defining the channel L.

4.2. **Additivity and Rényi entropy bounds**

We start our discussion with the proof of the fact the quantities $B_{\cdot}(L)$ introduced previously (4.9)-(4.13) are *multiplicative* with respect to the tensor product operation. Later, in order to be consistent with the entropic quantities, we shall take logarithms of these quantities, making them *additive*.

Lemma 4.2. *Let Q, R, S be arbitrary operations chosen from the set $\{\mathrm{id}, \mathrm{Tr}, \Theta\}$. Then, for any quantum channels L_1, L_2, the following multiplicativity relation holds:*

$$B_{Q,R,S}(L_1 \otimes L_2) = B_{Q,R,S}(L_1) \cdot B_{Q,R,S}(L_2). \tag{4.14}$$

Proof. Let $V_{1,2}$ the isometries defining the quantum channels $L_{1,2}$. It is immediate that $V_1 \otimes V_2$ is a Stinespring isometry for $L_1 \otimes L_2$; the same holds for the vectorized versions $v_{1,2}$. Hence,

$$
\begin{aligned}
B_{Q,R,S}(L_1 \otimes L_2) &= \|[Q \otimes R \otimes S] \left((v_1 \otimes v_2)(v_1 \otimes v_2)^* \right) \| \\
&= \|[Q \otimes R \otimes S] \left(v_1 v_1^* \otimes v_2 v_2^* \right) \| \\
&= \|[Q \otimes R \otimes S](v_1 v_1^*) \otimes [Q \otimes R \otimes S](v_2 v_2^*)\| \\
&= \|[Q \otimes R \otimes S](v_1 v_1^*)\| \cdot \|[Q \otimes R \otimes S](v_2 v_2^*)\| \\
&= B_{Q,R,S}(L_1) \cdot B_{Q,R,S}(L_2), \tag{4.15}
\end{aligned}
$$

where we have used the multiplicativity of the maps $T \in \{\mathrm{id}, \mathrm{Tr}, \Theta\}$: $T(X \otimes Y) = T(X) \otimes T(Y)$. $\square$

We continue with a useful linear algebra lemma, relating the (partially transposed) Choi matrix of a channel with that of the dual channel. Recall that the dual map L^* of a quantum channel $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ is the unital, completely positive map $L^* : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ which satisfies the following duality relation with respect to the Hilbert-Schmidt scalar product

$$\forall X \in \mathcal{M}_k(\mathbb{C}), Y \in \mathcal{M}_d(\mathbb{C}), \quad \langle X, L(Y) \rangle = \langle L^*(X), Y \rangle. \tag{4.16}$$

Lemma 4.3. *Let $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ be a linear map and $L^* : \mathcal{M}_k(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ be its dual with respect to the usual, Hilbert-Schmidt, scalar product. Then,*

$$C_{L^*} = F_{k,d} C_L^\top F_{k,d}^* \tag{4.17}$$

$$C_{L^*}^\Gamma = F_{k,d}(C_L^\Gamma)^\top F_{k,d}^*, \tag{4.18}$$

where $F_{k,d} : \mathbb{C}^k \otimes \mathbb{C}^d \to \mathbb{C}^d \otimes \mathbb{C}^k$ is the flip operator, i.e. $F_{k,d}(a \otimes b) = b \otimes a$. In particular, the matrices C_L and C_{L^*} (resp. C_L^Γ and $C_{L^*}^\Gamma$) have the same spectrum.

Proof. We are going to show the first equality, the proof for the partial transpositions being similar. A simple proof of the claim can be obtained using the graphical notation for tensors, see Figure 4.2: the diagram on

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the right can be obtained by flipping horizontally ($\top$) and vertically ($F_{k,d}$) the input and output labels of the diagram on the left.
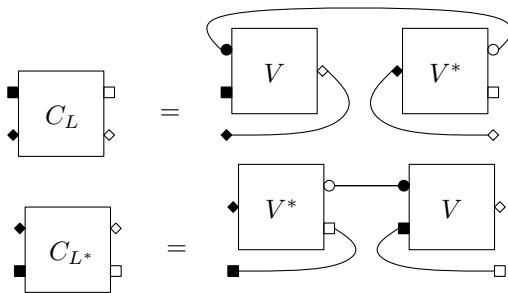

FIGURE 4.2. Diagrams for the Choi matrix of a channel L (left) and for the Choi matrix of the adjoint channel L^* (right). The channel L is given by a Stinespring isometry $V : \mathbb{C}^d \to \mathbb{C}^n \otimes \mathbb{C}^k$. The round, square, and resp. diamond shapes denote the vector spaces $\mathbb{C}^n$, $\mathbb{C}^k$ and resp. $\mathbb{C}^d$.

Let us now give a detailed, algebraic proof. Consider orthonormal bases $\{e_i\}_{i=1}^d$, $\{f_x\}_{x=1}^k$ of $\mathbb{C}^d$, resp. $\mathbb{C}^k$, and compute

$$
\begin{aligned}
e_i^* \otimes f_x^* C_{L^*} e_j \otimes f_y &= e_i^* L^*(f_x f_y^*) e_j = \mathrm{Tr}[e_j e_i^* L^*(f_x f_y^*)] \\
&= \mathrm{Tr}[L(e_j e_i^*) f_x f_y^*] = f_y^* L(e_j e_i^*) f_x \\
&= f_y^* \otimes e_j^* C_L f_x \otimes e_i = f_x^* \otimes e_i^* C_L^\top f_y \otimes e_j \\
&= e_i^* \otimes f_x^* F_{k,d} C_L^\top F_{k,d}^* e_j \otimes f_y,
\end{aligned}
\tag{4.19}
$$

proving the claim. $\qquad\square$

We can now prove the main result of this section, the fact that the quantities in (4.9)-(4.13) are lower bounds for the minimum output p-Rényi entropy of the quantum channel L.

Proposition 4.4. *Let $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ be a quantum channel. Then, for all integers $r \geq 1$,*

$$H_2^{\min}(L^{\otimes r}) \geq -r \log \|C_L\| \quad {}^{*} \tag{4.20}$$

$$H_2^{\min}(L^{\otimes r}) \geq -r \log \|C_L^{\Gamma}\| \tag{4.21}$$

$$H_2^{\min}(L^{\otimes r}) \geq -r \log \|C_{L^c}^{\Gamma}\| \tag{4.22}$$

$$H_\infty^{\min}(L^{\otimes r}) \geq -r \log \|M_L^{\Gamma}\| \tag{4.23}$$

$$H_\infty^{\min}(L^{\otimes r}) \geq -r \log \|L(I)\|. \tag{4.24}$$

Proof. Note that, using the multiplicativity result of Lemma 4.2, we only need to show that the inequalities hold for $r = 1$. Let us start with the first two inequalities, involving the Choi matrix C_L. The proof begins with a straightforward "linearization trick" and proceeds by linear algebra manipulations. For a fixed quantum state $X \in \mathcal{M}_d^{1,+}(\mathbb{C})$, we have

$$
\begin{aligned}
\mathrm{Tr}[L(X)^2] &= \mathrm{Tr}[L(X) \otimes L(X) \cdot F_{k,k}] \\
&= \mathrm{Tr}\left[(X \otimes L(X)) \cdot [L^* \otimes \mathrm{id}](F_{k,k})\right] \\
&= \mathrm{Tr}\left[(X \otimes L(X)) \cdot C_{L^*}^{\Gamma}\right] \\
&\leq \lambda_{\max}(C_{L^*}^{\Gamma}) = \lambda_{\max}(C_L^{\Gamma}) \leq \|C_L^{\Gamma}\|,
\end{aligned}
\tag{4.25}
$$

where we have used the isospectral property proved in Lemma 4.3 and the fact that $X \otimes L(X)$ is positive semidefinite and has unit trace. Taking the supremum over all input states X yields (4.21). To show (4.20), apply in (4.25) the transposition operation Θ on the second factor of the tensor product; this will remove the partial transposition on the matrix $C_{L^*}^{\Gamma}$, and the claim will follow.

The inequality (4.22) follows from (4.21) and the fact that $H_2^{\min}(L) = H_2^{\min}(L^c)$. Finally, (4.23) has been proved in [44, Fact 1 and Proposition 4], while (4.24) follows trivially from the matrix inequality $L(X) \leq L(I)$ (see also [26]). $\qquad\square$

Remark 4.5. The bounds above are tight: the maximally depolarizing channel $\Delta(X) = \mathrm{Tr}(X)I/d$ saturates bounds (4.20) and (4.21), while the identity channel $\mathrm{id}(X) = X$ saturates bounds (4.22), (4.23), and (4.24).

---

*Actually, the right hand side is a lower bound for $H_\infty^{\min}(L^{\otimes r})$. This can be seen by considering the complementary channel. We thank David Reeb for pointing this out.

4.3. **Bounds for additivity rate**

The additive bounds derived above can be used to bound the p-additivity rates of channels, as we show in the following result. Note that our starting point is the value at $p = 2$ (see Proposition 4.4) so our method will give interesting results for values of p in the interval $[0, 2]$. The extension of the inequalities for $p > 2$ is less interesting, since the derived inequalities contain additional factors which depend on p. Indeed, for any quantum state ρ and $p > 2$,

$$\frac{p}{2(p-1)} H_2(\rho) \leq H_p(\rho) \leq H_2(\rho), \tag{4.26}$$

which is the best general bound for the p-Rényi entropy in terms of the 2-Rényi entropy.

Proposition 4.6. *For a quantum channel L having no pure outputs, the p-additivity rate* (3.1) *is lower bounded, for all $p \in [0, 2]$, by the following quantity*

$$\alpha_p(L) \geq \hat{\alpha}_p(L) := \frac{-\log B}{H_p^{\min}(L)}, \tag{4.27}$$

where $B = \min\{\|C_L\|, \|C_L^\Gamma\|, \|C_{L^c}^\Gamma\|, \|M_L^\Gamma\|, \|L(I)\|\}$.

Remark 4.7. When $p > 2$, the quantity B above has to be replaced by

$$\min\{\|C_L\|^{c_p}, \|C_L^\Gamma\|^{c_p}, \|C_{L^c}^\Gamma\|^{c_p}, \|M_L^\Gamma\|, \|L(I)\|\}, \tag{4.28}$$

where $c_p = p/(2p - 2)$ is a correction exponent which appears because one has to use in this case (4.26).

We investigate next how the lower bound for additivity rates behaves with respect to tensor products.

Proposition 4.8. *Let L and K be two quantum channels with the property that their tensor product $L \otimes K$ has no pure output. Then, for all $p \in [0, 2]$, the following inequality holds:*

$$\hat{\alpha}_p(L \otimes K) \leq v_p(L, K) \left[t\hat{\alpha}_p(L) + (1 - t)\hat{\alpha}_p(K) \right], \tag{4.29}$$

where $v_p(L, K)$ is the relative violation of the minimum p-output entropy (2.12) *and $t = H_p^{\min}(L)/[H_p^{\min}(L) + H_p^{\min}(K)] \in [0, 1]$ (note that $H_p^{\min}(L)$ and $H_p^{\min}(K)$ cannot be both null).*

Proof. Let B_L, B_K, and $B_{L\otimes K}$ be the bounds associated to the channels L, K, and $L\otimes K$, as in Proposition 4.6; they obviously satisfy $B_L B_K \leq B_{L\otimes K}$. Starting from the right-hand-side of the inequality to be proved, we have then

$$v_p(L, K) \left[t\hat{\alpha}_p(L) + (1 - t)\hat{\alpha}_p(K) \right]$$
$$= \frac{-\log(B_L B_K)}{H_p^{\min}(L \otimes K)} \geq \frac{-\log B_{L\otimes K}}{H_p^{\min}(L \otimes K)} = \hat{\alpha}_p(L \otimes K). \tag{4.30}$$

$\square$

4.4. **Examples: the Werner-Holevo and the antisymmetric channels**

In this section we compute the bounds derived earlier for the Werner-Holevo channel discussed in Example 3.5 and for the antisymmetric channel from Examples 3.6.

Recall that the Werner-Holevo channel $W_d : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_d(\mathbb{C})$ is defined by

$$W_d(X) = \frac{1}{d-1}[\text{Tr}(X)I_d - X^\top]. \tag{4.31}$$

First, note that $W_d(I) = I$, hence $\|W_d(I)\| = 1$. The Choi matrix and its partial transpose read

$$C_{W_d} = \frac{1}{d-1}[I_{d^2} - F_d] \tag{4.32}$$

$$C_{W_d}^\Gamma = \frac{1}{d-1}[I_{d^2} - E_d], \tag{4.33}$$

where E_d is the (un-normalized) maximally entangled state (2.5) and F_d denotes the flip operator $F_d(x \otimes y) = y \otimes x$, for $x, y \in \mathbb{C}^d$. The corresponding bounds are easily computed from the above relations: $\|C_{W_d}\| = 2/(d-1)$ and $\|C_{W_d}^\Gamma\| = 1$; in particular, note that the matrix $C_{W_d}^\Gamma$ is not positive semidefinite, i.e. the Werner-Holevo channels is not PPT. A minimal purification of the Choi matrix C_{W_d} is given by the four-partite vector $v_{W_d} \in (\mathbb{C}^d)^{\otimes 4}$ (see Figure 4.3 for a graphical representation)

$$v_{W_d} = \frac{1}{\sqrt{2(d-1)}}v = \frac{1}{\sqrt{2(d-1)}}\sum_{i,j=1}^{d} e_i \otimes e_j \otimes e_j \otimes e_i - e_i \otimes e_j \otimes e_i \otimes e_j. \tag{4.34}$$

FIGURE 4.3. A minimal purification of the Choi matrix of a Werner-Holevo channel.

Using the above purification, the remaining two bounds are seen to be equal: $\|C_{W_d^c}^{\Gamma}\| = \|M_{W_d}^{\Gamma}\| = 2/(d-1)$. Note that for the vector v_{W_d}, the top two factors of the four-partite tensor product correspond to the "environment dimension", the third one corresponds to the output, while the fourth one corresponds to the input. In conclusion, the additivity rates of the Werner-Holevo channel are bounded, for all $p \in [0, 2]$, as follows:

$$\alpha_p(W_d) \geq \frac{-\log(2/(d-1))}{\log(d-1)} = 1 - \frac{\log 2}{\log(d-1)}. \qquad (4.35)$$

Let us now move to the case of the antisymmetric channel A_d introduced by Grudka, Horodecki, and Pankowski in [31]. If $P : \mathbb{C}^d \otimes \mathbb{C}^d \to \mathbb{C}^{\binom{d}{2}}$ is the projection operator on the antisymmetric subspace, i.e. $P = Q^*$, where $Q : \Lambda^2(\mathbb{C}^d) \cong \mathbb{C}^{\binom{d}{2}} \to \mathbb{C}^d \otimes \mathbb{C}^d$ is the canonical embedding of the antisymmetric space $\Lambda^2(\mathbb{C}^d)$ into $\mathbb{C}^d \otimes \mathbb{C}^d$, then

$$v_{A_d} = \frac{1}{\sqrt{2}}(I_d \otimes I_d \otimes P)v = \frac{1}{\sqrt{2}}(I_d \otimes I_d \otimes P)\sum_{i,j=1}^{d} e_i \otimes e_j \otimes e_j \otimes e_i - e_i \otimes e_j \otimes e_i \otimes e_j.$$
$$(4.36)$$

Noticing the similarities between the formula above and (4.34), one can easily show that $\|C_{A_d}^{\Gamma}\| = \|C_{A_d^c}^{\Gamma}\| = 1$, while $\|A_d(I)\| = \|C_{A_d}\| = \|M_{A_d}^{\Gamma}\| = (d-1)/2$. We conclude that the lower bounds discussed in this paper are trivial for the antisymmetric channel and we leave the question of determining the additivity rates of this channel open.

5. Partially transposed random Choi matrices and their norm

In this section and in the next one, we analyze the behavior of the bounds (4.9)-(4.13) for random quantum channels L. Here, we compute the bound

(4.10), while in the next section we compute the four others (when possible) and we compare them, in the spirit of Proposition 4.6.

This section contains one of the main results of our work, Theorem 5.5, which contains a formula for the asymptotic operator norm of the partially transposed Choi matrices of random quantum channels. More precisely, we consider a sequence of random quantum channels $L_n : \mathcal{M}_{d_n}(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ where $d_n \sim tkn$, defined by

$$L_n(X) = [\mathrm{Tr}_{\mathbb{C}^n} \otimes \mathrm{id}_k](V_n X V_n^*), \tag{5.1}$$

where $V_n : \mathbb{C}^{d_n} \to \mathbb{C}^n \otimes \mathbb{C}^k$ is the random isometry. We write $C_n := C_{L_n}$ for the Choi matrix of the quantum channel L_n. The proof is split into three parts, delimited by subsections, and uses the moment method. We start by computing the asymptotic moments of the random matrices C_n^Γ, using the graphical Weingarten calculus introduced in Section 2.2. Then, in the second subsection, we identify a probability measure having the exact moments computed in the first part; the measure is given in terms of the free additive convolution operation from free probability theory. Finally, we show that the random matrices C_n^Γ converge strongly, which gives us the desired norm convergence.

Let us first start with some general considerations about the random matrix we are studying. Associated to the channel L is the *partially transposed Choi matrix* of L,

$$C_L^\Gamma = [\mathrm{id}_k \otimes \Theta_d][L \otimes \mathrm{id}_d](E_d) = [L \otimes \mathrm{id}_d](F_d), \tag{5.2}$$

which is the random matrix we are interested in (see Figure 5.1 for a graphical representation). Before investigating the eigenvalue distribution of the random matrix C_L^Γ, let us first comment on its distribution as a matrix. Since, in this paper, Choi matrices are not normalized ($\mathrm{Tr} C_L^\Gamma = \mathrm{Tr} C_L = d$), the matrix $d^{-1} C_L$ is a (random) density matrix (mixed quantum state). In the literature, several ensembles of random density matrices have been considered: the induced measures [56], the Bures measure [32], or measures associated to graphs [18], just to name a few (see also [55] for a random matrix theory perspective). We would like to argue at this point that the distribution of the Choi matrix we consider is not related to the induced measures introduced in [56]. Indeed, it is easy to see that the distribution of C_L involves more than one column of the unitary matrix U, while, in order to define the induced measures, one needs just one column of a unitary operator (or a random point on the unit sphere, or a normalized

Gaussian vector). A rigorous argument for this fact is given in Section 10, Remark 10.3, using PPT thresholds.

5.1. **Exact moments**

In the next proposition, we compute the moments of the partially transposed Choi matrix C_n^Γ, for a fixed value of n. We make use of the graphical calculus formalism introduced in Section 2.2.

Proposition 5.1. *For any integer dimensions n, k, d, the moments of the random matrix $C_n^\Gamma \in \mathcal{M}_{kd}^{sa}(\mathbb{C})$ are given by, $\forall p \geq 1$,*

$$\mathbb{E}\frac{1}{kd}\operatorname{Tr}(C_n^\Gamma)^p = (kd)^{-1}\sum_{\alpha,\beta\in\mathcal{S}_p} n^{\#\alpha}k^{\#(\gamma^{-1}\alpha)}d^{\#(\gamma\beta)}\operatorname{Wg}_{nk}(\alpha^{-1}\beta). \qquad (5.3)$$

In the above equation, $\#(\cdot)$ denotes the number of cycles of a permutation, $\gamma = (p\cdots 3\,2\,1) \in \mathcal{S}_p$ denotes the full cycle permutation and Wg is the Weingarten function [12], see Definition 2.3.

Proof. The proof is an application of the graphical Weingarten formula from Theorem 2.4. We have depicted in Figure 5.1 the diagram for the random matrix C_n^Γ, which we are investigating. Note that we have replaced the random isometry V_n by a random, Haar-distributed, unitary matrix $U_n \in \mathcal{U}_{nk}$. Moreover, we choose, for the sake of simplicity, not to represent labels corresponding to the added dimensions nk/d, which do not play any role in the moment computations which follow, since the contractions of the corresponding wires multiply the result by the scalar 1. We are
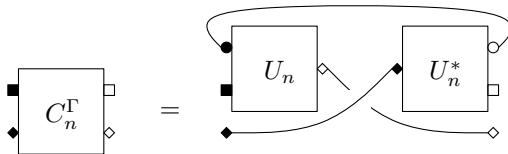

FIGURE 5.1. Diagram for the random matrix C_n^Γ. The box U_n corresponds to a random Haar unitary matrix. We do not depict the labels corresponding to the environment of size kn/d, which does not play any role in the computations.

interested in computing, for all integer $p \geq 1$, the moment $\mathbb{E}\operatorname{Tr}(C_n^\Gamma)^p$. The

diagram corresponding to this real number is depicted in Figure 5.2. We now use formula (2.18) to compute the expectation, with respect to the random unitary matrix U_n:

$$\mathbb{E}\,\mathrm{Tr}(C_n^\Gamma)^p = \sum_{\alpha,\beta \in \mathcal{S}_p} \mathcal{D}_{\alpha,\beta}\,\mathrm{Wg}_{kn}(\alpha^{-1}\beta), \qquad (5.4)$$

where $\mathcal{D}_{\alpha,\beta}$ is the diagram obtained by erasing the U_n and $\bar{U}_n$ boxes and connecting the black (resp. white) decorations of the i-th U_n box with the corresponding black (resp. white) decorations of the $\alpha(i)$-th (resp. $\beta(i)$-th) $\bar{U}_n$ box. The resulting diagram $\mathcal{D}_{\alpha,\beta}$ is a collection of loops corresponding to different vector spaces, as follows (see Figure 5.3):

(1) $\#\alpha$ loops of dimension n, corresponding to round-shaped labels. The round decorations are initially connected with the identity permutation and the graphical expansion connects them with the permutation α. The resulting number of loops is $\#\alpha = \#(\mathrm{id}^{-1}\alpha)$;

(2) $\#(\gamma^{-1}\alpha)$ loops of dimension k, corresponding to square-shaped labels. The square decorations are initially connected with the permutation γ (that is, $i \mapsto i-1$) and the graphical expansion connects them with the permutation α. The resulting number of loops is $\#(\gamma^{-1}\alpha)$;

(3) $\#(\gamma\beta)$ loops of dimension d, corresponding to diamond-shaped labels. The diamond decorations are initially connected with the permutation γ^{-1} (i.e. $i \mapsto i+1$) and the graphical expansion connects them with the permutation β. The resulting number of loops is $\#((\gamma^{-1})^{-1}\beta) = \#(\gamma\beta)$.
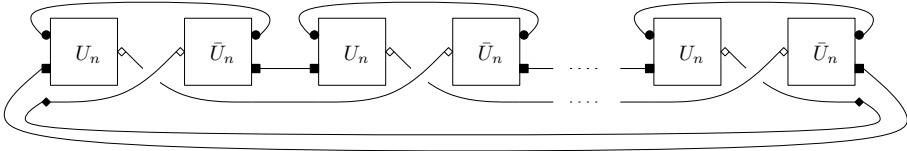

FIGURE 5.2. Diagram for the trace of the p-th power of the partially transposed Choi matrix C_n^Γ. The diagram contains p copies of the matrix from Figure 5.1.
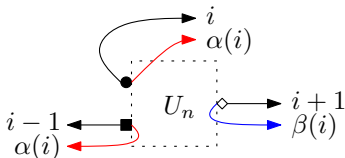
$\square$

FIGURE 5.3. Diagram representing the i-th U_n box in the graphical expansion. The box itself no longer exists and the diagram $\mathcal{D}_{\alpha,\beta}$ consists of a collection of loops.

5.2. **Limiting spectral distribution**

We consider now the asymptotic behavior of the partially transposed Choi matrix C_n^Γ, in the following regime:

Asymptotic regime

- k is fixed

- $n \to \infty$

- $d \to \infty$ with $d \sim tkn$ for some fixed ratio $t \in (0,1)$.

In this regime (which we call the "fixed k regime"), the output dimension k and the input/(total space) ratio t are treated as parameters that we shall consider fixed in what follows. We compute next the asymptotic moments of the random matrix C_n^Γ and we identify a probability measure having these exact moments. Recall that the dilation operator D_t is defined by $D_t[\mu] = (x \mapsto tx)_\#\mu$.

Theorem 5.2. *In the fixed k asymptotic regime, the moments of the random matrix C_n^Γ converge towards the moments of the following probability measure $\mu_{k,t}^{(C\Gamma)}$:*

$$\mu_{k,t}^{(C\Gamma)} = D_t \left[((1-s)\,\delta_{-1} + s\,\delta_{+1})^{\boxplus 1/t} \right], \tag{5.5}$$

where $D.$ is the dilation operator, $\boxplus$ is the additive free convolution and $s := (k+1)/(2k)$. More explicitly, we can write

$$t \cdot \mu_{k,t}^{(C\Gamma)} = \max(t-s,0)\delta_{-1} + \max(s+t-1,0)\delta_{+1} + \tag{5.6}$$

$$\frac{\sqrt{(2\varphi^+(s,t)-1-x)(x-2\varphi^-(s,t)+1)}}{2\pi(1-x)(x+1)} \mathbf{1}_{[2\varphi^-(s,t)-1,2\varphi^+(s,t)-1]}(x)dx,$$

where $\varphi^\pm(s,t)$ were defined in (2.28).

Proof. First, start from the exact moment formula (5.3) and isolate the powers of the parameter $n \to \infty$, using also the Weingarten function asymptotic (2.16). The exponent of n in the fixed k asymptotic regime is:

$$
\begin{aligned}
\text{power of } n \text{ in (5.3)} &= -1 + \#\alpha + \#(\gamma\beta) - p - |\alpha^{-1}\beta| \\
&= p - 1 - (|\alpha| + |\alpha^{-1}\beta| + |\beta^{-1}\gamma^{-1}|) \qquad (5.7) \\
&\leq p - 1 - |\gamma^{-1}| = 0,
\end{aligned}
$$

where we have only made use of the geodesic inequality

$$
|\alpha| + |\alpha^{-1}\beta| + |\beta^{-1}\gamma^{-1}| \geq |\gamma^{-1}|, \qquad (5.8)
$$

which is saturated for permutations α, β lying on the geodesic $\mathrm{id} - \alpha - \beta - \gamma^{-1}$. Then,

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbb{E}\frac{1}{kd}\operatorname{Tr}(C_n^\Gamma)^p \\
&= (tk^2)^{-1} \sum_{\mathrm{id}-\alpha-\beta-\gamma^{-1}} k^{\#(\gamma^{-1}\alpha)}(tk)^{\#(\gamma\beta)}k^{-p-|\alpha^{-1}\beta|}\operatorname{Mob}(\alpha^{-1}\beta) \\
&= (tk^2)^{-1} \sum_{\mathrm{id}-\alpha-\beta-\gamma^{-1}} k^{1+e(\alpha)}(tk)^{1+|\beta|}k^{-\#\alpha-|\beta|}\operatorname{Mob}(\alpha^{-1}\beta) \qquad (5.9) \\
&= \sum_{\mathrm{id}-\beta-\gamma^{-1}} t^{|\beta|} \sum_{\mathrm{id}-\alpha-\beta} k^{e(\alpha)-\#\alpha}\operatorname{Mob}(\alpha^{-1}\beta).
\end{aligned}
$$

Above, $e(\cdot)$ denotes the number of cycles of even size of a permutation, and we use the fact [6, Lemma 2.1] that

$$
1 + e(\alpha) = \#(\gamma^{-1}\alpha) \qquad (5.10)
$$

for permutations $\alpha \in \mathcal{S}_p$ lying on the geodesic $\mathrm{id} - \alpha - \gamma^{-1}$.

Next, we shall identify a probability measure $\mu_{k,t}^{(C\Gamma)}$ having moments as in equation (5.9). The main tool here will be the free moment-cumulant formulas [45, Lecture 11]. Consider the following probability measure

$$
\nu_k := \frac{k-1}{2k}\delta_{-1} + \frac{k+1}{2k}\delta_{+1}. \qquad (5.11)
$$

Since the α-moment of ν_k is ($\|c\|$ denotes the number of elements in a cycle c of a permutation)

$$m_\alpha(\nu_k) = \prod_{c \in \alpha} m_{\|c\|}(\nu_k) = \prod_{c \in \alpha} \left(\frac{k-1}{2k}(-1)^{\|c\|} + \frac{k+1}{2k} \right) = k^{e(\alpha) - \#\alpha},$$

(5.12)

we have that

$$(5.9) = \sum_{\mathrm{id}\,-\beta-\gamma^{-1}} t^{|\beta|} \sum_{\mathrm{id}\,-\alpha-\beta} m_\alpha(\nu_k) \operatorname{Mob}(\alpha^{-1}\beta)$$

$$= t^p \sum_{\mathrm{id}\,-\beta-\gamma^{-1}} t^{-\#\beta} \kappa_\beta(\nu_k) = t^p m_p \left(\nu_k^{\boxplus 1/t} \right), \qquad (5.13)$$

which is precisely the p-th moment of the probability measure

$$\mu_{k,t}^{(C\Gamma)} = D_t \left[\nu_k^{\boxplus 1/t} \right]. \tag{5.14}$$

This proves the first statement.

We now show the second statement. By using Lemma 2.8 we have (recall that $b_s = (1-s)\delta_0 + s\delta_1$ is the Bernoulli distribution)

$$\mu_{k,t}^{(C\Gamma)} = \{x \mapsto tx\}_\# \left(\{x \mapsto 2x - 1\}_\# b_s \right)^{\boxplus 1/t}$$

$$= \{x \mapsto tx\}_\# \{x \mapsto 2x - 1/t\}_\# \left(b_s \right)^{\boxplus 1/t} \tag{5.15}$$

$$= \{x \mapsto 2tx - 1\}_\# \left(b_s \right)^{\boxplus 1/t}$$

Therefore, by using Proposition 2.9 with $T = 1/t$ we have

$$\mu_{k,t}^{(C\Gamma)} = \max(0, 1 - s/t)\delta_{-1} + \max(0, 1 - (1-s)/t)\delta_1$$

$$+ \frac{1/t\sqrt{(\gamma_+ - \frac{x+1}{2t})(\frac{x+1}{2t} - \gamma_-)}}{2\pi(\frac{x+1}{2t})(\frac{1}{t} - \frac{x+1}{2t})} \mathbf{1}_{[2t\gamma_- - 1, 2t\gamma_+ - 1]} \frac{dx}{2t} \quad (5.16)$$

where $\varphi^\pm(s,t) = t\gamma_\pm$. This completes the proof. $\qquad\square$

Remark 5.3. The atoms (possibly) appearing in equation (5.6) can be interpreted as follows. First, note that the partially transposed Choi matrix C_L^Γ is equal, up to a unitary conjugation, to the matrix $(P_n \otimes I_k)(I_n \otimes F_{k,k})(P_n \otimes I_k)$, where $P_n \in \mathcal{M}_{nk}(\mathbb{C})$ is an orthogonal projection of rank $d \sim tnk$ and $F_{k,k}$ is the flip operator. Since the eigenvalues of the flip operator are $+1$, resp. -1, with multiplicities $k(k+1)/2$, resp. $k(k-1)/2$,

by the interlacing theorem for eigenvalues of Hermitian matrices, we get that the matrix has (among others) eigenvalues

$$+1 \quad \text{with multiplicity at least} \quad nk^2 \cdot \max\left(t + \frac{k+1}{2k} - 1, 0\right)$$

$$-1 \quad \text{with multiplicity at least} \quad nk^2 \cdot \max\left(t + \frac{k-1}{2k} - 1, 0\right)$$

values which corresponds, at the limit, to the atoms at ± 1 of the measure $\mu_{k,t}^{(C\Gamma)}$ from (5.6).

5.3. **Strong convergence**

In the previous subsection, we showed that the random variable C_n^Γ has the same asymptotic moments as the probability measure $\mu_{k,t}^{(C\Gamma)}$. This type of convergence is not sufficient for our purposes, since it does not deal with extremal eigenvalues. We will improve this result with the following proposition. In particular, we will obtain the convergence of the *norm* of the random matrix C_n^Γ, which is, ultimately, the quantity we are interested in (see Proposition 4.4). For an arbitrary probability measure μ, we denote by $\|\mu\|$ its L^∞ norm, i.e. the L^∞ norm of any random variable X having distribution μ.

We start with a technical lemma, showing the strong convergence (in the sense of Definition 2.5) for a family of deterministic matrices of growing dimension.

Lemma 5.4. *Let $\{E_{ij}\}_{i,j=1}^k$ be the matrix units of $\mathcal{M}_k(\mathbb{C})$ and define the orthogonal projection*

$$\mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C}) \ni P_n = \mathrm{diag}(\underbrace{1, \ldots, 1}_{d_n \text{ times}}, \underbrace{0, \ldots, 0}_{nk - d_n \text{ times}}), \qquad (5.17)$$

where $k \in \mathbb{N}$ and $t \in (0,1)$ are fixed and d_n is a multiple of k such that $d_n \sim tnk$ when $n \to \infty$. Then, the $k^2 + 1$ tuple of random variables $P_n, \{I_n \otimes E_{ij}\}_{i,j=1}^k$ converge strongly, as $n \to \infty$, towards respectively $p, \{e_{ij}\}_{i,j=1}^k \in \mathcal{P}_t \otimes \mathcal{M}_k(\mathbb{C})$, where $p = \pi_t \otimes I_k$, $e_{ij} = 1 \otimes E_{ij}$, and $\mathcal{P}_t$ is the algebra generated by a projection π_t of trace t and the identity 1.

Proof. The result follows from the block structure of the matrices $I_n \otimes E_{ij}$ and the fact that P_n respects this block structure (d_n is a multiple of k). We first show the convergence in distribution. In this proof, we denote by tr

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the *normalized* trace, e.g. $\operatorname{tr}_k I_k = 1$. For a fixed monomial $F \in \mathbb{C}\langle p, \{e_{ij}\}\rangle$ in $1 + k^2$ non-commutative random variables, we have

$$
\begin{aligned}
F(P_n, \{I_n \otimes E_{ij}\}) &= F(I_k^{\oplus d_n/k} \oplus 0_k^{\oplus n - d_n/k}, \{E_{ij}^{\oplus n}\}) \\
&= F(I_k, \{E_{ij}\})^{\oplus d_n/k} \oplus F(0_k, \{E_{ij}\})^{\oplus n - d_n/k}, \quad (5.18)
\end{aligned}
$$

and thus

$$
\begin{aligned}
\lim_{n \to \infty} [\operatorname{tr}_n \otimes \operatorname{tr}_k] F(P_n, \{E_{ij}\}) \\
= t \operatorname{tr}_k F(I_k, \{E_{ij}\}) + (1 - t) \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \notin F}. \quad (5.19)
\end{aligned}
$$

Note that since F is a monomial, $p \notin F$ means that F does not contain a factor p.

On the other hand, we have

$$
F(p, \{e_{ij}\}) = \begin{cases} \pi_t \otimes F(I_k, \{E_{ij}\}), & \text{if } p \in F, \\ 1 \otimes F(I_k, \{E_{ij}\}), & \text{if } p \notin F. \end{cases} \quad (5.20)
$$

If τ is the trace of $\mathcal{P}_t$, then we have

$$
\begin{aligned}
[\tau \otimes \operatorname{tr}_k] F(p, \{e_{ij}\}) &= t \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \in F} + \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \notin F} \\
&= t \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \in F} + t \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \notin F} \\
&\qquad + (1 - t) \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \notin F} \\
&= t \operatorname{tr}_k F(I_k, \{E_{ij}\}) + (1 - t) \operatorname{tr}_k F(I_k, \{E_{ij}\}) \mathbf{1}_{p \notin F} \\
&\qquad\qquad (5.21)
\end{aligned}
$$

which, together with (5.19), allows to conclude the proof of the convergence in distribution for monomials, and, using linearity, for arbitrary non-commutative polynomials.

Let us now show the norm convergence in Definition 2.5, for a fixed polynomial F. Using again the block structure of the matrices $I_n \otimes E_{ij}$ and of P_n (recall that d_n is a multiple of k), we get that

$$
\begin{aligned}
\|F(P_n, \{I_n \otimes E_{ij}\})\| &= \max(\|F(I_k, \{E_{ij}\})\|, \|F(0_k, \{E_{ij}\})\|) \quad (5.22) \\
&= \max(\|(G + H)(I_k, \{E_{ij}\})\|, \|G(I_k, \{E_{ij}\})\|),
\end{aligned}
$$

where $F = G + H$ is the decomposition of F into monomials which do not contain the first variable (G) and monomials which do (H). Using the

same decomposition, we have

$$\|F(p, \{e_{ij}\})\| = \|1 \otimes G(I_k, \{E_{ij}\}) + \pi_t \otimes H(I_k, \{E_{ij}\})\|$$
$$= \|\pi_t \otimes (G+H)(I_k, \{E_{ij}\}) + (1-\pi_t) \otimes G(I_k, \{E_{ij}\})\|$$
$$= \max(\|(G+H)(I_k, \{E_{ij}\})\|, \|G(I_k, \{E_{ij}\})\|), \qquad (5.23)$$

finishing the proof. $\qquad\square$

Theorem 5.5. *The random matrix C_n^{Γ} converges* strongly *towards an element having distribution $\mu_{k,t}^{(C\Gamma)}$ defined in (5.5)-(5.6). Hence, we have the following norm convergence: almost surely,*

$$\lim_{n\to\infty} \|C_n^{\Gamma}\| = \|\mu_{k,t}^{(C\Gamma)}\| = \begin{cases} 2\varphi^+(s,t) - 1, & \text{if } t+s < 1 \\ 1, & \text{if } t+s \geq 1, \end{cases} \qquad (5.24)$$

where $s = (k+1)/(2k)$ and $\varphi^+(s,t)$ was defined in (2.28). More explicitly, the above quantity is written as

$$2\varphi^+(s,t) - 1 = \frac{1-2t}{k} + 2\sqrt{\left(1 - \frac{1}{k^2}\right)t(1-t)}. \qquad (5.25)$$

Proof. Firstly, let $\varepsilon_n = |\frac{d_n}{nk} - t|$ so that

$$(t - \varepsilon_n)nk \leq d_n \leq (t + \varepsilon_n)nk. \qquad (5.26)$$

Then, we set $d_n^- = \lfloor n(t - \varepsilon_n)\rfloor k$ and $d_n^+ = \lceil n(t + \varepsilon_n)\rceil k$ so that we can use Lemma 5.4 for d_n^- and d_n^+. Hence, the following proofs are applied to d_n^- and d_n^+ but on the other hand by using interlacing theorem, we can obtain the desired statement for the original sequence d_n.

Secondly, we prove the strong convergence. Recall that the channel L has the following Stinespring representation

$$L(X) = [\mathrm{id}_k \otimes \mathrm{Tr}_n](VXV^*)$$
$$= [\mathrm{id}_k \otimes \mathrm{Tr}_n](UWXW^*U^*), \qquad (5.27)$$

where $U \in \mathcal{U}(nk)$ is a unitary operator and $W : \mathbb{C}^d \to \mathbb{C}^k \otimes \mathbb{C}^n$ is the isometric embedding: $W = [I_d\, 0_{d\times(nk-d)}]^\top$. With this notation, we have

$$C_n^{\Gamma} = \sum_{i,j=1}^{d} [\mathrm{id}_k \otimes \mathrm{Tr}_n](UWE_{ij}W^*U^*) \otimes E_{ji} \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C}). \quad (5.28)$$

Define for $D_n \in \mathcal{M}_n(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_k(\mathbb{C})$

$$
\begin{aligned}
D_n &= (P_d U^* \otimes I_k) \left[I_n \otimes \sum_{i,j=1}^{k} E_{ij} \otimes E_{ji} \right] (U P_d \otimes I_k) \\
&= \sum_{i,j=1}^{k} [P_d U^* (I_n \otimes E_{ij}) U P_d] \otimes E_{ji},
\end{aligned}
\tag{5.29}
$$

where $P_d = W W^* = I_d \oplus 0_{nk-d}$ is an orthogonal projection of rank d. It is easy to see that the matrices C_n^Γ and D_n have the same spectrum, up to null eigenvalues, so we will focus on showing the strong convergence property for D_n. Note that the expression in (5.29) is a polynomial with $\mathcal{M}_k(\mathbb{C})$ coefficients, in the variables P_d, U and $I_n \otimes E_{ij}$. Using Lemma 5.4 and Proposition 2.6, we conclude that the tuple $(U, U^*, P_d, \{I_n \otimes E_{ij}\}_{i,j=1}^{k})$ converges strongly, as $n \to \infty$, to a limit that we choose not to specify. We then apply Proposition 2.7 to obtain the strong convergence of D_n to a limit element x. Since the distribution of the random matrix C_n^Γ has been shown to converge to the measure $\mu_{k,t}^{(C\Gamma)}$, we conclude that, almost surely

$$
\lim_{n \to \infty} \left\| C_n^\Gamma \right\|_\infty = \| \mu_{k,t}^{(C\Gamma)} \|.
\tag{5.30}
$$

Thirdly, notice that the norm $\|C_n^\Gamma\|$ is the maximum between the largest element of the support of the measure $\mu_{k,t}^{(C\Gamma)}$ and minus the smallest element of the support of $\mu_{k,t}^{(C\Gamma)}$. Let us show now that the latter quantity is smaller than or equal to the former, finishing the proof. Indeed, in the case where $\mu_{k,t}^{(C\Gamma)}$ has an atom at -1, t should be larger than s. But in that case, we also have $s + t - 1 > 2s - 1 > 0$, since $s = (k+1)/(2k) > 1/2$, so $\mu_{k,t}^{(C\Gamma)}$ also has an atom at 1. In the case when $t < 1 - s$, we have that $t \leq 1/2$ and thus $2\varphi^+(s,t) - 1 \geq -(2\varphi^-(s,t) - 1)$, showing that the maximum is also attained on the positive part of the support. $\qquad \square$

6. **Other bounds for random quantum channels**

In this section, we compute the remaining four bounds from Section 4, in the case of random quantum channels. In the first three subsections, we study respectively the bounds (4.9), (4.11), (4.12), while in the last one we compare these three bounds with (4.10), which was computed in the

previous section. Our conclusion is that the bound (4.10) coming from the partially transposed Choi matrix of the channel L seems to give the sharpest estimate for minimum p-Rényi output entropies. Moreover, from a practical standpoint, we have a closed formula for the bound in (4.10) and we have shown strong convergence of the random matrices towards the corresponding probability distribution.

The computations in this section are similar to the ones in Section 5, so we refer the reader to that section for some of the details. We would like to mention already that in some cases, we are not able to obtain such precise estimates as in the previous section; in such situations, we state only some partial results, in the asymptotic limit where the output dimension k is large.

We consider first the bound (4.13) and we show that, in the case of random quantum channels, it is trivial. Hence, we shall not mention it in the remainder of the paper. Indeed, the (random) channels we are interested in have input dimension larger than output dimension: $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$, with $d \to \infty$ and k fixed. Using $d \geq k$, we have (see also [26])

$$\|L(I_d)\| \geq \frac{\mathrm{Tr}L(I_d)}{k} = \frac{\mathrm{Tr}I_d}{k} = d/k \geq 1, \tag{6.1}$$

and thus the bound in (4.13) reads $H_\infty^{\min}(L^{\otimes r}) \geq 0 \geq -r \log \|L(I_d)\|$, which is a trivial statement.

6.1. **Choi matrices**

In this section, we discuss on the limiting eigenvalue distributions of the Choi matrix C_L when taken randomly. As before, we consider a sequence of random quantum channels L_n as in (5.1), where the parameters scale as in Section 5.2. We denote by $C_n \in \mathcal{M}_k(\mathbb{C}) \otimes \mathcal{M}_d(\mathbb{C})$ the Choi matrix of L_n, which is a random quantum channel.

Proposition 6.1. *The random Choi matrix C_n converges* strongly *towards an element having distribution*

$$\mu_{k,t}^{(C)} = D_{kt} \left[b_{k^{-2}}^{\boxplus 1/t} \right]. \tag{6.2}$$

Hence, we have the following norm convergence: almost surely,

$$\lim_{n \to \infty} \|C_n\| = \|\mu_{k,t}^{(C)}\| = \begin{cases} k\varphi^+(k^{-2}, t) & \text{if } t + k^{-2} < 1 \\ k & \text{if } t + k^{-2} \geq 1, \end{cases} \tag{6.3}$$

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where the function φ^+ was defined in (2.28).

Proof. The starting point of the proof is the following moment formula for the random matrix C_n, obtained by graphical Weingarten calculus (see Theorem 2.4). The powers of n, $d \sim tkn$ and k in the formula below count loops and can be inferred from Figure 6.1. For any integer $p \geq 1$, we have (remember that $\gamma \in \mathcal{S}_p$ denotes the full cycle permutation $i \mapsto i - 1$)
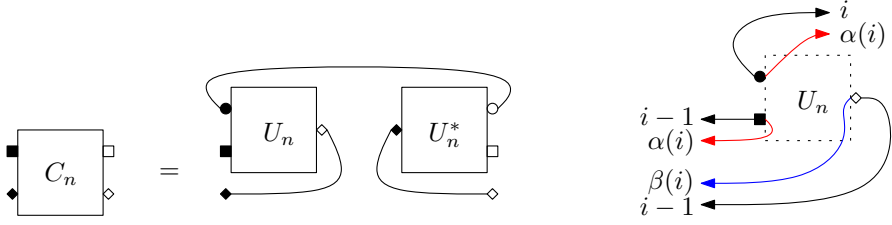

FIGURE 6.1. Diagrams for the Choi matrix C_n and for the i-th U_n box in the graphical expansion of $\mathrm{Tr}\,[C_n^p]$.

$$(kd)^{-1} \mathbb{E}\,\mathrm{Tr}\,[C_n^p] = (kd)^{-1} \sum_{\alpha,\beta \in \mathcal{S}_p} n^{\#\alpha} k^{\#(\gamma^{-1}\alpha)} d^{\#(\gamma^{-1}\beta)}\,\mathrm{Wg}_{kn}(\alpha^{-1}\beta). \quad (6.4)$$

In the above equation, the surviving terms as $n \to \infty$ are the ones which contain the largest power of n (we use below $d \sim tkn$ and the Weingarten function asymptotic from (2.16)):

$$\begin{aligned}
\text{power of } n \text{ in } (6.4) &= -1 + \#\alpha + \#(\gamma^{-1}\beta) - p - |\alpha^{-1}\beta| \\
&\leq p - 1 - (|\alpha| + |\alpha^{-1}\beta| + |\beta^{-1}\gamma|) \leq 0. \quad (6.5)
\end{aligned}$$

The above bound is saturated if and only if the triangle inequality $|\alpha| + |\alpha^{-1}\beta| + |\beta^{-1}\gamma| \geq |\gamma|$ is saturated, i.e. the permutations α, β are on the geodesic $\mathrm{id} - \alpha - \beta - \gamma$. This implies that

$$\begin{aligned}
&\lim_{n \to \infty} (kd)^{-1} \mathbb{E}\,\mathrm{Tr}\,[C_n^p] \\
&= (tk^2)^{-1} \sum_{\mathrm{id}-\alpha-\beta-\gamma} t^{p-|\gamma^{-1}\beta|} k^{2p-|\gamma^{-1}\alpha|-|\gamma^{-1}\beta|-p-|\alpha^{-1}\beta|}\,\mathrm{Mob}(\alpha, \beta) \\
&= k^{-p} \sum_{\mathrm{id}-\alpha-\beta-\gamma} t^{|\beta|} k^{2|\alpha|}\,\mathrm{Mob}(\alpha, \beta) \\
&= (tk)^p \sum_{\mathrm{id}-\alpha-\beta-\gamma} t^{-\#\beta} k^{-2\#\alpha}\,\mathrm{Mob}(\alpha, \beta). \quad (6.6)
\end{aligned}$$

Above, we have used some properties of geodesics, for example, $|\gamma^{-1}\beta| = |\gamma| - |\beta| = p - 1 - |\beta|$.

Next, we fix $\beta \in \mathcal{S}_p$ and use the moment-cumulant formula [45, Proposition 11.4 (2)] in free probability:

$$\sum_{\mathrm{id}-\alpha-\beta} k^{-2\#\alpha} \mathrm{Mob}(\alpha, \beta) = \sum_{\mathrm{id}-\alpha-\beta} m_\alpha(b_{k^{-2}}) \mathrm{Mob}(\alpha, \beta) = \kappa_\beta(b_{k^{-2}}), \quad (6.7)$$

where $b_{k^{-2}}$ is the Bernoulli distribution $b_{k^{-2}} = (1 - k^{-2})\delta_0 + k^{-2}\delta_1$. Hence, using the multi-linearity of the free cumulant, we have

$$\lim_{n\to\infty} (kd)^{-1} \mathbb{E}\,\mathrm{Tr}\,[C_n^p] = (tk)^p \sum_{\mathrm{id}-\beta-\gamma} t^{-\#\beta} \kappa_\beta(b_{k^{-2}})$$

$$= (tk)^p \sum_{\mathrm{id}-\beta-\gamma} \kappa_\beta(b_{k^{-2}}^{\boxplus 1/t})$$

$$= (tk)^p m_p \left(b_{k^{-2}}^{\boxplus 1/t} \right) = m_p \left(D_{kt} \left[b_{k^{-2}}^{\boxplus 1/t} \right] \right), \quad (6.8)$$

which shows that the random matrices C_n converge in moments to the probability measure $\mu_{k,t}^{(C)}$ defined in (6.2). The statement about the support of $\mu_{k,t}^{(C)}$ follows from Proposition 2.9 with $T = 1/t$ and $s = k^{-2}$. The proof of the strong convergence is identical to the one from Theorem 5.5, up to the occasional flipping of indices, due to the partial transposition (e.g. in (5.29), one should replace the matrix element E_{ji} by E_{ij}); we leave the details to the reader. $\square$

Note that the limiting probability measure $\mu_{k,t}^{(C)}$ from (6.2) is supported on $[0, \infty)$; this is a consequence of the fact that the Choi matrices C_n are positive semidefinite, since the quantum channels L_n are completely positive.

6.2. **Partially transposed Choi matrices of complementary channels**

In this subsection, we discuss the limiting eigenvalue distributions of the partially transposed Choi matrix of the complementary channel $C_{L^c}^\Gamma$, our goal being to estimate the quantity (4.11) in the case where L is a random quantum channel. More precisely, we consider a sequence of random quantum channels L_n and we denote by $C_n^{c\Gamma}$ the corresponding partially transposed Choi matrices of the channels L_n^c. Unfortunately, we are not

able to identify, at fixed k and t, the limiting eigenvalue distribution of the random matrix $C_n^{c\Gamma}$ when $n \to \infty$; we have to settle in this case for another asymptotic regime, where $1 \ll k \ll n$. This regime is obtained by taking first the limit $n \to \infty$, followed by the limit $k \to \infty$.

Before stating the result, we recall the notion of *Kreweras complement* for non-crossing partitions. The Kreweras complement of $\alpha \in NC(p)$ is another non-crossing partition, denoted $\alpha^{\mathrm{Kr}} \in NC(p)$, defined in the following way [45, Definition 9.21]. First, expand the domain of partitions to $\{\bar{1}, 1, \bar{2}, 2 \ldots, \bar{p}, p\}$; let then $\alpha^{\mathrm{Kr}} \in NC(\bar{1}, \bar{2}, \ldots, \bar{p}) \cong NC(p)$ be the *largest* non-crossing partition such that $\alpha \cup \alpha^{\mathrm{Kr}}$ is still a non-crossing partition on $\{\bar{1}, 1, \bar{2}, 2 \ldots, \bar{p}, p\}$. Given a geodesic permutation $\mathrm{id} - \alpha - \gamma$, we define the geodesic permutation $\alpha^{\mathrm{Kr}} \in \mathcal{S}_p$ by identifying, as usual, geodesic permutations with non-crossing partitions; more precisely, we have $\alpha^{\mathrm{Kr}} = \alpha^{-1}\gamma$ (see [45, Remark 23.24]). Note that the above construction of Kreweras complement is slightly different from the one in [45, Definition 9.21] in that $\bar{i}$ is left to i for $i = 1, \ldots, p$ because $\gamma = (p, p-1, \ldots, 1)$ in our paper.

Lemma 6.2. *For $\gamma_{2q} = (2q, 2q - 1, \ldots, 1) \in \mathcal{S}_{2q}$, $\gamma_1 = (2q - 1, 2q - 3, \ldots, 1)$ and $\gamma_2 = (2q, 2q - 2, \ldots, 2)$, take $\tilde{\alpha}_i \in \mathcal{S}_{2q}$ two permutations on the geodesics $\mathrm{id} - \alpha_i - \gamma_i$ with $i = 1, 2$ (in particular, α_1 acts only on odd numbers, while α_2 acts only on even numbers). Then,*

$$\# \left(\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) \right) = \#(\alpha_1 \gamma_q^{-1} \alpha_2) = \# \left(\alpha_1 \left(\alpha_2^{\mathrm{Kr}} \right)^{-1} \right), \qquad (6.9)$$

where the permutations $\alpha_{1,2} \in \mathcal{S}_q$ are associated with $\tilde{\alpha}_{1,2}$ by deleting the trivial fixed points, via the correspondences $(2i - 1) \to i$, resp. $2i \to i$, for $i = 1, 2, \ldots, q$.

Proof. We show $\# \left(\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2) \right) = \#(\alpha_1 \gamma_q^{-1} \alpha_2)$, the other equality following from the definition of the Kreweras complement for geodesic permutations. First, note that the respective actions on $\tilde{\alpha}_{1,2}$ are $\tilde{\alpha}_1(2i) = 2i$, $\tilde{\alpha}_1(2i - 1) = 2\alpha_1(i) - 1$, resp. $\tilde{\alpha}_2(2i) = 2\alpha_2(i)$, $\tilde{\alpha}_2(2i - 1) = 2i - 1$, for $i = 1, 2, \ldots, q$. An arbitrary element $i \in [q]$ is mapped by the permutation $\alpha_1 \gamma_q^{-1} \alpha_2$ to $\alpha_1(\alpha_2(i) + 1)$. Let us compute the image of the corresponding element $2i \in [2q]$ through the permutation $\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2)$:

$$2i \xrightarrow{\tilde{\alpha}_1 \oplus \tilde{\alpha}_2} 2\alpha_2(i) \xrightarrow{\gamma_{2q}^{-1}} 2\alpha_2(i) + 1, \qquad (6.10)$$

which is an odd number. Another application of $\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2)$ yields

$$2\alpha_2(i) + 1 = 2(\alpha_2(i) + 1) - 1 \xrightarrow{\tilde{\alpha}_1 \oplus \tilde{\alpha}_2} 2\alpha_1(\alpha_2(i) + 1) - 1$$
$$\xrightarrow{\gamma_{2q}^{-1}} 2\alpha_1(\alpha_2(i) + 1), \tag{6.11}$$

which establishes a bijection between the cycles of $\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2)$ and $\alpha_1 \gamma_q^{-1} \alpha_2$, finishing the proof. $\qquad\square$

Remark 6.3. In the proof of Lemma 6.2, $\gamma_{2q}^{-1}(\tilde{\alpha}_1 \oplus \tilde{\alpha}_2)$ is the number of loops in meanders [24] constructed by the two non-crossing partitions $\alpha_{1,2}$, see [29, 25].

To state the next result, we introduce the "symmetric square root" operator R acting on probability measures supported on the positive real line:

$$R[\mu] := \frac{1}{2}(\sqrt{x})_{\#}\mu + \frac{1}{2}(-\sqrt{x})_{\#}\mu. \tag{6.12}$$

Proposition 6.4. *In the asymptotic regime $1 \ll k \ll n$, the random matrix $C_n^{c\Gamma}$ converges in moments to the probability measure*

$$\mu_t^{(C c\Gamma)} = (R \circ D_t) \left[b_t^{\boxplus 1/t} \right], \tag{6.13}$$

for which we have

$$\|\mu_t^{(Cc\Gamma)}\| = \begin{cases} 2\sqrt{t(1-t)} & \text{if } t \leq 1/2 \\ 1 & \text{if } t > 1/2. \end{cases} \tag{6.14}$$

Proof. As usual, the first step in the proof is a moment formula, valid at any fixed dimensions n, k, d. Using the graphical Weingarten formula (see Figure 6.2), we have

$$\frac{1}{nd} \mathbb{E} \operatorname{Tr} \left[\left(C_n^{c\Gamma} \right)^p \right] = \frac{1}{nd} \sum_{\alpha, \beta \in \mathcal{S}_p} n^{\#(\gamma^{-1}\alpha)} k^{\#\alpha} d^{\#(\gamma\beta)} \operatorname{Wg}_{kn}(\alpha^{-1}\beta). \tag{6.15}$$

We compute the asymptotic moments of $C_n^{c\Gamma}$ by taking two successive limits, first in n and then in k. The power of n in the moment formula above reads

$$\text{power of } n \text{ in (6.15)} = -2 + p - |\gamma^{-1}\alpha| + p - |\gamma\beta| - p - |\alpha^{-1}\beta|$$
$$= p - 2 - (|\gamma\beta| + |\beta^{-1}\alpha| + |\alpha^{-1}\gamma|)$$
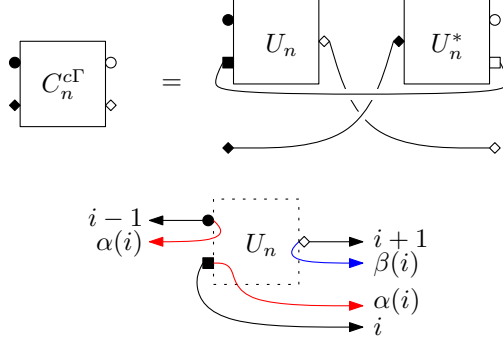$$\leq p - 2 - |\gamma^2| \leq 0, \tag{6.16}$$

Figure 6.2. Diagrams for the partially transposed Choi matrix $C_n^{c\Gamma}$ of the complementary channel and for the i-th U_n box in the graphical expansion of $\mathrm{Tr}\left[(C_n^{c\Gamma})^p\right]$.

where we have used the triangle inequality and the fact that γ^2 has either one or two cycles, depending on whether p is respectively odd or even. The above bound is saturated if and only if p is an even number and the permutations α, β lie on the geodesic $\gamma^{-1} - \beta - \alpha - \gamma$. Hence, for even integers $p = 2q$, $q \geq 1$, we have

$$
\lim_{n \to \infty} (nd)^{-1} \, \mathbb{E} \, \mathrm{Tr}\left[\left(C_n^{c\Gamma}\right)^p\right]
$$
$$
= (tk)^{-1} \sum_{\gamma^{-1}-\beta-\alpha-\gamma} k^{2p-|\alpha|-|\gamma\beta|-p-|\alpha^{-1}\beta|} \, t^{p-|\gamma\beta|} \, \mathrm{Mob}(\alpha, \beta)
$$
$$
= (tk)^{-1} \sum_{\gamma^{-1}-\beta-\alpha-\gamma} k^{p-|\alpha|-|\gamma\alpha|} \, t^{p-|\gamma\beta|} \, \mathrm{Mob}(\alpha, \beta)
$$
$$
= (tk)^{-1} \sum_{\gamma^{-1}-\beta-\alpha-\gamma} k^{\#\alpha-|\gamma\alpha|} \, t^{\#(\gamma\beta)} \, \mathrm{Mob}(\alpha, \beta). \tag{6.17}
$$

Then, we shift our permutations and work on the geodesic: $\mathrm{id} - \gamma\beta - \gamma\alpha - \gamma^2$. Importantly, γ^2 decomposes as $\gamma^2 = \gamma_1 \oplus \gamma_2$ with $\gamma_1 = (2q-1, 2q-3, \ldots, 1)$ and $\gamma_2 = (2q, 2q-2, \ldots, 2)$. Permutations on the geodesic $\mathrm{id} - \gamma\beta - \gamma\alpha - \gamma^2$ admit the same decomposition, so we write $\gamma\alpha = \alpha_1 \oplus \alpha_2$ and $\gamma\beta = \beta_1 \oplus \beta_2$. Using the moment-cumulant formula, we obtain (6.17)

as

$$\frac{1}{tk} \sum_{\substack{\mathrm{id}-\beta_1-\alpha_1-\gamma_1 \\ \mathrm{id}-\beta_2-\alpha_2-\gamma_2}} k^{\#(\gamma^{-1}(\alpha_1\oplus\alpha_2))-|\alpha_1\oplus\alpha_2|}\, t^{\#(\beta_1\oplus\beta_2)}\, \mathrm{Mob}(\alpha_1,\beta_1)\, \mathrm{Mob}(\alpha_2,\beta_2)$$

$$= \frac{1}{tk} \sum_{\substack{\mathrm{id}-\alpha_1-\gamma_1 \\ \mathrm{id}-\alpha_2-\gamma_2}} k^{\#(\gamma^{-1}(\alpha_1\oplus\alpha_2))-|\alpha_1|-|\alpha_2|} \left[\sum_{\mathrm{id}-\beta_1-\alpha_1} t^{\#\beta_1}\, \mathrm{Mob}(\alpha_1,\beta_1) \right]$$

$$\times \left[\sum_{\mathrm{id}-\beta_2-\alpha_2} t^{\#\beta_2}\, \mathrm{Mob}(\alpha_2,\beta_2) \right]$$

$$= \frac{1}{tk} \sum_{\substack{\mathrm{id}-\alpha_1-\gamma_1 \\ \mathrm{id}-\alpha_2-\gamma_2}} k^{\#(\gamma^{-1}(\alpha_1\oplus\alpha_2))-|\alpha_1|-|\alpha_2|}\, \kappa_{\alpha_1}(b_t)\, \kappa_{\alpha_2}(b_t). \qquad (6.18)$$

Unfortunately, we are not able to identify a probability measure having these moments. One of the reasons for this is the relation between the sum above and the combinatorics of meanders [24], see Remark 6.3. We are thus taking the second limit, $k \to \infty$. To do so, we calculate the power of k in (6.18) by using Lemma 6.2. In what follows, we abuse notation by writing, as in Lemma 6.2, $\alpha_{1,2}$ for the permutations in $\mathcal{S}_q$ obtained by restricting the previous $\alpha_{1,2} \in \mathcal{S}_{2q}$ on odd, resp. even numbers; note that by doing this, the quantities $|\alpha_{1,2}|$ remain invariant. We get:

$$\begin{aligned} \text{power of } k \text{ in (6.18)} \quad &= \quad -1 + \#(\alpha_1(\alpha_2^{\mathrm{Kr}})^{-1}) - |\alpha_1| - |\alpha_2| \\ &= \quad |\alpha_2^{\mathrm{Kr}}| - (|\alpha_1| + |\alpha_1^{-1}\alpha_2^{\mathrm{Kr}}|) \leq 0, \qquad (6.19) \end{aligned}$$

where the bound is saturated if and only if $\alpha_1, \alpha_2 \in \mathcal{S}_q$ are on the geodesic $\mathrm{id} - \alpha_2 - \alpha_1^{\mathrm{Kr}}$. Therefore, using repeatedly the moment cumulant formula,

we obtain

$$\lim_{k \to \infty} (6.18) = t^{-1} \sum_{\mathrm{id}-\alpha_1-\alpha_2^{\mathrm{Kr}}-\gamma_q} \kappa_{\alpha_1}(b_t)\, \kappa_{\alpha_2}(b_t) \qquad (6.20)$$

$$= t^{-1} \sum_{\mathrm{id}-\alpha_2^{\mathrm{Kr}}-\gamma_q} m_{\alpha_2^{\mathrm{Kr}}}(b_t)\, \kappa_{\alpha_2}(b_t)$$

$$= t^q \sum_{\mathrm{id}-\alpha_2^{\mathrm{Kr}}-\gamma_q} t^{-\#(\alpha_2)}\, \kappa_{\alpha_2}(b_t)$$

$$= t^q \sum_{\mathrm{id}-\alpha_2^{\mathrm{Kr}}-\gamma_q} \kappa_{\alpha_2}(b_t^{\boxplus 1/t})$$

$$= m_q \left(D_t \left[b_t^{\boxplus 1/t} \right] \right).$$

Finally, we have, for all integers $q \geq 1$,

$$\lim_{k \to \infty} \lim_{n \to \infty} (nd)^{-1}\, \mathbb{E}\, \mathrm{Tr} \left[\left(C_n^{c\Gamma} \right)^{2q} \right] = m_q \left(D_t \left[b_t^{\boxplus 1/t} \right] \right), \qquad (6.21)$$

and

$$\lim_{k \to \infty} \lim_{n \to \infty} (nd)^{-1}\, \mathbb{E}\, \mathrm{Tr} \left[\left(C_n^{c\Gamma} \right)^{2q+1} \right] = 0. \qquad (6.22)$$

To identify the distribution $\mu_t^{(Cc\Gamma)}$ as in (6.13), we have to modify $D_t \left[b_t^{\boxplus 1/t} \right]$ by $(x \to \sqrt{x})_\#$ to have right powers for the moments and make it symmetric as the odd moments vanish. This process is done by (6.12). To specify the support of $D_t \left[b_t^{\boxplus 1/t} \right]$, we use Proposition 2.9 with $T = 1/t$ and $s = t$, and since $\varphi^+(t,t) = 4t(1-t)$, we obtain the conclusion about the support of $\mu_t^{(Cc\Gamma)}$, equation (6.14). $\qquad \square$

6.3. **Partially transposed random projections**

In this subsection, we turn to the study of the bound (4.12) for random quantum channels. Using the fact that $M = VV^*$ is a (random) projection, we are actually interested in the operator norm of the partial transposition of a random projection – this was Montanaro's point of view in [44], where this question was studied, in a different asymptotic regime. For a sequence of random quantum channels L_n, we compute the limiting eigenvalue distribution $\mu_{k,t}^{(M\Gamma)}$ of the partially transposed random projection M_n^Γ, which is expressed in a free probabilistic framework. Unfortunately,

we are not able to compute the support of $\mu_{k,t}^{(M\Gamma)}$, so we have to settle with some partial information: as in the previous section, we consider the limit $\mu_t^{(M\Gamma)} = \lim_{k\to\infty} \mu_{k,t}^{(M\Gamma)}$, and we compute the support of this simpler measure. Below, we write $\mu \boxminus \nu := \mu \boxplus D_{-1}[\nu]$.

Recall that the *semicircular* probability distribution of mean m and standard deviation σ is given by

$$\mathrm{SC}_{m,\sigma} = \frac{\sqrt{4\sigma^2 - (x-m)^2}}{2\pi\sigma^2} \mathbf{1}_{[m-2\sigma, m+2\sigma]}(x)dx. \tag{6.23}$$

Proposition 6.5. *The partially transposed random projection M_n^Γ converges, in moments, towards the probability measure*

$$\mu_{k,t}^{(M\Gamma)} = D_{1/k}\left[b_t^{\boxplus \frac{k(k+1)}{2}} \boxminus b_t^{\boxplus \frac{k(k-1)}{2}}\right]. \tag{6.24}$$

The probability measures $\mu_{k,t}^{(M\Gamma)}$ converge, in distribution, as $k \to \infty$, towards

$$\lim_{k\to\infty} \mu_{k,t}^{(M\Gamma)} = \mu_t^{(M\Gamma)} = \mathrm{SC}_{t,\sqrt{t(1-t)}}, \tag{6.25}$$

a semi-circular distribution with mean t and varaince $t(1-t)$. In particular, we have

$$\|\mu_t^{(M\Gamma)}\| = t + 2\sqrt{t(1-t)}. \tag{6.26}$$

Proof. We start with a moment formula, obtained via graphical Weingarten calculus (see Figure 6.3 for an explanation of the exponents appearing below): for any integer $p \geq 1$,

$$\frac{1}{nk} \mathbb{E}\,\mathrm{Tr}\left[(M_n^\Gamma)^p\right] = \frac{1}{nk} \sum_{\alpha, \beta \in \mathcal{S}_p} n^{\#(\gamma^{-1}\alpha)} k^{\#(\gamma\alpha)} d^{\#\beta} \,\mathrm{Wg}(\alpha^{-1}\beta). \tag{6.27}$$

To obtain the surviving terms as $n \to \infty$, we maximize the power of n for each term in the equation above (recall that $d \sim tkn$)

$$\begin{aligned}
\text{power of } n \text{ in (6.27)} &= -1 + \#(\gamma^{-1}\alpha) + \#\beta - p - |\alpha^{-1}\beta| \\
&= p - 1 - (|\beta| + |\beta^{-1}\alpha| + |\alpha^{-1}\gamma|) \\
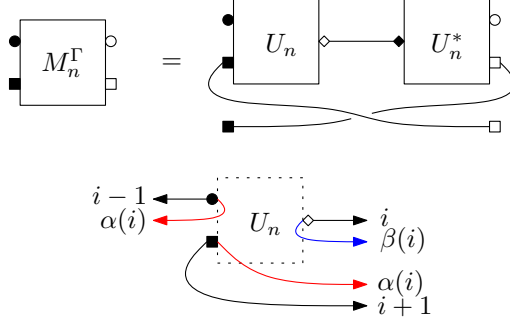&\leq p - 1 - |\gamma| = 0.
\end{aligned} \tag{6.28}$$

FIGURE 6.3. Diagrams for the partially transposed projection M_n^Γ and for the i-th U_n box in the graphical expansion of its p-th moment.

The bound above is saturated if and only if the permutations $\alpha, \beta \in \mathcal{S}_p$ lie on the geodesic $\text{id} - \beta - \alpha - \gamma$. This implies that

$$\lim_{n \to \infty} \frac{1}{nk} \mathbb{E} \operatorname{Tr}\left[\left(M_n^\Gamma\right)^p\right] = \frac{1}{k} \sum_{\text{id} - \beta - \alpha - \gamma} k^{\#(\gamma\alpha)}(tk)^{\#\beta} k^{-p-|\alpha^{-1}\beta|} \operatorname{Mob}(\alpha, \beta)$$

$$= k^{-p} \sum_{\text{id} - \beta - \alpha - \gamma} k^{e(\alpha) + \#\alpha} t^{\#(\beta)} \operatorname{Mob}(\alpha, \beta)$$

$$= k^{-p} \sum_{\text{id} - \alpha - \gamma} k^{e(\alpha) + \#\alpha} \kappa_\alpha(b_t), \tag{6.29}$$

where we have made use again of the fact that, for a geodesic permutation $\text{id} - \alpha - \gamma$, we have $\#(\gamma\alpha) = 1 + e(\alpha)$, where $e(\alpha)$ denotes the number of cycles of α having even length (see [6, Lemma 2.1] for a proof). Importantly, the general term in the sum above is a function which is multiplicative on the cycles of α:

$$k^{e(\alpha) + \#(\alpha)} \kappa_\alpha(b_t) = \prod_{c \in \alpha} f(\|c\|) \tag{6.30}$$

where $\|c\|$ is the length of a cycle c, and f is defined by

$$f(r) = \kappa_r(b_t) \times \begin{cases} k & \text{if } r \text{ is odd} \\ k^2 & \text{if } r \text{ is even.} \end{cases} \tag{6.31}$$

On the other hand, given two probability measures ν_1 and ν_2, we have

$$\kappa_r(\nu_1 \boxminus \nu_2) = \kappa_r(\nu_1) + (-1)^r \kappa_r(\nu_2) \tag{6.32}$$

and we notice in fact that

$$f(r) = \kappa_r \left(b_t^{\boxplus \frac{k(k+1)}{2}} \right) + (-1)^r \kappa_r \left(b_t^{\boxplus \frac{k(k-1)}{2}} \right). \tag{6.33}$$

Using the moment-cumulant formula, we prove our first claim:

$$\lim_{n \to \infty} (nk)^{-1} \mathbb{E} \operatorname{Tr} \left[\left(M_n^\Gamma \right)^p \right] = m_p \left(D_{1/k} \left[b_t^{\boxplus \frac{k(k+1)}{2}} \boxminus b_t^{\boxplus \frac{k(k-1)}{2}} \right] \right). \tag{6.34}$$

Since we are not able to analytically describe the support of the measure $\mu_{k,t}^{(M\Gamma)}$ above, we take limit $k \to \infty$ in (6.29). For each geodesic permutation $\mathrm{id} - \alpha - \gamma$, we claim that

$$\text{power of } k \text{ in (6.29)} = -p + e(\alpha) + \#(\alpha) \leq 0.$$

Indeed, assume that α has q fixed points, $0 \leq q \leq p$. Then, $e(\alpha) \leq (p-q)/2$ and $\#\alpha \leq q + (p-q)/2$, which proves the inequality. Permutations α saturating the bound must have *exactly q* fixed points and $(p-q)/2$ cycles of even length, which implies that the non-trivial cycles have length 2. We denote the set of non-crossing partitions having only blocks of length 1 and 2 by $NC_{1,2}(p)$ (see also [3] for another instance where this set was related to non-centered semicircular distributions). Then, $\lim_{k \to \infty}$ (6.29) is

$$\sum_{\alpha \in NC_{1,2}(p)} \kappa_\alpha(b_t) = \sum_{\alpha \in NC(p)} \kappa_\alpha \left(\mathrm{SC}_{t, \sqrt{t(1-t)}} \right) = m_p \left(\mathrm{SC}_{t, \sqrt{t(1-t)}} \right), \tag{6.35}$$

because the first two free cumulants are respectively

$$\kappa_1(b_t) = \kappa_1 \left(\mathrm{SC}_{t, \sqrt{t(1-t)}} \right) = t \text{ and } \kappa_2(b_t) = \kappa_2 \left(\mathrm{SC}_{t, \sqrt{t(1-t)}} \right) = t - t^2. \quad \square$$

Remark 6.6. The convergence in distribution in the result above was also found in [2], using operator-valued free probabilistic methods.

Remark 6.7. Equation (6.25) can also be proved using the free Central Limit Theorem [45, Theorem 8.10]. Indeed, reorder the terms in $\mu_{k,t}^{(M\Gamma)}$ to write

$$\mu_{k,t}^{(M\Gamma)} = D_{1/\sqrt{2}} D_{1/\sqrt{k(k-1)/2}} \left[(b_t \boxminus b_t)^{\boxplus \frac{k(k-1)}{2}} \right] \boxplus D_{1/k} [b_t^{\boxplus k}]. \tag{6.36}$$

The first part above is responsible for the centered semicircular part of (6.25), while the second term is responsible for the shift t.

6.4. **Comparing the bounds**

In the previous three subsections and in Section 5, we have computed the asymptotic limits of the bounds (4.9)-(4.12) in the case of large dimensional random quantum channels (recall that the bound (4.13) is always trivial). Since our ultimate goal is to use these quantities and Proposition 4.4 to lower bound minimum output Rényi entropies of quantum channels, we analyze in this subsection which of the four quantities yields the tightest bounds. Note that the quantity $\|M_L^\Gamma\|$ is different from the other three, since it provides a lower bound for the $p = \infty$ minimum output Rényi entropy of L; however, using the inequality $H_2 \geq H_\infty$ (see Lemma 2.1), we shall consider it here as a lower bound for $H_2^{\min}(L)$.

Let us start with the two bounds arising from the Choi matrix of the channel L.

Proposition 6.8. *For a sequence of random quantum channels L_n, the following inequality holds almost surely:*

$$\lim_{n \to \infty} \|C_n^\Gamma\| = \|\mu_{k,t}^{(C\Gamma)}\| \leq \|\mu_{k,t}^{(C)}\| = \lim_{n \to \infty} \|C_n\|. \tag{6.37}$$

Proof. First, note that at fixed k, the two norms in question, (5.24) and (6.3), are increasing functions of t. Then, note that the value of t where the norm of the Choi matrix becomes constant ($= 1$) is smaller: $t + k^{-2} \leq t + (k+1)/(2k)$. Finally, the following remarkable identity holds: $\forall t < 1 - k^{-2}$,

$$\|\mu_{k,t}^{(C)}\| - \|\mu_{k,t}^{(C\Gamma)}\| = k\varphi^+(k^{-2}, t) - 2\varphi^+\left(\frac{k+1}{2k}, t\right) + 1 = kt \geq 0. \tag{6.38}$$

The proof follows now from the three facts above, see also Figure 6.4 for the case $k = 2$. $\qquad\square$

In the previous two subsections, we were unfortunately not able to compute in full generality the asymptotic operator norm for the other two bounds, $\|C_{L^c}^\Gamma\|$ and $\|M_L^\Gamma\|$. We have to settle thus for a partial result, concerning their asymptotic behaviour in the case $1 \ll k \ll n$ (this regime corresponds to first taking the limit $n \to \infty$, followed by the limit $k \to \infty$).

Proposition 6.9. *For a sequence of random quantum channels L_n, in the asymptotical regime $1 \ll k \ll n$, the following inequalities hold:*

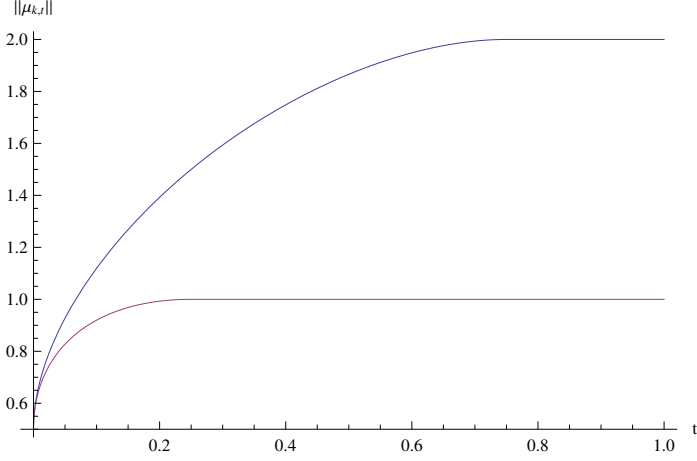$$\|\mu_t^{(C\Gamma)}\| = \|\mu_t^{(Cc\Gamma)}\| \leq \|\mu_t^{(M\Gamma)}\|, \tag{6.39}$$

FIGURE 6.4. For large random quantum channels, the norm of the Choi matrix (blue) is larger than the norm of the partially transposed Choi matrix (red), here for $k = 2$.

where

$$\mu_t^{(C\Gamma)} = \lim_{k \to \infty} \mu_{k,t}^{(C\Gamma)} = D_t \left[\left(\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{+1} \right)^{\boxplus 1/t} \right]. \tag{6.40}$$

In the case of the Choi matrices,

$$\lim_{k \to \infty} \|\mu_{k,t}^{(C)}\| = +\infty. \tag{6.41}$$

Proof. First, (5.5) shows the equality in (6.40). Then, taking $k \to \infty$ in Theorem 5.5, $\|\mu_t^{(C\Gamma)}\|$ turns out to be the same as $\|\mu_t^{(Cc\Gamma)}\|$, which is given in (6.14). Next, (6.26) results in the inequality in (6.39). Finally, (6.41) is obtained by (6.3). $\qquad\square$

Remark 6.10. Note that, although the probability measures $\mu_t^{(C\Gamma)}$ and $\mu_t^{(Cc\Gamma)}$ are both symmetric and have the same support upper bound, they are different. Indeed, we have

$$\text{Var}[\mu_t^{(C\Gamma)}] = t^2 \kappa_2 \left[\left(\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{+1} \right)^{\boxplus 1/t} \right] = t \text{Var} \left[\frac{1}{2} \delta_{-1} + \frac{1}{2} \delta_{+1} \right] = t,$$
$$\tag{6.42}$$

while

$$\mathrm{Var}[\mu_t^{(Cc\Gamma)}] = t^2 \kappa_1 \left[b_t^{\boxplus 1/t} \right] = t^2. \tag{6.43}$$

Finally, we present in Figure 6.5 some numerical result in the case $k = 2$, $t = 0.1$, for all the bounds. In the case of C_L, C_L^Γ, and M_L^Γ, the 10 random isometries with $n = 2000$ were used to produce the eigenvalue plot, while for $C_{L^c}^\Gamma$ we used 10 random isometries with $n = 100$ (note that we removed from the graphs some Dirac masses at zero corresponding to rank-deficient matrices). The approximations we deduce for the norms of the matrices are presented in Table 6.1. In the case we consider $(k = 2)$, it seems that the bound corresponding to $C_{L^c}^\Gamma$ is the tightest. However, this might be due simply to the rather small value of n compared to the other cases, so we do not wish to make any conjectures at this time.
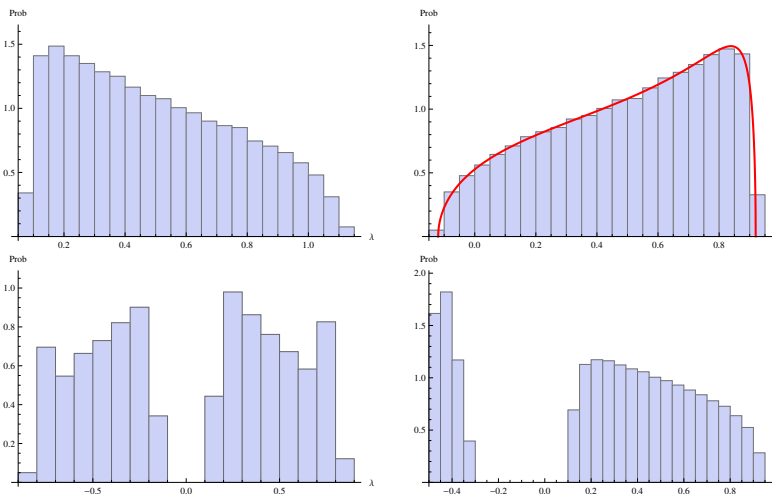

FIGURE 6.5. Histograms for the eigenvalue distributions of, respectively, C_L, C_L^Γ, and $C_{L^c}^\Gamma$, and M_L^Γ. For each histogram, we have chosen 10 random matrices with $k = 2$, $t = 0.1$ and $n = 2000$ (except for $C_{L^c}^\Gamma$, where $n = 100$).

We would like to conclude this section with a discussion on the optimality of the four bounds, in the case of random quantum channels. From a practical point of view, note that we have explicit formulas, at fixed k and t, only for the two bounds corresponding to Choi matrices, $\|C_L\|$ and

TABLE 6.1. Numerical results for the four bounds, in the case $k = 2$ and $t = 0.1$.

Bound	Numerical estimate	Theoretical value
$\|C_L\|$	1.12360	1.11962
$\|C_L^\Gamma\|$	0.91990	0.91961
$\|C_{L^c}^\Gamma\|$	0.85758	–
$\|M_L^\Gamma\|$	0.94548	–

$\|C_L^\Gamma\|$; among the two, Proposition 6.8 shows that the bound for the partial transposition is always tighter. In the asymptotical regime where t is fixed and $k \to \infty$, in Proposition 6.9 we show that the bound $\|C_L^\Gamma\|$ is tighter than $\|M_L^\Gamma\|$ (which is the bound used by Montanaro in [44]), while the two bounds corresponding to partially transposed Choi matrices, $\|C_L^\Gamma\|$ and $\|C_{L^c}^\Gamma\|$, perform equally well. Numerical simulations for $k = 2$ and $t = 0.1$ seem to suggest that $\|C_{L^c}^\Gamma\|$ performs better in this particular case; however, we do not consider this numerical data conclusive, because of the small value of the parameter n that was used to obtained them, due to machine memory limitations.

For these reasons, we choose to work in the next sections with the bound $\|C_L^\Gamma\|$, corresponding to the partial transposition of the Choi matrix, see (5.24)-(5.25).

7. Minimum output entropies for a single random quantum channel

In this section we recall some upper bounds for minimum output entropies of random quantum channels we shall use in what follows. The following fact is a collection of results from [9, 17]:

Theorem 7.1. *For all $p \in [0, \infty]$ and for almost all sequences of random quantum channels $(L_n)_n$, we have*

$$\limsup_{n \to \infty} H_p^{\min}(L_n) \leq H_p(x_{k,t}) =: h_{p,k,t}, \qquad (7.1)$$

where

$$x_{k,t} = \left(y, \underbrace{\frac{1-y}{k-1}, \ldots, \frac{1-y}{k-1}}_{k-1 \ times} \right) \tag{7.2}$$

with

$$y = \max \operatorname{supp}(b_t \boxtimes b_{1/k}) = \max(1, \varphi^+(t, 1/k))$$

$$= \min \left[1, t + \frac{1}{k} - 2\frac{t}{k} + 2\sqrt{t(1-t)\frac{1}{k}\left(1 - \frac{1}{k}\right)} \right]. \tag{7.3}$$

This statement also holds for the sequence of complementary channels $(L_n^C)_n$.

Moreover, for $p \in [1, \infty]$, the above inequality is an equality, and $\limsup$ can be replaced by $\lim$ in (7.1).

Proof. In [17, Theorem 4.1] it is shown that the largest eigenvalue of an output of a random quantum channel is at most y, and that the value y is almost surely attained. Given this partial information, the upper bound (7.1) follows from the concavity of p-Rényi entropies for $p \in [0, \infty]$ (to maximize entropy, the smaller eigenvalues should be identical). The second part of the statement follows from the finer analysis in [8, Theorem 5.2], where it is shown that the eigenvalue vector $x_{k,t}$ above is the one which achieves the minimum entropy among outputs of the random quantum channel, in the case where $p \geq 1$. □

Let us now study the asymptotics of the above upper bound, in the regime $k \to \infty$. Let us remind the reader that these results concern quantities for which the limit $n \to \infty$ has already been taken; in other words, we are considering the asymptotical regime $1 \ll k \ll n$.

Corollary 7.2. *In the setting of Theorem 7.1, for fixed $p \in [0, \infty]$, asymptotically as $k \to \infty$, we have:*

i) *When $0 < t < 1$ is a fixed constant,*

$$h_{p,k,t} = o(1) + \begin{cases} \dfrac{p}{1-p} \log t, & if \ p > 1 \\ (1-t) \log k - t \log t - (1-t) \log(1-t), & if \ p = 1 \\ \log k + \dfrac{p}{1-p} \log(1-t), & if \ 0 \le p < 1. \end{cases}$$

(7.4)

ii) *When $t \asymp k^{-\tau}$ for some constant $\tau > 0$,*

$$h_{p,k,t} = o(1) + \begin{cases} \dfrac{\tau p}{p-1} \log k, & if \ 0 < \tau \le 1 - \dfrac{1}{p} \\ \log k, & if \ 1 - \dfrac{1}{p} < \tau. \end{cases}$$

(7.5)

Proof. First, note that since we are in the large k regime, $y = \varphi^+(t, 1/k) < 1$. In the case where t is fixed, we have $y = t + O(k^{-1/2})$ so that, depending on the value of p, the main contribution to the quantity $\|x_{k,t}\|_p^p$ is given either by y (when $p > 1$), $(1-y)/(k-1)$ (when $p < 1$), or by the whole vector $x_{k,t}$ (when $p = 1$). In the second case, where t scales like $k^{-\tau}$, we have that $y \asymp k^{-\tau} + k^{-1}$, which implies that

$$\|x_{k,t}\|_p^p \asymp k^{-p\tau} + k^{-p+1}.$$

(7.6)

We conclude by taking logarithms of the expressions above. $\qquad\square$

Remark 7.3. In Corollary 7.2 the case when $\tau = 1$ can be derived from [5]. Also, the case $\tau \ge 1$ can be treated via $\max_{X \in \mathcal{M}_d^{1,+}} \|L(X) - I/k\|_2 \asymp \sqrt{\frac{t}{k}} + t$ [28] which means that all the output states are highly mixed.

8. **Additivity rates of random quantum channels**

This section contains one of the main results of our work, lower bounds for the additivity rates of random quantum channels. We shall use Proposition 4.4 and the estimates from Sections 5. Indeed, in the following sections we will only consider the bound given by the operator norm of the partially transposed Choi matrix of a quantum channel, for the reasons discussed in Section 6.4.

8.1. **Minimum output Rényi entropy**

The following result is a direct consequence of the bound in Proposition 4.4, the strong convergence result in Theorem 5.5 and Lemma 2.1.

Theorem 8.1. *Fix a positive integer r and a Rényi entropy parameter $p \in [0, 2]$. Then, almost surely, as $n \to \infty$,*

$$\lim_{n \to \infty} H_p^{\min}(L_n^{\otimes r}) \geq -r \log \|\mu_{k,t}^{(C\Gamma)}\|, \tag{8.1}$$

where $\|\mu_{k,t}^{(C\Gamma)}\|$ was given in (5.24)-(5.25). This statement also holds for the sequence of complementary channels $(L_n^C)_n$.

Remark 8.2. In the case $p > 2$, the inequality $\|x\|_p \leq \|x\|_2$ must be used, and thus a correction factor appears in the bound:

$$\lim_{n \to \infty} H_p^{\min}(L_n^{\otimes r}) \geq -r \log \|\mu_{k,t}^{(C\Gamma)}\| \cdot \frac{p}{2(p-1)}. \tag{8.2}$$

Next, we investigate how the quantity $\|\mu_{k,t}^{(C\Gamma)}\|$ behaves when $k \to \infty$ (recall that we write $x_n \asymp y_n$ when $\lim_{n \to \infty} x_n/y_n \in (0, \infty)$).

Corollary 8.3. *In the setting of Theorem 5.5 we have the following asymptotic behaviors as $k \to \infty$.*

i) *When $t \geq 1/2$ is a constant, we have, for any k,*

$$\|\mu_{k,t}^{(C\Gamma)}\| = 1. \tag{8.3}$$

ii) *When $0 < t < 1/2$ is a constant, we have*

$$\|\mu_{k,t}^{(C\Gamma)}\| = 2\sqrt{t(1-t)} + o(1). \tag{8.4}$$

iii) *When $t \asymp k^{-\tau}$ and $0 < \tau \leq 2$, we have*

$$\|\mu_{k,t}^{(C\Gamma)}\| \asymp k^{-\frac{\tau}{2}}. \tag{8.5}$$

iv) *When $t \asymp k^{-\tau}$ and $\tau > 2$, we have*

$$\|\mu_{k,t}^{(C\Gamma)}\| \asymp k^{-1}. \tag{8.6}$$

Finally, Theorems 8.1, 7.1, and Corollaries 8.3, 7.2 immediately give one of the main results of this paper. The constants $\alpha_{p,k,t}^\Gamma$ below are almost sure limits of the lower bounds $\alpha_p^\Gamma(L_n)$ defined in Proposition 4.6.

Theorem 8.4. *For any $p \in [0, 2]$, almost surely as $n \to \infty$, the p-additivity rates of random quantum channels L_n are lower bounded by the constants*

$$\alpha_p(L_n) \geq \alpha^{\Gamma}_{p,k,t} := \frac{-\log \|\mu^{(C\Gamma)}_{k,t}\|}{h_{p,k,t}}. \tag{8.7}$$

where $\|\mu^{(C\Gamma)}_{k,t}\|$ and $h_{p,k,t}$ are given in (5.24)-(5.25) and (7.1). For example, in the case of the von Neumann entropy ($p = 1$), we obtain

$$\alpha_1(L_n) \geq \frac{-\log\left[\frac{1-2t}{k} + 2\sqrt{\left(1 - \frac{1}{k^2}\right) t(1-t)}\right]}{-y \log y - (1-y) \log \frac{1-y}{k-1}} \mathbf{1}_{t < (k-1)/(2k)}, \tag{8.8}$$

where $y = \varphi_+(t, 1/k)$.

Again, these statements hold for the sequence of complementary channels $(L_n^C)_n$.

Corollary 8.5. *The additivity rate lower bounds $\alpha^{\Gamma}_{p,k,t}$ obtained in the theorem above have the following behaviour as $k \to \infty$:*

I) *When $t \geq 1/2$ is a constant, then $\alpha^{\Gamma}_{p,k,t} = 0$ for all p and, actually, for all k.*

II) *When $0 < t < 1/2$ is a constant,*

$$\alpha^{\Gamma}_{p,k,t} = o(1) + \frac{p-1}{2p}\left[1 + \frac{2\log 2 + \log(1-t)}{\log t}\right] \cdot \mathbf{1}_{(1,2]}. \tag{8.9}$$

III) *When $t \asymp k^{-\tau}$ with $\tau > 0$,*

$$\alpha^{\Gamma}_{p,k,t} = o(1) + \begin{cases} \frac{p-1}{2p} & \text{if } 0 < \tau \leq 1 - 1/p \\ \tau/2 & \text{if } 1 - 1/p \leq \tau \leq 2 \\ 1 & \text{if } \tau \geq 2. \end{cases} \tag{8.10}$$

The above result can be summarized using the "phase diagram" in Figure 8.1, in which the asymptotical behavior ($k \to \infty$) of the lower bound $\alpha^{\Gamma}_{p,k,t}$ is presented as a function of p and $\tau = \log(1/t)/\log k$.

Remark 8.6. The above lower bound is non-trivial (i.e. $\alpha^{\Gamma}_{p,k,t} > 0$) if and only if $\|\mu^{(C\Gamma)}_{k,t}\| < 1$, which is equivalent to the condition $t < 1 - s = (k-1)/(2k) < 1/2$.
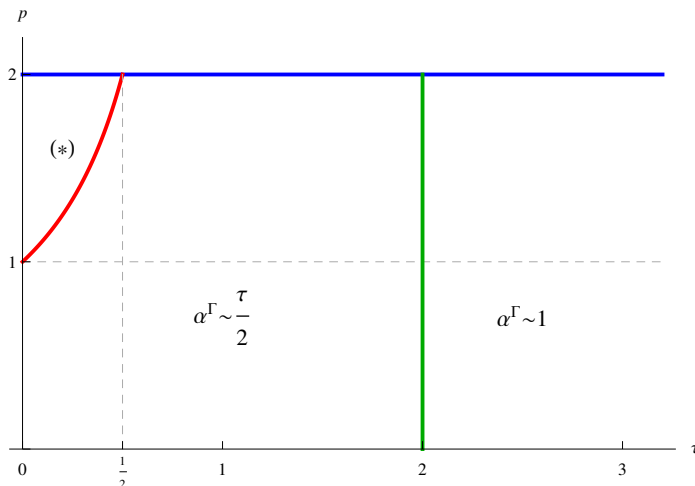
Figure 8.1. Phase diagram for the additivity rate lower bound $\alpha_{p,k,t}^{\Gamma}$ as a function of p and the scaling parameter τ, for large k. In the region denoted by $(*)$, $\alpha_{p,k,t}^{\Gamma}$ behaves like $(p-1)/(2p)$, while the red curve in the plot is defined by $p = 1/(1-\tau)$.

8.2. **Additivity rates versus weak multiplicativity exponents**

In this section, we compare the additivity rates from Theorem 8.4 with Montanaro's results from [44]. First, let us comment on some major differences between the two approaches. First and foremost, the asymptotic regimes for random quantum channels are different: whereas we consider sequences of random quantum channels with *fixed* output dimension, in [44] the author assumes that both the input and the output dimensions of the channels grow to infinity; for this reason, it is impossible to compare the results from a strictly mathematical perspective. Second, Montanaro's approach is to bound the additivity rate of a channel in terms of the matrix M^{Γ} defined in (4.12); in this work, we introduce three additional bounds and we argue that in the fixed output dimension regime, the bound corresponding to the partial transposition of the Choi matrix outperforms the one from [44]. The new quantities we introduce are interesting also for the

reason that they bound the 2 Rényi entropy, whereas the quantity from [44] bounds the ∞-Rényi entropy. Finally, we not only compute the limiting operator norms of the relevant random matrices, but in most cases we compute also the limiting eigenvalue distributions (see Section 10 for an application).

In spite of the fact that the current paper and [44] consider different asymptotic regimes, we would like to perform a heuristic benchmark of the additivity rates obtained. The following result is an adaptation of [44, Theorem 3] for the $p = 2$ Rényi entropy.

Proposition 8.7. *We follow the notations in Section 2.1. Suppose $k \leq n$, $\min\{d, k\} \geq 2(\log_2 n)^{3/2}$ and $d = o(kn)$. Then, typically the additivity rate is bounded: $\frac{1}{r} H_2^{\min}(L^{\otimes r}) \geq \beta H_2^{\min}(L)$ where*

$$\beta \sim \begin{cases} 1/4 & \text{if } d \geq n/k \\ 1/2 & \text{if } d \leq n/k \end{cases} \tag{8.11}$$

In this proposition, if we take $n = 2^{(k/2)^{2/3}}$ this brings it to our setting $1 \ll k \ll n$; note however that we are taking the two limits separately (first $n \to \infty$ and then $k \to \infty$) whereas in [44] the author considers the more general situation where both k and n grow at the same time (but at different speeds). By solving the equation (we have $t = k^{-\tau}$ in our mind)

$$d = \frac{n}{k} = k^{-\tau} kn \tag{8.12}$$

we get $\tau = 2$. Hence, $d \geq n/k$ and $d \leq n/k$ can be compared with our regimes $0 < \tau \leq 2$ and $\tau \geq 2$, respectively. Our method yields an improvement by a factor of two in this case.

In the case of the von Neumann entropy ($p = 1$), Montanaro obtains in [44, Section 1.3] additivity rates of 1 for $\tau = 1$ and $1/2$ for $\tau \geq 2$, which are precisely the values we obtain in Corollary 8.5. This can be explained by the fact that in the case where $t \sim k^{-\tau}$, the norm $\|M^\Gamma\|$ from equation (6.26) behaves like $2\sqrt{t}$, which is also the value from [44, Theorem 8]. Using the monotonicity of the Rényi entropies as functions of p, one could improve the results in [44] for other values of p, as in the case $p = 1$.

8.3. **Additivity rates for tensor products of conjugate random channels**

Besides the constructive counterexamples already discussed in the current work (see Examples 3.5, 3.6), the most successful technique to construct quantum channels that violate the additivity relation consists of taking *random conjugate channels* (recall that for a channel L as in (2.3), its conjugate channel is obtained by replacing V by $\bar{V}$). In this subsection, we investigate how the additivity violations of the minimum output p-entropy allow to improve the additivity rate of the product channel $\alpha_p(L \otimes \bar{L})$ with respect to the additivity rate of one channel $\alpha_p(L)$. These considerations are related to a conjecture by Hastings [33] that the quantum channels $L \otimes \bar{L}$ should be generically additive. Although there is some evidence supporting this claim [30], the conjecture is open to this date. We would like to bring further evidence to support this claim, by showing that, in some cases, the additivity rate of $L \otimes \bar{L}$ is larger than that of L, for random quantum channels L.

The idea of considering conjugate channels when trying to expose additivity violations is due to Hayden and Winter [34, Lemma 3.3], and it revolves around using the maximally entangled state (2.5) as a test input for the tensor product between a channel and its conjugate:

Lemma 8.8. *Consider a quantum channel* $L : \mathcal{M}_d(\mathbb{C}) \to \mathcal{M}_k(\mathbb{C})$ *which is defined via an isometry* $V : \mathbb{C}^d \to \mathbb{C}^n \otimes \mathbb{C}^k$. *Then, the output state* $[L \otimes \bar{L}](E_d)$ *cannot be too mixed, in the sense that*

$$\|[L \otimes \bar{L}](E_d)\| \geq \frac{d}{nk}. \tag{8.13}$$

The same bound holds for the complementary settings: $[L^C \otimes \overline{L^C}](E_d)$.

Since, obviously, the channels L and $\bar{L}$ have the same additivity rates, the additivity rates of their product can be lower bounded using Proposition 4.8 as follows:

$$\alpha_p^\Gamma(L \otimes \bar{L}) \geq v_p(L, \bar{L})\alpha_p^\Gamma(L). \tag{8.14}$$

It is intuitive now that the larger the entropy violation quotient v_p is, the larger the additivity rate of the product channel will be, when compared to that of a single channel. Moreover, in order to get explicit lower bounds on the additivity rates $\alpha_p(L \otimes \bar{L})$, we can lower bound the relative violation

of additivity v_p as follows:

$$v_p(L, \bar{L}) \geq \frac{2H_p^{\min}(L)}{H_p([L \otimes \bar{L}](E_d))}. \tag{8.15}$$

It follows that the quality of the lower bound $\alpha_p^{\Gamma}(L \otimes \bar{L})$ can be in principle improved by a factor as large as 2 when comparing to the single channel bound $\alpha_p^{\Gamma}(L)$.

We study next this phenomenon in the case of random quantum channels L_n described in Section 5. For our model of random quantum channels, the Hayden-Winter bound in Lemma 8.8 was refined in [22, Theorem 6.5]:

Lemma 8.9. *Consider a sequence of random quantum channels L_n as in Section 5. Then, almost surely as $n \to \infty$, the eigenvalues of the random quantum state $[L \otimes \bar{L}](E_d)$ converge to the following deterministic vector:*

$$\gamma_{k,t} = \left(t + \frac{1-t}{k^2}, \frac{1-t}{k^2}, \ldots, \frac{1-t}{k^2}\right) \in \mathcal{M}_{k^2}^{1,+}(\mathbb{C}). \tag{8.16}$$

Using this lemma, we are able to quantify the improvement in the lower bound of the product channel which is provided by additivity violations. The following proposition is a direct consequence of the above discussion and the asymptotic relations already used in this section.

Proposition 8.10. *Consider a sequence L_n of random quantum channels as in Section 5. Almost surely as $n \to \infty$, the p-additivity rates of random quantum channels $L_n \otimes \bar{L}_n$ are lower bounded by the constants*

$$\alpha_p(L_n \otimes \bar{L}_n) \geq v_{p,k,t}\alpha_{p,k,t}^{\Gamma}, \tag{8.17}$$

where $\alpha_{p,k,t}^{\Gamma}$ are the single channel bounds from Theorem 8.4, and

$$v_{p,k,t} = \frac{2h_{p,k,t}}{H_p(\gamma_{k,t})}, \tag{8.18}$$

where the vector $\gamma_{k,t}$ was defined in (8.16). The same bound holds for the complementary settings: $L^C \otimes \overline{L^C}$.

In particular, the parameter $v_{p,k,t}$ behaves like $1 + o(1)$ when $k \to \infty$, except in the following cases:

I) *When $0 < t < 1/2$ is a constant and $p > 1$, we have $v_{p,k,t} = 2 + o(1)$.*

II) *When $t \asymp k^{-\tau}$ with $p > 1$ and $0 < \tau < 1 - 1/p$, we have $v_{p,k,t} = 2 + o(1)$.*

III) *When $t \asymp k^{-\tau}$ with $p > 1$ and $1 - 1/p \leq \tau \leq 2 - 2/p$, we have*

$$v_{p,k,t} = \frac{2p-2}{\tau p} + o(1). \tag{8.19}$$

9. **Classical capacity for random quantum channels**

In this section, we discuss how the additivity rate bounds derived earlier yield interesting upper bounds for the classical capacity of (random) quantum channels. In general, it is difficult to calculate the classical capacity for a given quantum channel [10], so upper bounds are important in this situation.

Let us start with an important relation between the additivity rate for the von Neumann entropy α_1 and the classical capacity $\mathcal{C}_{cl}$ of a quantum channel.

Proposition 9.1. *For any quantum channel L,*

$$\mathcal{C}_{cl}(L) \leq H_1^{\max}(L) - \alpha_1(L) H_1^{\min}(L). \tag{9.1}$$

In particular,

$$\mathcal{C}_{cl}(L) \leq H_1^{\max}(L) + \log B, \tag{9.2}$$

where $B = \min\{\|C_L\|, \|C_L^\Gamma\|, \|C_{L^c}^\Gamma\|, \|M_L^\Gamma\|\}$.

Proof. The first inequality follows from (1.8) and Proposition 3.2, whereas the second one follows from Proposition 4.4. $\square$

Let us now analyze in detail the corresponding bounds for random quantum channels by using estimates developed in the previous sections. In our previous work [21], we have investigated the Holevo quantity $\chi(\cdot)$ for random quantum channels, but we were not able to analyze $\mathcal{C}_{cl}(\cdot)$ because we did not have the techniques to treat output entropy of tensor powers of quantum channels. However, now we use Proposition 9.1 and get bounds for the classical capacity, as follows.

Theorem 9.2. *For random quantum channels defined by (2.3), we have, almost surely*

$$\limsup_{n \to \infty} \mathcal{C}_{cl}(L_n) \leq \log k - \log \|\mu_{k,t}\|. \tag{9.3}$$

In particular, the capacity admits the following asymptotic bounds as $k \to \infty$.

i) *When $0 < t < 1/2$ is a constant, we have*
$$t \log k - h(t) \leq \liminf_{n\to\infty} \mathcal{C}_{cl}(L_n)$$

$$\leq \limsup_{n\to\infty} \mathcal{C}_{cl}(L_n) \leq \log k + \log 2 + \frac{1}{2} \log t(1-t) + o(1), \tag{9.4}$$

   where $h(t) = -t \log t - (1-t) \log(1-t)$ is the binary entropy.

ii) *When $t \asymp k^{-\tau}$ and $0 < \tau \leq 2$, we have*
$$\limsup_{n\to\infty} \mathcal{C}_{cl}(L_n) \leq \left(1 - \frac{\tau}{2}\right) \log k + c \tag{9.5}$$

   for some constant $c > 0$.

iii) *When $t \asymp k^{-\tau}$ and $\tau > 2$, we have*
$$\limsup_{n\to\infty} \mathcal{C}_{cl}(L_n) \leq c \tag{9.6}$$

   for some constant $c > 0$.

Proof. For the upper bound, one can use Theorem 8.1 and Corollary 8.3 together with (1.8). To show the lower bound, we claim that for our random channel L_n almost surely as $n \to \infty$
$$\lim_{n\to\infty} \chi(L_n) = \log k - \lim_{n\to\infty} H_1^{\min}(L_n) = \log k - h_{1,k,t}. \tag{9.7}$$
Indeed, the almost-sure limit image of L_n is unitarily invariant (see [8, 21] for details). Hence, we can rotate and average an optimal output to get the maximally mixed state, whose entropy is $\log k$. Then, Corollary 7.2 gives the lower bounds. $\qquad\square$

Note that the constants c appearing in the result above could have been explicitly computed using Theorem 8.1 and Corollary 8.3.

10. **PPT properties for random quantum channels**

In this section, we investigate the sequence of random quantum channels L_n defined in (2.3) and find the threshold for PPT/non-PPT property. Also, we show existence of PPT channels which violate additivity of Rényi p entropy.

Recall that a quantum channel L is said to have the *PPT property* if its Choi matrix is PPT, i.e. $C_L^\Gamma \geq 0$. This is equivalent to the fact that, for any bi-partite input state x, the output $[\mathrm{id} \otimes L](xx^*)$ is a PPT quantum state. It follows that the class of PPT channels contains as a (strict, for large enough dimensions) subclass the set of entanglement breaking channels (for which the Choi matrix is separable).

10.1. **Thresholds for PPT property**

Theorem 10.1. *Let L_n be a sequence of random quantum channels of parameters k, t as in (2.3), and let $s = (k+1)/(2k)$ and*

$$t_{PPT} = \frac{1}{2} - \sqrt{s - s^2} = \frac{1}{2}\left(1 - \sqrt{1 - \frac{1}{k^2}}\right). \tag{10.1}$$

If $t \in (0, t_{PPT})$ then, almost surely, the sequence $\lambda_{\min}(C_n^\Gamma)$ converges to a positive limit (the channels being asymptotic PPT), whereas if $t \in (t_{PPT}, 1]$, then, almost surely, the sequence $\lambda_{\min}(C_n^\Gamma)$ converges to a negative limit (the channels being asymptotic non-PPT). In other words, the value t_{PPT} is a threshold *for PPT channels: a random quantum channel is PPT if and only if its relative dimension t of the input space is smaller than t_{PPT}.*

Proof. We use the strong convergence proved in Theorem 5.5. Since the convergence is strong, the extremal eigenvalues of the partially transposed Choi matrix C_n^Γ converge to the edges of the support of the limiting measure $\mu_{k,t}$ defined in equations (5.5)-(5.6). Since we are interested in the positivity of the support, we only look at the smallest eigenvalue. We have that, almost surely,

$$\lim_{n \to \infty} \lambda_{\min}(C_n^\Gamma) = \begin{cases} 2\varphi^-(s,t) - 1, & \text{if } t < s \\ -1, & \text{if } t \geq s. \end{cases} \tag{10.2}$$

So, for the limiting quantity to be strictly positive, both conditions $t < s$ and $\varphi^-(s,t) > 1/2$ need to be satisfied. In order to conclude, we need to show that these conditions are equivalent to the ones in the statement.

For a fixed value of k (and thus s), the function $t \mapsto \varphi^-(s,t)$ is convex on $[0,1]$ and the equation $\varphi^-(s,t) = 1/2$ has solutions $t_\pm = 1/2 \pm \sqrt{s - s^2}$. Obviously, $\varphi^-(s,t) < 1/2 \iff t \in [0, t_-) \cup (t_+, 1]$. A direct computation shows that $t_+ > s$ iff $s > 1/2 + \sqrt{2}/4$, which is equivalent to $k > \sqrt{2}$,

which is true for all integer $k > 1$. Hence, $s \in (t_-, t_+)$ for the relevant values of k, and the conclusion follows. $\square$

Remark 10.2. In the above result, the threshold value t_{PPT} behaves as $k^{-2}/4 + o(k^{-2})$ as $k \to \infty$. In that range of parameters ($t \asymp k^{-2}$ or smaller), we have also shown in Theorem 8.4 that the channels L_n are almost additive, i.e. their additivity rate is $1 + o(1)$, for any $p \in [0, 2]$.

Remark 10.3. The above value of the threshold t_{PPT} proves also the claim that the distribution of the random matrix C_L^Γ is not that of a (rescaled) induced random density matrix [56]. Indeed, for random induced density matrices, the threshold for the PPT property has been computed in [6, Theorem 6.2]: $t_{PPT,induced} = 1/[4k(k-1)]$. In general, the value for random Choi matrices is smaller than the value for the induced ensemble, $t_{PPT} < t_{PPT,induced}$, proving that the two probability distributions are different. However, let us note that the two thresholds have the same asymptotic behaviour as $k \to \infty$.

Remark 10.4. Let us make one final remark concerning the case of the complementary channels. Indeed, it has been shown in Section 6.2 that the limiting spectral distribution of the Choi matrix of the complementary channel L_n^c is *symmetric* in the regime $1 \ll k \ll n$. Hence, in that regime, the complementary channel cannot be PPT. This illustrates the fact that, in general, a quantum channel and its complementary do not share the PPT property.

10.2. **PPT channels violating additivity**

In this subsection, we show existence of PPT channels which violate the additivity of Rényi p entropy with p large. There are two theorems presented below. Theorem 10.5 looks for the smallest possible dimension k, while Theorem 10.6 does for the minimum number of p.

Theorem 10.5. *Take $k \geq 76$ and set $d = \frac{n}{4k}$, then for large enough n and p, with high probability, L_n are PPT and we have additivity violation:*

$$H_p^{\min}(L_n \otimes \bar{L}_n) < 2H_p^{\min}(L_n). \tag{10.3}$$

Proof. First,

$$t_{PPT} = \frac{1}{2}\left[1 - \sqrt{1 - \frac{1}{k^2}}\right] > \frac{1}{4k^2} \tag{10.4}$$

So, set $t = 1/(4k^2)$ so that typical random channels are PPT by Theorem 10.1.

Next, by using the result in [22], almost surely

$$\lim_{n\to\infty} \max_X \left\| (L_n \otimes \bar{L}_n)(B_n) \right\|_\infty = t + \frac{1-t}{k^2} = \frac{5}{4k^2} - \frac{1}{4k^4} \qquad (10.5)$$

where B_n are Bell states on $\mathbb{C}^n \otimes \mathbb{C}^n$. On the other hand, by using the result in [22], almost surely

$$\lim_{n\to\infty} \max_X \|L_n(X)\|_\infty = t + \frac{1}{k} - 2\frac{t}{k} + 2\sqrt{t(1-t)\frac{1}{k}\left(1 - \frac{1}{k}\right)} \qquad (10.6)$$

$$\leq \frac{1}{k} + \frac{1}{k\sqrt{k}} + \frac{1}{4k^2} - \frac{1}{2k^3} \qquad (10.7)$$

This implies that, almost surely,

$$\lim_{n\to\infty} H_\infty^{\min}(L_n \otimes \bar{L}_n) < 2 \lim_{n\to\infty} H_\infty^{\min}(L_n) \qquad (10.8)$$

for large enough k. Indeed, for $76 \geq k$, we have $(10.5) > (10.7)^2$. Note that since $H_\infty^{\min}(\cdot) = \lim_{p\to\infty} H^{\min(\cdot)}$ we can extend the above violation of additivity to large p. The proof is complete by taking the intersection of two large-probability events. $\qquad \square$

Theorem 10.6. *Set $d = \frac{n}{4k}$. Then, for large enough k and n, with high probability L_n are PPT and we have additivity violation for all $p \geq 30.95$:*

$$H_p^{\min}(L_n \otimes \bar{L}_n) < 2H_p^{\min}(L_n). \qquad (10.9)$$

Proof. First, by using the second statement of Theorem 7.1, almost surely

$$\lim_{n\to\infty} H_p^{\min}(L_n) = h_{p,k,t} \qquad (10.10)$$

which was obtained by the output distribution $x_{k,t}$ in (7.2). For this distribution with $t = 1/(4k^2)$, we have

$$\left(\|x_{k,t}\|_p^p \right)^2 = \frac{k^2}{k^{2p}} + (p^2 - p)\frac{1}{k^{2p}} + O\left(\frac{1}{k^{2p+1/2}} \right) \qquad (10.11)$$

where the approximation is given by Wolfram Mathematica. Similarly, for $\gamma_{k,t}$ in (8.16),

$$\|\gamma_{k,t}\|_p^p = \frac{1}{(k^2-1)^p}\left[k^2 - 1 - \frac{5p}{4} \right] + \frac{1}{k^{2p}}\left(\frac{5}{4} \right)^p + O\left(\frac{1}{k^{2p+2}} \right)$$

$$= \frac{1}{k^{2p}}\left[k^2 - 1 - \frac{p}{4} + \left(\frac{5}{4} \right)^p \right] + O\left(\frac{1}{k^{2p+1}} \right) \qquad (10.12)$$

A sufficient condition for additivity violation is that (10.11) < (10.12), which is equivalent. in the regime $k \to \infty$, to

$$p^2 - \frac{3p}{4} + 1 < \left(\frac{5}{4}\right)^p \tag{10.13}$$

Again, solving the equation gives $p = 30.9441$ via Wolfram Mathematica. As in the proof of Theorem 10.5, L_n are typically PPT as $n \to \infty$ when $d = \frac{n}{4k}$. Therefore, taking the intersection of two high-probability events completes our proof. □

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