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Universal edge fluctuations of discrete interlaced particle systems

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Abstract

We impose the uniform probability measure on the set of all discrete Gelfand–Tsetlin patterns of depth n with the particles on row n in deterministic positions. These systems equivalently describe a broad class of random tilings models, and are closely related to the eigenvalue minor processes of a broad class of random Hermitian matrices. They have a determinantal structure, with a known correlation kernel. We rescale the systems by $\frac{1}{n}$, and examine the asymptotic behaviour, as $n \to \infty$, under weak asymptotic assumptions for the (rescaled) particles on row n: The empirical distribution of these converges weakly to a probability measure with compact support, and they otherwise satisfy mild regulatory restrictions.

We prove that the correlation kernel of particles in the neighbourhood of "typical edge points" convergences to the extended Airy kernel. To do this, we first find an appropriate scaling for the fluctuations of the particles. We give an explicit parameterisation of the asymptotic edge, define an analogous non-asymptotic edge curve (or finite n-deterministic equivalent), and choose our scaling such that the particles fluctuate around this with fluctuations of order $O(n^{-\frac{1}{3}})$ and $O(n^{-\frac{2}{3}})$ in the tangent and normal directions respectively. While the final results are quite natural, the technicalities involved in studying such a broad class of models under such weak asymptotic assumptions are unavoidable and extensive.

Universalité au bord pour la fluctuation de systèmes discrets de particules entrelacées

Résumé

Nous considérons la mesure uniforme sur l'ensemble des configurations de Gelfand–Tsetlin de profondeur n après avoir fixé la position des particules de la n-ième ligne. De maniere équivalente, ces systèmes décrivent une grande classe de modèles de pavages aléatoires et ont un rapport étroit avec les processus de valeurs propres de mineurs d'une grande classe de matrices aléatoires hermitiennes. Ils ont une structure déterminantale et leur noyau de corrélation est connu. Nous redimensionnons le système par un facteur $1/n$, et examinons son comportement asymptotique lorsque $n \to \infty$, sous l'hypothèse faible pour les particules sur la rangée n, que la distribution empirique redimensionnée de ces dernières converge faiblement vers une mesure de probabilité avec support compact, et que cette dernière satisfasse un minimum de régularité.

Nous prouvons que le noyau de corrélation des particules dans le voisinage d'un « point typique du bord » converge vers le noyau de Airy étendu. A cette fin, nous trouvons dans un premier temps un dimensionnement adéquat pour la fluctuation des particules. Nous donnons une paramétrisation explicite du noyau asymptotique, définissons une courbe non-asymptotique analogue (et son équivalent en dimension n), et choisissons notre scaling de telle sorte que les particules fluctuent autour de cette courbe avec des ordres $O(n^{-\frac{1}{3}})$ et $O(n^{-\frac{2}{3}})$, dans les directions respectivement tangentes et normales. Bien que les résultats de l'article soient naturels, les difficultés techniques liées Ã l'etude d'une si grande classe de modèles sous des hypothèses si faibles sont substantielles et inévitables.

1. **Introduction**

1.1. **Overview of the model, the asymptotic assumptions, and results**

In this paper we consider universal edge behaviour of random systems of discrete interlaced particles referred to as *Gelfand–Tsetlin patterns*. A discrete Gelfand–Tsetlin pattern of depth n is an n-tuple, $(y^{(1)}, y^{(2)}, \ldots, y^{(n)}) \in \mathbb{Z} \times \mathbb{Z}^2 \times \cdots \times \mathbb{Z}^n$, which satisfies the interlacing constraint,

$$y_1^{(r+1)} \geq y_1^{(r)} > y_2^{(r+1)} \geq y_2^{(r)} > \cdots \geq y_r^{(r)} > y_{r+1}^{(r+1)}, \qquad (1.1)$$

for all $r \in \{1, \ldots, n-1\}$, denoted $y^{(r+1)} > y^{(r)}$. Equivalently this can be considered as an interlaced configuration of $\frac{1}{2}n(n+1)$ particles in $\mathbb{Z} \times \{1, 2, \ldots, n\}$ by placing a particle at position $(u, r) \in \mathbb{Z} \times \{1, 2, \ldots, n\}$ whenever u is an element of $y^{(r)}$. A Gelfand–Tsetlin pattern of depth 4 is shown on the left of Figure 1.1.

For each $n \geq 1$, fix $x^{(n)} \in \mathbb{Z}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$. Consider the uniform probability measure, ν_n, on the set of discrete Gelfand–Tsetlin patterns of depth n with the particles on row n in the deterministic positions defined by $x^{(n)}$:

$$\nu_n[(y^{(1)}, y^{(2)}, \ldots, y^{(n)})] := \frac{1}{Z_n} \cdot \begin{cases} 1 & \text{when } x^{(n)} = y^{(n)} > y^{(n-1)} > \cdots > y^{(1)}, \\ 0 & \text{otherwise}, \end{cases} \qquad (1.2)$$

where $Z_n > 0$ is a normalisation constant. This measure, and the equivalent description of Gelfand–Tsetlin patterns given above, induces a random point process on interlaced configurations of particles in $\mathbb{Z} \times \{1, 2, \ldots, n\}$. In [8], we show that this process is determinantal. See Johansson, [14], for an introduction to such processes. (1.3), below, recalls our expression for the correlation kernel of this process, denoted $K_n : (\mathbb{Z} \times \{1, 2, \ldots, n\})^2 \to \mathbb{C}$. Loosely speaking, K_n is a function on pairs of particle positions which conveniently encodes the densities and correlations of the particles. K_n was also, independently, obtained by Petrov, [23].

As discussed in Sections 1.1 and 1.2 of Duse and Metcalfe, [8], our motivation for studying these processes is that they are an equivalent description of uniform random tilings of "half-hexagons" with lozenges, and of perfect matchings of dimer configurations of honeycomb lattices. The set of "half-hexagons" is a class of polygons with quite general boundaries. The boundary is determined by the (arbitrary) choice of the above $x^{(n)} \in \mathbb{Z}^n$, and particular choices of $x^{(n)}$ recover well-studied models. For example, if we fix $p, q, r \in \mathbb{Z}^+$, and choose $n = p + r$ and $x^{(n)} = (p+q+r, p+q+r-1, \ldots, p+q+1, p, p-1, \ldots, 1)$, then we recover the uniform random tiling of a hexagon with sides of length p, q, r, p, q, r. Cohn et al., [5], studied the asymptotic shape of a "typical" such tiling as $n \to \infty$, under the assumption that $\frac{p}{q}$ and $\frac{p}{r}$ converge. Analogous limit shapes for other random tilings models have been studied, for example, in [17] and [16].
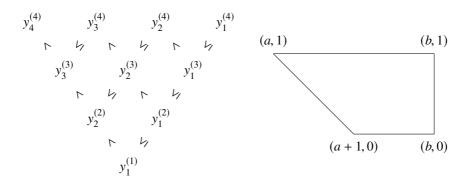
$y_4^{(4)}$ $y_3^{(4)}$ $y_2^{(4)}$ $y_1^{(4)}$

$(a, 1)$ $(b, 1)$

$y_3^{(3)}$ $y_2^{(3)}$ $y_1^{(3)}$

$y_2^{(2)}$ $y_1^{(2)}$

$(a + 1, 0)$ $(b, 0)$

$y_1^{(1)}$

FIGURE 1.1. Left: A visualisation of a Gelfand–Tsetlin pattern of depth 4. Right: $\{(\chi, \eta) \in [a, b] \times [0, 1] : b \geq \chi \geq \chi + \eta - 1 \geq a\}$. Assumption 1.2 implies that the bulk of the rescaled particles of the Gelfand–Tsetlin patterns lie asymptotically in this region as $n \to \infty$.

As we will discuss in more detail below, we rescale our systems by $\frac{1}{n}$, and examine the asymptotic behaviour, as $n \to \infty$, under weak asymptotic assumptions: We assume that the empirical distribution of $\frac{1}{n} x^{(n)}$ converges weakly to a probability measure, μ, with compact support (Assumption 1.2), and that $\frac{1}{n} x^{(n)}$ otherwise satisfies only mild regulatory restrictions (Assumptions 1.7 and 2.4). We avoid only that degenerate case where μ is Lebesgue measure on a single interval of length 1, and allow all other μ which can be obtained via the weak convergence. Note, as $n \to \infty$, the interlacing constraint implies that the bulk of the rescaled particles of the Gelfand–Tsetlin patterns lie asymptotically in the polygon shown on the right of Figure 1.1. The technicalities involved in studying the asymptotic behaviours of these systems, under such weak asymptotic assumptions, are unavoidable and extensive. A positive aspect of this is that we uncovered many unexpected situations. Indeed, we have written 4 papers on these rich systems of models. Papers [8, 9] explore the possible global asymptotic shapes, and this paper and [7] examine the local asymptotic fluctuations of the particles in neighbourhoods of the possible edges. Paper [8] examines "classic" global asymptotic shapes, and [9] finds novel global asymptotic behaviours. Figure 1.2 depicts some example asymptotic shapes obtained using the results of those papers. In [7], we find novel local asymptotic edge fluctuations.

In this paper, we examine *universal* local asymptotic fluctuations at "typical edge points". To do this, we must first find an appropriate scaling for the fluctuations of the particles. We start with the explicit parameterisation of the asymptotic edge obtained in [8] (see (1.15)), and a define a natural subset of the asymptotic edge called the set of *typical edge points* (see Definition 1.5). This set is always non-empty, and the difference between

it and the whole edge is either empty or discrete. Next, we use the parameterisation to define an analogous *non-asymptotic edge curve for each n*, sometimes referred to in the literature as the *finite n-deterministic equivalent* (see Definition 1.8). We fix a typical edge point, denoted by (χ, η), and (for each n) we let (χ_n, η_n) denote the analogous point on the non-asymptotic edge. Our asymptotic assumptions imply that $(\chi_n, \eta_n) \to (\chi, \eta)$ as $n \to \infty$, but we have no control of the rate of convergence. Nevertheless, Theorems 1.10 and 1.11 essentially prove the following:

Theorem 1.1. *Let $\{(u_n, r_n)\}_{n \geq 1}, \{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z} \times \{1, 2, \ldots, n\}$ be sequences of particle positions chosen as follows: For all n sufficiently large, both $\frac{1}{n}(u_n, r_n)$ and $\frac{1}{n}(v_n, s_n)$ fluctuate around (χ_n, η_n), with fluctuations of order $O(n^{-\frac{1}{3}})$ and $O(n^{-\frac{2}{3}})$ respectively in the tangent and normal directions of the non-asymptotic edge curve. Then the asymptotic behaviour of $n^{\frac{1}{3}} K_n((u_n, r_n), (v_n, s_n))$ is governed by the extended Airy kernel as $n \to \infty$.*

A steepest descent analysis of an equivalent contour integral expression for $K_n((u_n, r_n), (v_n, s_n))$ (see (1.7), below) is used to prove the asymptotics in Theorems 1.10 and 1.11. The length of this paper reflects the extensive technicalities needed to do the analysis rigorously under our weak assumptions. In Section 2, we examine the roots of the derivatives of the appropriate steepest descent functions (see (1.8), (1.9) and (1.10)). In Section 3, we perform the steepest descent analysis. In particular, we highlight Lemmas 3.5 and 3.6, which prove the existence of appropriate contours of descent/ascent. The proofs of these are given in Section 4. This is, by far, the most difficult part of the paper. Indeed, our weak assumptions necessitate that we need to prove existence of different contours for 12 distinct cases (see Lemma 2.2).

Note, while convergence to the extended Airy kernel is the only case which we consider rigorously in this paper, we will briefly discuss other natural asymptotic situations in Section 1.5. We will discuss known results in the literature, and conjecture analogous asymptotic results for this model. Unfortunately, the length of this paper necessitates that we do not attempt to study these situations in greater detail here.

We end this section by comparing our result with analogous results in the literature. Perhaps the closest result to ours is in Petrov, [23]. Theorem 8.1 of [23] proves a similar asymptotic result for the special case where μ is given by Lebesgue measure on a disjoint union, $[a_1, b_1] \cup [a_2, b_2] \cup \cdots \cup [a_k, b_k]$, where $k \geq 2$ and $\sum_i (b_i - a_i) = 1$. By contrast, here, Assumption 1.2 avoids only that degenerate case where μ is Lebesgue measure on a single interval of length 1, and allows all other μ with compact support which can be obtained via the weak convergence. Petrov further specialises by assuming that $\frac{1}{n} x^{(n)}$ is essentially densely packed in the above disjoint union of intervals. By contrast, here, Assumption 1.2 implies that the empirical distribution of $\frac{1}{n} x^{(n)}$ converges weakly to μ, and Assumptions 1.7 and 2.4 otherwise impose only mild regulatory restrictions on $\frac{1}{n} x^{(n)}$.

The stronger assumptions of Petrov give a fast rate of convergence, and indeed it can be shown that $(\chi_n, \eta_n) = (\chi, \eta) + O(n^{-1})$ for all n sufficiently large under these. Petrov, therefore, can ignore (χ_n, η_n) (the non-asymptotic edge), and fluctuate simply around (χ, η) (the asymptotic edge). By contrast, we have no control of the rate of convergence. The fast rate of convergence in [23] also allows Petrov to avoid many subtle technical points.

Theorem 8.1 of [23] furthermore chooses a somewhat different scaling for the sequences of particle positions: Petrov fixes parameters $(\tau_1, \sigma_1) \in \mathbb{R}^2$ and $(\tau_2, \sigma_2) \in \mathbb{R}^2$, and fluctuates around the asymptotic edge. For all n sufficiently large, the fluctuations have order $O(n^{-\frac{1}{3}})$ and $O(n^{-\frac{2}{3}})$ respectively in the tangent direction (of the asymptotic edge) and the direction $(1, 0)$, τ_1 and τ_2 measure the size of the $O(n^{-\frac{1}{3}})$ fluctuations, and $\sigma_1 - \tau_1^2$ and $\sigma_2 - \tau_2^2$ measure the size of the $O(n^{-\frac{2}{3}})$ fluctuations. Petrov chooses this scaling to ensure convergence to the form of the extended Airy kernel defined in [24], evaluated at (τ_1, σ_1) and (τ_2, σ_2). A contour integral expression for that kernel was obtained in [2]. By contrast, here, we fix parameters $(u, r) \in \mathbb{R}^2$ and $(v, s) \in \mathbb{R}^2$, and fluctuate around the non-asymptotic edge. For all n sufficiently large, the fluctuations have order $O(n^{-\frac{1}{3}})$ and $O(n^{-\frac{2}{3}})$ respectively in the tangent and normal directions of the non-asymptotic edge, u and v measure the size of the $O(n^{-\frac{1}{3}})$ fluctuations, and r and s measure the size of the $O(n^{-\frac{2}{3}})$ fluctuations (see (1.28), (1.29)). The scaling is thus naturally related to the geometric behaviour of the edge. We use the scaling in Lemma 2.9 to show that the relevant roots and derivatives of the steepest descent functions have well-behaved asymptotic behaviours, and these result in simple Taylor expansions for the steepest descent functions give in Corollary 3.3. We then use these in Theorems 1.10 and 1.11 to prove convergence to the form of the extended Airy kernel given in (1.21), evaluated at (u, r) and (v, s). Note that this expression is simpler than that given in [23], but equivalent, as can be seen via a change of variables and the removal of a conjugation factor.

Our results also have interesting connections with those of Kenyon et al., [17, 16]. Those papers study the global asymptotic shapes of random tilings of a class of polygons. In papers [8, 9], we explore the global asymptotic shapes of random tilings of a more restricted class of polygons, but we allow more general boundary/asymptotic conditions which results in some important differences. For example, the asymptotic boundaries in [17, 16] are shown to be *algebraic*, and this is not necessarily true in [8]. Moreover, in [9], we find novel global asymptotic behaviours. Also, in [8, 9], we obtain parameterisations of the boundaries. This enables us to prove the asymptotic fluctuations seen in this paper and [7], in neighbourhoods of the edges. It is intuitively clear that these universal fluctuations will also appear in the models of [17, 16], under analogous conditions. For example, in Figure 1 of [16], the asymptotic frozen boundary of the polygon in seen to

be a cardioid. In [7], we consider the asymptotic fluctuations in neighbourhoods of an analogous cusp, and show that they are governed by a novel point process, which we call the *Cusp Airy* process.

More generally, we believe that our techniques can be of use in other random tiling models, or random perfect matchings, or systems of random non-intersecting paths, etc., that can equivalently be described as interacting particle systems. For example, [3] is a recent work concerning the asymptotics of random domino tilings of rectangular Aztec diamonds. In it, the authors find explicit parameterisations of the possible asymptotic boundaries using almost identical methods to those that we used in [8], and find analogous results. It is therefore reasonable to expect that the results we found in [9], this paper, and in [7], also have natural analogues for random domino tilings, and the techniques of those papers would be sufficient to prove these results. There has been significant interest in related models. See, for example, [2, 4, 6, 12, 13].

Other closely connected models arise from random matrices. For each $n \geq 1$, let $A_n \in \mathbb{C}^{n \times n}$ be a random Hermitian matrix whose distribution is unitarily invariant. For each $r \leq n$, let $\lambda^{(r)} \in \mathbb{R}^r$ be the eigenvalues (in decreasing order) of the rth principal sub-matrix of A_n consisting of the first r rows and columns. The asymptotic behaviour of $\lambda^{(n)}$ (the eigenvalues of A_n) as $n \to \infty$ has been studied for many different ensembles of random matrices (i.e. for different choices of A_n), and universal behaviours have been found. See, for example, [1, 22] for reviews of known results. See also the recent work of Hachem et al., [10], which studies the asymptotic behaviour of the edge of $\lambda^{(n)}$ as $n \to \infty$ when A_n is a complex correlated Wishart matrix. In [10], the model is shown to be determinantal, the edge asymptotic behaviour is examined via a closely related saddle point problem to that seen in this paper, and it is shown that this behaviour is governed by the standard Airy kernel.

Note, for general Hermitian A_n, an elementary result from matrix analysis shows that $(\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(n)})$ is a Gelfand–Tsetlin pattern of depth n, where now the particles on each level take positions in $\mathbb{R}$ rather than $\mathbb{Z}$. Such models often display a similar determinantal structure to the discrete Gelfand–Tsetlin patterns of this paper. Perhaps the best studied determinantal minor process is that of the *Gaussian Unitary Ensemble* (GUE), where the entries of A_n are random independent Gaussians. See, for example, Mehta, [18], and Johansson and Nordenstam, [15].

Perhaps the most similar minor process to ours is that studied by the author Metcalfe in [19]. There, the eigenvalues of A_n are deterministic: $\lambda^{(n)} = x^{(n)}$ for some fixed $x^{(n)} \in \mathbb{R}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$, and the eigenvalue minor process induces the uniform probability distribution on the set of Gelfand–Tsetlin patterns of depth n with the particles on the top row in the deterministic positions defined by $x^{(n)} \in \mathbb{R}^n$. This is very similar to the measure defined in (1.2). Metcalfe showed that the process is determinantal, and

found a correlation kernel. This kernel can, in fact, be shown to be a limit of the kernel in (1.3), below, where we scale the discrete particle positions on each level such that they become continuous. In [19], Metcalfe proved universal bulk asymptotic behaviour under the assumption that the empirical distribution of $x^{(n)}$ converges weakly to a probability measure with compact support, similar to Assumption 1.2, below. The edge asymptotic behaviour has not yet been studied. It is clear to the authors, however, that the techniques of this paper would be sufficient to study this. Also, it is worth noting that there are interesting asymptotic situations there that have no natural analogues here. In particular, the limit measure may have atoms in [19], which is not possible here. The related asymptotic situations have also not yet been studied, and we believe that our techniques would help in studying these. More generally, we hope that our techniques would help to study the determinantal minor processes of other ensembles of random matrices.

1.2. **The determinantal structure of discrete Gelfand–Tsetlin patterns**

As in the previous section, define interlacing as in (1.1), fix $x^{(n)} \in \mathbb{Z}^n$ with $x_1^{(n)} > x_2^{(n)} > \cdots > x_n^{(n)}$, and define ν_n as in (1.2). Recall that ν_n induces a random point process on interlaced configurations of particles in $\mathbb{Z} \times \{1, 2, \ldots, n\}$. In Section 4.1 of [8], we showed that this process is determinantal, and we found an expression for a correlation kernel, denoted by $K_n : (\mathbb{Z} \times \{1, 2, \ldots, n\})^2 \to \mathbb{C}$. Note, ignoring the deterministic particles on row n, interlacing implies that we need only consider those particle positions, $(u, r), (v, s) \in \mathbb{Z} \times \{1, 2, \ldots, n-1\}$, which satisfy $u \geq x_n^{(n)} + n - r$ and $v \geq x_n^{(n)} + n - s$. For all such $(u, r), (v, s)$,

$$
K_n((u, r), (v, s)) = -\phi_{r,s}(u, v)
$$
$$
+ \frac{(n-s)!}{(n-r-1)!} \sum_{k=1}^{n} 1_{(x_k^{(n)} \geq u)} \sum_{l=v+s-n}^{v} \frac{\prod_{j=u+r-n+1}^{u-1} (x_k^{(n)} - j)}{\prod_{j=v+s-n, j\neq l}^{v} (l - j)} \prod_{i=1, i\neq k}^{n} \left(\frac{l - x_i^{(n)}}{x_k^{(n)} - x_i^{(n)}} \right), \quad (1.3)
$$

where

$$
\phi_{r,s}(u, v) := \begin{cases} 0 & \text{when } v < u \text{ or } s \leq r, \\ 1 & \text{when } v \geq u \text{ and } s = r + 1, \quad (1.4) \\ \frac{1}{(s-r-1)!} \prod_{j=1}^{s-r-1} (v - u + s - r - j) & \text{when } v \geq u \text{ and } s > r + 1. \end{cases}
$$

Note, correlation kernels are not uniquely defined. Indeed, $\mathcal{K}_n : (\mathbb{Z} \times \{1, 2, \ldots, n\})^2 \to \mathbb{C}$ is an equivalent kernel if $\det \left[\mathcal{K}_n((u_i, r_i), (u_j, r_j)) \right]_{i,j=1}^{m} = \det[K_n((u_i, r_i), (u_j, r_j))]_{i,j=1}^{m}$ for all $m \geq 1$, and for all particle positions $\{(u_1, r_1), \ldots, (u_m, r_m)\} \subset \mathbb{Z} \times \{1, 2, \ldots, n\}$. An

equivalent kernel which will prove useful in our asymptotic analysis is the following:

$$\mathcal{K}_n((u,r),(v,s)) := K_n((v,s),(u,r))B_n(s,r)^{-1}A_{t,n}((v,s),(u,r))^{-1}, \qquad (1.5)$$

for all $(u,r),(v,s) \in \mathbb{Z} \times \{1,2,\ldots,n\}$, where $t \in \mathbb{R}$ is that fixed value in (1.18), $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ is defined in Lemma 2.13, and $B_n : \mathbb{Z}_+^2 \to (0,+\infty)$ is defined by,

$$B_n(r,s) := \frac{(n-s)!}{(n-r)!}n^{s-r}. \qquad (1.6)$$

1.3. The asymptotic "shape" of discrete Gelfand–Tsetlin patterns

In [8] and [9] we consider the asymptotic "shape" of the systems of Gelfand–Tsetlin patterns of the previous sections, under some natural asymptotic assumptions. In this section, we recall the relevant asymptotic assumptions, definitions, and results of those papers. We state these without motivation or proof, and refer the interested reader to those papers.

Assumption 1.2. *Let μ be a probability measure on $\mathbb{R}$ with $\mu \leq \lambda$, where λ is Lebesgue measure. Assume that there is a compact interval $[a,b] \subset \mathbb{R}$ with $b - a > 1$, $\mathrm{Supp}(\mu) \subset [a,b]$ and $\{a,b\} \subset \mathrm{Supp}(\mu)$. Moreover, assume that,*

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i^{(n)}/n} \to \mu,$$

as $n \to \infty$, in the sense of weak convergence of measures.

Then, rescaling the sides of the Gelfand–Tsetlin patterns by $\frac{1}{n}$, the bulk of the rescaled particles asymptotically lie in the polygon on the right of Figure 1.1 as $n \to \infty$, i.e., $\{(\chi,\eta) \in [a,b] \times [0,1] : b \geq \chi \geq \chi + \eta - 1 \geq a\}$. The local asymptotic behaviour of particles near a fixed point, (χ,η), in this polygon is studied by considering $K_n((u_n,r_n),(v_n,s_n))$ as $n \to \infty$, where $\{(u_n,r_n)\}_{n\geq1} \subset \mathbb{Z}^2$ and $\{(v_n,s_n)\}_{n\geq1} \subset \mathbb{Z}^2$ satisfy $\frac{1}{n}(u_n,r_n) \to (\chi,\eta)$ and $\frac{1}{n}(v_n,s_n) \to (\chi,\eta)$ as $n \to \infty$. First note, (1.3) and the Residue theorem give

$$K_n((u_n,r_n),(v_n,s_n)) = -\phi_{r_n,s_n}(u_n,v_n)$$

$$+ \frac{(n-s_n)!}{(n-r_n-1)!}\frac{n^{s_n-r_n-1}}{(2\pi i)^2}\int_{c_n}\mathrm{d}w\int_{C_n}\mathrm{d}z\frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(z-\frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w-\frac{j}{n})}\frac{1}{w-z}\prod_{i=1}^{n}\left(\frac{w-\frac{x_i}{n}}{z-\frac{x_i}{n}}\right), \quad (1.7)$$

for all $n \geq 1$. Here, we omit the superscript from $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)})$ for simplicity, C_n is any counter-clockwise simple closed contour which contains all of $\{\frac{1}{n}x_j : x_j \geq u_n\}$ but none of $\{\frac{1}{n}x_j : x_j \leq u_n + r_n - n\}$, and c_n is any counter-clockwise simple closed

contour which contains $\frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$ and C_n. Next note, the above integrand equals $\frac{1}{w-z} \exp(nf_n(w) - n\tilde{f}_n(z))$, where

$$f_n(w) := \frac{1}{n}\sum_{i=1}^{n} \log\left(w - \frac{x_i}{n}\right) - \frac{1}{n}\sum_{j=v_n+s_n-n}^{v_n} \log\left(w - \frac{j}{n}\right), \tag{1.8}$$

$$\tilde{f}_n(z) := \frac{1}{n}\sum_{i=1}^{n} \log\left(z - \frac{x_i}{n}\right) - \frac{1}{n}\sum_{j=u_n+r_n-n+1}^{u_n-1} \log\left(z - \frac{j}{n}\right), \tag{1.9}$$

for all $w, z \in \mathbb{C} \setminus \mathbb{R}$, and log denotes the principal logarithm. Finally note, since $\frac{1}{n}\sum_i \delta_{x_i/n} \to \mu$ weakly as $n \to \infty$, and $\frac{1}{n}(u_n, r_n), \frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ as $n \to \infty$, it is natural to define the following asymptotic function:

$$f_{(\chi,\eta)}(w) := \int_a^b \log(w - x)\mu[\mathrm{d}x] - \int_{\chi+\eta-1}^{\chi} \log(w - x)\,\mathrm{d}x, \tag{1.10}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$.

Steepest descent analysis, and the above structure, intuitively suggests that the behaviour of $K_n((u_n, r_n), (v_n, s_n))$ as $n \to \infty$ depends on the roots of $f'_{(\chi,\eta)}$. Recall that $b \geq \chi \geq \chi + \eta - 1 \geq a$, and (see Assumption 1.2) that μ and $\lambda - \mu$ are positive measures. Thus, for all $w \in \mathbb{C} \setminus \mathbb{R}$, it is natural to write,

$$f_{(\chi,\eta)}(w) \tag{1.11}$$
$$= \int_{S_1} \log(w - x)\mu[\mathrm{d}x] - \int_{S_2} \log(w - x)(\lambda - \mu)[\mathrm{d}x] + \int_{S_3} \log(w - x)\mu[\mathrm{d}x],$$

where $S_i := S_i(\chi, \eta)$ for all $i \in \{1, 2, 3\}$ are defined by:

$$S_1 := \mathrm{Supp}(\mu|_{[\chi,b]}), \tag{1.12}$$
$$S_2 := \mathrm{Supp}((\lambda - \mu)|_{[\chi+\eta-1,\chi]}),$$
$$S_3 := \mathrm{Supp}(\mu|_{[a,\chi+\eta-1]}).$$

Then, for all $w \in \mathbb{C} \setminus \mathbb{R}$,

$$f'_{(\chi,\eta)}(w) = \int_{S_1} \frac{\mu[\mathrm{d}x]}{w - x} - \int_{S_2} \frac{(\lambda - \mu)[\mathrm{d}x]}{w - x} + \int_{S_3} \frac{\mu[\mathrm{d}x]}{w - x}. \tag{1.13}$$

Thus $f'_{(\chi,\eta)}$ has a unique analytic extension to $\mathbb{C} \setminus S$, where $S := S_1 \cup S_2 \cup S_3$.

In [8, 9], we used the possible behaviours of the roots of $f'_{(\chi,\eta)}$ in the above domain to examine the asymptotic shapes. First we defined:

Definition 1.3. The liquid region, $\mathcal{L}$, is the set of all $(\chi, \eta) \in [a, b] \times [0, 1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$, for which $f'_{(\chi,\eta)}$ has a root in $\mathbb{H} := \{w \in \mathbb{C} : \mathrm{Im}(w) > 0\}$.

We showed that $f'_{(\chi,\eta)}$ has a unique root in $\mathbb{H}$ whenever $(\chi,\eta) \in \mathcal{L}$, and this root is of multiplicity 1. Moreover, we showed that the resulting map from $\mathcal{L}$ to $\mathbb{H}$ is a homeomorphism, and so $\mathcal{L}$ is a non-empty, open, connected subset of the interior of the polygon on the right of Figure 1.1. We used the homeomorphism to study $\partial \mathcal{L}$. In [8], we obtained a complete parameterisation of $\partial \mathcal{L}$ for μ in a broad class. In [9], we examined the highly non-trivial behaviours of $\partial \mathcal{L}$ that can occur when μ is not restricted to this class. $\mathcal{L}$ and $\partial \mathcal{L}$ for some example measures, μ, studied in [8, 9], are given in Figure 1.2.

For the purposes of this paper, we only need the results of [8], which we now recall in more detail. We showed that the inverse of the above homeomorphism, from $\mathbb{H}$ to $\mathcal{L}$, has a unique continuous extension to a natural, non-empty, open subset of $\mathbb{R}$ which depends on μ. We denoted this set by $R \subset \mathbb{R}$, and showed that the extension is an injective smooth curve, parameterised over R. We defined the *edge*, denoted $\mathcal{E} \subset \partial \mathcal{L}$, to be the image of this curve, and referred to this curve as the *edge curve*, denoted

$$(\chi_{\mathcal{E}}(\cdot), \eta_{\mathcal{E}}(\cdot)) : R \to \mathcal{E} \subset \partial \mathcal{L}.$$

In [8], we found an alternative definition of $\mathcal{E}$ which is analogous to that of $\mathcal{L}$:

Definition 1.4. The edge, $\mathcal{E}$, is the union $\mathcal{E} := \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^- \cup \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$, where

- $\mathcal{E}_\mu^+$ is the set of all $(\chi,\eta) \in [a,b] \times [0,1]$ with $b \geq \chi \geq \chi + \eta - 1 \geq a$, for which $f'_{(\chi,\eta)}$ has a repeated root in $(\chi, +\infty) \setminus \text{Supp}(\mu)$.

- $\mathcal{E}_{\lambda-\mu}$ is the set of all (χ,η) for which $f'_{(\chi,\eta)}$ has a repeated root in $(\chi + \eta - 1, \chi) \setminus \text{Supp}(\lambda - \mu)$.

- $\mathcal{E}_\mu^-$ is the set of all (χ,η) for which $f'_{(\chi,\eta)}$ has a repeated root in $(-\infty, \chi + \eta - 1) \setminus \text{Supp}(\mu)$.

- $\mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ is the set of (χ,η) for which $f'_{(\chi,\eta)}$ has a root in $\{\chi, \chi + \eta - 1\}$.

For clarity, in [8], we state that we denoted $\mathcal{E}_\mu = \mathcal{E}_\mu^+ \cup \mathcal{E}_\mu^-$. This decomposition is more convenient here. Moreover, we defined $\mathcal{E}_0$ and $\mathcal{E}_1$ and $\mathcal{E}_2$ exactly in [8], but do not do so here for brevity. We showed that the above two definitions of $\mathcal{E}$ are equivalent: We started with Definition 1.4, showed that the sets in this definition are mutually disjoint, $f'_{(\chi,\eta)}$ has a unique real-valued repeated root in $\mathbb{R} \setminus \{\chi, \chi + \eta - 1\}$ when $(\chi,\eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$, and $f'_{(\chi,\eta)}$ has a unique root in $\{\chi, \chi + \eta - 1\}$ when $(\chi,\eta) \in \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$. We showed that the resulting map from $\mathcal{E}$ to $\mathbb{R}$ is injective, has image space R, and has inverse equal to the edge curve discussed above. Therefore the definitions are trivially equivalent. We also showed that the multiplicity of the unique root determines the geometric behaviour of the edge curve. Indeed, denoting the multiplicity by $m = m(\chi,\eta) \geq 1$, we showed that:

- The edge curve behaves like a parabola in a neighbourhood of (χ, η) when $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ and $m = 2$, and when $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ and $m = 1$.

- The edge curve behaves like an algebraic cusp of first order in a neighbourhood of (χ, η) when $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ and $m = 3$, and when $(\chi, \eta) \in \mathcal{E}_1 \cup \mathcal{E}_2$ and $m = 2$.

For clarity we state that m takes no other values. Examples edge curves, with the above sets clearly labelled, are depicted in Figure 1.2. Finally, we showed that $\mathcal{E} \setminus \{(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^- : m = 2\}$ is either empty or discrete. We therefore now define:

Definition 1.5. The set of typical edge points is $\{(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^- : m = 2\}$.

We now consider the subset $\mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^- \subset \mathcal{E}$, in more detail. Let $C : \mathbb{C} \setminus \mathrm{Supp}(\mu) \to \mathbb{C}$ denote the *Cauchy* transform of μ,

$$C(w) := \int_a^b \frac{\mu[\mathrm{d}x]}{w - x}, \tag{1.14}$$

for all $w \in \mathbb{C} \setminus \mathrm{Supp}(\mu)$. In [8], we showed that:

Lemma 1.6. *Recall that the edge curve, $(\chi_\mathcal{E}(\cdot), \eta_\mathcal{E}(\cdot)) : R \to \mathcal{E}$, is bijective. Define:*

- $R_\mu^+ := \{t \in \mathbb{R} \setminus \mathrm{Supp}(\mu) : C(t) > 0\}$.

- $R_{\lambda-\mu} := \mathbb{R} \setminus \mathrm{Supp}(\lambda - \mu)$.

- $R_\mu^- := \{t \in \mathbb{R} \setminus \mathrm{Supp}(\mu) : C(t) < 0\}$.

Then, these are disjoint open subsets of R, and the image spaces of these under the bijection are (respectively) $\mathcal{E}_\mu^+$, $\mathcal{E}_{\lambda-\mu}$, $\mathcal{E}_\mu^-$. Moreover, for all $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$,

$$\chi_\mathcal{E}(t) = t + \frac{e^{C(t)} - 1}{e^{C(t)} C'(t)} \qquad \text{and} \qquad \eta_\mathcal{E}(t) = 1 + \frac{(e^{C(t)} - 1)^2}{e^{C(t)} C'(t)}. \tag{1.15}$$

Finally, $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t)) \in (a, b) \times (0, 1)$ and $b > \chi_\mathcal{E}(t) > \chi_\mathcal{E}(t) + \eta_\mathcal{E}(t) - 1 > a$ for all $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, i.e., $(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ is in the interior of the polygon shown on the right of Figure 1.1.

Note, (1.15) is well-defined whenever $t \in R_\mu^+ \cup R_\mu^-$, since $R_\mu^+ \cup R_\mu^- \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)$. Indeed, (1.14) gives

$$C(t) = \int_a^b \frac{\mu[\mathrm{d}x]}{t - x} \qquad \text{and} \qquad C'(t) = -\int_a^b \frac{\mu[\mathrm{d}x]}{(t - x)^2}, \tag{1.16}$$
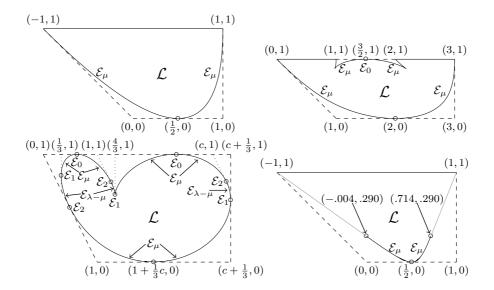
FIGURE 1.2. $\mathcal{L}$ and $\partial\mathcal{L}$ for some example measures, μ, with density $\varphi : \mathbb{R} \to [0,1]$. Top left: $\varphi(x) = \frac{1}{2}\forall x \in [-1,1]$, all points of $\mathcal{E}$ are in $\mathcal{E}_\mu$ with $m = 2$. Top right: $\varphi(x) = \frac{1}{2}\forall x \in [0,1] \cup [2,3]$, the cusps are in $\mathcal{E}_\mu$ with $m = 3$, all other points of $\mathcal{E}$ are in $\mathcal{E}_\mu$ with $m = 2$. Lower left: $\varphi(x) = 1\forall x \in [0,\frac{1}{3}] \cup [1,\frac{4}{3}] \cup [c, c + \frac{1}{3}]$, where $c := \frac{1}{12}(23 + \sqrt{217})$, the cusp is in $\mathcal{E}_1$ with $m = 2$, all other points in $\mathcal{E}_1 \cup \mathcal{E}_2$ have $m = 1$. Lower right: $\varphi(x) = \frac{15}{16}(x-1)^2(x+1)^2\forall x \in [-1,1]$, the grey parts of $\partial\mathcal{L}$ are not in $\mathcal{E}$ but follow from the analysis in [9]. For general μ, there is always an analogous lower convex part of $\mathcal{E}$ which is contained in $\{(\chi, \eta) \in \mathcal{E}_\mu : m = 2\}$, and the lower tangent point always equals $(\frac{1}{2} + \int x\mu[\mathrm{d}x], 0)$.

for all $t \in R_\mu^+ \cup R_\mu^-$. Moreover, it is well-defined whenever $t \in R_{\lambda-\mu}$. Indeed, since $\mu = \lambda$ in $R_{\lambda-\mu} = \mathbb{R} \setminus \mathrm{Supp}(\lambda - \mu)$, Lemma 2.2 of [8] implies that $e^{C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ and $C'(\cdot) : \mathbb{C} \setminus R \to \mathbb{C}$ have the following unique analytic extensions to $R_{\lambda-\mu}$:

$$e^{C(t)} = e^{C_I(t)}\left(\frac{t - t_2}{t - t_1}\right) \qquad \text{and} \qquad C'(t) = C_I'(t) - \frac{1}{t - t_1} + \frac{1}{t - t_2}, \qquad (1.17)$$

for all $t \in R_{\lambda-\mu}$, where $I = (t_2, t_1)$ is any interval with $t \in I \subset R_{\lambda-\mu}$, and $C_I(t) := \int_{[a,b]\setminus I} \frac{\mu[\mathrm{d}x]}{t-x}$. These expressions are independent of the choice of I.

Finally, as discussed above, the set of typical edge points is $\{(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^- : m = 2\}$, and the edge curve behaves like a parabola in a neighbourhood of each (χ, η) in this set. Fix the corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$. Define the (un-normalised) orthogonal vectors $\mathbf{x}(t) := (1, e^{C(t)} - 1)$ and $\mathbf{y}(t) := (e^{C(t)} - 1, -1)$. In [8], we show that $\mathbf{x}(t)$ and $\mathbf{y}(t)$ are (respectively) tangent and normal to the edge curve at (χ, η).

1.4. **Motivation and statement of main results**

In this paper, we consider the universal asymptotic behaviour, as $n \to \infty$, of the systems introduced in the last section, in the neighbourhood of typical edge points (see Definition 1.5). More specifically, we study the asymptotic behaviour of $K_n((u_n, r_n), (v_n, s_n))$ as $n \to \infty$, where K_n is the correlation kernel of the system (see (1.3)), and $\{(u_n, r_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ are sequence of particle positions which satisfy:

> *Fix the corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, where t is a root of $f'_{(\chi,\eta)}$ of multiplicity 2, and take,*

$$(u_n, r_n) = n(\chi, \eta) + o(n) \ \ and \ \ (v_n, s_n) = n(\chi, \eta) + o(n) \ \ as \ \ n \to \infty. \quad (1.18)$$

A steepest descent analysis of (1.7) for $K_n((u_n, r_n), (v_n, s_n))$ is used to consider the asymptotic behaviour. Since t is a root of $f'_{(\chi,\eta)}$ of multiplicity 2, (1.7) and steepest descent analysis intuitively imply that universal edge asymptotic behaviour should be observed. The main results of this paper, Theorems 1.10 and 1.11, puts this intuition on a rigorous footing. We show, under natural conditions, that the asymptotic behaviour of $n^{\frac{1}{3}} K_n((u_n, r_n), (v_n, s_n))$ is governed by the *extended Airy kernel*, $K_{\mathrm{Ai}} : (\mathbb{R}^2)^2 :\to \mathbb{R}$: First define $\widetilde{K}_{\mathrm{Ai}} : (\mathbb{R}^2)^2 :\to \mathbb{R}$ by,

$$\widetilde{K}_{\mathrm{Ai}}((u, r), (v, s)) := \frac{1}{(2\pi i)^2} \int_l dw \int_L dz \frac{1}{w - z} \frac{\exp(wr + w^2 u + \frac{1}{3} w^3)}{\exp(zs + z^2 v + \frac{1}{3} z^3)}, \quad (1.19)$$

for all $(u, r), (v, s) \in \mathbb{R}^2$, where l and L are the contours in Figure 1.3. Note that the above integrals are finite since $w^3 = -|w|^3$ and $z^3 = |z|^3$ for all w on l and z on L respectively. Next define $\Phi : (\mathbb{R}^2)^2 \to \mathbb{R}$ by,

$$\Phi((u, r), (v, s)) := 1_{(u>v)} \frac{1}{2\sqrt{\pi(u - v)}} \exp\left(-\frac{1}{4} \frac{(s - r)^2}{u - v}\right), \quad (1.20)$$

for all $(u, r), (v, s) \in \mathbb{R}^2$. Finally define $K_{\mathrm{Ai}} : (\mathbb{R}^2)^2 :\to \mathbb{R}$ by,

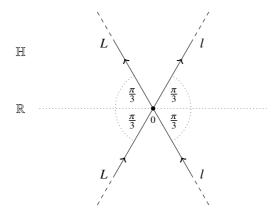$$K_{\mathrm{Ai}} := \widetilde{K}_{\mathrm{Ai}} - \Phi. \quad (1.21)$$

FIGURE 1.3. The contours l and L of (1.19). l is the (straight) lines from $\infty e^{-i\frac{\pi}{3}}$ to 0, and from 0 to $\infty e^{i\frac{\pi}{3}}$. l is the lines from $\infty e^{-i\frac{2\pi}{3}}$ to 0, and from 0 to $\infty e^{i\frac{2\pi}{3}}$.

We begin the analysis with a technical assumption that arises from steepest descent considerations. First, for all $n \geq 1$, define

$$P_n := \tfrac{1}{n}\{x_1, x_2, \ldots, x_n\} \quad \text{and} \quad H_n := \tfrac{1}{n}(\mathbb{Z} \setminus \{x_1, x_2, \ldots, x_n\}). \tag{1.22}$$

Above, we again omit the superscript from $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)})$ for simplicity. P_n is referred to as the set of *particles*, and H_n as the set of *holes*. Note, an element of these sets may act as a pole for the contour integral expression of (1.7), and so a problem may arise in the steepest descent analysis if these are not *eventually isolated* from the root, t, in (1.18). We therefore assume:

Assumption 1.7. *Assume that $d(P_n, \mathrm{Supp}(\mu)) \to 0$ and $d(H_n, \mathrm{Supp}(\lambda - \mu)) \to 0$ as $n \to \infty$, where d represents the Hausdorff distance.*

To see that this assumption has the desired effect, recall that $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, a union of mutually disjoint open sets. Thus there exists a fixed $\epsilon > 0$ for which:

$$t \in \begin{cases} R_\mu^+(\epsilon) := \{s \in R_\mu^+ : (s - \epsilon, s + \epsilon) \subset R_\mu^+\} & \text{when } t \in R_\mu^+. \\ R_{\lambda-\mu}(\epsilon) := \{s \in R_{\lambda-\mu} : (s - \epsilon, s + \epsilon) \subset R_{\lambda-\mu}\} & \text{when } t \in R_{\lambda-\mu}. \\ R_\mu^-(\epsilon) := \{s \in R_\mu^- : (s - \epsilon, s + \epsilon) \subset R_\mu^-\} & \text{when } t \in R_\mu^-. \end{cases} \tag{1.23}$$

Also, since $R_\mu^+ \cup R_\mu^- \subset \mathbb{R} \setminus \mathrm{Supp}(\mu)$ and $R_{\lambda-\mu} = \mathbb{R} \setminus \mathrm{Supp}(\lambda - \mu)$, Assumption 1.7 implies that $P_n \subset \mathbb{R} \setminus (R_\mu^+(\epsilon) \cup R_\mu^-(\epsilon))$ and $H_n \subset \mathbb{R} \setminus R_{\lambda-\mu}(\epsilon)$ for all n sufficiently large, as desired.

Finally note that we can equivalently write,

$$P_n \cap R_\mu^+(\epsilon) = \emptyset \quad \text{and} \quad \frac{\mathbb{Z}}{n} \cap R_{\lambda-\mu}(\epsilon) \subset P_n \quad \text{and} \quad P_n \cap R_\mu^-(\epsilon) = \emptyset, \qquad (1.24)$$

for all n sufficiently large. Indeed, the second part follows since $H_n = \frac{\mathbb{Z}}{n} \setminus P_n$ (see (1.22)), and it implies that particles are *eventually densely packed* in $R_{\lambda-\mu}(\epsilon)$.

Note, Assumption 1.7 imposes mild regulatory restrictions on $x^{(n)}$. Our final assumption, Assumption 2.4, similarly imposes mild regulatory restrictions on $x^{(n)}$. Assumption 2.4 is more subtle, however, and applies only in specific cases, and so we leave the statement of this to Section 2.1. The regularity effect of Assumption 2.4 can be seen, for example, in part (ii) in the proof of Lemma 3.9. For the relevant cases of Assumption 2.4, part (ii) is not necessarily true if Assumption 2.4 does not hold. Note also, Assumptions 1.7 and 2.4 are sufficient for Theorems 1.10 and 1.11 to be satisfied for *all typical edge points*, i.e., for all corresponding points t and (χ, η) chosen as in (1.18). Though we do not discuss this further, weaker forms of these assumptions can be used if we are only interested in specific edge points.

Next, we motivate the choice of the $o(n)$ terms in (1.18). Note, since the convergence in Assumption 1.2 is weak, there is no control of the rate of convergence. It is therefore not natural to simply consider fluctuations of the particles around the asymptotic edge curve. Instead, we consider fluctuations around analogous *non-asymptotic edge curves*. Intuitively, we could define these by replacing all Cauchy transforms (see (1.14)) in (1.15) with the following non-asymptotic analogue inspired by Assumption 1.2: $w \mapsto \frac{1}{n} \sum_{i=1}^n (w - \frac{x_i}{n})^{-1} = \frac{1}{n} \sum_{x \in P_n} (w - x)^{-1}$ for all $n \geq 1$ and $w \in \mathbb{C} \setminus \mathbb{R}$. However, since it is desirable that the non-asymptotic edge curves and the asymptotic edge curve have approximately the same domain of definition, we use a modified non-asymptotic Cauchy transform. First, fixing $\epsilon > 0$ as in (1.23), define a new non-asymptotic measure by,

$$\mu_n := \frac{1}{n} \sum_{x \in P_n; x \notin R_{\lambda-\mu}(\epsilon)} \delta_x + \lambda|_{R_{\lambda-\mu}(\epsilon)}, \qquad (1.25)$$

for all $n \geq 1$. Assumption 1.2 and (1.22) then imply that $\mu_n \to \mu$ weakly as $n \to \infty$. Next, let $C_n : \mathbb{C} \setminus \text{Supp}(\mu_n) \to \mathbb{C}$ denote the Cauchy Transform of μ_n:

$$C_n(w) := \frac{1}{n} \sum_{x \in P_n; x \notin R_{\lambda-\mu}(\epsilon)} \frac{1}{w - x} + \int_{R_{\lambda-\mu}(\epsilon)} \frac{dx}{w - x}, \qquad (1.26)$$

for all $n \geq 1$ and $w \in \mathbb{C} \setminus \text{Supp}(\mu_n)$. Note that $R_\mu(\epsilon)^+ \cup R_\mu(\epsilon)^- \subset \mathbb{C} \setminus \text{Supp}(\mu_n)$ for all n sufficiently large, since $P_n \cap (R_\mu(\epsilon)^+ \cup R_\mu(\epsilon)^-) = \emptyset$ (see (1.24)), and since $R_\mu(\epsilon)^+ \cup R_\mu(\epsilon)^-$ and $R_{\lambda-\mu}(\epsilon)$ are disjoint (see (1.23)). Therefore C_n is well-defined and analytic in $R_\mu(\epsilon)^+ \cup R_\mu(\epsilon)^-$. Also, since $\mu_n = \lambda$ in $R_{\lambda-\mu}(\epsilon)$ for all n sufficiently large,

Lemma 2.2 of [8] shows that e^{C_n} and C_n' have unique analytic extensions to $R_{\lambda-\mu}(\epsilon)$. Finally, inspired by (1.15), define:

Definition 1.8. Fix $\epsilon > 0$ as in (1.23), and define C_n as in (1.26). Then, for all n sufficiently large, the non-asymptotic edge curves, $(\chi_n(\cdot), \eta_n(\cdot)) : R_\mu^+(\epsilon) \cup R_{\lambda-\mu}(\epsilon) \cup R_\mu^-(\epsilon) \to \mathbb{R}^2$, are defined by,

$$\chi_n(s) := s + \frac{e^{C_n(s)} - 1}{e^{C_n(s)} C_n'(s)} \qquad \text{and} \qquad \eta_n(s) := 1 + \frac{(e^{C_n(s)} - 1)^2}{e^{C_n(s)} C_n'(s)},$$

for all $s \in R_\mu^+(\epsilon) \cup R_{\lambda-\mu}(\epsilon) \cup R_\mu^-(\epsilon)$. Moreover, we define $(\chi_n, \eta_n) := (\chi_n(t), \eta_n(t))$.

Note, since $\mu_n \to \mu$ weakly as $n \to \infty$,

$$e^{C_n(s)} \to e^{C(s)},$$
$$C_n'(s) \to C'(s), \tag{1.27}$$
$$(\chi_n(s), \eta_n(s)) \to (\chi_\mathcal{E}(s), \eta_\mathcal{E}(s)),$$

for all $s \in R_\mu^+(\epsilon) \cup R_{\lambda-\mu}(\epsilon) \cup R_\mu^-(\epsilon)$. As observed above, however, we have no control of the rate of convergence.

Remark 1.9. Note, the above definition depends on an arbitrary $\epsilon > 0$. Suppose $\epsilon' > 0$ is any other value which also satisfies (1.23), and let (χ_n', η_n') denote the analogous non-asymptotic edge obtained using ϵ'. Then, (1.23), (1.24), (1.26), Definition 1.8, and Riemann sum approximations imply that $(\chi_n', \eta_n') = (\chi_n, \eta_n) + O(n^{-1})$ for all n sufficiently large. This error is absorbed into the errors of order $O(1)$ in (1.28) and (1.29), below, and possibly effects the asymptotic behaviour in Theorems 1.10 and 1.11 when $(u, r) = (v, s)$. The asymptotic behaviour when $(u, r) \neq (v, s)$ is unaffected.

Next recall (see end of last section), that the asymptotic edge curve behaves like a parabola in a neighbourhood of (χ, η) with tangent vector $\mathbf{x}(t) := (1, e^{C(t)} - 1)$ and normal vector $\mathbf{y}(t) := (e^{C(t)} - 1, -1)$. This was proven in Lemma 2.9 of [8]. Proceeding similarly for the non-asymptotic edge curves for all n sufficiently large, we can show that these also behave like a parabola in a neighbourhood of (χ_n, η_n), with tangent vector $\mathbf{x}_n(t) := (1, e^{C_n(t)} - 1)$ and normal vector $\mathbf{y}_n(t) := (e^{C_n(t)} - 1, -1)$. Finally, we choose the $o(n)$ terms in (1.18) as follows: Fix $(u, r) \in \mathbb{R}^2$ and $(v, s) \in \mathbb{R}^2$, and let $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$ be sequences in $\mathbb{Z}^2$ which satisfy,

$$(u_n, r_n) = n(\chi_n, \eta_n) + n^{\frac{2}{3}} m_n \mathbf{x}_n u + n^{\frac{1}{3}} p_n \mathbf{y}_n r + O(1), \tag{1.28}$$

$$(v_n, s_n) = n(\chi_n, \eta_n) + n^{\frac{2}{3}} m_n \mathbf{x}_n v + n^{\frac{1}{3}} p_n \mathbf{y}_n s + O(1), \tag{1.29}$$

for all n sufficiently large, where $\{m_n\}_{n \geq 1} = \{m_n(t)\}_{n \geq 1}$ and $\{p_n\}_{n \geq 1} = \{p_n(t)\}_{n \geq 1}$ are those convergent sequences of real numbers with non-zero limits given in Definition 2.8.

In words, $(\chi_n, \eta_n) \to (\chi, \eta)$ as $n \to \infty$ at an indeterminate rate, and $\{\frac{1}{n}(u_n, r_n)\}_{n \geq 1}$ and $\{\frac{1}{n}(v_n, s_n)\}_{n \geq 1}$ fluctuate around (χ_n, η_n). The fluctuations are of order $O(n^{-\frac{1}{3}})$ and $O(n^{-\frac{2}{3}})$ respectively in the tangent and normal directions of the non-asymptotic edge curve, u and v measure the size of the $O(n^{-\frac{1}{3}})$ fluctuations, and r and s measure the size of the $O(n^{-\frac{2}{3}})$ fluctuations.

We finally state the main results of this paper:

Theorem 1.10. *Assume Assumptions 1.2 and 1.7. Fix any corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, where t is a root of $f'_{(\chi, \eta)}$ of multiplicity 2. Assume Assumption 2.4 if one of the relevant cases is satisfied, and choose $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$ as in (1.28) and (1.29). Additionally assume that either $(u, r) \neq (v, s)$, or $(u, r) = (v, s)$ and $r_n = s_n$ for all n sufficiently large.*

Let $\mathcal{K}_n$ be the equivalent correlation kernel defined in (1.5), and K_{Ai} be the extended Airy kernel defined in (1.21). Define $C'(t)$ as in (1.16), (1.17), note that $C'(t) \neq 0$ (see Lemma 2.3, below), and define $\beta(t) > 0$ by $\beta(t) := 2^{\frac{1}{3}} |f'''_{(\chi, \eta)}(t)|^{-\frac{1}{3}} |C'(t)|$. Then, when $t \in R_\mu^+ \cup R_\mu^-$,

$$n^{\frac{1}{3}} \beta(t)^{-1} \mathcal{K}_n((u_n, r_n), (v_n, s_n)) \to K_{Ai}((u, r), (v, s)),$$

as $n \to \infty$. Moreover, when $t \in R_{\lambda-\mu}$,

$$n^{\frac{1}{3}} \beta(t)^{-1} (1_{(u_n = v_n)} - \mathcal{K}_n((u_n, r_n), (v_n, s_n))) \to K_{Ai}((u, r), (v, s)),$$

as $n \to \infty$.

In other words, we let (χ, η) be any typical edge point (see Definition 1.5), we let (χ_n, η_n) denote the analogous point on the non-asymptotic edge for each n (see Definition 1.8), and we choose $\frac{1}{n}(u_n, r_n)$ and $\frac{1}{n}(v_n, s_n)$ (the rescaled particle positions) to fluctuate around (χ_n, η_n) as described by (1.28) and (1.29). Then, when $t \in R_\mu^+ \cup R_\mu^-$, Theorem 1.10 shows that $n^{\frac{1}{3}} \mathcal{K}_n((u_n, r_n), (v_n, s_n))$ (the rescaled correlation kernel of the particles) converges to the extended Airy kernel as $n \to \infty$. When $t \in R_{\lambda-\mu}$, Theorem 1.10 shows that $n^{\frac{1}{3}} (1_{(u_n = v_n)} - \mathcal{K}_n((u_n, r_n), (v_n, s_n)))$ (the rescaled correlation kernel of the "holes") converges to the extended Airy kernel as $n \to \infty$.

Note that Theorem 1.10 does not give the asymptotic behaviour when $(u, r) = (v, s)$ and $r_n \neq s_n$. The following theorem completes the result by giving the asymptotic behaviour for all cases. We prove the existence of a term, α_n, which has a well-defined asymptotic behaviour for the cases of Theorem 1.10, but (possibly) has no well-defined asymptotic behaviour when $(u, r) = (v, s)$ and $r_n \neq s_n$. Theorem 1.10 is, in fact, a trivial corollary of Theorem 1.11. We will prove Theorem 1.11, and leave the deduction of Theorem 1.10 to the interested reader:

Theorem 1.11. *Assume Assumptions 1.2 and 1.7. Fix any corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, where t is a root of $f'_{(\chi, \eta)}$ of multiplicity 2. Assume Assumption 2.4 if one of the relevant cases is satisfied, and choose $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$ as in (1.28) and (1.29).*

Define $\widetilde{K}_{Ai} : (\mathbb{R}^2)^2 :\to \mathbb{R}$ as in (1.19), $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ as in Lemma 2.13, $B_n : \mathbb{Z}_+^2 \to (0, +\infty)$ as in (1.6), $\beta(t) > 0$ as in Theorem 1.10, $c_n > 0$ by $c_n := |A_{t,n}((v_n, s_n), (u_n, r_n))| B_n(s_n, r_n)$, and $\alpha_n \geq 0$ by,

$$
\alpha_n := \frac{1}{c_n} \cdot \begin{cases} \dfrac{1_{(u_n \geq v_n, r_n > s_n)}(u_n + r_n - v_n - s_n - 1)!}{(r_n - s_n - 1)!(u_n - v_n)!} & \text{when } t \in R_\mu^+, \\[3mm] \dfrac{1_{(u_n \geq v_n, u_n + r_n \leq v_n + s_n, r_n \leq s_n)}(s_n - r_n)!}{(u_n - v_n)!(v_n + s_n - u_n - r_n)!} & \text{when } t \in R_{\lambda-\mu}, \\[3mm] \dfrac{1_{(u_n + r_n \leq v_n + s_n, r_n > s_n)}(v_n - u_n - 1)!}{(r_n - s_n - 1)!(v_n + s_n - u_n - r_n)!} & \text{when } t \in R_\mu^-. \end{cases}
$$

Then, when $t \in R_\mu^+ \cup R_\mu^-$,

$$
\mathcal{K}_n((u_n, r_n), (v_n, s_n)) = -\alpha_n + n^{-\frac{1}{3}}\beta(t)\widetilde{K}_{Ai}((u, r), (v, s)) + o(n^{-\frac{1}{3}}),
$$

as $n \to \infty$. Moreover, when $t \in R_{\lambda-\mu}$,

$$
\mathcal{K}_n((u_n, r_n), (v_n, s_n)) = +\alpha_n - n^{-\frac{1}{3}}\beta(t)\widetilde{K}_{Ai}((u, r), (v, s)) + o(n^{-\frac{1}{3}}),
$$

as $n \to \infty$. Also, for all n sufficiently large, $\alpha_n = 0$ when $(u, r) = (v, s)$ and $r_n = s_n$ and $t \in R_\mu^+ \cup R_\mu^-$, $\alpha_n = 1_{(u_n = v_n)}$ when $(u, r) = (v, s)$ and $r_n = s_n$ and $t \in R_{\lambda-\mu}$, and $\alpha_n = O(1)$ when $(u, r) = (v, s)$ and $r_n \neq s_n$. Finally,

$$
\alpha_n = n^{-\frac{1}{3}} 1_{(u > v)}\beta(t)\Phi((u, r), (v, s)) + o(n^{-\frac{1}{3}}),
$$

as $n \to \infty$ when $(u, r) \neq (v, s)$, where $\Phi : (\mathbb{R}^2)^2 :\to \mathbb{R}$ is defined in (1.20).

Note that Theorems 1.10 and 1.11 prove pointwise convergence. However, our methods also give the following extension:

Theorem 1.12. *Assume Assumptions 1.2 and 1.7. Fix any corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, where t is a root of $f'_{(\chi, \eta)}$ of multiplicity 2. Assume Assumption 2.4 if one of the relevant cases is satisfied. Finally, fix compact sets $U, R, V, S \subset \mathbb{R}$. Then the convergence in Theorems 1.10 and 1.11 holds uniformly for $(u, r) \in U \times R$ and $(v, s) \in V \times S$.*

For clarity, we state that Theorem 1.12 follows simply by taking the appropriate uniform bounds at every step of our proof. We do not attempt to prove Theorem 1.12 directly for the sake of readability.

We end this section by stating that Theorem 1.12 does not truly prove *edge universality*: It does not yet prove that the (rescaled) Gelfand–Tsetlin particle process at the edge converges to the extended Airy kernel particle process. Convergence of the respective Fredholm determinants of the processes remains to be shown. To do this, one could attempt to find an additional uniform bound of $|K_n((u_n, r_n), (v_n, s_n))|$, similar to that seen in Lemma 3.1 of [13]. This step is often overlooked in the literature, and we do not attempt to prove this here due to the length and complexity of the paper. Nevertheless, the results of this paper are an important step towards proving edge universality for this broad class of models.

1.5. **Other asymptotic situations and conjectures**

Recall the discussions given in Sections 1.1 and 1.3. Assume Assumption 1.2, fix (χ, η) in the polygon on the right of Figure 1.1, and fix sequences of particle positions, $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$, which satisfy $\frac{1}{n}(u_n, r_n) \to (\chi, \eta)$ and $\frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ as $n \to \infty$. In (1.18), we fixed the corresponding points $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$, and $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, where t is a root of $f'_{(\chi, \eta)}$ of multiplicity 2 ($m = 2$). In this section, instead of (1.18), we briefly discuss other natural asymptotic situations.

First, suppose that $(\chi, \eta) \in \mathcal{L}$, and let $w_0 \in \mathbb{H}$, denote the corresponding root of $f'_{(\chi, \eta)}$ of multiplicity 1. In [8], for general μ, we mapped $\mathcal{L}$ to $\mathbb{H}$ by mapping to the unique root, showed that this is a homeomorphism (indeed, a diffeomorphism), and found an expression for the inverse of this map. In other words, we wrote (χ, η) as an explicit function of w_0. Steepest descent analysis, combined with the above root behaviour and (1.7), intuitively suggests that $K_n((u_n, r_n), (v_n, s_n))$ should converge to the *Sine* kernel as $n \to \infty$ (see, for example, [11, 21]). When $r_n = s_n$, convergence to the standard Sine kernel was shown in Metcalfe, [19], for the analogous interlaced particle system where the particles on each row of the Gelfand–Tsetlin patterns take positions in $\mathbb{R}$. To our knowledge, this situation has not yet been studied in this new setting.

Next suppose that $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$, and the corresponding root, $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, of $f'_{(\chi, \eta)}$ is of multiplicity 3 ($m = 3$). Recall that the set of all such points is discrete, and they define algebraic cusps of first order in the edge curve. Examples are shown in the top right of Figure 1.2. Steepest descent analysis, combined with the above root behaviour and (1.7), intuitively suggests that $K_n((u_n, r_n), (v_n, s_n))$ should converge to the *Pearcey* kernel as $n \to \infty$ (see, for example, Tracy and Widom, [25]). To our knowledge, this situation has not yet been studied in this new setting.

Next suppose that $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$. Recall that the set of all such points is discrete, $m = 1$ when $(\chi, \eta) \in \mathcal{E}_0$, $m = 1$ or $m = 2$ when $(\chi, \eta) \in \mathcal{E}_1 \cup \mathcal{E}_2$, the edge curve behaves like a parabola when $m = 1$, and the edge curve behaves like an algebraic cusp of first

order when $m = 2$. Examples are shown in the lower left of Figure 1.2. In [7], we examined the local asymptotic behaviour when $(\chi, \eta) \in \mathcal{E}_1 \cup \mathcal{E}_2$ and $m = 2$. Scaling $\{(u_n, r_n)\}_{n \geq 1}$ and $\{(v_n, s_n)\}_{n \geq 1}$ appropriately, we showed that $K_n((u_n, r_n), (v_n, s_n))$ converges to a novel kernel which we called the *Cusp-Airy* kernel. To our knowledge, the situation when $(\chi, \eta) \in \mathcal{E}_0 \cup \mathcal{E}_1 \cup \mathcal{E}_2$ and $m = 1$ has not yet been studied. However, steepest descent analysis, combined with the above root behaviour and (1.7), intuitively suggests a universal asymptotic behaviour.

Finally suppose that (χ, η) lies on that grey part of $\partial \mathcal{L}$ for the example depicted on the bottom right of Figure 1.2. In [9], we showed that the behaviour of the roots of $f'_{(\chi, \eta)}$ is identical for every point on these sections. Therefore these sections cannot be parameterised by defining unique roots for each point, as we did for $\mathcal{E}$. In [9], we instead made heavy use of the theory of singular integrals to parameterise these sections. We also examined similar, surprisingly subtle, situations. We now conjecture that the local asymptotic behaviour of particles in neighbourhoods of (χ, η), when (χ, η) lies on such sections of $\partial \mathcal{L}$, are non-universal.

1.6. **Notation and terminology**

We end the introduction by discussing the notation and terminology that will be in use for the rest of the paper. The arguments are quite technical, and we use these for simplicity, and to avoid needless repetition. First recall (see (1.18)) than we fix the corresponding points $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$. From now on, we simply label $f_{(\chi, \eta)}$ with t instead of (χ, η), i.e., we define,

$$f_t := f_{(\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))} = f_{(\chi, \eta)}. \tag{1.30}$$

Next, fix $a \in \mathbb{C}$, and a measure ν on $\mathbb{R}$. Also, for all $n \geq 1$, fix $a_n \in \mathbb{C}$ and $b_n \in \mathbb{C}$, and a measure ν_n on $\mathbb{R}$. Then:

- "$a_n \to a$" means "$a_n \to a$ as $n \to \infty$".

- "$\nu_n \to \nu$ weakly" means "$\nu_n \to \nu$ as $n \to \infty$ in the sense of weak convergence of measures".

- "$a_n = b_n$" means "$a_n = b_n$ for all n sufficiently large". In other words, if we do not explicitly state those $n \geq 1$ for which $a_n = b_n$, then we implicitly understand that the equality holds for all n sufficiently large. Similarly for *any other expression or statement* involving n.

Next, for all $n \geq 1$, fix $c_n \in \mathbb{R}$ with $c_n > 0$. Then:

- "$o(c_n)$" denotes a complex-number which is well-defined for all n sufficiently large. Moreover, $|o(c_n)|/c_n \to 0$ as $n \to \infty$.

- "$O(c_n)$" denotes a complex-number which is well-defined for all n sufficiently large. Moreover, there exists a $C > 0$ for which $|O(c_n)|/c_n \leq C$ for all such n.

Next, fix $A \subset \mathbb{C}$, $f : \mathbb{C} \to \mathbb{C}$, and $m \geq 1$. Also, for all $n \geq 1$, fix $A_n \subset \mathbb{C}$, $f_n : \mathbb{C} \to \mathbb{C}$ and $g_n : \mathbb{C} \to \mathbb{C}$. Then:

- "f has m roots in A" means "f has exactly m roots in A counting multiplicities". Similarly for "at least/most m roots".

- "f_n has m roots in A_n" means "f_n has exactly m roots in A_n for all n sufficiently large and counting multiplicities'. Similarly for "at least/most m roots".

- "$f_n \to f$ uniformly in A" means "$\sup_{w \in A} |f_n(w) - f(w)| \to 0$" as $n \to \infty$.

- "$|f_n(w)| > c_n$ uniformly for $w \in A$" means "$\inf_{w \in A} |f_n(w)| > c_n$ for all n sufficiently large".

- "$f_n(w) = g_n(w) + o(c_n)$ uniformly for $w \in A$" means "$\sup_{w \in A} |f_n(w) - g_n(w)|/c_n \to 0$" as $n \to \infty$.

- "$f_n(w) = g_n(w) + O(c_n)$ uniformly for $w \in A$" means "there exists a $C > 0$ for which $\sup_{w \in A} |f_n(w) - g_n(w)|/c_n \leq C$ for all n sufficiently large".

Finally, given $B \subset \mathbb{C}$, and $S \subset \mathbb{R}$ bounded:

- $\mathrm{cl}(B)$ is the closure of B.

- $\overline{S} := \sup S$ and $\underline{S} := \inf S$.

2. The roots of the steepest descent functions

In this section, we assume Assumptions 1.2 and 1.7, and that (1.18) is satisfied for some fixed $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, a root of f_t' of multiplicity 2. We consider the behaviour of the roots of f_t', f_n' and $\tilde{f}_n'$ (see (1.8), (1.9), (1.10), (1.30)). This results of this section enable us to perform the steepest descent analysis of Section 3.

2.1. **The roots of f_t'**

In this section, we consider the behaviour of the roots of f_t'. Recall the following basic facts which we make use of throughout this section: Assumption 1.2 implies that μ is a probability measure on $[a, b]$ with $\{a, b\} \in \text{Supp}(\mu)$ and $\mu \leq \lambda$. (1.18) implies that $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ and $(\chi, \eta) \in \mathcal{E}_\mu^+ \cup \mathcal{E}_{\lambda-\mu} \cup \mathcal{E}_\mu^-$ are the corresponding points with $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, and t is a root of $f_{(\chi,\eta)}'$ of multiplicity 2. Also, since $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, Definition 1.4 and Lemma 1.6 imply that $t \in \mathbb{R} \setminus \{\chi, \chi + \eta - 1\}$, $(\chi, \eta) \in (a, b) \times (0, 1)$, and $b > \chi > \chi + \eta - 1 > a$. Finally, (1.13) and (1.30) give

$$f_t'(w) = \int_{S_1} \frac{\mu[dx]}{w - x} - \int_{S_2} \frac{(\lambda - \mu)[dx]}{w - x} + \int_{S_3} \frac{\mu[dx]}{w - x}, \tag{2.1}$$

for all $w \in \mathbb{C} \setminus S$, where $S = S_1 \cup S_2 \cup S_3$, and S_1, S_2, S_3 are defined in (1.12).

First, note the following:

$$
\begin{aligned}
&S_1 \neq \emptyset, \mu[S_1] > 0, \quad S_2 \neq \emptyset, (\lambda - \mu)[S_2] > 0, \quad S_3 \neq \emptyset, \mu[S_3] > 0. \\
&\mu[S_1] - (\lambda - \mu)[S_2] + \mu[S_3] = \mu[a, b] - \lambda[\chi + \eta - 1, \chi] = \eta \in (0, 1). \\
&b = \overline{S_1} > \underline{S_1} \geq \chi \geq \overline{S_2} > \underline{S_2} \geq \chi + \eta - 1 \geq \overline{S_3} > \underline{S_3} = a. \\
&[\overline{S_i} - \epsilon, \overline{S_i}] \cup [\underline{S_i}, \underline{S_i} + \epsilon] \subset S_i \text{ for all } i \in \{1, 2, 3\} \text{ and some } \epsilon > 0.
\end{aligned}
\tag{2.2}
$$

Above, $\underline{S_1} := \inf S_1$, $\overline{S_1} := \sup S_1$, etc. (see Section 1.6). The first part, above, follows from part (a) of Corollary 3.2 of [8]. The second part follows since μ is a probability measure on $[a, b]$ and $\eta \in (0, 1)$. The third part follows from the first part and (1.12) since $\mu \leq \lambda$, $\{a, b\} \in \text{Supp}(\mu)$, and $b > \chi > \chi + \eta - 1 > a$. Finally, the fourth part follows since $\mu \leq \lambda$ and $\{a, b\} \in \text{Supp}(\mu)$.

Next note, (2.2) implies that $\mathbb{C} \setminus S$, the domain of f_t', can be partitioned as follows:

$$\mathbb{C} \setminus S = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K, \tag{2.3}$$

where $J := \bigcup_{i=1}^4 J_i$, $K := \bigcup_{i=1}^3 K^{(i)}$, and

- $J_1 := (\sup S_1, +\infty)$.

- $J_2 := (-\infty, \inf S_3)$.

- $J_3 := (\sup S_2, \inf S_1)$.

- $J_4 := (\sup S_3, \inf S_2)$.

- $K^{(i)} := [\inf S_i, \sup S_i] \setminus S_i$ for all $i \in \{1, 2, 3\}$.
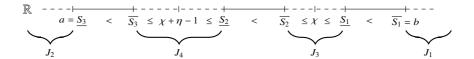
$$\mathbb{H}$$

FIGURE 2.1. The sets of (2.3), with $K^{(i)} = [\inf S_i, \sup S_i] \setminus S_i$ for all $i \in \{1, 2, 3\}$.

This partition is depicted in Figure 2.1. Note that J_1 and J_2 are non-empty, but that $J_3 = \emptyset$ when $\sup S_2 = \inf S_1$, $J_4 = \emptyset$ when $\sup S_3 = \inf S_2$, and $K^{(i)} = \emptyset$ when $S_i = [\inf S_i, \sup S_i]$ for each $K^{(i)}$. Note also that χ is contained in the domain of f_t' if and only if J_3 is non-empty, in which case $\chi \in J_3$. Similarly $\chi + \eta - 1$ is contained in the domain of f_t' if and only if J_4 is non-empty, in which case $\chi + \eta - 1 \in J_4$. Finally note that each $K^{(i)}$ is open, and so can be partitioned as a set of pairwise disjoint open intervals, which is unique up to order, and which is either empty or finite or countable. We denote this partition of open intervals as $\{K_1^{(i)}, K_2^{(i)}, \ldots\}$, and we note that any $I \in \{K_1^{(i)}, K_2^{(i)}, \ldots\}$ must satisfy $\{\inf I, \sup I\} \subset S_i$.

Next note, since t is real-valued root of f_t', (2.3) implies that $t \in \mathbb{R} \setminus S = J \cup K$. We let $L_t \subset \mathbb{R} \setminus S = J \cup K$ denote that open interval for which,

$$t \in L_t \qquad \text{and} \qquad L_t \in \{J_1, J_2, J_3, J_4\} \cup \bigcup_{i=1}^{3}\{K_1^{(i)}, K_2^{(i)}, \ldots\}. \tag{2.4}$$

In other words, L_t is the largest open sub-interval of $\mathbb{R} \setminus S = J \cup K$ which contains t. These observations, and Theorem 3.1 of [8], then immediately give the following, stated without proof:

Lemma 2.1. *Counting multiplicities:*

(1) *f_t' has a root of multiplicity 2 at $t \in L_t$, has 0 roots in $L_t \setminus \{t\}$ when $L_t \in \{J_1, J_2, J_3, J_4\}$, and has at most 1 root in $L_t \setminus \{t\}$ when $L_t \in \bigcup_{i=1}^{3}\{K_1^{(i)}, K_2^{(i)}, \ldots\}$.*

(2) *f_t' has 0 roots in $\mathbb{C} \setminus \mathbb{R}$, and in each of $\{J_1, J_2, J_3, J_4\} \setminus \{L_t\}$.*

(3) *f_t' has at most 1 root in each of $\bigcup_{i=1}^{3}\{K_1^{(i)}, K_2^{(i)}, \ldots\} \setminus \{L_t\}$.*

Next, we give the following useful result which describes the various situations of Theorem 1.11 in explicit detail. A separate steepest descent analysis must be performed for each case:

Lemma 2.2. *The following 12 cases exhaust all possibilities:*

(1) $t \in R_\mu^+$, $t > \chi$, $L_t = J_1$, $f_t'''(t) > 0$.

(2) $t \in R_\mu^+$, $t > \chi$, $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, $f_t'''(t) > 0$.

(3) $t \in R_\mu^+$, $t > \chi$, $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, $f_t'''(t) < 0$.

(4) $t \in R_\mu^+$, $t > \chi$, $\chi \in L_t$, $L_t = J_3$, $f_t'''(t) < 0$.

(5) $t \in R_{\lambda-\mu}$, $t \in (\chi + \eta - 1, \chi)$, $\chi \in L_t$, $L_t = J_3$, $f_t'''(t) < 0$.

(6) $t \in R_{\lambda-\mu}$, $t \in (\chi + \eta - 1, \chi)$, $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$, $f_t'''(t) < 0$.

(7) $t \in R_{\lambda-\mu}$, $t \in (\chi + \eta - 1, \chi)$, $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$, $f_t'''(t) > 0$.

(8) $t \in R_{\lambda-\mu}$, $t \in (\chi + \eta - 1, \chi)$, $\chi + \eta - 1 \in L_t$, $L_t = J_4$, $f_t'''(t) > 0$.

(9) $t \in R_\mu^-$, $t < \chi + \eta - 1$, $\chi + \eta - 1 \in L_t$, $L_t = J_4$, $f_t'''(t) > 0$.

(10) $t \in R_\mu^-$, $t < \chi + \eta - 1$, $L_t \in \{K_1^{(3)}, K_2^{(3)}, \ldots\}$, $f_t'''(t) > 0$.

(11) $t \in R_\mu^-$, $t < \chi + \eta - 1$, $L_t \in \{K_1^{(3)}, K_2^{(3)}, \ldots\}$, $f_t'''(t) < 0$.

(12) $t \in R_\mu^-$, $t < \chi + \eta - 1$, $L_t = J_2$, $f_t'''(t) < 0$.

Proof. First recall that $t \in \mathbb{R} \setminus \{\chi, \chi + \eta - 1\}$. Then, (2.2), (2.3), (2.4) imply that the following exhaust all possibilities (see, also, Figure 2.1):

- $t > \chi$, and $L_t = J_1$ or $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$ or $L_t = J_3$.

- $t \in (\chi + \eta - 1, \chi)$, and $L_t = J_3$ or $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$ or $L_t = J_4$.

- $t < \chi + \eta - 1$, and $L_t = J_4$ or $L_t \in \{K_1^{(3)}, K_2^{(3)}, \ldots\}$ or $L_t = J_2$.

We will show:

(i) Case (1) is satisfied when $t > \chi$ and $L_t = J_1$.

(ii) Case (2) or (3) is satisfied when $t > \chi$ and $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$.

(iii) Case (4) is satisfied when $t > \chi$ and $L_t = J_3$.

Similarly it can be shown that:

   (iv) (5) is satisfied when $t \in (\chi + \eta - 1, \chi)$ and $L_t = J_3$.

    (v) (6) or (7) is satisfied when $t \in (\chi + \eta - 1, \chi)$ and $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$.

   (vi) (8) is satisfied when $t \in (\chi + \eta - 1, \chi)$ and $L_t = J_4$.

  (vii) (9) is satisfied when $t < \chi + \eta - 1$ and $L_t = J_4$.

 (viii) (10) or (11) is satisfied when $t < \chi + \eta - 1$ and $L_t \in \{K_1^{(3)}, K_2^{(3)}, \ldots\}$.

   (ix) (12) is satisfied when $t < \chi + \eta - 1$ and $L_t = J_2$.

The required result follows from (i)–(ix).

Consider (i). Recall that $t > \chi$ and $L_t = J_1$. Also, since $t > \chi$ and $(\chi, \eta) = (\chi_\varepsilon(t), \eta_\varepsilon(t))$, Definition 1.4 and Lemma 1.6 imply that $(\chi, \eta) \in \mathcal{E}_\mu^+$ and $t \in R_\mu^+$. It thus remains to show that $f_t'''(t) > 0$. To see this, first note, (2.1) implies that $(f_t')|_{J_1}$ is continuous and real-valued. Moreover, since $L_t = J_1$, part (1) of Lemma 2.1 implies that $f_t'(t) = f_t''(t) = 0$, $f_t'''(t) \neq 0$, and $f_t'(s) \neq 0$ for all $s \in J_1 \setminus \{t\} = (\sup S_1, t) \cup (t, +\infty)$. Therefore, it is sufficient to show that there exists an $s_t \in (t, +\infty)$ with $f_t'(s_t) > 0$. To see this, note (2.1) gives

$$f_t'(w) = \left(\int_{S_1} \frac{\mu[dx]}{w} - \int_{S_2} \frac{(\lambda - \mu)[dx]}{w} + \int_{S_3} \frac{\mu[dx]}{w} \right) + O\left(\frac{1}{|w|^2} \right),$$

for all $w \in \mathbb{C}$ with $|w|$ sufficiently large. Therefore $f_t'(w) = (\mu[S_1] - (\lambda - \mu)[S_2] + \mu[S_3])w^{-1} + O(|w|^{-2}) = \eta w^{-1} + O(|w|^{-2})$ for all such w, where the last part follows from (2.2). Therefore, since $\eta > 0$, there exists an $s_t \in (t, +\infty)$ with $f_t'(s_t) > 0$. This proves (i).

Consider (ii). Recall that $t > \chi$ and $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$. Also, since $t > \chi$ and $(\chi, \eta) = (\chi_\varepsilon(t), \eta_\varepsilon(t))$, Definition 1.4 and Lemma 1.6 imply that $(\chi, \eta) \in \mathcal{E}_\mu^+$ and $t \in R_\mu^+$. Finally recall that t is a root of f_t' of multiplicity 2, and so $f_t'''(t) \neq 0$. This proves (ii).

Consider (iii). Recall that $t > \chi$ and $L_t = J_3$. Also, since $t > \chi$ and $(\chi, \eta) = (\chi_\varepsilon(t), \eta_\varepsilon(t))$, Definition 1.4 and Lemma 1.6 imply that $(\chi, \eta) \in \mathcal{E}_\mu^+$ and $t \in R_\mu^+$. Moreover, (2.2) and (2.3) imply that $\chi \in J_3(= L_t)$ (see, also, Figure 2.1). It thus remains to show that $f_t'''(t) < 0$. To see this, first note, (2.1) implies that $(f_t')|_{J_3}$ is continuous and real-valued. Moreover, since $L_t = J_3$, part (1) of Lemma 2.1 implies that $f_t'(t) = f_t''(t) = 0$, $f_t'''(t) \neq 0$, and $f_t'(s) \neq 0$ for all $s \in J_3 \setminus \{t\} = (\sup S_2, t) \cup (t, \inf S_1)$. Therefore, it is sufficient to show that there exists an $s_t \in (t, \inf S_t)$ with $f_t'(s_t) < 0$.

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To see the above, we use the notation and definitions and results of next section. This is not a circular argument since the current lemma is not used in the next section. First, fix $\xi > 0$ as in Lemma 2.6, and fix $s_t \in [t + \xi, \inf S_1)$. We have shown above that $f_t'(s_t) \neq 0$, and so either $f_t'(s_t) < 0$ or $f_t'(s_t) > 0$. Next note, (2.7) implies that $(f_n')|_{J_{3,n}}$ is continuous and real-valued. Moreover, since $L_n = J_{3,n}$ (see (2.12)), part (1) of Lemma 2.6 implies that $f_n'(s) \neq 0$ for all $s \in J_{3,n} \setminus (t - \xi, t + \xi) = (\max S_{2,n}, t - \xi] \cup [t + \xi, \min S_{1,n})$. Also, (2.7) gives

$$\lim_{s \in \mathbb{R}, s \uparrow \min S_{1,n}} f_n'(s) = -\infty.$$

The above observations imply that $f_n'(s) < 0$ for all $s \in [t + \xi, \min S_{1,n})$. In particular, since $\min S_{1,n} = \inf S_1 + o(1)$ (see (2.12) and note that $L_t = J_3$, $L_n = J_{3,n}$, $\inf S_1 = \sup J_3$, $\min S_{1,n} = \sup J_{3,n}$), and since $s_t \in [t + \xi, \inf S_1)$, this gives $f_n'(s_t) < 0$. Finally note, (2.1), (2.7) and (2.8) give $f_n'(s_t) \to f_t'(s_t)$. Therefore $f_t'(s_t) < 0$, as required. This proves (iii). □

Next recall that $e^{C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ and $C'(\cdot) : \mathbb{C} \setminus R \to \mathbb{C}$ have those unique analytic extensions to $R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ given in (1.16) and (1.17). We now consider some useful properties of these extensions for cases (1)–(12) of Lemma 2.2:

Lemma 2.3. *The following hold:*

- $e^{C(t)} > 1$ *and* $C'(t) < 0$ *when* $t \in R_\mu^+$, *i.e., for cases* (1)–(4).

- $e^{C(t)} < 0$ *and* $C'(t) > 0$ *when* $t \in R_{\lambda-\mu}$, *i.e., for cases* (5)–(8).

- $e^{C(t)} \in (0, 1)$ *and* $C'(t) < 0$ *when* $t \in R_\mu^-$, *i.e., for cases* (9)–(12).

Proof. First recall that $C(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ and $C'(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ have those unique analytic extensions to $\mathbb{R} \setminus \text{Supp}(\mu)$ given in (1.16). Moreover, Lemma 1.6 gives $R_\mu^+ = \{s \in \mathbb{R} \setminus \text{Supp}(\mu) : C(s) > 0\}$ and $R_\mu^- = \{s \in \mathbb{R} \setminus \text{Supp}(\mu) : C(s) < 0\}$. These easily give $e^{C(t)} > 1$ when $t \in R_\mu^+$, $e^{C(t)} \in (0, 1)$ when $t \in R_\mu^-$, and $C'(t) < 0$ when $t \in R_\mu^+ \cup R_\mu^-$. Next recall that $e^{C(\cdot)} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ and $C'(\cdot) : \mathbb{C} \setminus R \to \mathbb{C}$ have those unique analytic extensions to $R_{\lambda-\mu}$ given in (1.17). These easily give $e^{C(t)} < 0$ when $t \in R_{\lambda-\mu}$. Finally, we showed in Lemma 2.2 of [8] that $C'(t) > 0$ when $t \in R_{\lambda-\mu}$. □

We end this section with the final technical assumption of Theorem 1.11. First note the following (see Lemma 2.2):

- For case (4), $\chi < \inf S_1$ and $t \in (\chi, \inf S_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

- For case (5), $\sup S_2 < \chi$ and $t \in (\sup S_2, \chi) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.

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- For case (8), $\chi + \eta - 1 < \inf S_2$ and $t \in (\chi + \eta - 1, \inf S_2) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.

- For case (9), $\chi + \eta - 1 > \sup S_3$ and $t \in (\sup S_3, \chi + \eta - 1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

We assume:

Assumption 2.4. *If one of the cases* (4), (5), (8) *and* (9) *of Lemma 2.2 is satisfied, assume the following:*

- $\chi \notin \text{Supp}(\mu)$ *for case* (4), *i.e.,* $[\chi, \inf S_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

- $\chi \notin \text{Supp}(\lambda - \mu)$ *for case* (5), *i.e.,* $(\sup S_2, \chi] \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.

- $\chi + \eta - 1 \notin \text{Supp}(\lambda - \mu)$ *for case* (8), *i.e.,* $[\chi + \eta - 1, \inf S_2) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$.

- $\chi + \eta - 1 \notin \text{Supp}(\mu)$ *for case* (9), *i.e.,* $(\sup S_3, \chi + \eta - 1] \subset \mathbb{R} \setminus \text{Supp}(\mu)$.

Note, with the above assumption, (1.12) implies that $\chi = \sup S_2$ for case (4), $\chi = \inf S_1$ for case (5), $\chi + \eta - 1 = \sup S_3$ for case (8), and $\chi + \eta - 1 = \inf S_2$ for case (9). Another effect of Assumption 2.4 can be seen, for example, in part (ii) in the proof of Lemma 3.9. Part (ii) is not necessarily true if Assumption 2.4 does not hold.

2.2. **The roots of f_n' and $\tilde{f}_n'$**

In this section, we assume Assumptions 1.2 and 1.7, that (1.18) is satisfied for some fixed $t \in R_\mu^+ \cup R_{\lambda - \mu} \cup R_\mu^-$ (a root of f_t' of multiplicity 2), and Assumption 2.4. We consider the behaviour of the roots of f_n' and $\tilde{f}_n'$ as $n \to \infty$. We concentrate mainly on f_n', since $\tilde{f}_n'$ has a similar behaviour.

First note, (1.8) and (1.9) give

$$f_n(w) = \frac{1}{n} \sum_{x \in S_{1,n}} \log(w - x) - \frac{1}{n} \sum_{x \in S_{2,n}} \log(w - x) + \frac{1}{n} \sum_{x \in S_{3,n}} \log(w - x), \qquad (2.5)$$

$$\tilde{f}_n(w) = \frac{1}{n} \sum_{x \in \tilde{S}_{1,n}} \log(w - x) - \frac{1}{n} \sum_{x \in \tilde{S}_{2,n}} \log(w - x) + \frac{1}{n} \sum_{x \in \tilde{S}_{3,n}} \log(w - x), \qquad (2.6)$$

where the branch cuts are either $(-\infty, 0]$ or $[0, +\infty)$, and where:

- $S_{1,n} := \frac{1}{n} \{x_i : x_i > v_n\}$.

- $S_{2,n} := \frac{1}{n} \{v_n, v_n - 1, \ldots, v_n + s_n - n\} \setminus \{x_1, x_2, \ldots, x_n\}$.

- $S_{3,n} := \frac{1}{n} \{x_i : x_i < v_n + s_n - n\}$.

- $\tilde{S}_{1,n} := \frac{1}{n}\{x_i : x_i \geq u_n\}$.

- $\tilde{S}_{2,n} := \frac{1}{n}\{u_n - 1, u_n - 2, \ldots, u_n + r_n - n + 1\} \setminus \{x_1, x_2, \ldots, x_n\}$.

- $\tilde{S}_{3,n} := \frac{1}{n}\{x_i : x_i \leq u_n + r_n - n\}$.

Consider f_n. First note, irrespective of the choices of the branches of the logarithm in (2.5), that

$$f_n'(w) = \frac{1}{n}\sum_{x \in S_{1,n}} \frac{1}{w - x} - \frac{1}{n}\sum_{x \in S_{2,n}} \frac{1}{w - x} + \frac{1}{n}\sum_{x \in S_{3,n}} \frac{1}{w - x}. \tag{2.7}$$

Also note, Assumption 1.2, (1.12) and (1.18) imply the following:

$$\frac{1}{n}\sum_{x \in S_{1,n}} \delta_x \to \mu|_{S_1} \text{ weakly,}$$

$$\frac{1}{n}\sum_{x \in S_{2,n}} \delta_x \to (\lambda - \mu)|_{S_2} \text{ weakly,} \tag{2.8}$$

$$\frac{1}{n}\sum_{x \in S_{3,n}} \delta_x \to \mu|_{S_3} \text{ weakly.}$$

Moreover, Assumption 1.7, (2.2) and (2.8) imply the following:

$$S_{1,n} \neq \emptyset, \quad \frac{1}{n}|S_{1,n}| = \mu[S_1] + o(1) = \mu[\chi, b] + o(1) > 0.$$

$$S_{2,n} \neq \emptyset, \quad \frac{1}{n}|S_{2,n}| = (\lambda - \mu)[S_2] + o(1) = (\lambda - \mu)[\chi + \eta - 1, \chi] + o(1) > 0.$$

$$S_{3,n} \neq \emptyset, \quad \frac{1}{n}|S_{3,n}| = \mu[S_3] + o(1) = \mu[a, \chi + \eta - 1] + o(1) > 0. \tag{2.9}$$

$$\frac{1}{n}x_1^{(n)} = \overline{S_{1,n}} > \underline{S_{1,n}} > \overline{S_{2,n}} > \underline{S_{2,n}} > \overline{S_{3,n}} > \underline{S_{3,n}} = \frac{1}{n}x_n^{(n)}.$$

$$\overline{S_{1,n}} = \overline{S_1} + o(1) \quad \text{and} \quad \underline{S_{3,n}} = \underline{S_3} + o(1).$$

Next note that f_n' extends analytically to $\mathbb{C} \setminus S_n$, where $S_n := S_{1,n} \cup S_{2,n} \cup S_{3,n}$. Then, in analogy with (2.3), partition the domain of f_n' as follows:

$$\mathbb{C} \setminus S_n = (\mathbb{C} \setminus \mathbb{R}) \cup J_n \cup K_n, \tag{2.10}$$

where $J_n := \bigcup_{i=1}^4 J_{i,n}$, $K_n := \bigcup_{i=1}^3 K_n^{(i)}$, and

- $J_{1,n} := (\max S_{1,n}, +\infty)$.

- $J_{2,n} := (-\infty, \min S_{3,n})$.

- $J_{3,n} := (\max S_{2,n}, \min S_{1,n})$.

- $J_{4,n} := (\max S_{3,n}, \min S_{2,n})$.

- $K_n^{(i)} := [\min S_{i,n}, \max S_{i,n}] \setminus S_{i,n}$ for all $i \in \{1, 2, 3\}$.

Note that each $K_n^{(i)}$ is open, and so can be partitioned as a set of pairwise disjoint open intervals, which is unique up to order. We denote this partition of open intervals as $\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$, and we note that $I_n \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$ if and only if $\inf I_n$ and $\sup I_n$ are two consecutive elements of $S_{i,n}$.

Next, recall (see (1.18), (2.1)) that $t \in \mathbb{R} \setminus S$ is a root of f_t', and (see (2.4)) that $L_t \subset \mathbb{R} \setminus S = J \cup K$ is that interval for which $t \in L_t$ and $L_t \in \{J_1, J_2, J_3, J_4\} \cup \bigcup_{i=1}^{3} \{K_1^{(i)}, K_2^{(i)}, \ldots\}$. Therefore, for all $\xi > 0$ sufficiently small,

$$(t - 4\xi, t + 4\xi) \subset L_t, \ B(t, 4\xi) \subset \mathbb{C} \setminus S, \text{ and } t \text{ is the unique root of } f_t' \text{ in } B(t, 4\xi). \quad (2.11)$$

Fix such an $\xi > 0$. Then:

Lemma 2.5. *There exists an open interval,* $L_n \in \{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\} \cup \bigcup_{i=1}^{3} \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$, *which satisfies,*

$$
\begin{aligned}
&\sup L_n = \sup L_t + o(1) \qquad and \qquad \inf L_n = \inf L_t + o(1), \\
&(t - 2\xi, t + 2\xi) \subset L_n \qquad and \qquad B(t, 2\xi) \subset \mathbb{C} \setminus S_n, \\
&L_n = J_{i,n} \Leftrightarrow L_t = J_i \text{ for all } i \in \{1, 2, 3, 4\}, \\
&L_n \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \Leftrightarrow L_t \in \{K_1^{(i)}, K_2^{(i)}, \ldots\} \text{ for all } i \in \{1, 2, 3\}.
\end{aligned}
\qquad (2.12)
$$

Proof. Throughout this proof, it is helpful to refer to Figure 2.1 to visualise the sets in question. Recall the exhaustive cases (1)–(12) of Lemma 2.2. We will prove the result for cases (1), (2), (4) and (7). The proof for case (12) is similar to case (1), the proof for cases (3), (10) and (11) are similar to case (2), the proof for case (6) is similar to case (7), and the proof for cases (5), (8) and (9) are similar to case (4).

For case (1), recall that $L_t = J_1 = (\sup S_1, +\infty)$, $J_{1,n} = (\max S_{1,n}, +\infty)$, and $\max S_{1,n} \to \sup S_1$ (see Lemma 2.2, and (2.9) and (2.10)). The result for case (1) then follows from (2.11).

For case (2), recall that $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, $L_t \in \mathbb{R} \setminus \mathrm{Supp}(\mu)$, $\sup S_1 > \sup L_t$ and $\inf L_t > \chi$, $\{\inf L_t, \sup L_t\} \subset S_1 = \mathrm{Supp}(\mu|_{[\chi, b]})$, and S_1 is entirely contained in $[\sup L_t, \sup S_1] \cup [\chi, \inf L_t]$ (see Lemma 2.2, and (1.12), (2.2) and (2.3)). Moreover, recall that $S_{1,n} = \frac{1}{n}\{x_i : x_i > v_n\}$, $\max S_{1,n} \to \sup S_1$ and $\frac{v_n}{n} \to \chi$, and $I_n \in \{K_{1,n}^{(1)}, K_{2,n}^{(1)}, \ldots\}$ if and only if $\inf I_n$ and $\sup I_n$ are two consecutive elements of $S_{1,n}$ (see (1.18), (2.7) and (2.10)). The result for case (2) then follows from Assumption 1.7 and (2.11).

For case (7), recall that $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$, $L_t \in \mathbb{R} \setminus \mathrm{Supp}(\lambda - \mu)$, $\chi > \sup L_t$ and $\inf L_t > \chi + \eta - 1$, $\{\inf L_t, \sup L_t\} \subset S_2 = \mathrm{Supp}((\lambda - \mu)|_{[\chi + \eta - 1, \chi]})$, and S_2 is entirely

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contained in $[\sup L_t, \chi] \cup [\chi + \eta - 1, \inf L_t]$ (see Lemma 2.2, and (1.12), (2.2) and (2.3)). Moreover, recall that $S_{2,n} = \frac{1}{n}\{v_n, v_n - 1, \ldots, v_n + s_n - n\} \setminus \{x_1, x_2, \ldots, x_n\}$, $\frac{v_n}{n} \to \chi$ and $\frac{v_n + s_n - n}{n} \to \chi + \eta - 1$, and $I_n \in \{K_{1,n}^{(2)}, K_{2,n}^{(2)}, \ldots\}$ if and only if $\inf I_n$ and $\sup I_n$ are two consecutive elements of $S_{2,n}$ (see (1.18), (2.7) and (2.10)). The result for case (2) then follows from Assumption 1.7 and (2.11).

For case (4), recall that $L_t = J_3 = (\sup S_2, \inf S_1)$, $\sup L_t \subset S_1 = \text{Supp}(\mu|_{[\chi,b]})$, and $\inf L_t \subset S_2 = \text{Supp}((\lambda - \mu)|_{[\chi+\eta-1,\chi]})$ (see Lemma 2.2, and (1.12), (2.2) and (2.3)). Moreover, Assumption 2.4 implies that $\sup S_2 = \chi$, i.e., that $\inf L_t = \chi$, and $[\chi, \inf S_1) \subset \mathbb{R} \setminus \text{Supp}(\mu)$. Therefore $(\chi - \delta, \inf S_1 - \delta) \subset \mathbb{R} \setminus \text{Supp}(\mu)$ for all $\delta > 0$ sufficiently small. Then, for all such $\delta > 0$, Assumption 1.7 implies that there exists an $n(\delta) \geq 1$ for which $(\chi - \delta, \inf S_1 - \delta) \cap (\frac{1}{n}\{x_1, x_2, \ldots, x_n\}) = \emptyset$ for all $n > n(\delta)$. Finally recall that $J_{3,n} = (\max S_{2,n}, \min S_{1,n})$, $S_{1,n} = \frac{1}{n}\{x_i : x_i > v_n\}$ and $S_{2,n} = \frac{1}{n}\{v_n, v_n - 1, \ldots, v_n + s_n - n\} \setminus \{x_1, x_2, \ldots, x_n\}$, and $\frac{v_n}{n} \to \chi$ (see (1.18), (2.7) and (2.10)). The above observations imply that $\sup J_{3,n} = \min S_{1,n} \geq \inf S_1 - \delta$ and $\inf J_{3,n} = \max S_{2,n} = \frac{v_n}{n}$ for all $n > n(\delta)$. Then, letting $\delta \downarrow 0$, the result for case (4) follows from (2.11). $\qquad \square$

Moreover:

Lemma 2.6. *The following hold:*

(1) f_n' *has 2 roots in* $B(t, \xi)$*, has 0 roots in* $L_n \setminus (t - \xi, t + \xi)$ *when* $L_n \in \{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$*, and has 1 root in* $L_n \setminus (t - \xi, t + \xi)$ *when* $L_n \in \bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$*.*

(2) f_n' *has 0 roots in* $\mathbb{C} \setminus (\mathbb{R} \cup B(t, \xi))$*, and in each of* $\{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\} \setminus \{L_n\}$*.*

(3) f_n' *has 1 root in each of* $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \setminus \{L_n\}$*.*

Next, denote the roots of f_n' *in* $B(t, \xi)$ *by* $\{t_{1,n}, t_{2,n}\}$*, with the understanding that* $t_{1,n} = t_{2,n}$ *means that* $t_{1,n}$ *is a root of multiplicity* 2*. Then we can always choose the labelling such that one of the following possibilities is satisfied:*

(a) $t_{1,n} \in (t - \xi, t + \xi)$ *and* $t_{1,n} = t_{2,n}$*.*

(b) $\{t_{1,n}, t_{2,n}\} \subset (t - \xi, t + \xi)$ *and* $t_{1,n} > t_{2,n}$*.*

(c) $t_{1,n} \in B(t, \xi) \cap \mathbb{H}$ *and* $t_{2,n}$ *is the complex conjugate of* $t_{1,n}$*.*

Proof. Consider (1)–(3). We will show that:

(i) f_n' has $\sum_{i=1}^3 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}| + 2$ roots in $\mathbb{C} \setminus S_n = (\mathbb{C} \setminus \mathbb{R}) \cup J_n \cup K_n$.

(ii) f_n' an odd number of roots in each of $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$.

(iii) f_n' has 2 roots in $B(t, \xi)$. Moreover, either both are in $(t - \xi, t + \xi)$, or both are in $B(t, \xi) \setminus (t - \xi, t + \xi)$.

Next recall (see (2.12)) that either $L_n \in \{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$ for all n sufficiently large, or $L_n \in \bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$ for all n sufficiently large. We consider both cases separately. First suppose that $L_n \in \{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$. Recall (see (2.12)) that $B(t, \xi) \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_n$. Then parts (i), (ii) and (iii) and a simple counting argument imply that the following is satisfied: f_n' has 2 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup L_n$, 0 roots in $\{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\} \setminus \{L_n\}$, and 1 root in each of $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$. Moreover, since $B(t, \xi) \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_n$, part (iii) implies that the 2 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup L_n$ must be in $B(t, \xi)$. This proves (1)–(3) when $L_n \in \{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$. Next suppose that $L_n \in \bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$. Recall that $B(t, \xi) \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_n$. Then parts (i), (ii) and (iii) and a simple counting argument imply that the following is satisfied: f_n' has 3 roots in $(\mathbb{C} \setminus \mathbb{R}) \cup L_n$, 0 roots in $\{J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$, and 1 root in each of $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \setminus \{L_n\}$. Moreover, since $B(t, \xi) \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_n$, parts (ii) and (iii) imply that 2 of the roots in $(\mathbb{C} \setminus \mathbb{R}) \cup L_n$ must be in $B(t, \xi)$, and 1 of the roots must be in $L_n \setminus (t - \xi, t + \xi)$. This proves (1)–(3) when $L_n \in \bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$.

Consider (i). First recall, (see (2.9)) that $S_n = S_{1,n} \cup S_{2,n} \cup S_{3,n}$, where $\{S_{1,n}, S_{2,n}, S_{3,n}\}$ are mutually disjoint, each set is non-empty and composed of singletons, and $\min S_{1,n} > \max S_{2,n}$ and $\min S_{2,n} > \max S_{3,n}$. Note, for all $w \in \mathbb{C} \setminus S_n$, (2.7) gives $f_n'(w) = \frac{1}{n}(\prod_{x \in S_n} \frac{1}{w-x})Q_n(w)$, where Q_n is the polynomial,

$$Q_n(w) = \sum_{x \in S_{1,n} \cup S_{3,n}} \left(\prod_{y \in S_n \setminus \{x\}} (w - y) \right) - \sum_{x \in S_{2,n}} \left(\prod_{y \in S_n \setminus \{x\}} (w - y) \right).$$

Note that Q_n is a polynomial of degree $|S_n| - 1$. Also note that Q_n has no roots in S_n, and so the roots of Q_n and f_n' coincide. Therefore f_n' has $|S_n| - 1$ roots in $\mathbb{C} \setminus S_n$. Finally recall (see (2.10) and the subsequent remarks) that $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$ is a set of pairwise disjoint open intervals, and that $I_n \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$ if and only if $\inf I_n$ and $\sup I_n$ are two consecutive elements of $S_{i,n}$. Therefore $|S_n| = \sum_{i=1}^3 |\{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}| + 3$, which proves (i).

Consider (ii). Fix $i \in \{1, 2, 3\}$, and any interval $I_n \in \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$, and recall that $\inf I_n$ and $\sup I_n$ are consecutive elements of $S_{i,n}$. Note, when $i = 1$, (2.7) gives

$$\lim_{w \in \mathbb{R}, w \uparrow \sup I_n} f_n'(w) = -\infty \qquad \text{and} \qquad \lim_{w \in \mathbb{R}, w \downarrow \inf I_n} f_n'(w) = +\infty.$$

Thus f_n' has an odd number of roots in I_n. Similarly whenever $i \in \{2, 3\}$. This proves (ii).

Consider (iii). First note, part (1) of Lemma 2.1 and (2.11) imply that f_t' has 2 roots in $B(t, \xi)$. Next note, (2.7) implies that non-real roots of f_n' occur in complex conjugate pairs. Part (iii) thus follows from Rouché's theorem if, we can show that $|f_t'(w)| > |f_t'(w) - f_n'(w)|$ for all $w \in \partial B(t, \xi)$. We shall show:

$$\inf_{w \in \partial B(t,\xi)} |f_t'(w)| > 0 \qquad \text{and} \qquad \lim_{n \to \infty} \sup_{w \in \mathrm{cl}(B(t,\xi))} |f_t'(w) - f_n'(w)| = 0.$$

The first part follows from the extreme value theorem, since f_t' is analytic in $B(t, 2\xi)$ (see (2.11)). We prove the second part via contradiction: Assume that the second part does not hold. Then there exists a $\delta > 0$ for which, for all $n \geq 1$, there exists some $p_n \geq n$ and $z_n \in \mathrm{cl}(B(t, \xi))$ with $\delta < |f_t'(z_n) - f_{p_n}'(z_n)|$. Choosing $\{z_n\}_{n \geq 1}$ to be convergent, and denoting the limit by z, the triangle inequality gives

$$\delta < |f_t'(z_n) - f_t'(z)| + |f_t'(z) - f_{p_n}'(z)| + |f_{p_n}'(z) - f_{p_n}'(z_n)|. \tag{2.13}$$

Note, $|f_t'(z_n) - f_t'(z)| \to 0$ since $z_n \to z$, $\{z, z_1, z_2 \ldots\} \subset \mathrm{cl}(B(t, \xi))$, and f_t' is analytic in $B(t, 2\xi)$. Also, since $z \in \mathrm{cl}(B(t, \xi))$, and $B(t, 2\xi) \subset \mathbb{C} \setminus S$ and $B(t, 2\xi) \subset \mathbb{C} \setminus S_n$ (see (2.11), (2.12)), (2.1), (2.7) and (2.8)) imply that $|f_t'(z) - f_{p_n}'(z)| \to 0$. Finally, (2.7) implies that,

$$|f_{p_n}'(z) - f_{p_n}'(z_n)| \leq \sum_{i=1}^{3} \left(\frac{1}{p_n} |S_{i,p_n}| \right) \sup_{x \in S_{i,p_n}} \left| \frac{1}{z - x} - \frac{1}{z_n - x} \right|.$$

This implies that $|f_{p_n}'(z) - f_{p_n}'(z_n)| \to 0$, since $z_n \to z$, $\{z, z_1, z_2 \ldots\} \subset \mathrm{cl}(B(t, \xi))$, $B(t, 2\xi) \subset \mathbb{C} \setminus S_{p_n}$, and $\frac{1}{p_n} |S_{i,p_n}| = O(1)$ for all $i \in \{1, 2, 3\}$ (see (2.9)). The above observations contradict (2.13), and so our assumption is false. This proves (iii).

Finally, we show that one of the possibilities, (a), (b) or (c), must be satisfied. First recall, part (1) implies that f_n' has 2 roots in $B(t, \xi)$. Next note, (2.7) implies that non-real roots of f_n' occur in complex conjugate pairs. Possibilities (a), (b) and (c) easily follow. $\square$

2.3. **The rates of convergence**

In the previous sections, we assumed Assumptions 1.2 and 1.7, and that (1.18) is satisfied for some fixed $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$, a root of f_t' of multiplicity 2, and we considered the behaviour of the roots of f_t' and f_n' and $\tilde{f}_n'$. We saw that 2 roots of f_n' (and $\tilde{f}_n'$) converge to t as $n \to \infty$. We did not, however, discuss the rate of convergence.

In this section we assume the above, and additionally assume that the sequences of particle positions, $\{(u_n, r_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z}^2$, are chosen as in (1.28) and (1.29). Recall that these are defined in terms of sequences, $\{m_n\}_{n \geq 1} \subset \mathbb{R}$ and $\{p_n\}_{n \geq 1} \subset \mathbb{R}$, which we have yet to define. In this section, we define these in a natural way such that the rate of convergence of the roots is sufficiently fast to allow a steepest descent

analysis of the correlation kernel. The sequences are defined in Definition 2.8, and the rate of convergence of the roots is given in part (4) of Lemma 2.9. In that lemma, we also examine the asymptotic behaviour of $f_n'(t)$ and $\tilde{f}_n'(t)$, and their higher order derivatives. We concentrate mainly on f_n', since $\tilde{f}_n'$ has a similar behaviour.

We begin by writing convenient expressions for f_t' in neighbourhoods of t. First recall that $\mu \leq \lambda$ (see Assumption 1.2), and $(\chi, \eta) = (\chi_{\mathcal{E}}(t), \eta_{\mathcal{E}}(t))$ where $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ (see (1.18)). Next recall (see Definition 1.4 and Lemma 1.6) that $t \in (\chi, +\infty) \setminus \mathrm{Supp}(\mu)$ when $t \in R_\mu^+$, $t \in (\chi + \eta - 1, \chi) \setminus \mathrm{Supp}(\lambda - \mu)$ when $t \in R_{\lambda-\mu}$, and $t \in (-\infty, \chi + \eta - 1) \setminus \mathrm{Supp}(\mu)$ when $t \in R_\mu^-$. Thus the following are satisfied for all $\xi > 0$ sufficiently small:

$$\mu|_{(t-4\xi, t+4\xi)} = 0, (t - 4\xi, t + 4\xi) \subset (\chi, +\infty) \text{ when } t \in R_\mu^+.$$

$$\mu|_{(t-4\xi, t+4\xi)} = \lambda|_{(t-4\xi, t+4\xi)}, (t - 4\xi, t + 4\xi) \subset (\chi + \eta - 1, \chi) \text{ when } t \in R_{\lambda-\mu}. \quad (2.14)$$

$$\mu|_{(t-4\xi, t+4\xi)} = 0, (t - 4\xi, t + 4\xi) \subset (-\infty, \chi + \eta - 1) \text{ when } t \in R_\mu^-.$$

Then, fixing such an $\xi > 0$, (1.10) and (1.30) imply the following, which are well-defined and analytic for all $w \in B(t, 4\xi)$: When $t \in R_\mu^+ \cup R_\mu^-$,

$$f_t'(w) = \int_a^b \frac{\mu[\mathrm{d}x]}{w - x} - \int_{\chi+\eta-1}^\chi \frac{\mathrm{d}x}{w - x}. \quad (2.15)$$

When $t \in R_{\lambda-\mu}$,

$$f_t'(w) = \left[\int_a^{t-4\xi} + \int_{t+4\xi}^b\right]\frac{\mu[\mathrm{d}x]}{w - x} - \left[\int_{\chi+\eta-1}^{t-4\xi} + \int_{t+4\xi}^\chi\right]\frac{\mathrm{d}x}{w - x}. \quad (2.16)$$

Now, we use the above to inspire the definition of natural non-asymptotic functions which have a similar root behaviour in $B(t, 4\xi)$. First, fix $\epsilon > 0$ sufficiently small such that (1.23) is satisfied, and recall that $R_\mu^+(\epsilon) = \{s \in R_\mu^+ : (s - \epsilon, s + \epsilon) \subset R_\mu^+\}$, etc.. Next, fix the above $\xi > 0$ sufficiently small such that the following are also satisfied:

$$(t - 4\xi, t + 4\xi) \subset \begin{cases} R_\mu^+(\epsilon) & \text{when } t \in R_\mu^+. \\ R_{\lambda-\mu}(\epsilon) & \text{when } t \in R_{\lambda-\mu}. \\ R_\mu^-(\epsilon) & \text{when } t \in R_\mu^-. \end{cases} \quad (2.17)$$

Next, define μ_n as in (1.25), and recall that $\mu_n \to \mu$ weakly. Note that $\mathrm{Supp}(\mu_n) \subset P_n \cup \mathrm{cl}(R_{\lambda-\mu}(\epsilon))$, where $P_n = \frac{1}{n}\{x_1, \ldots, x_n\}$. Therefore $\mathrm{Supp}(\mu_n) \subset (a - \epsilon, b + \epsilon)$, since $\mathrm{Supp}(\mu) \subset [a, b]$ (see Assumption 1.2), since $d(P_n, \mathrm{Supp}(\mu)) \to 0$ (see Assumption 1.7), and since $R_{\lambda-\mu}(\epsilon) \subset (a, b)$ (indeed, $R_{\lambda-\mu}(\epsilon) \subset R_{\lambda-\mu} = \mathbb{R} \setminus \mathrm{Supp}(\lambda - \mu) \subset \mathrm{Supp}(\mu)^\circ \subset (a, b)$). Next, define (χ_n, η_n) as in Definition 1.8, and recall that $(\chi_n, \eta_n) \to (\chi, \eta)$ (see (1.27)). Finally, inspired by (2.15) and (2.16), define: When $t \in R_\mu^+ \cup R_\mu^-$,

$$f_{t,n}'(w) := \int_{a-\epsilon}^{b+\epsilon} \frac{\mu_n[\mathrm{d}x]}{w - x} - \int_{\chi_n+\eta_n-1}^{\chi_n} \frac{\mathrm{d}x}{w - x}. \quad (2.18)$$

When $t \in R_{\lambda-\mu}$,

$$f'_{t,n}(w) := \left[\int_{a-\epsilon}^{t-4\xi} + \int_{t+4\xi}^{b+\epsilon} \right] \frac{\mu_n[\mathrm{d}x]}{w-x} - \left[\int_{\chi_n+\eta_n-1}^{t-4\xi} + \int_{t+4\xi}^{\chi_n} \right] \frac{\mathrm{d}x}{w-x}. \qquad (2.19)$$

The function $f_{t,n}$ is unimportant and left unspecified. Note that these functions are well-defined and analytic in $B(t, 4\xi)$. Indeed, when $t \in R_{\mu}^{+}$, the second term on the RHS of (2.18) is well-defined and analytic since $(\chi_n, \eta_n) \to (\chi, \eta)$ and $(t-4\xi, t+4\xi) \subset (\chi, +\infty)$ (see (2.14)). Moreover, when $t \in R_{\mu}^{+}$, note that $(t-4\xi, t+4\xi) \subset \mathbb{R} \setminus \mathrm{Supp}(\mu_n)$, since $\mathrm{Supp}(\mu_n) \subset P_n \cup \mathrm{cl}(R_{\lambda-\mu}(\epsilon))$, since $(t-4\xi, t+4\xi) \subset R_{\mu}^{+}(\epsilon) \subset \mathbb{R} \setminus P_n$ (see (1.24) and (2.17)), and since $(t-4\xi, t+4\xi) \subset R_{\mu}^{+}(\epsilon) \subset \mathbb{R} \setminus \mathrm{cl}(R_{\lambda-\mu}(\epsilon))$ (see (1.23) and (2.17)). Thus the first term on the RHS of (2.18) is also well-defined and analytic in $B(t, 4\xi)$. Similarly the terms on the RHS of (2.18) are well-defined and analytic in $B(t, 4\xi)$ when $t \in R_{\mu}^{-}$, and the terms on the RHS of (2.19) are well-defined and analytic in $B(t, 4\xi)$ when $t \in R_{\lambda-\mu}$. Moreover:

Lemma 2.7. $f'_{t,n}(t) = f''_{t,n}(t) = 0$. Moreover, $f'''_{t,n}(t) \to f'''_{t}(t)$, where $f'''_{t}(t) \neq 0$ (see (1.18)). Finally, t is the unique root of $f'_{t,n}$ in $B(t, \xi)$.

Proof. First suppose that $t \in R_{\mu}^{+}$. Recall, Definition 1.8 gives

$$\chi_n = t + \frac{e^{C_n(t)} - 1}{e^{C_n(t)} C'_n(t)} \qquad \text{and} \qquad \chi_n + \eta_n - 1 = t + \frac{e^{C_n(t)} - 1}{C'_n(t)},$$

where C_n is the Cauchy transform of μ_n given in (1.26). Recall that $(\chi_n, \eta_n) \to (\chi, \eta)$. (2.14) thus gives $(t-4\xi, t+4\xi) \subset (\chi_n, +\infty)$. (1.26) and (2.18) then give $f'_{t,n}(w) = C_n(w) + \log(w - \chi_n) - \log(w - \chi_n - \eta_n + 1)$ for all $w \in B(t, 4\xi)$, where the logarithms use $(-\infty, 0)$ as the branch cut. Finally, taking $w = t$, the above expressions give $f'_{t,n}(t) = f''_{t,n}(t) = 0$. Moreover, recalling that $\mu_n \to \mu$ weakly and $(\chi_n, \eta_n) \to (\chi, \eta)$, (2.15) and (2.18) give $f'''_{t,n}(t) \to f'''_{t}(t) \neq 0$. Finally, using those equations, we can proceed as in part (iii) in the proof of Lemma 2.6 to show that $f'_{t,n}$ has 2 roots in $B(t, \xi)$. Thus t is the unique root of $f'_{t,n}$ in $B(t, \xi)$. This proves the result when $t \in R_{\mu}^{+}$.

Next suppose that $t \in R_{\lambda-\mu}$. Note that the above expressions for χ_n and $\chi_n + \eta_n - 1$ also hold in this case, where now $e^{C_n(t)}$ and $C'_n(t)$ are defined by analytic extensions. More exactly, since $\mu_n = \lambda$ in $(t - 4\xi, t + 4\xi)$ (see (1.25) and (2.17)), Lemma 2.2 of [8] gives

$$e^{C_n(w)} = e^{B_n(w)} \left(\frac{w - t + 4\xi}{w - t - 4\xi} \right), \quad C'_n(w) = B'_n(w) - \frac{1}{w - t - 4\xi} + \frac{1}{w - t + 4\xi},$$

for all $w \in B(t, 4\xi)$, where $B_n(w) := \int_{[a,b]\setminus(t-4\xi,t+4\xi)} \frac{\mu_n[\mathrm{d}x]}{w-x}$. Recall that $(\chi_n, \eta_n) \to (\chi, \eta)$. (2.14) thus gives $(t - 4\xi, t + 4\xi) \subset (\chi_n + \eta_n - 1, \chi_n)$. (2.19) then gives $f'_{t,n}(w) = B_n(w) + \log(w - \chi_n) - \log(w - t - 2\xi) + \log(w - t + 2\xi) - \log(w - \chi_n - \eta_n + 1)$ for all

$w \in B(t, 2\xi)$, where the first and second logarithms use $(0, +\infty)$ as the branch cut, and the third and fourth logarithms use $(-\infty, 0)$ as the branch cut. Finally, taking $w = t$, the above expressions give $f'_{t,n}(t) = f''_{t,n}(t) = 0$. Moreover, recalling that $\mu_n \to \mu$ weakly and $(\chi_n, \eta_n) \to (\chi, \eta)$, (2.16) and (2.19) give $f'''_{t,n}(t) \to f'''_t(t) \neq 0$. Finally, using those equations, we can proceed as in part (iii) in the proof of Lemma 2.6 to show that $f'_{t,n}$ has 2 roots in $B(t, \xi)$. Thus t is the unique root of $f'_{t,n}$ in $B(t, \xi)$. This proves the result when $t \in R_{\lambda-\mu}$. Similarly when $t \in R_\mu^-$. $\qquad\square$

Next, we write similar convenient expressions for f'_n. Recall the definition for f_n given in (1.8). Note, (1.22), (1.24) and (2.17) give $P_n = \frac{1}{n}\{x_1, x_2, \ldots, x_n\}$, $(t-4\xi, t+4\xi) \cap P_n = \emptyset$ when $t \in R_\mu^+ \cup R_\mu^-$, and $(t - 4\xi, t + 4\xi) \cap \frac{\mathbb{Z}}{n} \subset P_n$ when $t \in R_{\lambda-\mu}$. Next define $V_n := \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$. Then, since $\frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ (see (1.18)), (2.14) gives $(t - 4\xi, t + 4\xi) \cap V_n = \emptyset$ when $t \in R_\mu^+ \cup R_\mu^-$, and $(t - 4\xi, t + 4\xi) \cap \frac{\mathbb{Z}}{n} \subset V_n$ when $t \in R_{\lambda-\mu}$. Finally, (1.8) and the above observations imply the following, which are well-defined and analytic for all $w \in B(t, 4\xi)$: When $t \in R_\mu^+ \cup R_\mu^-$,

$$f'_n(w) = \frac{1}{n} \sum_{x \in P_n} \frac{1}{w - x} - \frac{1}{n} \sum_{x \in V_n} \frac{1}{w - x}. \tag{2.20}$$

When $t \in R_{\lambda-\mu}$,

$$f'_n(w) = \frac{1}{n} \sum_{x \in P_n; x \notin (t-4\xi, t+4\xi)} \frac{1}{w - x} - \frac{1}{n} \sum_{x \in V_n; x \notin (t-4\xi, t+4\xi)} \frac{1}{w - x}. \tag{2.21}$$

Finally recall that the sequences $\{(u_n, r_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ of (1.28) and (1.29), depend on sequences $\{m_n\}_{n \geq 1} \subset \mathbb{R}^2$ and $\{p_n\}_{n \geq 1} \subset \mathbb{R}^2$ which we have yet to define. We now define these, along with some other useful sequences:

Definition 2.8. First, recall that $f'''_{t,n}(t) \to f'''_t(t) \neq 0$ (see Lemma 2.7), and define $q_n := q_n(t)$ such that,

$$\tfrac{1}{6} q_n^3 f'''_{t,n}(t) = \tfrac{1}{3}.$$

Next, define $q_{1,n} := q_{1,n}(t)$ and $q_{2,n} := q_{2,n}(t)$ such that,

$$q_n q_{1,n} = \tfrac{1}{2} q_n^2 q_{2,n} = 1.$$

Finally, recall that $e^{C_n(t)} \to e^{C(t)} \notin \{0, 1\}$ (see (1.27) and Lemma 2.3), and $\chi_n \to \chi \neq t$ (see (1.27)), and define $m_n := m_n(t)$ and $p_n := p_n(t)$ such that,

$$q_{1,n} = -\frac{p_n((e^{C_n(t)} - 1)^2 + 1)}{(t - \chi_n)e^{C_n(t)}} \qquad \text{and} \qquad q_{2,n} = \frac{m_n(e^{C_n(t)} - 1)}{(t - \chi_n)^2 e^{C_n(t)}}.$$

Note that each $\{m_n\}_{n \geq 1}$, $\{p_n\}_{n \geq 1}$, $\{q_n\}_{n \geq 1}$, $\{q_{1,n}\}_{n \geq 1}$, $\{q_{2,n}\}_{n \geq 1}$, are convergent sequences of real-numbers with non-zero limits. Moreover:

Lemma 2.9. *Assume that $\{(u_n, r_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ are chosen as in (1.28) and (1.29). Then:*

(1) $f_n'(t) = n^{-\frac{2}{3}} s q_{1,n} + O(n^{-1})$ *and* $\tilde{f}_n'(t) = n^{-\frac{2}{3}} r q_{1,n} + O(n^{-1})$.

(2) $f_n''(t) = n^{-\frac{1}{3}} v q_{2,n} + O(n^{-\frac{2}{3}})$ *and* $\tilde{f}_n''(t) = n^{-\frac{1}{3}} u q_{2,n} + O(n^{-\frac{2}{3}})$.

(3) $f_n'''(t) = f_{t,n}'''(t) + O(n^{-\frac{1}{3}})$ *and* $\tilde{f}_n'''(t) = f_{t,n}'''(t) + O(n^{-\frac{1}{3}})$.

(4) $t = t_{1,n} + O(n^{-\frac{1}{3}}) = t_{2,n} + O(n^{-\frac{1}{3}})$ *and* $t = \tilde{t}_{1,n} + O(n^{-\frac{1}{3}}) = \tilde{t}_{2,n} + O(n^{-\frac{1}{3}})$.

Above, u, v, r, s are the parameters in (1.28) and (1.29), $\{t_{1,n}, t_{2,n}\}$ denotes the set of roots of f_n' in $B(t, \xi)$ (see Lemma 2.6), and $\{\tilde{t}_{1,n}, \tilde{t}_{2,n}\}$ denotes the analogous roots of $\tilde{f}_n'$.

Proof. We prove the results for f_n, and state that the results for $\tilde{f}_n$ follow similarly. Consider (1), (2) and (3) when $t \in R_\mu^+$. First note, (1.25), (2.18) and (2.20) give

$$f_{t,n}'(w) - f_n'(w) = \int_{R_{\lambda-\mu}(\epsilon)} \frac{dx}{w - x} - \frac{1}{n} \sum_{x \in P_n; x \in R_{\lambda-\mu}(\epsilon)} \frac{1}{w - x}$$
$$- \int_{\chi_n + \eta_n - 1}^{\chi_n} \frac{dx}{w - x} + \frac{1}{n} \sum_{x \in V_n} \frac{1}{w - x},$$

for all $w \in B(t, 4\xi)$, where $V_n = \frac{1}{n} \{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$. Next recall (see (2.17)) that $(t - 4\xi, t + 4\xi) \subset R_\mu^+(\epsilon) \subset \mathbb{R} \setminus R_{\lambda-\mu}(\epsilon)$, and (see (1.24)) that $\frac{\mathbb{Z}}{n} \cap R_{\lambda-\mu}(\epsilon) \subset P_n$. Riemann approximations thus give

$$f_{t,n}'(w) - f_n'(w) = \int_{\frac{v_n}{n} + \frac{s_n}{n} - 1}^{\frac{v_n}{n}} \frac{dx}{w - x} - \int_{\chi_n + \eta_n - 1}^{\chi_n} \frac{dx}{w - x} + O(n^{-1}),$$

$$f_{t,n}''(w) - f_n''(w) = -\int_{\frac{v_n}{n} + \frac{s_n}{n} - 1}^{\frac{v_n}{n}} \frac{dx}{(w - x)^2} + \int_{\chi_n + \eta_n - 1}^{\chi_n} \frac{dx}{(w - x)^2} + O(n^{-1}), \quad (2.22)$$

$$f_{t,n}'''(w) - f_n''(w) = \int_{\frac{v_n}{n} + \frac{s_n}{n} - 1}^{\frac{v_n}{n}} \frac{2dx}{(w - x)^3} - \int_{\chi_n + \eta_n - 1}^{\chi_n} \frac{2dx}{(w - x)^3} + O(n^{-1}),$$

uniformly for $w \in B(t, \xi)$. Recall that $\frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ and $(\chi_n, \eta_n) \to (\chi, \eta)$, and $t \in (\chi, +\infty)$. Therefore,

$$f'_{t,n}(t) - f'_n(t) = -\log\left(\frac{t - \frac{v_n}{n}}{t - \chi_n}\right) + \log\left(\frac{t - \frac{v_n}{n} - \frac{s_n}{n} + 1}{t - \chi_n - \eta_n + 1}\right) + O(n^{-1}),$$

$$f''_{t,n}(t) - f''_n(t) = -\frac{\frac{v_n}{n} - \chi_n}{(t - \chi_n)(t - \frac{v_n}{n})}$$
$$+ \frac{\frac{v_n}{n} + \frac{s_n}{n} - \chi_n - \eta_n}{(t - \chi_n - \eta_n + 1)(t - \frac{v_n}{n} - \frac{s_n}{n} + 1)} + O(n^{-1}),$$

$$f'''_{t,n}(t) - f'''_n(t) = \frac{(\frac{v_n}{n} - \chi_n)((t - \chi_n) + (t - \frac{v_n}{n}))}{(t - \chi_n)^2(t - \frac{v_n}{n})^2}$$
$$- \frac{(\frac{v_n}{n} + \frac{s_n}{n} - \chi_n - \eta_n)((t - \chi_n - \eta_n + 1) + (t - \frac{v_n}{n} - \frac{s_n}{n} + 1))}{(t - \chi_n - \eta_n + 1)^2(t - \frac{v_n}{n} - \frac{s_n}{n} + 1)^2} + O(n^{-1}),$$

where log now represents the natural logarithm. (1.29) thus gives

$$f'_{t,n}(t) - f'_n(t) = -\log\left(1 - \frac{n^{-\frac{1}{3}}m_n v + n^{-\frac{2}{3}}p_n s(e^{C_n(t)} - 1) + O(n^{-1})}{t - \chi_n}\right)$$
$$+ \log\left(1 - \frac{n^{-\frac{1}{3}}m_n v e^{C_n(t)} + n^{-\frac{2}{3}}p_n s(e^{C_n(t)} - 2) + O(n^{-1})}{t - \chi_n - \eta_n + 1}\right) + O(n^{-1}),$$

$$f''_{t,n}(t) - f''_n(t) = -\frac{n^{-\frac{1}{3}}m_n v + O(n^{-\frac{2}{3}})}{(t - \chi_n)((t - \chi_n) + O(n^{-\frac{1}{3}}))}$$
$$+ \frac{n^{-\frac{1}{3}}m_n v e^{C_n(t)} + O(n^{-\frac{2}{3}})}{(t - \chi_n - \eta_n + 1)((t - \chi_n - \eta_n + 1) + O(n^{-\frac{1}{3}}))} + O(n^{-1}),$$

$$f'''_{t,n}(t) - f'''_n(t) = O(n^{-\frac{1}{3}}),$$

The third equation gives (3) when $t \in R_\mu^+$. Also note, that $t - \chi_n - \eta_n + 1 = (t - \chi_n)e^{C_n(t)}$ (see Definition 1.8). The first and second equations thus give

$$f'_{t,n}(t) - f'_n(t) = -\log\left(1 - \frac{n^{-\frac{1}{3}}m_n v}{t - \chi_n} - \frac{n^{-\frac{2}{3}}p_n s(e^{C_n(t)} - 1)}{t - \chi_n} + O(n^{-1})\right)$$
$$+ \log\left(1 - \frac{n^{-\frac{1}{3}}m_n v}{t - \chi_n} - \frac{n^{-\frac{2}{3}}p_n s(e^{C_n(t)} - 2)}{(t - \chi_n)e^{C_n(t)}} + O(n^{-1})\right) + O(n^{-1}),$$

$$f''_{t,n}(t) - f''_n(t) = -\frac{n^{-\frac{1}{3}}m_n v}{(t - \chi_n)^2} + \frac{n^{-\frac{1}{3}}m_n v}{(t - \chi_n)^2 e^{C_n(t)}} + O(n^{-\frac{2}{3}}).$$

Then, since $f''_{t,n}(t) = 0$ (see Lemma 2.7) the second equation and Definition 2.8 give (2) when $t \in R_\mu^+$. Also, since $f'_{t,n}(t) = 0$ (see Lemma 2.7) the first equation, Definition 2.8

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and a Taylor expansion give (1) when $t \in R_\mu^+$. We can similarly prove (1), (2) and (3) when $t \in R_\mu^-$.

Consider (1), (2) and (3) when $t \in R_{\lambda-\mu}$. First recall (see (1.24) and (2.17)) that $(t - 4\xi, t + 4\xi) \subset R_{\lambda-\mu}(\epsilon)$ and $\frac{\mathbb{Z}}{n} \cap R_{\lambda-\mu}(\epsilon) \subset P_n$. Also, recall (see (2.14)) that $(t - 4\xi, t + 4\xi) \subset (\chi + \eta - 1, \chi)$, and that $\frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ and $(\chi_n, \eta_n) \to (\chi, \eta)$. Next note, (1.25), (2.19) and (2.21) give

$$f'_{t,n}(w) - f'_n(w) = \int_{R_{\lambda-\mu}(\epsilon)\backslash(t-4\xi,t+4\xi)} \frac{\mathrm{d}x}{w - x} - \frac{1}{n} \sum_{x \in P_n; x \in R_{\lambda-\mu}(\epsilon)\backslash(t-4\xi,t+4\xi)} \frac{1}{w - x}$$
$$- \int_{[\chi_n+\eta_n-1,\chi_n]\backslash(t-4\xi,t+4\xi)} \frac{\mathrm{d}x}{w - x} + \frac{1}{n} \sum_{x \in V_n; x \notin (t-4\xi,t+4\xi)} \frac{1}{w - x},$$

for all $w \in B(t, 4\xi)$, where $V_n = \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$. Riemann approximations thus give

$$f'_{t,n}(w) - f'_n(w) = -\int_{[\chi_n+\eta_n-1,\chi_n]\backslash(t-4\xi,t+4\xi)} \frac{\mathrm{d}x}{w - x}$$
$$+ \int_{[\frac{v_n}{n}+\frac{s_n}{n}-1, \frac{v_n}{n}]\backslash(t-4\xi,t+4\xi)} \frac{\mathrm{d}x}{w - x} + O(n^{-1}), \quad (2.23)$$

uniformly for $w \in B(t, \xi)$. We can then proceed similarly to above to prove (1), (2) and (3) when $t \in R_{\lambda-\mu}$.

Consider (4). First recall, that f'_n and $f'_{t,n}$ are well-defined and analytic in $B(t, 4\xi)$. Also, Lemma 2.7 implies that t is a root of $f'_{t,n}$ of multiplicity 2, and t is the unique root of $f'_{t,n}$ in $B(t, \xi)$. We will show that there exists constants $c_1, c_2 > 0$ for which, given any $\{\xi_n\}_{n\geq 1}$ with $\xi_n \downarrow 0$,

$$\inf_{w \in \partial B(t,\xi_n)} |f'_{t,n}(w)| > c_1 \xi_n^2,$$
$$\sup_{w \in \mathrm{cl}(B(t,\xi_n))} |f'_{t,n}(w) - f'_n(w)| < c_2(\xi_n n^{-\frac{1}{3}} + n^{-\frac{2}{3}}). \quad (2.24)$$

For clarity we state that c_1, c_2 are independent of the choice of $\{\xi_n\}_{n\geq 1}$. Thus there exists a choice of $\{\xi_n\}_{n\geq 1}$ with $\xi_n \sim O(n^{-\frac{1}{3}})$ for which, for all n sufficiently large, $|f'_{t,n}(w)| > |f'_{t,n}(w) - f'_n(w)|$ for all $w \in \partial B(t, \xi_n)$. Part (4) then follows from Rouché's theorem.

Fix $\{\xi_n\}_{n\geq 1}$ with $\xi_n \downarrow 0$. Recall, Lemma 2.7 implies that $f'_{t,n}(t) = f''_{t,n}(t) = 0$, and $f'''_{t,n}(t) \to f'''_t(t) \neq 0$. Also note, (2.18) and (2.19) give $|f^{(4)}_{t,n}(w)| = O(1)$ uniformly for $w \in B(t, \xi)$. Taylor's theorem thus gives $f'_{t,n}(w) = \frac{1}{2}(w - t)^2 f'''_{t,n}(t) + O(\xi_n^3)$ uniformly for

$w \in \mathrm{cl}(B(w, \xi_n))$. Therefore,

$$f_{t,n}''(w) = \xi_n^2 \frac{1}{2} e^{i2 \operatorname{Arg}(w-t)} f_{t,n}'''(t) + O(\xi_n^3),$$

uniformly for $w \in \partial B(t, \xi_n)$. This proves the first part of (2.24).

Consider the second part of (2.24) when $t \in R_\mu^+$. Note, (2.22) and Taylor's theorem give

$$f_{t,n}'(w) - f_n'(w) = -\log\left(\frac{t - \frac{v_n}{n}}{t - \chi_n}\right) + \log\left(\frac{t - \frac{v_n}{n} - \frac{s_n}{n} + 1}{t - \chi_n - \eta_n + 1}\right)$$
$$+ O\left(|w - t|\left|\chi_n - \frac{v_n}{n}\right| + |w - t|\left|\eta_n - \frac{s_n}{n}\right| + n^{-1}\right),$$

uniformly for $w \in B(t, \xi)$, where log now represents the natural logarithm. Proceeding similarly to part (1) then gives

$$f_{t,n}'(w) - f_n'(w) = O\left(|w - t|n^{-\frac{1}{3}} + n^{-\frac{2}{3}}\right),$$

uniformly for $w \in B(t, \xi)$. This proves the second part of (2.24) when $t \in R_\mu^+$. Similarly when $t \in R_\mu^-$. Finally, the second part of (2.24) when $t \in R_{\lambda - \mu}$ can be shown using similar arguments and (2.23). $\square$

2.4. The asymptotic behaviour of $f_n - \tilde{f}_n$

In this section, we assume Assumptions 1.2 and 1.7, that (1.18) is satisfied for some fixed $t \in R_\mu^+ \cup R_{\lambda - \mu} \cup R_\mu^-$ (a root of f_t' of multiplicity 2), and that $\{(u_n, r_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n \geq 1} \subset \mathbb{Z}^2$ are chosen as in (1.28) and (1.29). Define,

$$F_n := f_n - \tilde{f}_n. \tag{2.25}$$

This function will be useful for the steepest descent analysis. In this section, we examine the roots of F_n', and the asymptotic behaviour of F_n as $n \to \infty$. Note, since much of the analysis of this section is similar to that of the previous two sections, we do not go into as much detail here.

First, it is useful to examine the following functions: Define $G_t, G_{t,n} : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ as,

$$G_t(w) := -\log(w - \chi) + e^{C(t)} \log(w - \chi - \eta + 1),$$
$$G_{t,n}(w) := -\log(w - \chi_n) + e^{C_n(t)} \log(w - \chi_n - \eta_n + 1), \tag{2.26}$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where C and C_n are defined in (1.14) and (1.26) respectively, $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$ and $(\chi_n, \eta_n) = (\chi_n(t), \eta_n(t))$ (see (1.18) and Definition 1.8), and the branch cuts are chosen as follows:

- For cases (1)–(4) of Lemma 2.2, all branch cuts are $(-\infty, 0]$.

- For cases (5)–(8) of Lemma 2.2, the branch cut in the 1st terms on the RHS is $[0, +\infty)$, and branch cut in the 2nd terms is $(-\infty, 0]$.

- For cases (9)–(12) of Lemma 2.2, all branch cuts are $[0, +\infty)$.

Recall that $e^{C_n(t)} \to e^{C(t)}$ and $(\chi_n, \eta_n) \to (\chi, \eta)$ (see (1.27)). The above choices for the branches, Lemma 2.2 and (2.14), imply that G_t and $G_{t,n}$ both extend analytically to $(\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$.

Next note, irrespective of the choices of the branches of the logarithms,

$$G_t'(w) = -\frac{1}{w - \chi} + \frac{e^{C(t)}}{w - \chi - \eta + 1}, \tag{2.27}$$

$$G_{t,n}'(w) = -\frac{1}{w - \chi_n} + \frac{e^{C_n(t)}}{w - \chi_n - \eta_n + 1}, \tag{2.28}$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$. Therefore, G_t' and $G_{t,n}'$ extend analytically to $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$ and $\mathbb{C} \setminus \{\chi_n, \chi_n + \eta_n - 1\}$, respectively. Moreover:

Lemma 2.10. G_t' *has a root of multiplicity* 1 *at* t, *and no other roots in* $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$. *Moreover,* $G_{t,n}'$ *has a root of multiplicity* 1 *at* t, *and no other roots in* $\mathbb{C} \setminus \{\chi_n, \chi_n + \eta_n - 1\}$. *Finally,* $G_{t,n}''(t) \to G_t''(t)$.

Proof. Consider G_t'. Note, since $(\chi, \eta) = (\chi_\mathcal{E}(t), \eta_\mathcal{E}(t))$, (1.15) and (2.27) imply that t is the only root of G_t' in $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$. These also give

$$G_t''(t) = \frac{e^{C(t)} C'(t)^2}{e^{C(t)} - 1}.$$

Lemma 2.3 then implies that $G_t''(t) \neq 0$.

Consider $G_{t,n}'$. Note, since $(\chi_n, \eta_n) = (\chi_n(t), \eta_n(t))$, Definition 1.8 and (2.28) imply that t is the only root of $G_{t,n}'$ in $\mathbb{C} \setminus \{\chi_n, \chi_n + \eta_n - 1\}$. These also give

$$G_{t,n}''(t) = \frac{e^{C_n(t)} C_n'(t)^2}{e^{C_n(t)} - 1}.$$

(1.27) then gives $G_{t,n}''(t) \to G_t''(t) \neq 0$. $\qquad\square$

Next consider F_n. Define, for convenience,

$$\begin{aligned} U_n &:= \tfrac{1}{n}\{u_n + r_n - n + 1, u_n + r_n - n + 2, \dots, u_n - 1\}, \\ V_n &:= \tfrac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \dots, v_n\}. \end{aligned} \tag{2.29}$$

Thus, since $\eta \in (0, 1)$ and $\frac{1}{n}(u_n, r_n) \to (\chi, \eta)$ and $\frac{1}{n}(v_n, s_n) \to (\chi, \eta)$ (see (1.18)), $\min\{u_n - 1, v_n\} > \max\{u_n + r_n - n + 1, v_n + s_n - n\}$. Moreover,

$$V_n \setminus U_n = (VU^{(n)}) \cup (VU_{(n)}) \qquad \text{and} \qquad U_n \setminus V_n = (UV^{(n)}) \cup (UV_{(n)}), \tag{2.30}$$

where we define:

- $VU^{(n)} := \frac{1}{n}\{u_n, u_n + 1, \ldots, v_n\}$ when $v_n \geq u_n$.

- $UV^{(n)} := \frac{1}{n}\{v_n + 1, v_n + 2, \ldots, u_n - 1\}$ when $v_n + 1 \leq u_n - 1$.

- $VU_{(n)} := \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, u_n + r_n - n\}$ when $v_n + s_n \leq u_n + r_n$.

- $UV_{(n)} := \frac{1}{n}\{u_n + r_n - n + 1, u_n + r_n - n + 2, \ldots, v_n + s_n - n - 1\}$ when $v_n + s_n - 1 \geq u_n + r_n + 1$.

Note, implicit in the above definitions is that $VU^{(n)} := \emptyset$ when $v_n < u_n$, etc. Finally, fixing $\xi > 0$ sufficiently small such that (2.11), (2.14) and (2.17) are satisfied, and such that $\chi - 4\xi > \chi + \eta - 1 + 4\xi$, (1.18) gives:

$$VU^{(n)} \subset (\chi - 2\xi, \chi + 2\xi) \quad \text{and} \quad VU_{(n)} \subset (\chi + \eta - 1 - 2\xi, \chi + \eta - 1 + 2\xi). \quad (2.31)$$

Similarly for $UV^{(n)}$ and $UV_{(n)}$. Finally note that (1.8), (1.9), (2.25), (2.29) and (2.30) give

$$nF_n(w) = \left(1_{(v_n+1 \leq u_n-1)} \sum_{x \in UV^{(n)}} -1_{(v_n \geq u_n)} \sum_{x \in VU^{(n)}}\right) \log(w - x)$$

$$+ \left(1_{(v_n+s_n-1 \geq u_n+r_n+1)} \sum_{x \in UV_{(n)}} -1_{(v_n+s_n \leq u_n+r_n)} \sum_{x \in VU_{(n)}}\right) \log(w - x), \quad (2.32)$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where the branch cuts are chosen as follows:

- For cases (1)–(4) of Lemma 2.2, all branch cuts are $(-\infty, 0]$.

- For cases (5)–(8) of Lemma 2.2, the branch cuts in the 1st and 2nd sums on the RHS are all $[0, +\infty)$, and the branch cuts in the 3rd and 4th sums are all $(-\infty, 0]$.

- For cases (9)–(12) of Lemma 2.2, all branch cuts are $[0, +\infty)$.

Note, the above branch cut choices are consistent with those made in (1.8) and (1.9) (see the discussion given before Lemma 3.1). Also note, Lemma 2.2, and (2.11), (2.14), (2.17) and (2.31), imply that F_n extends analytically to $(\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$.

Note, irrespective of the choices of the branches of the logarithms,

$$nF_n'(w) = \left(1_{(v_n+1 \leq u_n-1)} \sum_{x \in UV^{(n)}} -1_{(v_n \geq u_n)} \sum_{x \in VU^{(n)}}\right) \frac{1}{w - x}$$

$$+ \left(1_{(v_n+s_n-1 \geq u_n+r_n+1)} \sum_{x \in UV_{(n)}} -1_{(v_n+s_n \leq u_n+r_n)} \sum_{x \in VU_{(n)}}\right) \frac{1}{w - x}, \quad (2.33)$$

for all $(\mathbb{C}\setminus\mathbb{R})\cup(t-2\xi, t+2\xi)$. Therefore F_n' extends analytically to $\mathbb{C}\setminus((V_n\setminus U_n)\cup(U_n\setminus V_n))$. Next we investigate the relationships between G_t', $G_{t,n}'$ and F_n' in $B(t,\xi)$:

Lemma 2.11. *Fix $\xi > 0$ as above. Then:*

(1) $G_{t,n}'(w) = G_t'(w) + o(1)$ *uniformly for $w \in B(t,\xi)$. Similarly for $G_{t,n}''$, G_t''.*

Next, fix $\{(u_n, r_n)\}_{n\geq1}$, $\{(v_n, s_n)\}_{n\geq1}$, $\{m_n\}_{n\geq1}$, u, v as in (1.28) and (1.29). Additionally assume that $u \neq v$. Then:

(2) $n^{\frac{1}{3}} F_n'(w) = m_n(v - u)G_{t,n}'(w) + O(n^{-\frac{1}{3}})$ *uniformly for $w \in B(t,\xi)$. Similarly for $G_{t,n}''$, F_n''.*

Proof. Consider (1). We will prove (1) only for $G_{t,n}'$, G_t'. Part (1) for $G_{t,n}''$, G_t'' follows similarly. First note, (2.27) and (2.28) give

$$G_{t,n}'(w) = G_t'(w) - \frac{\chi_n - \chi}{(w - \chi)(w - \chi_n)}$$
$$+ \frac{e^{C_n(t)} - e^{C(t)}}{w - \chi_n - \eta_n + 1} + \frac{e^{C(t)}(\chi_n + \eta_n - \chi - \eta)}{(w - \chi - \eta + 1)(w - \chi_n - \eta_n + 1)},$$

for all $w \in B(t,\xi)$. Recall (see (1.27)) that $e^{C_n(t)} \to e^{C(t)}$ and $(\chi_n, \eta_n) \to (\chi, \eta)$. Also recall (see (2.14)) that $|w - \chi| > 3\xi$ and $|w - \chi - \eta + 1| > 3\xi$ for all $w \in B(t,\xi)$. Combined, the above prove (1) for $G_{t,n}'$, G_t'.

Consider (2). Note, since $u \neq v$, (1.28), (1.29) and (2.30) imply that either $VU^{(n)} \neq \emptyset$ and $UV^{(n)} = \emptyset$ for all n sufficiently large, or $VU^{(n)} = \emptyset$ and $UV^{(n)} \neq \emptyset$ for all n sufficiently large. Similarly for $VU_{(n)}$ and $UV_{(n)}$. Moreover, these sets contain at least 2 distinct elements, whenever they are non-empty.

First suppose that $UV^{(n)} \neq \emptyset$ and $VU_{(n)} \neq \emptyset$ and $UV_{(n)} = VU^{(n)} = \emptyset$. For this case, (2.33) gives

$$nF_n'(w) = \sum_{x \in UV^{(n)}} \frac{1}{w - x} - \sum_{x \in VU_{(n)}} \frac{1}{w - x},$$

for all $w \in B(t,\xi)$. We write this as,

$$nF_n'(w) = \sum_{x \in UV^{(n)}} \left(\frac{1}{w - \chi_n} - \frac{\chi_n - x}{(w - \chi_n)(w - x)} \right)$$
$$- \sum_{x \in VU_{(n)}} \left(\frac{1}{w - \chi_n - \eta_n + 1} - \frac{\chi_n + \eta_n - 1 - x}{(w - \chi_n - \eta_n + 1)(w - x)} \right),$$

for all $w \in B(t,\xi)$. Note, (1.27) and (2.14) imply that $|w - \chi_n| > \xi$ and $|w - \chi_n - \eta_n + 1| > \xi$ uniformly for $w \in B(t,\xi)$. Also note, (2.14) and (2.31) imply that $|w - x| > \xi$ uniformly

for $w \in B(t, \xi)$ and $x \in UV^{(n)}$, and $|w - x| > \xi$ uniformly for $w \in B(t, \xi)$ and $x \in VU_{(n)}$. Moreover, since $u \neq v$, (1.28), (1.29) and (2.30), imply the following:

- $\chi_n - x = O(n^{-\frac{1}{3}})$ uniformly for $x \in UV^{(n)}$.

- $\chi_n + \eta_n - 1 - x = O(n^{-\frac{1}{3}})$ uniformly for $x \in VU_{(n)}$.

- $|UV^{(n)}| = O(n^{\frac{2}{3}})$ and $|VU_{(n)}| = O(n^{\frac{2}{3}})$.

Combined, the above give

$$nF'_n(w) = \sum_{x \in UV^{(n)}} \frac{1}{w - \chi_n} - \sum_{x \in VU_{(n)}} \frac{1}{w - \chi_n - \eta_n + 1} + O(n^{\frac{1}{3}}),$$

uniformly for $w \in B(t, \xi)$. Finally note, (1.28), (1.29) and (2.30) give $|UV^{(n)}| = n^{\frac{2}{3}} m_n(u - v) + O(n^{\frac{1}{3}})$ and $|VU_{(n)}| = n^{\frac{2}{3}} m_n e^{C_n(t)}(u - v) + O(n^{\frac{1}{3}})$. Therefore,

$$nF'_n(w) = \frac{n^{\frac{2}{3}} m_n(u - v)}{w - \chi_n} - \frac{n^{\frac{2}{3}} m_n e^{C_n(t)}(u - v)}{w - \chi_n - \eta_n + 1} + O(n^{\frac{1}{3}}),$$

uniformly for $w \in B(t, \xi)$. (2.28) finally proves (2) for $F'_n, G'_{t,n}$, when $UV^{(n)} \neq \emptyset$ and $VU_{(n)} \neq \emptyset$ and $UV_{(n)} = VU^{(n)} = \emptyset$. Part (2) for the other cases follows similarly. $\square$

Next we use the previous lemmas to investigate the roots of F'_n when $u \neq v$.

Lemma 2.12. *Fix $\xi > 0$ as above, and assume that $u \neq v$. Then:*

(1) *F'_n has 1 root in $(t - \xi, t + \xi)$.*

(2) *F'_n has 0 roots in $\mathbb{C} \setminus \mathbb{R}$.*

(3) *F'_n has 1 root in each interval of the form (x, y), when x and y are any two consecutive elements of either $VU^{(n)}$ or $UV^{(n)}$ or $VU_{(n)}$ or $UV_{(n)}$.*

(4) *F'_n has no other roots in $\mathbb{R}$ excluding those listed in parts (1,3).*

(5) *$w_n = t + O(n^{-\frac{1}{3}})$, where w_n denotes the root in part (1), above.*

Proof. Consider (1). First, recall that $u \neq v$, and let $m_t \neq 0$ denote the non-zero limit of the sequence $\{m_n\}_{n \geq 1}$ of real-numbers (see Definition 2.8). Next recall (see Lemma 2.10) that t is the only root of G'_t in $B(t, \xi)$. The extremal value theorem thus gives

$$\inf_{w \in \partial B(t, \xi)} |(v - u) m_t G'_t(w)| > 0.$$

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Next note, parts (1) and (2) of Lemma 2.11 give $n^{\frac{1}{3}} F_n'(w) = (v - u) m_t G_t'(w) + o(1)$ uniformly for $w \in B(t, \xi)$. Combined, the above imply that $|(v - u) m_t G_t'(w)| > |(v - u) m_t G_t'(w) - n^{\frac{1}{3}} F_n'(w)|$ for all $w \in \partial B(t, \xi)$. Rouché's theorem thus implies that G_t' and F_n' have the same number of roots in $B(t, \xi)$. Lemma 2.10 thus implies that F_n' has 1 root in $B(t, \xi)$. This root is necessarily real-valued since non-real roots of F_n' occur in complex conjugate pairs (see (2.33)). This proves (1).

Consider (2)–(4). As in the proof of part (2) of Lemma 2.11, we will prove these only when $UV^{(n)} \neq \emptyset$ and $VU_{(n)} \neq \emptyset$ and $UV_{(n)} = VU^{(n)} = \emptyset$. Part (2-4) for the other cases follows similarly. Note, for the above case, (2.33) gives

$$ nF_n'(w) = \sum_{x \in UV^{(n)}} \frac{1}{w - x} - \sum_{x \in VU_{(n)}} \frac{1}{w - x}, \tag{2.34} $$

for all $w \in \mathbb{C} \setminus ((UV^{(n)}) \cup (VU_{(n)}))$. Recall, since $u \neq v$, that $UV^{(n)}$ and $VU_{(n)}$ both contain at least 2 elements. Also recall (see (2.14) and (2.31)) that $(t - \xi, t + \xi)$ and $UV^{(n)}$ and $VU_{(n)}$ are mutually disjoint. We will show:

  (i) F_n' has $|UV^{(n)}| + |VU_{(n)}| - 1$ roots in $\mathbb{C} \setminus ((UV^{(n)}) \cup (VU_{(n)}))$.

  (ii) F_n' has at least 1 root in each interval of the form (x, y), where x and y are any two consecutive elements of either $UV^{(n)}$ or $VU_{(n)}$.

Finally recall (see part (1)) that F_n' has 1 root in $(t - \xi, t + \xi)$. Combined, the above imply that F_n' has 1 root in $(t - \xi, t + \xi)$, 1 in each of the intervals listed in part (ii), and no other roots. This proves parts (2)–(4) in this case.

Consider (i). First note (2.34) gives

$$ F_n'(w) = \frac{1}{n} \left(\prod_{y \in (UV^{(n)}) \cup (VU_{(n)})} \frac{1}{w - y} \right) P_n(w), $$

for all $w \in \mathbb{C} \setminus ((UV^{(n)}) \cup (VU_{(n)}))$, where P_n is the polynomial,

$$ P_n(w) = \sum_{x \in UV^{(n)}} \left(\prod_{y \in ((UV^{(n)}) \setminus \{x\}) \cup VU_{(n)}} (w - y) \right) $$

$$ - \sum_{x \in VU_{(n)}} \left(\prod_{y \in (UV^{(n)}) \cup ((VU_{(n)}) \setminus \{x\})} (w - y) \right). $$

Note that P_n has degree at most $|UV^{(n)}| + |VU_{(n)}| - 1$. Moreover, since $UV^{(n)}$ and $VU_{(n)}$ are disjoint, P_n has no roots in $(UV^{(n)}) \cup (VU_{(n)})$. Therefore the roots of P_n and F_n' coincide. This proves (i).

Consider (ii). Let x and y denote any two consecutive elements of $UV^{(n)}$. (2.34) then implies that F'_n is a real-valued continuous function on (x, y), and

$$\lim_{w \in (x,y), w \uparrow y} F'_n(w) = -\infty \qquad \text{and} \qquad \lim_{w \in (x,y), w \downarrow x} F'_n(w) = +\infty.$$

The intermediate value theorem thus implies that F'_n has a root in (x, y). Similarly F'_n has a root in (x, y), when x and y denote any two consecutive elements of $VU_{(n)}$. This proves (ii).

Consider (5). First recall (see Lemma 2.10) that $G'_{t,n}$ has a root of multiplicity 1 at t, and no other roots in $\mathbb{C} \setminus \{\chi_n, \chi_n + \eta_n - 1\}$, and $G''_{t,n}(t) \to G''_t(t) \neq 0$. Next, recall that $v - u \neq 0$ (by assumption) and that $\{m_n\}_{n \geq 1}$ is a convergent sequence of real-numbers with a non-zero limit (see Definition 2.8). Then, using part (2) of Lemma 2.11, we can proceed similarly to the proof of part (4) of Lemma 2.9 to show the following: There exists a sequence $\{\xi_n\}_{n \geq 1}$ of positive numbers for which $\xi_n = O(n^{-\frac{1}{3}})$ and $|(v - u)m_n G'_{t,n}(w)| > |(v - u)m_n G'_{t,n}(w) - n^{\frac{1}{3}} F'_n(w)|$ for all $w \in \partial B(t, \xi_n)$. Rouché's theorem thus implies that F'_n and $G'_{t,n}$ have the same number of roots in $B(t, \xi_n)$ for this choice of ξ_n, i.e., 1 root. Thus the root w_n of F'_n, discovered in part (1), must satisfy $w_n \in B(t, \xi_n)$. This proves (5). □

We end this section by examining the asymptotic behaviour, as $n \to \infty$, of $F_n(t)$:

Lemma 2.13. *We have,*

$$\exp(nF_n(t)) = \frac{A_{t,n}((u_n, r_n), (v_n, s_n))}{(t - \chi_n)(t - \chi_n - \eta_n + 1)} \exp(O(n^{-\frac{1}{3}})),$$

where $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ is defined by:

$$A_{t,n}((U, R), (V, S)) := (t - \chi_n)^{-(V-U)}(t - \chi_n - \eta_n + 1)^{V+S-U-R}$$
$$\times \exp\left[\frac{n}{2} \frac{(\frac{V}{n} - \chi_n)^2 - (\frac{U}{n} - \chi_n)^2}{t - \chi_n} + \frac{n}{6} \frac{(\frac{V}{n} - \chi_n)^3 - (\frac{U}{n} - \chi_n)^3}{(t - \chi_n)^2}\right.$$
$$- \frac{n}{2} \frac{(\frac{V}{n} + \frac{S}{n} - \chi_n - \eta_n)^2 - (\frac{U}{n} + \frac{R}{n} - \chi_n - \eta_n)^2}{t - \chi_n - \eta_n + 1}$$
$$\left. - \frac{n}{6} \frac{(\frac{V}{n} + \frac{S}{n} - \chi_n - \eta_n)^3 - (\frac{U}{n} + \frac{R}{n} - \chi_n - \eta_n)^3}{(t - \chi_n - \eta_n + 1)^2}\right],$$

for all $(U, R), (V, S) \in \mathbb{Z}^2$.

Proof. We will prove this result when $VU^{(n)} \neq \emptyset$ and $VU_{(n)} \neq \emptyset$ and $UV^{(n)} = UV_{(n)} = \emptyset$, and state that the result for the other cases follows from similar considerations.

Assume the above case. Then, irrespective of the choices of the branches of the logarithms, (2.30) and (2.32) give

$$\exp(nF_n(t)) = \left[\prod_{j=u_n}^{v_n} (t - \tfrac{j}{n}) \right]^{-1} \left[\prod_{k=v_n+s_n-n}^{u_n+r_n-n} (t - \tfrac{k}{n}) \right]^{-1}. \qquad (2.35)$$

To examine this, first write,

$$\prod_{j=u_n}^{v_n} (t - \tfrac{j}{n}) = (t - \chi_n)^{v_n-u_n+1} \prod_{j=u_n}^{v_n} \left(1 - \frac{\tfrac{j}{n} - \chi_n}{t - \chi_n} \right).$$

Note that $\tfrac{j}{n} - \chi_n = O(n^{-\frac{1}{3}})$ uniformly for $j \in \{u_n, u_n + 1, \ldots, v_n\}$ (see (1.28) and (1.29)) and that $|t - \chi_n| > 2\xi > 0$ (see (2.14), and recall that $\chi_n \to \chi$). Thus, we can write,

$$\prod_{j=u_n}^{v_n} (t - \tfrac{j}{n}) = (t - \chi_n)^{v_n-u_n+1} \exp\left[\sum_{j=u_n}^{v_n} \log\left(1 - \frac{\tfrac{j}{n} - \chi_n}{t - \chi_n} \right) \right],$$

where log denotes the natural logarithm. Moreover, Taylor's theorem gives

$$\log\left(1 - \frac{\tfrac{j}{n} - \chi_n}{t - \chi_n} \right) = -\frac{\tfrac{j}{n} - \chi_n}{t - \chi_n} - \frac{1}{2}\frac{(\tfrac{j}{n} - \chi_n)^2}{(t - \chi_n)^2} + O(n^{-1}),$$

uniformly for $j \in \{u_n, u_n + 1, \ldots, v_n\}$. Therefore, since $v_n = n\chi_n + O(n^{\frac{2}{3}})$ and $u_n = n\chi_n + O(n^{\frac{2}{3}})$ (see (1.28) and (1.29)),

$$\prod_{j=u_n}^{v_n} (t - \tfrac{j}{n}) = (t - \chi_n)^{v_n-u_n+1} \exp\left[-\frac{n}{2}\frac{(\tfrac{v_n}{n} - \chi_n)^2 - (\tfrac{u_n}{n} - \chi_n)^2}{t - \chi_n} \right.$$
$$\left. -\frac{n}{6}\frac{(\tfrac{v_n}{n} - \chi_n)^3 - (\tfrac{u_n}{n} - \chi_n)^3}{(t - \chi_n)^2} + O(n^{-\frac{1}{3}}) \right].$$

Similarly we can show that,

$$\prod_{k=v_n+s_n-n}^{u_n+r_n-n} (t - \tfrac{k}{n}) = (t - \chi_n - \eta_n + 1)^{u_n+r_n-v_n-s_n+1}$$
$$\times \exp\left[-\frac{n}{2}\frac{(\tfrac{u_n}{n} + \tfrac{r_n}{n} - \chi_n - \eta_n)^2 - (\tfrac{v_n}{n} + \tfrac{s_n}{n} - \chi_n - \eta_n)^2}{t - \chi_n - \eta_n + 1} \right.$$
$$\left. -\frac{n}{6}\frac{(\tfrac{u_n}{n} + \tfrac{r_n}{n} - \chi_n - \eta_n)^3 - (\tfrac{v_n}{n} + \tfrac{s_n}{n} - \chi_n - \eta_n)^3}{(t - \chi_n - \eta_n + 1)^2} + O(n^{-\frac{1}{3}}) \right].$$

(2.35) then gives the required result. $\qquad\square$

3. **Steepest descent analysis**

In this section we prove Theorem 1.11 via steepest descent analysis. Assume the conditions of that theorem: Assume Assumptions 1.2 and 1.7, that (1.18) is satisfied for some fixed $t \in R_\mu^+ \cup R_{\lambda-\mu} \cup R_\mu^-$ (a root of f_t' of multiplicity 2), Assumption 2.4, and that $\{(u_n, r_n)\}_{n\geq 1} \subset \mathbb{Z}^2$ and $\{(v_n, s_n)\}_{n\geq 1} \subset \mathbb{Z}^2$ are chosen as in (1.28) and (1.29).

3.1. **Local asymptotic behaviour**

In this section we examine the local behaviour of f_t, f_n and $\tilde{f}_n$ in neighbourhoods of t. We begin by using Lemma 2.2, which describes the various situations of Theorem 1.11 in explicit detail, to choose the branches of the logarithms in (1.11) and (2.5) so that both f_t and f_n are well-defined and analytic in convenient open subsets of $\mathbb{C}$ which contain t. We similarly choose the branches of the logarithms in (2.6), for $\tilde{f}_n$. Recall (see (2.3)) that $\mathbb{C} \setminus S = (\mathbb{C} \setminus \mathbb{R}) \cup J \cup K$, is the domain of f_t', $S = S_1 \cup S_2 \cup S_3 \subset \mathbb{R}$ (see (1.12)), $J = \bigcup_{i=1}^4 J_i$, $K = \bigcup_{i=1}^3 K^{(i)}$, $K^{(i)}$ is partitioned as $\{K_1^{(i)}, K_2^{(i)}, \ldots\}$ for all $i \in \{1, 2, 3\}$, and $\{J_1, J_2, J_3, J_4\} \cup \bigcup_{i=1}^3 \{K_1^{(i)}, K_2^{(i)}, \ldots\}$ is a set of pairwise disjoint open intervals. These sets are depicted in Figure 2.1, and properties of S_1, S_2, S_3 are discussed in (2.2). Also recall (see (2.4)) that $L_t \in \{J_1, J_2, J_3, J_4\} \cup \bigcup_{i=1}^3 \{K_1^{(i)}, K_2^{(i)}, \ldots\}$ denotes that open interval with $t \in L_t$. We write (see (1.11) and (1.30)),

$$f_t(w) = \int_{S_1^+} \log(w - x)\mu[\mathrm{d}x] - \int_{S_2^+} \log(w - x)(\lambda - \mu)[\mathrm{d}x] + \int_{S_3^+} \log(w - x)\mu[\mathrm{d}x]$$
$$+ \int_{S_1^-} \log(w - x)\mu[\mathrm{d}x] - \int_{S_2^-} \log(w - x)(\lambda - \mu)[\mathrm{d}x] + \int_{S_3^-} \log(w - x)\mu[\mathrm{d}x],$$

for all $w \in \mathbb{C} \setminus \mathbb{R}$, where $S_i^+ = S_i \cap [\sup L_t, +\infty)$ and $S_i^- = S_i \cap (-\infty, \inf L_t]$. Thus f_t is analytic in $(\mathbb{C} \setminus \mathbb{R}) \cup L_t$ if we choose the branch cuts of all the logarithms in the first three terms on the RHS to be $[0, +\infty)$, and the branch cuts of all the logarithms in the last three terms to be $(-\infty, 0]$. Next, define $L_n \subset \mathbb{R} \setminus S_n$ as in (2.12). Then we can similarly choose the branches of the logarithms in (2.5) such that f_n is well-defined and analytic in $\mathbb{C} \setminus L_n$. Finally, fix $\xi > 0$ sufficiently small such that (2.11) is satisfied, and define

$$\mathbb{C}_\xi := \{w \in \mathbb{C} : \inf L_t + 2\xi < \mathrm{Re}(w) < \sup L_t - 2\xi \text{ or } |\mathrm{Im}(w)| > \xi^4\}. \tag{3.1}$$

The use of ξ^4, above, is a choice of convenience which will simplify some calculations later. Note, (2.11) and (3.1) imply that $B(t, 2\xi) \subset \mathbb{C}_\xi$. Also note that $\mathbb{C}_\xi \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_t$, and so f_t is well-defined and analytic in $\mathbb{C}_\xi$. Also, (2.12) implies that $\mathbb{C}_\xi \subset (\mathbb{C} \setminus \mathbb{R}) \cup L_n$, and so f_n is well-defined and analytic in $\mathbb{C}_\xi$. Similarly for $\tilde{f}_n$. Moreover:

Lemma 3.1. *Fix $\xi > 0$ as above, fix $r > 0$, and fix an integer $k \geq 0$. Then:*

(1) *Then there exists a positive constant, $C = C(t, \xi, k, r)$, for which*

$$\sup_{w \in \mathbb{C}_\xi \cap B(0,r)} |f_t^{(k)}(w)| < C \quad and \quad \sup_{w \in \mathbb{C}_\xi \cap B(0,r)} |f_n^{(k)}(w)| < C.$$

(2) $f_n^{(k)} \to f_t^{(k)}$ *uniformly in $\mathbb{C}_\xi \cap B(0,r)$.*

Similarly for $\tilde{f}_n$.

Proof. Consider (1). First recall that $S_n = S_{1,n} \cup S_{2,n} \cup S_{3,n}$, and $\frac{1}{n}|S_{i,n}| = O(1)$ for all $i \in \{1, 2, 3\}$ (see (2.5) and (2.9)). (2.5) then implies that there exists a constant, $c > 0$, for which

$$\sup_{w \in \mathbb{C}_\xi \cap B(0,r)} |f_n(w)| < c \sup_{(w,x) \in (\mathbb{C}_\xi \cap B(0,r)) \times S_n} |\log(w - x)|.$$

Next note, (2.9) gives

$$\sup_{(w,x) \in (\mathbb{C}_\xi \cap B(0,r)) \times S_n} |w - x| < r + 2\max\{|\inf S_3|, |\sup S_1|\}.$$

Moreover,

$$\inf_{(w,x) \in (\mathbb{C}_\xi \cap B(0,r)) \times S_n} |w - x| > \min\{\xi, \xi^4\}.$$

Indeed, the above follows since either $|\operatorname{Im}(w - x)| > \xi^4$ (see (3.1)) or $|\operatorname{Re}(w - x)| > \min\{(\inf L_t + 2\xi) - \inf L_n, \sup L_n - (\sup L_t - 2\xi)\}$ (see (2.12) and (3.1)) and note that $L_n \subset \mathbb{R} \setminus S_n$ and $\{\sup L_n, \inf L_n\} \subset S_n$), and since $\sup L_n = \sup L_t + o(1)$ and $\inf L_n = \inf L_t + o(1)$ (see (2.12)). Combined, the above three inequalities prove part (1) for f_n. Part (1) for $f_n^{(k)}$ for all $k \geq 1$ follows similarly. Also, part (1) for $f_t^{(k)}$ and $\tilde{f}_n^{(k)}$ for all $k \geq 0$ follows similarly.

Consider (2). First note, for all $k \geq 0$, (1.11), (1.30), (2.5) and (2.8) imply that $f_n^{(k)} \to f_t^{(k)}$ pointwise in $\mathbb{C}_\xi$. Moreover, for all $k \geq 0$, part (1) implies that $\{f_t^{(k)}, f_1^{(k)}, f_2^{(k)}, \ldots\}$ are equicontinuous in $\mathbb{C}_\xi \cap B(0,r)$. Part (2) trivially follows. $\square$

Next we examine the Taylor expansions of f_t and f_n and $\tilde{f}_n$ in neighbourhoods of t:

Lemma 3.2. *Fix $\xi > 0$ as above, $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ as in Definition 2.8, and $\{\xi_n\}_{n \geq 1} \subset \mathbb{R}$ such that $|\xi_n q_n| \leq \xi$ for all n. Recall that $f_{t,n}'''(t) \to f_t'''(t) \neq 0$ (see Lemma 2.7), and let u, r, v, s be the parameters in (1.28) and (1.29). Then, uniformly for $\alpha \in (-\pi, \pi]$:*

(1) $f_t(t + \xi_n q_n e^{i\alpha}) = f_t(t) + \frac{1}{3}\xi_n^3 e^{3i\alpha} + O(|\xi_n|^3 |f_{t,n}'''(t) - f_t'''(t)| + |\xi_n|^4),$

(2) $f_n(t + \xi_n q_n e^{i\alpha}) = f_n(t) + n^{-\frac{2}{3}}\xi_n e^{i\alpha} s + n^{-\frac{1}{3}}\xi_n^2 e^{2i\alpha} v + \frac{1}{3}\xi_n^3 e^{3i\alpha}$
$\qquad\qquad\qquad\qquad + O(n^{-1}|\xi_n| + n^{-\frac{2}{3}}|\xi_n|^2 + n^{-\frac{1}{3}}|\xi_n|^3 + |\xi_n|^4),$

(3) $\tilde{f}_n(t + \xi_n q_n e^{i\alpha}) = \tilde{f}_n(t) + n^{-\frac{2}{3}}\xi_n e^{i\alpha} r + n^{-\frac{1}{3}}\xi_n^2 e^{2i\alpha} u + \frac{1}{3}\xi_n^3 e^{3i\alpha}$
$$+ O(n^{-1}|\xi_n| + n^{-\frac{2}{3}}|\xi_n|^2 + n^{-\frac{1}{3}}|\xi_n|^3 + |\xi_n|^4).$$

Proof. Consider (1). First recall that $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) \neq 0$. Taylor's theorem and part (1) of Lemma 3.1 then give

$$f_t(t + \xi_n q_n e^{i\alpha}) = f_t(t) + \frac{1}{6}\xi_n^3 q_n^3 e^{3i\alpha} f_t'''(t) + O(|\xi_n q_n|^4),$$

uniformly for $\alpha \in (-\pi, \pi]$. Therefore,

$$f_t(t + \xi_n q_n e^{i\alpha}) = f_t(t) + \frac{1}{6}\xi_n^3 q_n^3 e^{3i\alpha} f_{t,n}'''(t) + O(|\xi_n q_n|^3 |f_t'''(t) - f_{t,n}'''(t)| + |\xi_n q_n|^4),$$

uniformly for $\alpha \in (-\pi, \pi]$. Finally recall (see Definition 2.8), $\frac{1}{6}q_n^3 f_{t,n}'''(t) = \frac{1}{3}$, and $\{q_n\}_{n\geq 1}$ is a convergent sequence with a non-zero limit. This proves (1).

Consider (2). Note, Taylor's theorem and part (1) of Lemma 3.1 give

$$f_n(t + \xi_n q_n e^{i\alpha}) = f_n(t) + \xi_n q_n e^{i\alpha} f_n'(t) + \frac{1}{2}\xi_n^2 q_n^2 e^{2i\alpha} f_n''(t)$$
$$+ \frac{1}{6}\xi_n^3 q_n^3 e^{3i\alpha} f_n'''(t) + O(|\xi_n q_n|^4),$$

uniformly for $\alpha \in (-\pi, \pi]$. Parts (1)–(3) of Lemma 2.9 then give

$$f_n(t + \xi_n q_n e^{i\alpha}) = f_n(t) + n^{-\frac{2}{3}}\xi_n q_n e^{i\alpha} s q_{1,n} + \frac{1}{2}n^{-\frac{1}{3}}\xi_n^2 q_n^2 e^{2i\alpha} v q_{2,n} + \frac{1}{6}\xi_n^3 q_n^3 e^{3i\alpha} f_{t,n}'''(t)$$
$$+ O(n^{-1}|\xi_n q_n| + n^{-\frac{2}{3}}|\xi_n q_n|^2 + n^{-\frac{1}{3}}|\xi_n q_n|^3 + |\xi_n q_n|^4).$$

Part (2) then follows from Definition 2.8. (3) follows similarly. □

A useful corollary is the following:

Corollary 3.3. *Fix $\{q_n\}_{n\geq 1} \subset \mathbb{R}$ as in Definition 2.8. Also fix $c > 0$ and $\theta \in (\frac{1}{4}, \frac{1}{3})$. Then, uniformly in the appropriate sets:*

(1) $f_n(w) = f_n(t) + O(n^{-1})$ *for* $w \in B(t, cn^{-\frac{1}{3}})$.

(2) $\tilde{f}_n(z) = \tilde{f}_n(t) + O(n^{-1})$ *for* $z \in B(t, cn^{-\frac{1}{3}})$.

(3) $nf_n(t + n^{-\frac{1}{3}}q_n w) = nf_n(t) + ws + w^2 v + \frac{1}{3}w^3 + O(n^{1-4\theta})$ *for* $w \in cl(B(0, n^{\frac{1}{3}-\theta}))$.

(4) $n\tilde{f}_n(t + n^{-\frac{1}{3}}q_n z) = n\tilde{f}_n(t) + zr + z^2 u + \frac{1}{3}z^3 + O(n^{1-4\theta})$ *for* $z \in cl(B(0, n^{\frac{1}{3}-\theta}))$.

Proof. First recall (see Definition 2.8) that $\{q_n\}_{n\geq 1} \subset \mathbb{R}$ is a convergent sequence with a non-zero limit. Then, (1) and (2) follow from parts (2) and (3) of Lemma 3.2 by choosing $\xi_n := cn^{-\frac{1}{3}}|q_n|^{-1}$. Next, choose $\xi_n := n^{-\theta}$. Part (2) of Lemma 3.2 then gives

$$nf_n(t + n^{-\theta}q_n e^{i\alpha}) = nf_n(t) + n^{\frac{1}{3}-\theta}e^{i\alpha}s + n^{\frac{2}{3}-2\theta}e^{2i\alpha}v + \frac{1}{3}n^{1-3\theta}e^{3i\alpha}$$
$$+ O(n^{-\theta} + n^{\frac{1}{3}-2\theta} + n^{\frac{2}{3}-3\theta} + n^{1-4\theta}),$$

uniformly for $\alpha \in (-\pi, \pi]$. Finally recall that $\theta \in (\frac{1}{4}, \frac{1}{3})$. Therefore $-\theta < \frac{1}{3} - 2\theta < \frac{2}{3} - 3\theta < 1 - 4\theta < 0$, and so $O(n^{-\theta} + n^{\frac{1}{3} - 2\theta} + n^{\frac{2}{3} - 3\theta} + n^{1 - 4\theta}) = O(n^{1-4\theta})$. This proves (3). Similarly, (4) follows from part (3) of Lemma 3.2. $\qquad\square$

3.2. **Contours of descent/ascent**

The main results of this section, Lemmas 3.5 and 3.6, prove the existence of appropriate contours of descent/ascent. These proofs are the most difficult part of the paper, and will be given in Section 4.

First we consider contours of steepest descent/ascent for f_t, f_n and $\tilde{f}_n$. We do not define these rigorously, and refer the interested reader to [20], for example, for more information. We consider these for f_t and for f_n for some fixed n, and state that $\tilde{f}_n$ can be treated similarly to f_n. Note, irrespective of the choices of the branches of the logarithms in (1.11) and (2.5), that the real-parts of f_t and f_n have unique continuous extensions to $\mathbb{C} \setminus S$ and $\mathbb{C} \setminus S_n$ respectively, denoted by R_t and R_n, and given by,

$$R_t(w) := \int_{S_1} \log|w - x|\mu[\mathrm{d}x] - \int_{S_2} \log|w - x|(\lambda - \mu)[\mathrm{d}x] + \int_{S_3} \log|w - x|\mu[\mathrm{d}x], \quad (3.2)$$

$$R_n(w) := \frac{1}{n} \sum_{x \in S_{1,n}} \log|w - x| - \frac{1}{n} \sum_{x \in S_{2,n}} \log|w - x| + \frac{1}{n} \sum_{x \in S_{3,n}} \log|w - x|, \quad (3.3)$$

where log now represents natural logarithm. Then:

Lemma 3.4. *Fix $n \geq 1$ and $z \in \mathbb{C} \setminus S_n$, and let $D_n, A_n \subset \mathbb{C} \setminus S_n$ denote contours of steepest descent and ascent (respectively) for f_n which pass through z. Also, let $m_n := m_n(z)$ denote the multiplicity of z as a root of f'_n (with the understanding that $m_n = 0$ means $f'_n(z) \neq 0$), and let $\alpha_n := \alpha_n(z) \in (-\pi, \pi]$ denote the principal value of the argument of $f_n^{(m_n(w)+1)}(z)$. Then:*

(1) *R_n strictly decreases along D_n, and strictly increases along A_n.*

(2) *The imaginary-part of f_n is constant along both D_n and A_n.*

(3) *There are $m_n + 1$ possible directions for both D_n and A_n at z, given by $((2i+1)\pi - \alpha_n)/(m_n + 1)$ and $(2i\pi - \alpha_n)/(m_n + 1)$ respectively for each $i \in \{0, 1, \ldots, m_n\}$.*

(4) *D_n is bounded, i.e., there exists a $C > 0$ for which $D_n \subset B(0, C)$.*

(5) *For all $x \in S_n$, there exists a $c(x) > 0$ for which D_n does not intersect $B(x, c(x))$ when $x \in S_{2,n}$, and A_n does not intersect $B(x, c(x))$ when $x \in S_{1,n} \cup S_{3,n}$.*

The equivalent objects for f_t, denoted D_t, A_t, m_t, α_t, also satisfy parts (1)–(4).

Proof. Parts (1)–(3) follow from general considerations about contours of steepest descent/ascent. Consider (4) for D_n. First recall that $|S_{1,n}| - |S_{2,n}| + |S_{3,n}| > 0$ (indeed, (2.2) and (2.9) give $\frac{1}{n}|S_{1,n}| + \frac{1}{n}|S_{3,n}| - \frac{1}{n}|S_{2,n}| \to \eta > 0$). (3.3) thus gives $R_n(w) \sim \frac{1}{n}(|S_{1,n}| - |S_{2,n}| + |S_{3,n}|) \log |w|$ for all $w \in \mathbb{C} \setminus S_n$ with $|w|$ sufficiently large. Then, letting $t_n \in \mathbb{C} \setminus S_n$ be the initial point of D_n, there exists a $C > 0$ for which $R_n(w) > R_n(t_n)$ for all $w \in \mathbb{C} \setminus S_n$ with $|w| \geq C$. Also, part (1) gives $R_n(w) \leq R_n(t_n)$ for all w on D_n. Part (4) for D_n easily follows. Part (4) for D_t follows similarly.

Consider (5). Fix $x \in S_n$. (3.3) gives $R_n(w) \sim s(x)\frac{1}{n} \log |w - x|$ for all $w \in \mathbb{C} \setminus S_n$ with $|w - x|$ sufficiently small, where $s(x) = -1$ whenever $x \in S_{2,n}$ and $s(x) = 1$ whenever $x \in S_{1,n} \cup S_{3,n}$. Thus, whenever $x \in S_{2,n}$, letting $t_n \in \mathbb{C} \setminus S_n$ be the initial point of D_n, there exists a $c(x) > 0$ for which $R_n(w) > R_n(t_n)$ for all $w \in \mathbb{C} \setminus S_n$ with $|w - x| < c(x)$. Also, part (1) gives $R_n(w) \leq R_n(t_n)$ for all w on D_n. Part (5) for D_n easily follows. Part (5) for A_n follows similarly. □

We now discuss natural extensions of the real and imaginary parts of f_t and f_n, from $\mathbb{H} = \{w \in \mathbb{C} : \text{Im}(w) > 0\}$, to $\mathbb{R} \setminus S$ and $\mathbb{R} \setminus S_n$ respectively. Our motivation is the following: In Section 4 we will be examining contours of steepest descent/ascent which are contained in $\mathbb{H}$ except (possibly) for the end-points. Part (2) of the previous lemma thus show that these extensions are natural to examine. First note, irrespective of the choices of the branches of the logarithms in (1.30) and (2.5), that

$$\text{Im}(f_t(w)) = \int_{S_1} \text{Arg}(w-x)\mu[dx] - \int_{S_2} \text{Arg}(w-x)(\lambda-\mu)[dx] + \int_{S_3} \text{Arg}(w-x)\mu[dx],$$

$$\text{Im}(f_n(w)) = \frac{1}{n} \sum_{x \in S_{1,n}} \text{Arg}(w - x) - \frac{1}{n} \sum_{x \in S_{2,n}} \text{Arg}(w - x) + \frac{1}{n} \sum_{x \in S_{3,n}} \text{Arg}(w - x),$$

(3.4)

for all $w \in \mathbb{H}$, where Arg represents the principal value of the argument. Note that these has unique extensions from $\mathbb{H}$ to $\mathbb{R}$, denoted by I_t and I_n respectively and given by,

$$I_t(s) := \pi\mu[\{x \in S_1 : x > s\}] - \pi(\lambda-\mu)[\{x \in S_2 : x > s\}] + \pi\mu[\{x \in S_3 : x > s\}], \quad (3.5)$$

$$I_n(s) := \frac{\pi}{n}|\{x \in S_{1,n} : x > s\}| - \frac{\pi}{n}|\{x \in S_{2,n} : x > s\}| + \frac{\pi}{n}|\{x \in S_{3,n} : x > s\}|, \quad (3.6)$$

for all $s \in \mathbb{R}$. Note, since $\mu \leq \lambda$ (see Assumption 1.2), that $I_t : \mathbb{R} \to \mathbb{R}$ is continuous. Also, (2.2) and (2.3) imply that $I_t : \mathbb{R} \to \mathbb{R}$ is constant in sub-intervals of $\mathbb{R} \setminus S = J \cup K$, is strictly decreasing in the interior of S_1 and S_3, and is strictly increasing in the interior of S_2. Similarly, (2.10) implies that $I_n : \mathbb{R} \to \mathbb{R}$, is constant in sub-intervals of $\mathbb{R} \setminus S_n = J_n \cup K_n$. Finally note that each discrete element of S_n acts as a point of discontinuity for I_n: I_n decreases by $\frac{\pi}{n}$ at each point of $S_{1,n}$ and $S_{3,n}$, and increases by $\frac{\pi}{n}$ at each point of $S_{2,n}$. These sets, and the above extensions, are depicted in Figure 3.1.
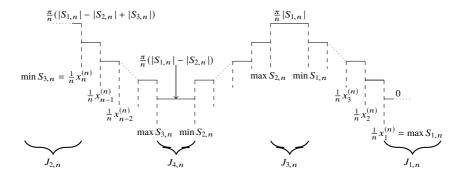
FIGURE 3.1. The functions given in (3.5) and (3.6), with I_t on the top and I_n on the bottom. The identity $\eta = \mu[S_1] - (\lambda - \mu)[S_2] + \mu[S_3]$ is given in (2.2). All jumps in I_n are of size $\frac{\pi}{n}$.

Next note, (3.2) and (3.3) give

$$R_t(s) = \int_{S_1} \log |s - x| \mu[dx] - \int_{S_2} \log |s - x| (\lambda - \mu)[dx] + \int_{S_3} \log |s - x| \mu[dx],$$

$$R_n(s) = \frac{1}{n} \sum_{x \in S_{1,n}} \log |s - x| - \frac{1}{n} \sum_{x \in S_{2,n}} \log |s - x| + \frac{1}{n} \sum_{x \in S_{3,n}} \log |s - x|,$$

for all $s \in \mathbb{R} \setminus S$ and $s \in \mathbb{R} \setminus S_n$ respectively. Thus the restrictions are real-valued, and we can regard them as functions from $\mathbb{R} \setminus S = J \cup K$ and $\mathbb{R} \setminus S_n = J_n \cup K_n$ (respectively) to $\mathbb{R}$. Moreover, note that,

$$(R_t|_{\mathbb{R} \setminus S})' = (f_t')|_{\mathbb{R} \setminus S} \qquad \text{and} \qquad (R_n|_{\mathbb{R} \setminus S_n})' = (f_n')|_{\mathbb{R} \setminus S_n}, \tag{3.7}$$

and similarly for the higher order derivatives. Above, the functions on the LHSs are the "real-derivative" of the real-valued restrictions, and the functions on the RHSs are those given in (2.1) and (2.7) (respectively) restricted to $\mathbb{R} \setminus S$ and to $\mathbb{R} \setminus S_n$.

We now state the main results of this section, which will be proven in Section 4. Recall that L_t is the largest open sub-interval of $\mathbb{R} \setminus S$ which contains t (see (2.4)). Also recall that Lemma 2.2 splits the conditions of Theorem 1.11 into 12 exhaustive cases. These lemmas prove the existence of appropriate contours of descent/ascent for each case:

Lemma 3.5. *Fix $\theta \in (\frac{1}{4}, \frac{1}{3})$, and $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied. Then $(t - 4\xi, t + 4\xi) \subset L_t$, and $B(t, n^{-\theta}|q_n|) \subset B(t, \xi)$, where $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ is given in Definition 2.8. Moreover, in each of the cases of Lemma 2.2, there exists simply contours as shown in Figure 3.2 with the following properties:*

(1) *$\gamma_{1,n}^+$ and $\Gamma_{1,n}^+$ both start at t, end in the interior of the intervals shown in Figure 3.2, are otherwise contained in $\mathbb{H}$, do not intersect except at t, and are independent of n outside $cl(B(t, \xi))$. $\gamma_{2,n}^+$ and $\Gamma_{2,n}^+$ either start in $(t + \xi, \sup L_t)$ or in $(\inf L_t, t - \xi)$ as shown in Figure 3.2, end in the interior of the intervals shown, are otherwise contained in $\mathbb{H}$, do not intersect $\gamma_{1,n}^+$ or $\Gamma_{1,n}^+$ or $cl(B(t, \xi))$, and are independent of n everywhere.*

(2) *$\gamma_{1,n}^+ \cap B(t, n^{-\theta}|q_n|)$ and $\Gamma_{1,n}^+ \cap B(t, n^{-\theta}|q_n|)$ are straight lines from t to points $d_{1,n} \in \partial B(t, n^{-\theta}|q_n|)$ and $\tilde{a}_{1,n} \in \partial B(t, n^{-\theta}|q_n|)$ (respectively) for which, letting $\text{Arg}(\cdot)$ be the principal value of the argument, $\text{Arg}(d_{1,n} - t) = \frac{\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ and $\text{Arg}(\tilde{a}_{1,n} - t) = \frac{2\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ in cases (1,2,7,8,9,10), and $\text{Arg}(d_{1,n} - t) = \frac{2\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ and $\text{Arg}(\tilde{a}_{1,n} - t) = \frac{\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ in cases (3,4,5,6,11,12).*

(3) *$Re(f_n(w)) \leq Re(f_n(d_{1,n}))$ for all $w \in \gamma_{1,n}^+ \setminus B(t, n^{-\theta}|q_n|)$ and $w \in \gamma_{2,n}^+$.*

(4) *$Re(\tilde{f}_n(z)) \geq Re(\tilde{f}_n(\tilde{a}_{1,n}))$ for all $z \in \Gamma_{1,n}^+ \setminus B(t, n^{-\theta}|q_n|)$ and $z \in \Gamma_{2,n}^+$.*

(5) *$|w - z|^{-1} = O(n^\theta)$ uniformly for $w \in \gamma_{1,n}^+$ and $w \in \gamma_{2,n}^+$, and uniformly for $z \in \Gamma_{1,n}^+ \setminus B(t, n^{-\theta}|q_n|)$ and $z \in \Gamma_{2,n}^+$. $|w - z|^{-1} = O(n^\theta)$ uniformly for $w \in \gamma_{1,n}^+ \setminus B(t, n^{-\theta}|q_n|)$ and $w \in \gamma_{2,n}^+$, and uniformly for $z \in \Gamma_{1,n}^+$ and $z \in \Gamma_{2,n}^+$.*

(6) *$|\gamma_{1,n}^+| = O(1)$ and $|\Gamma_{1,n}^+| = O(1)$, where $|\cdot|$ represents length.*
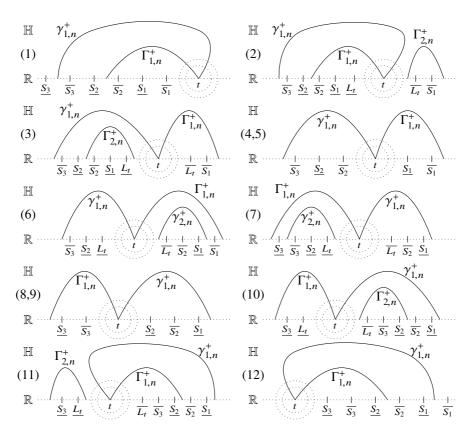
FIGURE 3.2. The contours described in Lemma 3.5, for the exhaustive cases (1)–(12) of Lemma 2.2. Above, $\underline{S_1} := \inf S_1$, $\overline{S_1} := \sup S_1$, etc.. The smaller circles represent $B(t, n^{-\theta}|q_n|)$, the larger circles represent $B(t, \xi)$.

Lemma 3.6. *Define F_n as in (2.25). Assume that $v > u$. Fix $\theta \in (\frac{1}{4}, \frac{1}{3})$, and $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied. Then, in each of the cases of Lemma 2.2, there exists a simple contour as shown in Figure 3.3 with the following properties:*

- *κ_n^+ starts at t, ends in the interior of the intervals shown in Figure 3.3, is otherwise contained in $\mathbb{H}$, and is independent of n outside $cl(B(t, \xi))$.*
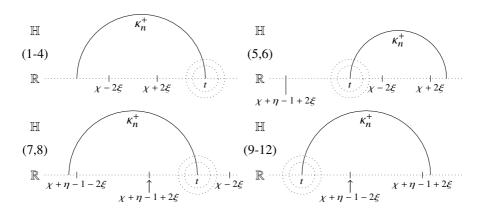
FIGURE 3.3. The contours described in Lemma 3.6 for the exhaustive cases (1)–(12) of Lemma 2.2, when $v > u$. The smaller circles represent $B(t, n^{-\theta}|q_n|)$, the larger circles represent $B(t, \xi)$.

- $\kappa_n^+ \cap B(t, n^{-\theta}|q_n|)$ is a straight line from t to $D_{1,n} \in \partial B(t, n^{-\theta}|q_n|)$ for which $\operatorname{Arg}(D_{1,n} - t) = \frac{\pi}{2} + O(n^{-\frac{1}{3}+\theta})$.

- $Re(F_n(w)) \le Re(F_n(D_{1,n}))$ for all $w \in \kappa_n^+ \setminus B(t, n^{-\theta}|q_n|)$.

- $|\kappa_n^+| = O(1)$, where $|\cdot|$ represents length.

3.3. Alternative contour integral expressions

The main results of the previous section, Lemmas 3.5 and 3.6, prove the existence of appropriate contours of descent/ascent for the cases (1)–(12) of Lemma 2.2. In this section, we will use these contours to find alternative contour integral expressions for the correlation kernel than that given in (1.7). These new expressions will allow us to perform a steepest descent analysis for each case. First, using Lemmas 3.5 and 3.6, we define:

Definition 3.7. For cases (1)–(12) of Lemma 2.2, define $\gamma_{1,n}$ to be the following simple closed contour with counter-clockwise orientation: $\gamma_{1,n} := \gamma_{1,n}^+ + \gamma_{1,n}^-$, where $\gamma_{1,n}^-$ is the reflection of $\gamma_{1,n}^+$ in $\mathbb{R}$. Similarly define $\gamma_{2,n}$, $\Gamma_{1,n}$, $\Gamma_{2,n}$. Similarly define κ_n when $v > u$. Finally, define $\gamma_n := \gamma_{1,n}$ when $\gamma_{2,n}$ does not exist, and $\gamma_n := \gamma_{1,n} + \gamma_{2,n}$ when $\gamma_{2,n}$ exists. Similarly define Γ_n.

The main result of this section is then:

Lemma 3.8. *Assume the conditions of Theorem 1.11. Recall that one of the cases* (1)–(12) *of Lemma 2.2 must be satisfied. Then,*

$$
(1) \qquad \beta_n K_n((u_n, r_n), (v_n, s_n)) =
\begin{cases}
J_n - \Phi_n & \text{for (1)–(4),} \\
J_n + \Phi_n & \text{for (5) and (6),} \\
-J_n - \Phi_n & \text{for (7) and (8),} \\
-J_n + \Phi_n & \text{for (9)–(12),}
\end{cases}
$$

where we define:

$$
\beta_n := \frac{(n - r_n - 1)!}{(n - s_n)!} n^{r_n + 1 - s_n},
$$

and

$$
J_n := \frac{1}{(2\pi i)^2} \int_{\gamma_n} dw \int_{\Gamma_n} dz \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(z - \frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w - \frac{j}{n})} \frac{1}{w - z} \prod_{i=1}^{n} \left(\frac{w - \frac{x_i}{n}}{z - \frac{x_i}{n}} \right),
$$

and

$$
\Phi_n :=
\begin{cases}
1_{(v_n \geq u_n, s_n > r_n)} \frac{(v_n + s_n - u_n - r_n - 1)!}{(s_n - r_n - 1)!(v_n - u_n)!} \beta_n & \text{for (1)–(4),} \\[2mm]
1_{(v_n \geq u_n, v_n + s_n \leq u_n + r_n, s_n \leq r_n)} \frac{(-1)^{v_n - u_n}(r_n - s_n)!}{(v_n - u_n)!(u_n + r_n - v_n - s_n)!} \beta_n & \text{for (5) and (6),} \\[2mm]
1_{(v_n \geq u_n, v_n + s_n \leq u_n + r_n, s_n \leq r_n)} \frac{(-1)^{v_n - u_n - 1}(r_n - s_n)!}{(v_n - u_n)!(u_n + r_n - v_n - s_n)!} \beta_n & \text{for (7) and (8),} \\[2mm]
1_{(v_n + s_n \leq u_n + r_n, s_n > r_n)} \frac{(-1)^{s_n - r_n - 1}(u_n - v_n - 1)!}{(s_n - r_n - 1)!(u_n + r_n - v_n - s_n)!} \beta_n & \text{for (9)–(12).}
\end{cases}
$$

Moreover, when $(u, r) \neq (v, s)$,

$$
(2) \qquad \Phi_n = 1_{(v > u)} \frac{1}{2\pi i} \int_{\kappa_n} dw \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(w - \frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w - \frac{j}{n})}.
$$

We will prove the above using a number of sub-results. First, we examine J_n using the Residue theorem. Note that J_n can be written as follows:

$$
J_n = \frac{1}{(2\pi i)^2} \int_{\gamma_n} dw \int_{\Gamma_n} dz \frac{\prod_{y \in U_n}(z - y)}{\prod_{x \in V_n}(w - x)} \frac{1}{w - z} \frac{\prod_{x \in P_n}(w - x)}{\prod_{y \in P_n}(z - y)}, \qquad (3.8)
$$

where:

- $P_n := \frac{1}{n}\{x_1^{(n)}, x_2^{(2)}, \ldots, x_n^{(n)}\}$ (as in (1.22)).

- $U_n := \frac{1}{n}\{u_n + r_n - n + 1, u_n + r_n - n + 2, \ldots, u_n - 1\}$ (as in (2.29)).

- $V_n := \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$ (as in (2.29)).

Also note, the above definitions, and (2.5) and (2.6), give

$$\tilde{S}_{1,n} = \{y \in P_n : y \geq \tfrac{u_n}{n}\} \quad \text{and} \quad \tilde{S}_{3,n} = \{y \in P_n : y \leq \tfrac{u_n + r_n - n}{n}\}, \qquad (3.9)$$

$$P_n \setminus U_n = \tilde{S}_{1,n} \cup \tilde{S}_{3,n} \quad \text{and} \quad V_n \setminus P_n = S_{2,n}. \qquad (3.10)$$

Finally recall the decomposition, $V_n \setminus U_n = (VU^{(n)}) \cup (VU_{(n)})$, given in (2.30). Then:

Lemma 3.9. *Assume the conditions of Theorem 1.11. Define U_n, V_n, P_n, $VU^{(n)}$, $VU_{(n)}$ as above. Let $A \in \{1, 2, \ldots, 12\}$ denote that case of Lemma 2.2 which is satisfied. Then, for cases (1)–(6),*

$$J_n = \sum_{y' \in \tilde{S}_{1,n}} \sum_{x' \in S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$+ 1_{(v_n \geq u_n)} 1_{(A \in \{1,2,3\})} \sum_{y' \in (VU^{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}$$

$$- 1_{(v_n \geq u_n)} 1_{(A=6)} \sum_{y' \in (VU^{(n)}) \setminus P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Moreover, for cases (7)–(12),

$$J_n = \sum_{y' \in \tilde{S}_{3,n}} \sum_{x' \in S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$+ 1_{(v_n + s_n \leq u_n + r_n)} 1_{(A \in \{10,11,12\})} \sum_{y' \in (VU_{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}$$

$$- 1_{(v_n + s_n \leq u_n + r_n)} 1_{(A=7)} \sum_{y' \in (VU_{(n)}) \setminus P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Proof. In this lemma, we let γ_n° and Γ_n° respectively denote the interiors of γ_n and Γ_n. Note, in (3.8), we perform the Γ_n integral first, and so we consider the $w \in \gamma_n$ in the integrand to be fixed. Also note that $w \notin P_n$ since γ_n can always be chosen so that it does not intersect P_n (see Remark 4.5), and that each element of P_n is distinct. The integrand of (3.8) thus has a simple pole at each distinct element of $P_n \setminus U_n = \tilde{S}_{1,n} \cup \tilde{S}_{3,n}$ (see (3.10)), a simple pole or removable singularity at w, and no other singularities. Therefore, the

Residue theorem gives

$$
\begin{aligned}
J_n &= \frac{1}{2\pi i}\int_{\gamma_n} dw \sum_{y'\in \tilde{S}_{1,n}\cup \tilde{S}_{3,n}} 1_{(y'\in\Gamma_n^\circ)} \frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n}(w-x)}\frac{1}{w-y'}\frac{\prod_{x\in P_n}(w-x)}{\prod_{y\in P_n\setminus y'}(y'-y)}\\
&\quad - \frac{1}{2\pi i}\int_{\gamma_n} dw 1_{(w\in\Gamma_n^\circ)}\frac{\prod_{y\in U_n}(w-y)}{\prod_{x\in V_n}(w-x)}\frac{\prod_{x\in P_n}(w-x)}{\prod_{y\in P_n}(w-y)}\\
&= \sum_{y'\in \tilde{S}_{1,n}\cup \tilde{S}_{3,n}} 1_{(y'\in\Gamma_n^\circ)}\frac{1}{2\pi i}\int_{\gamma_n} dw \frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n}(w-x)}\frac{\prod_{x\in P_n\setminus y'}(w-x)}{\prod_{y\in P_n\setminus y'}(y'-y)}\\
&\quad - \frac{1}{2\pi i}\int_{\gamma_n} dw 1_{(w\in\Gamma_n^\circ)}\frac{\prod_{y\in U_n}(w-y)}{\prod_{x\in V_n}(w-x)}.
\end{aligned}
\tag{3.11}
$$

Consider the first term on the RHS, above. Recall that $y'\in P_n$ is fixed, and (see (3.10)) $V_n\setminus P_n = S_{2,n}$. Therefore $V_n = S_{2,n}\cup (V_n\cap\{y'\})\cup (V_n\cap (P_n\setminus\{y'\}))$, a disjoint union. The integrand thus has a simple pole at each distinct element of $S_{2,n}\cup (V_n\cap\{y'\})$, and no other singularities. The Residue theorem thus implies that the first term equals,

$$
\begin{aligned}
&\sum_{y'\in \tilde{S}_{1,n}\cup \tilde{S}_{3,n}} 1_{(y'\in\Gamma_n^\circ)}\Bigg[\sum_{x'\in S_{2,n}} 1_{(x'\in\gamma_n^\circ)}\frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n\setminus x'}(x'-x)}\frac{\prod_{x\in P_n\setminus y'}(x'-x)}{\prod_{y\in P_n\setminus y'}(y'-y)}\\
&\qquad\qquad + 1_{(y'\in V_n, y'\in\gamma_n^\circ)}\frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n\setminus y'}(y'-x)}\frac{\prod_{x\in P_n\setminus y'}(y'-x)}{\prod_{y\in P_n\setminus y'}(y'-y)}\Bigg]\\
&= \sum_{y'\in \tilde{S}_{1,n}\cup \tilde{S}_{3,n}} 1_{(y'\in\Gamma_n^\circ)}\sum_{x'\in S_{2,n}} 1_{(x'\in\gamma_n^\circ)}\frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n\setminus x'}(x'-x)}\frac{\prod_{x\in P_n\setminus y'}(x'-x)}{\prod_{y\in P_n\setminus y'}(y'-y)}\\
&\qquad\qquad + \sum_{y'\in(\tilde{S}_{1,n}\cup\tilde{S}_{3,n})\cap V_n} 1_{(y'\in\gamma_n^\circ\cap\Gamma_n^\circ)}\frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n\setminus y'}(y'-x)}.
\end{aligned}
$$

Next consider the second term on the RHS of (3.11). Note, Definition 3.7 and Figure 3.2 imply that Γ_n contains $\gamma_{2,n}$ and none of $\gamma_{1,n}$ for cases (6) and (7), and Γ_n contains no parts of γ_n for all other cases. Therefore, the second term equals,

$$
-1_{(A\in\{6,7\})}\int_{\gamma_{2,n}} dw \frac{\prod_{y\in U_n}(w-y)}{\prod_{x\in V_n}(w-x)} = -1_{(A\in\{6,7\})}\sum_{y'\in V_n\setminus U_n} 1_{(y'\in\gamma_{2,n}^\circ)}\frac{\prod_{y\in U_n}(y'-y)}{\prod_{x\in V_n\setminus y'}(y'-x)}.
$$

where $A \in \{1, 2, \ldots, 12\}$ denotes that particular case of Lemma 2.2 which is satisfied. Combined the above give

$$J_n = \sum_{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n}} 1_{(y' \in \Gamma_n^\circ)} \sum_{x' \in S_{2,n}} 1_{(x' \in \gamma_n^\circ)} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$+ \sum_{y' \in (\tilde{S}_{1,n} \cup \tilde{S}_{3,n}) \cap V_n} 1_{(y' \in \gamma_n^\circ \cap \Gamma_n^\circ)} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}$$

$$- 1_{(A \in \{6,7\})} \sum_{y' \in V_n \setminus U_n} 1_{(y' \in \gamma_{2,n}^\circ)} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Finally recall that $V_n \setminus U_n = (VU^{(n)}) \cup (VU_{(n)})$ (see (2.30)), and $\tilde{S}_{1,n} \cap V_n = (VU^{(n)}) \cap P_n$ and $\tilde{S}_{3,n} \cap V_n = (VU_{(n)}) \cap P_n$ (see (2.29), (2.30) and (3.9)). The required result thus follows if we can show that:

(i) $\{x' \in S_{2,n} : x' \in \gamma_n^\circ\}$ equals $S_{2,n}$ for all cases (1)–(12) of Lemma 2.2.

(ii) $\{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n} : y' \in \Gamma_n^\circ\}$ equals $\tilde{S}_{1,n}$ for (1)–(6), and equals $\tilde{S}_{3,n}$ for (7)–(12).

(iii) $\{y' \in (\tilde{S}_{1,n} \cup \tilde{S}_{3,n}) \cap V_n : y' \in \gamma_n^\circ \cap \Gamma_n^\circ\}$ equals $(VU^{(n)}) \cap P_n$ for (1)–(3) and (6) when $v_n \geq u_n$, equals $(VU_{(n)}) \cap P_n$ for (7) and (10)–(12) when $v_n + s_n \leq u_n + r_n$, and is empty otherwise.

(iv) $\{y' \in V_n \setminus U_n : y' \in \gamma_{2,n}^\circ\}$ equals $VU^{(n)}$ for (6) when $v_n \geq u_n$, equals $VU_{(n)}$ for (7) when $v_n + s_n \leq u_n + r_n$, is empty for (6) when $u_n > v_n$, and is empty for (7) when $v_n + s_n > u_n + r_n$.

We will prove (i) only for cases (1)–(6). The proof of (i) for case (12) is similar to case (1), the proof of (i) for case (11) is similar to case (2), etc. We will prove (ii) only for case (4). The proof of (ii) for all other cases follows from similar considerations. We will not prove (iii) and (iv), but we state that their proofs also follow from similar considerations.

Consider (i) for cases (1)–(4). Recall (see (2.5)) that $S_{2,n} \subset \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\}$. Thus, fixing $\delta > 0$, (1.18) implies that $S_{2,n} \subset (\chi + \eta - 1 - \delta, \chi + \delta)$. Next recall (see Lemma 2.2 and (2.2)) that $t > \chi > \chi + \eta - 1 \geq \overline{S_3}$. Definition 3.7 and Figure 3.2 thus imply that we can choose the above $\delta > 0$ such that γ_n contains $(\chi + \eta - 1 - \delta, \chi + \delta)$. This proves (i) for cases (1)–(4).

Consider (i) for case (6). First recall (see Lemma 2.2) that $t \in L_t \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Thus, fixing $\delta > 0$ such that $\overline{L_t} - \delta > t > \underline{L_t} + \delta$, (1.24) gives $\frac{\mathbb{Z}}{n} \cap (\underline{L_t} + \delta, \overline{L_t} - \delta) \subset P_n$. Next recall that $S_{2,n} = \frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\} \setminus P_n$ (see (2.5)), that

$\frac{1}{n}\{v_n + s_n - n, v_n + s_n - n + 1, \ldots, v_n\} \subset (\chi + \eta - 1 - \delta, \chi + \delta)$ (see (1.18)). Therefore $S_{2,n} \subset (\chi + \eta - 1 - \delta, \underline{L_t} + \delta] \cup [\overline{L_t} - \delta, \chi + \delta)$. Next recall (see Lemma 2.2 and (2.2)) that $\underline{S_1} \geq \chi > \overline{L_t} > t > \underline{L_t} > \chi + \eta - 1 \geq \overline{S_3}$. Definition 3.7 and Figure 3.2 thus imply that we can choose the above $\delta > 0$ such that $\gamma_{1,n}$ contains $(\chi + \eta - 1 - \delta, \underline{L_t} + \delta]$ and $\gamma_{2,n}$ contains $[\overline{L_t} - \delta, \chi + \delta)$. This proves (i) for case (6).

Consider (i) for case (5). First recall (see Lemma 2.2) that $t \in (\overline{S_2}, \chi) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Also recall (see Assumption 2.4) that $\chi \in \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Thus we can fix a $\delta > 0$ such that $t > \overline{S_2} + \delta$ and $(\overline{S_2} + \delta, \chi + \delta) \subset \mathbb{R} \setminus \text{Supp}(\lambda - \mu)$. Then we can proceed similarly to case (6), above, to show that $S_{2,n} \subset (\chi + \eta - 1 - \delta, \overline{S_2} + \delta]$. Moreover, Definition 3.7 and Figure 3.2 imply that we can choose the $\delta > 0$ such that γ_n contains $(\chi + \eta - 1 - \delta, \overline{S_2} + \delta]$. This proves (i) for case (5).

Consider (ii) for case (4). We must show that Γ_n contains all of $\tilde{S}_{1,n}$ and none of $\tilde{S}_{3,n}$. First recall (see Lemma 2.2) that $t \in (\chi, \underline{S_1}) \subset \mathbb{R} \setminus \text{Supp}(\mu)$. Also recall (see Assumption 2.4) that $\chi \in \mathbb{R} \setminus \text{Supp}(\mu)$. Thus we can fix a $\delta > 0$ such that $t < \underline{S_1} - \delta$ and $(\chi - \delta, \underline{S_1} - \delta) \subset \mathbb{R} \setminus \text{Supp}(\mu)$. (1.24) thus gives $P_n \cap (\chi - \delta, \underline{S_1} - \delta) = \emptyset$. (1.18) and (3.9) then give $\tilde{S}_{1,n} = \{y \in P_n : y \geq \underline{S_1} - \delta\}$. (2.9) then gives $\tilde{S}_{1,n} \subset [\underline{S_1} - \delta, \overline{S_1} + \delta)$. Next recall (see (3.9)) that $\tilde{S}_{3,n} = \{y \in P_n : y \leq \frac{u_n + r_n - n}{n}\}$. (1.18) and (2.9) then give $\tilde{S}_{3,n} \subset (\underline{S_3} - \delta, \chi + \eta - 1 + \delta)$. Finally recall (see Lemma 2.2 and (2.2)) that $\underline{S_1} > t > \chi > \chi + \eta - 1$. Definition 3.7 and Figure 3.2 thus imply that we can choose the above $\delta > 0$ such that Γ_n contains all of $[\underline{S_1} - \delta, \overline{S_1} + \delta)$ and none of $(\underline{S_3} - \delta, \chi + \eta - 1 + \delta)$. This proves (ii) for case (4). □

Next we prove the following technical result:

Lemma 3.10. *Fix $k, i \geq 1$. Then, for all $x \in \mathbb{Z}$,*

(1)
$$\sum_{l=0}^{k} \frac{\prod_{j=1}^{i}(x + l - j)}{\prod_{j=0, j \neq l}^{k}(l - j)} = \begin{cases} \frac{i!}{k!(i-k)!} \prod_{j=1}^{i-k}(x - j) & \text{when } k < i, \\ 1 & \text{when } k = i, \\ 0 & \text{when } k > i. \end{cases}$$

Next, fix and $a, b \in \mathbb{Z}$. Then, for all $x \in \mathbb{Z}$ with $x + b, x + a \geq 0$,

(2) $$\sum_{l=0}^{k} \frac{(x + b + l)!}{(x + a + l)!} \frac{1}{\prod_{j=0, j \neq l}^{k}(l - j)}$$
$$= (b - a)(b - a - 1) \ldots (b - a - k + 1) \frac{1}{k!} \frac{(x + b)!}{(x + a + k)!}.$$

Proof. First note, letting $g : \mathbb{Z} \to \mathbb{R}$ be any function, and letting Δ denote the finite difference operator (i.e., $(\Delta g)(x) := g(x + 1) - g(x)$, $(\Delta^2 g)(x) := (\Delta(\Delta g))(x) = g(x + 2) - 2g(x + 1) + g(x)$, etc.), then

$$(\Delta^k g)(x) = \sum_{l=0}^{k} (-1)^{k-l} \binom{k}{l} g(x + l) = k! \sum_{l=0}^{k} \frac{g(x + l)}{\prod_{j=0, j \neq l}^{k} (l - j)}, \qquad (3.12)$$

for all $x \in \mathbb{Z}$. Above, the first step follows by induction, and the second step follows since $(-1)^{k-l} l! (k - l)! = \prod_{j=0, j \neq l}^{k} (l - j)$ for all $l \in \{0, \ldots, k\}$. We will use the above identity to prove (1) and (2).

Consider (1). In this case, take

$$g(x) := \prod_{j=1}^{i} (x - j),$$

for all $x \in \mathbb{Z}$. Then, induction gives

$$(\Delta^k g)(x) = \begin{cases} \frac{i!}{(i-k)!} \prod_{j=1}^{i-k} (x - j) & \text{when } k < i, \\ k! & \text{when } k = i, \\ 0 & \text{when } k > i, \end{cases}$$

Moreover, (3.12) gives

$$(\Delta^k g)(x) = k! \sum_{l=0}^{k} \frac{\prod_{j=1}^{i} (x + l - j)}{\prod_{j=0, j \neq l}^{k} (l - j)},$$

for all $x \in \mathbb{Z}$. Combined, the above prove (1).

Consider (2). In this case take,

$$g(x) := \frac{(x + b)!}{(x + a)!},$$

for all $x \in \mathbb{Z}$ with $x + b, x + a \geq 0$. Then, induction gives

$$(\Delta^k g)(x) = (b - a)(b - a - 1) \ldots (b - a - k + 1) \frac{(x + b)!}{(x + a + k)!},$$

for all $x \in \mathbb{Z}$ with $x + b, x + a \geq 0$. Moreover, (3.12) gives

$$(\Delta^k g)(x) = k! \sum_{l=0}^{k} \frac{(x + b + l)!}{(x + a + l)!} \frac{1}{\prod_{j=0, j \neq l}^{k} (l - j)},$$

for all $x \in \mathbb{Z}$ with $x + b, x + a \geq 0$. Combined, the above prove (2). $\qquad\square$

Next we examine $K_n((u_n, r_n), (v_n, s_n))$:

Lemma 3.11. *Assume the conditions of Theorem 1.11. Define the sets U_n, V_n, P_n, $VU^{(n)}$, $VU_{(n)}$ as above. Also define β_n as in Lemma 3.8. Then,*

(1) $\beta_n K_n((u_n, r_n), (v_n, s_n))$

$$
= \sum_{y' \in \tilde{S}_{1,n}} \sum_{x' \in S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}
$$

$$
+ 1_{(v_n \geq u_n, s_n \leq r_n)} \sum_{y' \in (VU^{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}
$$

$$
- 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in (VU^{(n)}) \setminus P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.
$$

Moreover,

(2) $\beta_n K_n((u_n, r_n), (v_n, s_n))$

$$
= - \sum_{y' \in \tilde{S}_{3,n}} \sum_{x' \in S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}
$$

$$
- 1_{(v_n + s_n \leq u_n + r_n, s_n \leq r_n)} \sum_{y' \in (VU_{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}
$$

$$
+ 1_{(v_n + s_n \leq u_n + r_n, s_n > r_n)} \sum_{y' \in (VU_{(n)}) \setminus P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.
$$

Proof. First note, (1.3) and the expression for β_n (see statement of Lemma 3.8) give

$$
\beta_n K_n((u_n, r_n), (v_n, s_n)) = -\beta_n \phi_{r_n, s_n}(u_n, v_n)
$$

$$
+ \sum_{k=1}^{n} 1_{(x_k^{(n)} \geq u_n)} \sum_{l=v_n+s_n-n}^{v_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(\frac{x_k^{(n)}}{n} - \frac{j}{n})}{\prod_{j=v_n+s_n-n, j \neq l}^{v_n}(\frac{l}{n} - \frac{j}{n})} \prod_{i=1, i \neq k}^{n} \left(\frac{\frac{l}{n} - \frac{x_i^{(n)}}{n}}{\frac{x_k^{(n)}}{n} - \frac{x_i^{(n)}}{n}} \right).
$$

(3.8) and (3.9) then give

$$
\beta_n K_n((u_n, r_n), (v_n, s_n)) = -\beta_n \phi_{r_n, s_n}(u_n, v_n)
$$

$$
+ \sum_{y' \in \tilde{S}_{1,n}} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}. \quad (3.13)
$$

First, we will show:

(i)
$$\sum_{y' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = \begin{cases} \frac{\beta_n \prod_{j=1}^{s_n - r_n - 1}(v_n - u_n + s_n - r_n - j)}{(s_n - r_n - 1)!} & \text{when } s_n > r_n + 1, \\ 1 & \text{when } s_n = r_n + 1, \\ 0 & \text{when } s_n \leq r_n. \end{cases}$$

Then, we will use this to show:

(ii)
$$\beta_n \phi_{r_n, s_n}(u_n, v_n) = 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Next, recalling that $S_{2,n} \subset V_n$ (see (3.10)), we will show:

(iii)
$$\sum_{y' \in \tilde{S}_{1,n}} \sum_{x' \in V_n \setminus S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$
$$= 1_{(v_n \geq u_n)} \sum_{y' \in (VU^{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Thus, since $S_{2,n} \subset V_n$, (3.13), and parts (ii) and (iii) prove (1). Next we will show:

(iv)
$$\sum_{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n}} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$
$$= 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}$$
$$+ 1_{(v_n + s_n \leq u_n + r_n, s_n > r_n)} \sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Finally we will show:

(v)
$$\sum_{y' \in \tilde{S}_{3,n}} \sum_{x' \in V_n \setminus S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$
$$= 1_{(v_n + s_n \leq u_n + r_n)} \sum_{y' \in (VU_{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Thus, since $S_{2,n} \subset V_n$, (3.13) and parts (ii), (iv) and (v) prove (2).

Consider (i). Note, taking $k := n - s_n$ and $i := n - r_n - 1$ and $x := v_n - u_n + s_n - r_n$, part (1) of Lemma 3.10 gives

$$\sum_{l=0}^{n-s_n} \frac{\prod_{j=1}^{n-r_n-1}(v_n - u_n + s_n - r_n + l - j)}{\prod_{j=0, j\neq l}^{n-s_n}(l - j)}$$

$$= \begin{cases} \frac{(n-r_n-1)!}{(n-s_n)!(s_n-r_n-1)!} \prod_{j=1}^{s_n-r_n-1}(v_n - u_n + s_n - r_n - j) & \text{when } s_n > r_n + 1, \\ 1 & \text{when } s_n = r_n + 1, \\ 0 & \text{when } s_n < r_n + 1. \end{cases}$$

Shifting the dummy variables implies that the above LHS equals,

$$\sum_{l=v_n+s_n-n}^{v_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(l - j)}{\prod_{j=v_n+s_n-n, j\neq l}^{v_n}(l - j)}.$$

(3.8) and the expression for β_n (see statement of Lemma 3.8) then prove (i).

Consider (ii). First note, (i), (1.4), and the expression for β_n (see statement of Lemma 3.8) prove the following:

$$\beta_n \phi_{r_n,s_n}(u_n, v_n) = 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Therefore,

$$\beta_n \phi_{r_n,s_n}(u_n, v_n) = 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in V_n \setminus U_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Next note, (2.30) implies that $V_n \setminus U_n = (VU^{(n)}) \cup (VU_{(n)})$. Finally note, (2.30) also implies that $VU^{(n)} \neq \emptyset$ and $VU_{(n)} = \emptyset$ when $v_n \geq u_n$ and $s_n > r_n$. This proves (ii).

Consider (iii). Recall (see (3.9) and (3.10)) that $\tilde{S}_{1,n} \subset P_n$ and $S_{2,n} = V_n \setminus P_n$. Thus, for all $y' \in \tilde{S}_{1,n}$, V_n equals the disjoint union $S_{2,n} \cup (V_n \cap \{y'\}) \cup (V_n \cap (P_n \setminus y'))$. Therefore,

$$\sum_{y' \in \tilde{S}_{1,n}} \sum_{x' \in V_n \setminus S_{2,n}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$= \sum_{y' \in \tilde{S}_{1,n} \cap V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} \frac{\prod_{x \in P_n \setminus y'}(y' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$= \sum_{y' \in \tilde{S}_{1,n} \cap (V_n \setminus U_n)} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)}.$$

Note that (2.30) and (3.8) give $\tilde{S}_{1,n} \cap (V_n \setminus U_n) = (VU^{(n)}) \cap P_n$ when $v_n \geq u_n$, and $\tilde{S}_{1,n} \cap (V_n \setminus U_n) = \emptyset$ when $v_n < u_n$. This proves (iii). (v) follows similarly.

Consider (iv). First recall (see (3.10)) that $P_n \setminus U_n = \tilde{S}_{1,n} \cup \tilde{S}_{3,n}$, a disjoint union. Therefore,

$$\sum_{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n}} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}$$

$$= \sum_{y' \in P_n} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)}.$$

Thus, since P_n and U_n are sets of distinct points, and since $|U_n| < |P_n|$ (see (3.8)), Lagrange interpolation gives

$$\sum_{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n}} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)} = \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(x' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)}.$$

(i) then implies (iv) when $s_n \le r_n$. To see part (iv) when $s_n > r_n$, first note the above equation gives

$$\sum_{y' \in \tilde{S}_{1,n} \cup \tilde{S}_{3,n}} \sum_{x' \in V_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)} \frac{\prod_{x \in P_n \setminus y'}(x' - x)}{\prod_{y \in P_n \setminus y'}(y' - y)} = \sum_{x' \in V_n \setminus U_n} \frac{\prod_{y \in U_n}(x' - y)}{\prod_{x \in V_n \setminus x'}(x' - x)}.$$

Next note, (2.30) implies that $V_n \setminus U_n = (VU^{(n)}) \cup (VU_{(n)})$. Finally note, (2.30) also implies the following:

- $VU^{(n)} \ne \emptyset$ and $VU_{(n)} = \emptyset$ when $s_n > r_n$ and $v_n \ge u_n$.

- $VU^{(n)} = \emptyset$ and $VU_{(n)} = \emptyset$ when $s_n > r_n$ and $v_n < u_n$ and $v_n + s_n > u_n + r_n$.

- $VU^{(n)} = \emptyset$ and $VU_{(n)} \ne \emptyset$ when $s_n > r_n$ and $v_n + s_n \le u_n + r_n$.

The above exhaust all possibilities when $s_n > r_n$. This proves (iv). $\qquad\square$

Next we examine Φ_n:

Lemma 3.12. *Assume the conditions of Theorem 1.11. Define the sets* U_n, V_n, P_n, $VU^{(n)}$, $VU_{(n)}$ *as above. Also define* Φ_n *and* β_n *as in Lemma 3.8. Recall that one of the cases* (1)–(12) *of Lemma 2.2 must be satisfied. Then:*

$$\Phi_n = \begin{cases} \mathbf{1}_{(v_n \ge u_n, s_n > r_n)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} & \text{for cases (1)–(4),} \\[2mm] \mathbf{1}_{(v_n \ge u_n, s_n \le r_n)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} & \text{for cases (5)–(6),} \\[2mm] \mathbf{1}_{(v_n + s_n \le u_n + r_n, s_n \le r_n)} \sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} & \text{for cases (7)–(8),} \\[2mm] \mathbf{1}_{(v_n + s_n \le u_n + r_n, s_n > r_n)} \sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} & \text{for cases (9)–(12).} \end{cases}$$

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Proof. We will prove the result for cases (5) and (6) of Lemma 2.2. The other cases follow from similar considerations.

Assume that (5) or (6) is satisfied. First note, the definition of Φ_n given in the statement of Lemma 3.8 gives $\Phi_n = 0$ when $v_n < u_n$ or $s_n > r_n$. This proves the result for these cases. Next note, the definition also gives $\Phi_n = 0$ when $v_n \geq u_n$ and $s_n \leq r_n$ and $v_n + s_n > u_n + r_n$. Thus, in this case, it is necessary to show that,

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = 0.$$

To see this, first note that (2.29) and (2.30) give

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n+1-s_n} \sum_{l=u_n}^{v_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(l - j)}{\prod_{j=v_n+s_n-n, j \neq l}^{v_n}(l - j)}.$$

Therefore, since $v_n + s_n > u_n + r_n$,

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n+1-s_n} \sum_{l=v_n+s_n-n}^{v_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(l - j)}{\prod_{j=v_n+s_n-n, j \neq l}^{v_n}(l - j)}.$$

Finally, since $s_n \leq r_n$, we can proceed as in (i) in the proof of Lemma 3.11 to show that the above RHS equals 0. This proves the result when $v_n \geq u_n$ and $s_n \leq r_n$ and $v_n + s_n > u_n + r_n$. It thus remains to prove the result when $v_n \geq u_n$ and $s_n \leq r_n$ and $v_n + s_n \leq u_n + r_n$.

Suppose first that $v_n = u_n$ and $s_n \leq r_n$ and $v_n + s_n \leq u_n + r_n$. Then, (2.29) and (2.30) give

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n+1-s_n} \sum_{l=u_n}^{u_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(l - j)}{\prod_{j=u_n+s_n-n, j \neq l}^{u_n}(l - j)}$$

$$= n^{r_n+1-s_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(u_n - j)}{\prod_{j=u_n+s_n-n}^{u_n-1}(u_n - j)}.$$

Thus, since $s_n \leq r_n$,

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n+1-s_n} \frac{1}{\prod_{j=u_n+s_n-n}^{u_n+r_n-n}(u_n - j)}$$

$$= n^{r_n+1-s_n} \frac{(n - r_n - 1)!}{(n - s_n)!}.$$

The definitions of Φ_n and β_n in the statement of Lemma 3.8 then prove the result when $v_n = u_n$ and $s_n \leq r_n$ and $v_n + s_n \leq u_n + r_n$.

Finally suppose that $v_n > u_n$ and $s_n \le r_n$ and $v_n + s_n \le u_n + r_n$. Then, (2.29) and (2.30) give

$$\frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n + 1 - s_n} \sum_{l=u_n}^{v_n} \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(l - j)}{\prod_{j=v_n+s_n-n, j \ne l}^{v_n}(l - j)}$$

$$= n^{r_n + 1 - s_n} \sum_{l=u_n}^{v_n} \frac{1}{\prod_{j=v_n+s_n-n}^{u_n+r_n-n}(l - j)} \frac{1}{\prod_{j=u_n, j \ne l}^{v_n}(l - j)}.$$

Note that $l - j > 0$ for all $l \in \{u_n, u_n + 1, \ldots, v_n\}$ and $j \in \{v_n + s_n - n, v_n + s_n - n + 1, \ldots, u_n + r_n - n\}$ (indeed, (1.18) gives $l - j = n(1 - \eta) + o(n)$ uniformly for l, j). We can thus write,

$$\frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = n^{r_n + 1 - s_n} \sum_{l=u_n}^{v_n} \frac{(l - (u_n + r_n - n + 1))!}{(l - (v_n + s_n - n))!} \frac{1}{\prod_{j=u_n, j \ne l}^{v_n}(l - j)}$$

$$= n^{r_n + 1 - s_n} \sum_{l=0}^{v_n - u_n} \frac{(l - (r_n - n + 1))!}{(l - (v_n - u_n + s_n - n))!} \frac{1}{\prod_{j=0, j \ne l}^{v_n - u_n}(l - j)}.$$

The last part, above, is obtained by simply shifting the dummy variables. The above, part (2) of Lemma 3.10 (take $k := v_n - u_n$ and $x := n$ and $b := -r_n - 1$ and $a := u_n - v_n - s_n$), and the definition of β_n in the statement of Lemma 3.8 give

$$\frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \setminus y'}(y' - x)} = \frac{(s_n - r_n)(s_n - r_n + 1) \ldots (v_n + s_n - u_n - r_n - 1)}{(v_n - u_n)!} \beta_n.$$

The definition of Φ_n in the statement of Lemma 3.8 then proves the result when $v_n > u_n$ and $s_n \le r_n$ and $v_n + s_n \le u_n + r_n$. $\qquad \square$

Next we prove the following technical result:

Lemma 3.13. *Fix $(u, r) \in \mathbb{R}^2$ and $(v, s) \in \mathbb{R}^2$, and define $\{(u_n, r_n)\}_{n \ge 1}$ and $\{(v_n, s_n)\}_{n \ge 1}$ as in (1.28) and (1.29). Recall that one of the cases (1)–(12) of Lemma 2.2 must be satisfied. Fix $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied. Assume that $(u, r) \ne (v, s)$, i.e., that either $v > u$, or $v < u$, or $v = u$ and $s \ne r$.*
When $v > u$, the following are satisfied:

- *For cases (1)–(4), $v_n > u_n$ (and so $VU^{(n)} \ne \emptyset$) and $v_n + s_n > u_n + r_n$ (and so $VU_{(n)} = \emptyset$) and $s_n > r_n + 1$. Moreover, $t - 2\xi > \chi + 2\xi > \max(VU^{(n)}) > \min(VU^{(n)}) > \chi - 2\xi > \chi + \eta - 1 + 2\xi$.*

- *For cases (5)–(8), $v_n > u_n$ (and so $VU^{(n)} \ne \emptyset$) and $v_n + s_n < u_n + r_n$ (and so $VU_{(n)} \ne \emptyset$) and $s_n < r_n + 1$. Moreover, $\min(VU^{(n)}) > \chi - 2\xi > t + 2\xi > t - 2\xi > \chi + \eta - 1 + 2\xi > \max(VU_{(n)}) > \min(VU_{(n)}) > \chi + \eta - 1 - 2\xi$.*

- *For cases (9)–(12), $v_n < u_n$ (and so $VU^{(n)} = \emptyset$) and $v_n + s_n < u_n + r_n$ (and so $VU_{(n)} \neq \emptyset$) and $s_n > r_n + 1$. Moreover, $\chi - 2\xi > \chi + \eta - 1 + 2\xi > \max(VU_{(n)}) > \min(VU_{(n)}) > \chi + \eta - 1 - 2\xi > t + 2\xi$.*

Moreover, when $v < u$:

- *For cases (1)–(4), $v_n < u_n$ and $v_n + s_n < u_n + r_n$ and $s_n < r_n + 1$.*

- *For cases (5)–(8), $v_n < u_n$ and $v_n + s_n > u_n + r_n$ and $s_n > r_n + 1$.*

- *For cases (9)–(12), $v_n > u_n$ and $v_n + s_n > u_n + r_n$ and $s_n < r_n + 1$.*

Finally, when $v = u$ and $s \neq r$:

- *For cases (1)–(4), $v_n - u_n$ and $s_n - (r_n + 1)$ have opposite signs.*

- *For cases (5)–(12), $v_n - u_n$ and $s_n - (r_n + 1)$ and $(v_n + s_n) - (u_n + r_n)$ have the same sign.*

Proof. First note, (1.28) and (1.29) give

$$v_n - u_n = n^{\frac{2}{3}} m_n(v - u) + n^{\frac{1}{3}} p_n(e^{C_n(t)} - 1)(s - r) + O(1),$$

$$s_n - r_n - 1 = n^{\frac{2}{3}} m_n(e^{C_n(t)} - 1)(v - u) + n^{\frac{1}{3}} p_n(-1)(s - r) + O(1),$$

$$v_n + s_n - u_n - r_n = n^{\frac{2}{3}} m_n e^{C_n(t)}(v - u) + n^{\frac{1}{3}} p_n(e^{C_n(t)} - 2)(s - r) + O(1),$$

where $\{m_n\}_{n \geq 1}$ and $\{p_n\}_{n \geq 1}$ are those convergent sequences of real numbers with non-zero limits given in Definition 2.8. Note that this definition also gives $m_n(e^{C_n(t)} - 1)/e^{C_n(t)} > 0$, and so m_n and $(e^{C_n(t)} - 1)/e^{C_n(t)}$ have the same sign. Next recall (see Lemma 2.2 and 2.3) that $e^{C(t)} > 1$ for cases (1)–(4), $e^{C(t)} < 0$ for cases (5)–(8), and $e^{C(t)} \in (0, 1)$ for cases (9)–(12). Also recall that $e^{C_n(t)} \to e^{C(t)}$ (see (1.27)).

Consider cases (1)–(4) with $v > u$. The above observations then imply that $\{m_n\}_{n \geq 1}$ is a convergent sequence of real numbers with a positive limit, and that $v_n > u_n$ and $v_n + s_n > u_n + r_n$ and $s_n > r_n + 1$. (2.30) then implies that $VU^{(n)} \neq \emptyset$ and $VU_{(n)} = \emptyset$. Next note, since $t > \chi$ (see cases (1)–(4) of Lemma 2.2), (2.14) and (2.31) give $t - 2\xi > \chi + 2\xi > \max(VU^{(n)}) > \min(VU^{(n)}) > \chi - 2\xi > \chi + \eta - 1 + 2\xi$. We have thus shown the required result for cases (1)–(4) when $v > u$. The other cases follow similarly. $\qquad \square$

Finally, we prove Lemma 3.8:

Proof of Lemma 3.8. We will prove part (1) for cases (1)–(4) of Lemma 2.2, and part (2) for cases (1)–(4) and (7)–(8). Parts (1) and (2) for the remaining cases follow from similar considerations.

Part (1) for cases (1)–(3) of Lemma 2.2 easily follows from Lemmas 3.9 and 3.11 and 3.12. Consider part (1) for case (4) of Lemma 2.2. Note, Lemmas 3.9 and 3.11 give

$$\beta_n K_n((u_n, r_n), (v_n, s_n)) = J_n + 1_{(v_n \geq u_n, s_n \leq r_n)} \sum_{y' \in (VU^{(n)}) \cap P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \backslash y'}(y' - x)}$$
$$- 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in (VU^{(n)}) \backslash P_n} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \backslash y'}(y' - x)}.$$

Lemma 3.12 thus implies that the result for case (4) follows if $(VU^{(n)}) \cap P_n = \emptyset$. To see this, fix $\delta > 0$ such that $(\chi - \delta, \chi + \delta) \subset \mathbb{R} \backslash \text{Supp}(\mu)$ (see Assumption 2.4). Assumption 1.7 and (1.24) then give $(\chi - \delta, \chi + \delta) \cap P_n = \emptyset$. Finally recall (see (1.18) and (2.30)) that $VU^{(n)} \subset (\chi - \delta, \chi + \delta)$. Therefore $(VU^{(n)}) \cap P_n = \emptyset$, as required.

Consider part (2) for cases (1)–(4) of Lemma 2.2. First recall (see Lemma 3.12) that,

$$\Phi_n = 1_{(v_n \geq u_n, s_n > r_n)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \backslash y'}(y' - x)}.$$

Next recall (see statement of Lemma 3.8) that $(u, r) \neq (v, s)$, i.e., that either $v > u$, or $v < u$, or $v = u$ and $s \neq r$. Moreover, Lemma 3.13 implies that $1_{(v_n \geq u_n, s_n > r_n)} = 1$ when $v > u$, $1_{(v_n \geq u_n, s_n > r_n)} = 0$ when $v < u$, and $1_{(v_n \geq u_n, s_n > r_n)} = 0$ when $v = u$ and $s \neq r$. Therefore,

$$\Phi_n = 1_{(v>u)} \sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \backslash y'}(y' - x)}.$$

It thus remains to show that,

$$\sum_{y' \in VU^{(n)}} \frac{\prod_{y \in U_n}(y' - y)}{\prod_{x \in V_n \backslash y'}(y' - x)} = \frac{1}{2\pi i} \int_{\kappa_n} dw \frac{\prod_{y \in U_n}(w - y)}{\prod_{x \in V_n}(w - x)},$$

when $v > u$. To see the above, first note that the integrand on the RHS has a simple pole at each distinct element of $V_n \backslash U_n$. Next note, since $v > u$ and one of cases (1)–(4) is satisfied, (2.30) and Lemma 3.13 imply that $V_n \backslash U_n = VU^{(n)} \subset (\chi - 2\xi, \chi + 2\xi)$. Finally, Lemma 3.6 and Definition 3.7 and Figure 3.3 clearly imply that κ_n contains $(\chi - 2\xi, \chi + 2\xi)$. The above equation thus follows from the Residue theorem. This proves part (2) for cases (1)–(4) of Lemma 2.2.

Consider part (2) for cases (7) and (8) of Lemma 2.2. First recall (see Lemma 3.12) that,

$$\Phi_n = 1_{(v_n+s_n \leq u_n+r_n, s_n \leq r_n)} \sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y'-y)}{\prod_{x \in V_n \setminus y'}(y'-x)}.$$

Next recall (see statement of Lemma 3.8) that $(u,r) \neq (v,s)$, i.e., that either $v > u$, or $v < u$, or $v = u$ and $s \neq r$. Moreover, Lemma 3.13 implies that $1_{(v_n+s_n \leq u_n+r_n, s_n \leq r_n)} = 1$ when $v > u$, and $1_{(v_n+s_n \leq u_n+r_n, s_n \leq r_n)} = 0$ when $v < u$. Finally, when $v = u$ and $s \neq r$, Lemma 3.13 implies that either of the following is satisfied:

- $v_n + s_n > u_n + r_n$ and $s_n > r_n + 1$ and $v_n > u_n$ for all n sufficiently large. In this case $1_{(v_n+s_n \leq u_n+r_n, s_n \leq r_n)} = 0$.

- $v_n + s_n < u_n + r_n$ and $s_n < r_n + 1$ and $v_n < u_n$ for all n sufficiently large. In this case (2.30) gives $V_n \setminus U_n = VU_{(n)}$, and part (i) in the proof of Lemma 3.11 gives

$$\sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y'-y)}{\prod_{x \in V_n \setminus y'}(y'-x)} = 0.$$

Combined, the above observations give

$$\Phi_n = 1_{(v>u)} \sum_{y' \in VU_{(n)}} \frac{\prod_{y \in U_n}(y'-y)}{\prod_{x \in V_n \setminus y'}(y'-x)}.$$

We can then proceed similar to above to prove part (2) for cases (7) and (8) of Lemma 2.2. □

3.4. **Proof of Theorem 1.11**

In this section we finally prove Theorem 1.11 using the results of the previous sections. We will prove the result only when $t \in R_\mu^+$, i.e, when one of cases (1)–(4) of Lemma 2.2 is satisfied. The results when $t \in R_{\lambda-\mu}$ (cases (5)–(8)), and when $t \in R_\mu^-$ (cases (9)–(12)), follow from similar considerations.

Assume the conditions of Theorem 1.11. Additionally assume that one of cases (1)–(4) of Lemma 2.2 is satisfied. Lemma 3.8 thus gives

$$\beta_n K_n((u_n, r_n), (v_n, s_n)) = J_n - \Phi_n. \tag{3.14}$$

We begin by using a steepest descent argument to examine the asymptotic behaviour of J_n. First, fix $\theta \in (\frac{1}{4}, \frac{1}{3})$, and $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ as in Definition 2.8. Next, using Lemma 3.5 and Definition 3.7, we partition γ_n as follows:

$$\gamma_n = \gamma_n^{(l)} + \gamma_n^{(r)} \qquad \text{and} \qquad \Gamma_n = \Gamma_n^{(l)} + \Gamma_n^{(r)}, \tag{3.15}$$

where $\gamma_n^{(l)}$ and $\Gamma_n^{(l)}$ are (respectively) those *local sections* of γ_n and Γ_n inside $B(t, n^{-\theta}|q_n|)$, and $\gamma_n^{(r)}$ and $\Gamma_n^{(r)}$ are (respectively) the *remaining sections* of γ_n and Γ_n outside $B(t, n^{-\theta}|q_n|)$. Then, the definition of J_n in the statement of Lemma 3.8 gives

$$J_n = J_n^{(l,l)} + J_n^{(l,r)} + J_n^{(r,l)} + J_n^{(r,r)}, \tag{3.16}$$

where

$$J_n^{(l,l)} := \frac{1}{(2\pi i)^2} \int_{\gamma_n^{(l)}} dw \int_{\Gamma_n^{(l)}} dz \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(z-\frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w-\frac{j}{n})} \frac{1}{w-z} \prod_{i=1}^{n} \left(\frac{w - \frac{x_i}{n}}{z - \frac{x_i}{n}} \right).$$

The other three terms on the RHS of (3.16) are defined analogously. As we shall see in the following lemmas, the asymptotic behaviour of $J_n^{(l,l)}$ dominates the other three terms:

Lemma 3.14. *Assume the conditions of Theorem 1.11. Additionally assume that one of cases* (1)–(4) *of Lemma 2.2 is satisfied. Fix* $\theta \in (\frac{1}{4}, \frac{1}{3})$, *and* $\{q_n\}_{n\geq 1} \subset \mathbb{R}$ *as in Definition 2.8. Define* $J_n^{(l,l)}$ *as in* (3.16), $\widetilde{K}_{Ai} : (\mathbb{R}^2)^2 \to \mathbb{R}$ *as in* (1.19), *and* $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ *as in Lemma 2.13. Then,*

$$\frac{n^{\frac{1}{3}} |q_n|^{-1}(t - \chi_n)(t - \chi_n - \eta_n + 1)}{A_{t,n}((u_n, r_n), (v_n, s_n))} J_n^{(l,l)} \to \widetilde{K}_{Ai}((v, s), (u, r)).$$

Proof. First note, (1.8), (1.9) and (3.16) give

$$J_n^{(l,l)} = \frac{1}{(2\pi i)^2} \int_{\gamma_n^{(l)}} dw \int_{\Gamma_n^{(l)}} dz \frac{\exp(n f_n(w) - n \tilde{f}_n(z))}{w - z}.$$

Define $d_{1,n}$ and $\tilde{a}_{1,n}$ as in Lemma 3.5. Also define,

$$
\begin{aligned}
\alpha_n &:= \begin{cases} \mathrm{Arg}(d_{1,n} - t) & \text{for cases (1) and (2) of Lemma 2.2,} \\ \mathrm{Arg}(\tilde{a}_{1,n} - t) & \text{for cases (3) and (4) of Lemma 2.2,} \end{cases} \\
\zeta_n &:= \begin{cases} \mathrm{Arg}(\tilde{a}_{1,n} - t) & \text{for cases (1) and (2) of Lemma 2.2,} \\ \mathrm{Arg}(d_{1,n} - t) & \text{for cases (3) and (4) of Lemma 2.2,} \end{cases}
\end{aligned}
\tag{3.17}
$$

where Arg represents the principal value of the argument, and note that part (2) of Lemma 3.5 gives $\alpha_n = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ and $\zeta_n = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$. Recall that $\gamma_n^{(l)}$ and $\Gamma_n^{(l)}$ are (respectively) those sections of γ_n and Γ_n inside $B(t, n^{-\theta}|q_n|)$ (see (3.15)), and γ_n and Γ_n are counter-clockwise (see Definition 3.7). Lemma 3.5 and Figure 3.2 then imply, for cases (1) and (2) of Lemma 2.2, that:

- $\gamma_n^{(l)}$ is the lines from $t + n^{-\theta}|q_n|e^{-i\alpha_n}$ to t, and from t to $t + n^{-\theta}|q_n|e^{i\alpha_n}$.

- $\Gamma_n^{(l)}$ is the lines from $t + n^{-\theta}|q_n|e^{-i\zeta_n}$ to t, and from t to $t + n^{-\theta}|q_n|e^{i\zeta_n}$.

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Moreover, for cases (3) and (4):

- $\gamma_n^{(l)}$ is the lines from $t + n^{-\theta}|q_n|e^{-i\zeta_n}$ to t, and from t to $t + n^{-\theta}|q_n|e^{i\zeta_n}$.

- $\Gamma_n^{(l)}$ is the lines from $t + n^{-\theta}|q_n|e^{i\alpha_n}$ to t, and from t to $t + n^{-\theta}|q_n|e^{-i\alpha_n}$.

For cases (1) and (2), a change of variables thus gives

$$J_n^{(l,l)} = \frac{n^{-\frac{1}{3}}|q_n|}{(2\pi i)^2} \int_{h_n} dw \int_{H_n} dz \frac{\exp(nf_n(t + n^{-\frac{1}{3}}|q_n|w) - n\tilde{f}_n(t + n^{-\frac{1}{3}}|q_n|z))}{w - z},$$

where:

- h_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\alpha_n}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\alpha_n}$.

- H_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\zeta_n}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\zeta_n}$.

These contours are shown on the left of Figure 3.4. Similarly, for cases (3) and (4),

$$J_n^{(l,l)} = \frac{n^{-\frac{1}{3}}|q_n|}{(2\pi i)^2} \int_{h_n} dw \int_{H_n} dz \frac{\exp(nf_n(t - n^{-\frac{1}{3}}|q_n|w) - n\tilde{f}_n(t - n^{-\frac{1}{3}}|q_n|z))}{w - z},$$

where h_n and H_n are defined as above.

Next recall that $q_n > 0$ for cases (1) and (2) of Lemma 2.2, and $q_n < 0$ for cases (3) and (4) (see Definition 2.8 and Lemma 2.2). Therefore, for cases (1)–(4),

$$J_n^{(l,l)} = \frac{n^{-\frac{1}{3}}|q_n|}{(2\pi i)^2} \int_{h_n} dw \int_{H_n} dz \frac{\exp(nf_n(t + n^{-\frac{1}{3}}q_n w) - n\tilde{f}_n(t + n^{-\frac{1}{3}}q_n z))}{w - z}.$$

Parts (3) and (4) of Corollary 3.3 then give

$$J_n^{(l,l)} = n^{-\frac{1}{3}}|q_n| \exp(nf_n(t) - n\tilde{f}_n(t) + O(n^{1-4\theta}))I_n,$$

for cases (1)–(4), where

$$I_n := \frac{1}{(2\pi i)^2} \int_{h_n} dw \int_{H_n} dz \frac{1}{w - z} \frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(zr + z^2u + \frac{1}{3}z^3)}.$$

(2.25) and Lemma 2.13 then give

$$J_n^{(l,l)} = \frac{n^{-\frac{1}{3}}|q_n|A_{t,n}((u_n, r_n), (v_n, s_n))}{(t - \chi_n)(t - \chi_n - \eta_n + 1)} \exp(O(n^{-\frac{1}{3}} + n^{1-4\theta}))I_n.$$

Next, recall that $\alpha_n = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ and $\zeta_n = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ for cases (1)–(4), and define:

- l_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\frac{\pi}{3}}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\frac{\pi}{3}}$. c_n is the smallest arcs of $\partial B(0, n^{\frac{1}{3}-\theta})$ from $n^{\frac{1}{3}-\theta}e^{-i\alpha_n}$ to $n^{\frac{1}{3}-\theta}e^{-i\frac{\pi}{3}}$, and from $n^{\frac{1}{3}-\theta}e^{i\frac{\pi}{3}}$ to $n^{\frac{1}{3}-\theta}e^{i\alpha_n}$.
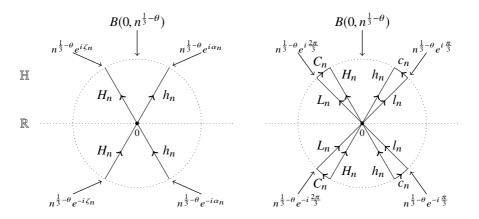
FIGURE 3.4. Left: The contours h_n and H_n. Recall $\theta \in (\frac{1}{4}, \frac{1}{3})$, $\alpha_n = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ and $\zeta_n = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ (see (3.17)). Right: The contours $h_n, H_n, l_n, L_n, c_n, C_n$.

- L_n is the lines from $n^{\frac{1}{3}-\theta} e^{-i\frac{2\pi}{3}}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta} e^{i\frac{2\pi}{3}}$. C_n is the smallest arcs of $\partial B(0, n^{\frac{1}{3}-\theta})$ from $n^{\frac{1}{3}-\theta} e^{-i\zeta_n}$ to $n^{\frac{1}{3}-\theta} e^{-i\frac{2\pi}{3}}$, and from $n^{\frac{1}{3}-\theta} e^{i\frac{2\pi}{3}}$ to $n^{\frac{1}{3}-\theta} e^{i\zeta_n}$.

These contours are shown on the right of Figure 3.4. Then, noting that h_n and $c_n + l_n$ have the same initial and final points, and similarly for H_n and $C_n + L_n$,

$$I_n = I_{1,n} + I_{2,n} + I_{3,n} + I_{4,n},$$

where

$$I_{1,n} := \frac{1}{(2\pi i)^2} \int_{l_n} dw \int_{L_n} dz \frac{1}{w-z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)},$$

$$I_{2,n} := \frac{1}{(2\pi i)^2} \int_{l_n} dw \int_{C_n} dz \frac{1}{w-z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)},$$

$$I_{3,n} := \frac{1}{(2\pi i)^2} \int_{c_n} dw \int_{L_n} dz \frac{1}{w-z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)},$$

$$I_{4,n} := \frac{1}{(2\pi i)^2} \int_{c_n} dw \int_{C_n} dz \frac{1}{w-z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)}.$$

(3.18)

Finally, we will show that,

(i) $I_{1,n} \to \widetilde{K}_{\text{Ai}}((v, s), (u, r))$.

(ii) $I_{2,n} \to 0$ and $I_{3,n} \to 0$ and $I_{4,n} \to 0$.

Thus, since $\theta \in (\frac{1}{4}, \frac{1}{3})$, the required result follows from parts (i,ii) and the above expression of $J_n^{(l,l)}$.

Consider (i). First define:

- r_n is the lines from $\infty e^{-i\frac{\pi}{3}}$ to $n^{\frac{1}{3}-\theta} e^{-i\frac{\pi}{3}}$, and from $n^{\frac{1}{3}-\theta} e^{i\frac{\pi}{3}}$ to $\infty e^{i\frac{\pi}{3}}$.

- R_n is the lines from $\infty e^{-i\frac{2\pi}{3}}$ to $n^{\frac{1}{3}-\theta} e^{-i\frac{2\pi}{3}}$, and from $n^{\frac{1}{3}-\theta} e^{i\frac{2\pi}{3}}$ to $\infty e^{i\frac{2\pi}{3}}$.

It thus follows from Figures 1.3 and 3.4 that $l = l_n + r_n$ and $L = L_n + R_n$. (1.19) and the definition of $I_{1,n}$, above, thus give

$$\widetilde{K}_{\text{Ai}}((v, s), (u, r)) = I_{1,n} + I'_{1,n} + I''_{1,n} + I'''_{1,n},$$

where

$$I'_{1,n} := \frac{1}{(2\pi i)^2} \int_{l_n} dw \int_{R_n} dz \frac{1}{w - z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)},$$

$$I''_{1,n} := \frac{1}{(2\pi i)^2} \int_{r_n} dw \int_{L_n} dz \frac{1}{w - z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)}, \qquad (3.19)$$

$$I'''_{1,n} := \frac{1}{(2\pi i)^2} \int_{r_n} dw \int_{R_n} dz \frac{1}{w - z} \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)}.$$

Part (i) thus follows if we can show that $I'_{1,n} = o(1)$, $I''_{1,n} = o(1)$, and $I'''_{1,n} = o(1)$.

Consider $I'_{1,n}$, defined in (3.19). The contours in this expression are given in Figure 3.5. Note that,

$$\frac{1}{|w - z|} \leq \frac{1}{n^{\frac{1}{3}-\theta} \cos(\frac{\pi}{3})} = \frac{2}{n^{\frac{1}{3}-\theta}}, \qquad (3.20)$$

uniformly for w on l_n and z on R_n. Also,

$$\left| \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)} \right| = \frac{\exp(\text{Re}(ws + w^2 v + \frac{1}{3}w^3))}{\exp(\text{Re}(zr + z^2 u + \frac{1}{3}z^3))},$$

for all w on l_n and z on R_n. Moreover, $|\text{Arg}(w)| = \frac{\pi}{3}$ and $|\text{Arg}(z)| = \frac{2\pi}{3}$ for all w on l_n and z on R_n, and so $\text{Re}(w^3) = -|w|^3$ and $\text{Re}(z^3) = |z|^3$. Therefore,

$$\left| \frac{\exp(ws + w^2 v + \frac{1}{3}w^3)}{\exp(zr + z^2 u + \frac{1}{3}z^3)} \right| \leq \frac{\exp(|w||s| + |w|^2|v| - \frac{1}{3}|w|^3)}{\exp(-|z||r| - |z|^2|u| + \frac{1}{3}|z|^3)}, \qquad (3.21)$$

for all w on l_n and z on R_n. (3.19), (3.20) and (3.21) then give

$$|I'_{1,n}| \leq 4 \frac{1}{(2\pi)^2} \int_0^\infty dy_1 \int_0^\infty dy_2 \frac{2}{n^{\frac{1}{3}-\theta}} \frac{\exp(y_1|s| + y_1^2|v| - \frac{1}{3}y_1^3)}{\exp(-y_2|r| - y_2^2|u| + \frac{1}{3}y_2^3)}.$$
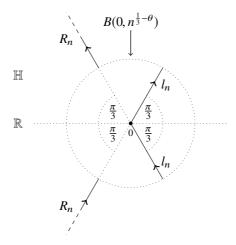
FIGURE 3.5. The contours l_n and R_n. l_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\frac{\pi}{3}}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\frac{\pi}{3}}$. R_n is the lines from $\infty e^{-i\frac{2\pi}{3}}$ to $n^{\frac{1}{3}-\theta}e^{-i\frac{2\pi}{3}}$, and from $n^{\frac{1}{3}-\theta}e^{i\frac{2\pi}{3}}$ to $\infty e^{i\frac{2\pi}{3}}$.

The above integral converges, and so $I'_{1,n} = O(n^{-(\frac{1}{3}-\theta)})$. Similarly it can be shown that $I''_{1,n} = O(n^{-(\frac{1}{3}-\theta)})$ and $I'''_{1,n} = O(n^{-(\frac{1}{3}-\theta)})$. Finally, since $\theta \in (\frac{1}{4}, \frac{1}{3})$, it follows that $I'_{1,n} = o(1)$ and $I''_{1,n} = o(1)$ and $I'''_{1,n} = o(1)$. This proves (i).

Consider (ii). Recall that $I_{2,n}$ is defined in (3.18). The contours in this expression are given in Figure 3.6. Also recall that $\zeta_n = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$. It thus follows that $|\operatorname{Arg}(w)| = \frac{\pi}{3}$ for all w on l_n, and $|\operatorname{Arg}(z)| = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ uniformly for z on C_n. Moreover,

$$\frac{1}{|w-z|} \le \frac{1}{n^{\frac{1}{3}-\theta}\cos(\frac{\pi}{3}+o(1))} < \frac{4}{n^{\frac{1}{3}-\theta}}, \tag{3.22}$$

uniformly for w on l_n and z on C_n. Also,

$$\left|\frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(zr + z^2u + \frac{1}{3}z^3)}\right| = \frac{\exp(\operatorname{Re}(ws + w^2v + \frac{1}{3}w^3))}{\exp(\operatorname{Re}(zr + z^2u + \frac{1}{3}z^3))},$$

for all w on l_n and z on C_n. Moreover, $|\operatorname{Arg}(w)| = \frac{\pi}{3}$ and $|\operatorname{Arg}(z)| = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ uniformly for w on l_n and z on C_n, and so $\operatorname{Re}(w^3) = -|w|^3$ and $\operatorname{Re}(z^3) > \frac{1}{2}|z|^3$ for all such w, z. Therefore,

$$\left|\frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(zr + z^2u + \frac{1}{3}z^3)}\right| < \frac{\exp(|w||s| + |w|^2|v| - \frac{1}{3}|w|^3)}{\exp(-|z||r| - |z|^2|u| + \frac{1}{6}|z|^3)}.$$
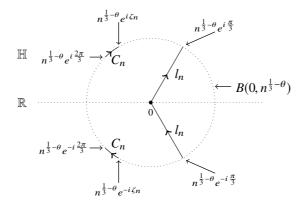
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FIGURE 3.6. The contours l_n and C_n. l_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\frac{\pi}{3}}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\frac{\pi}{3}}$. C_n is the smallest arcs of $\partial B(0, n^{\frac{1}{3}-\theta})$ from $n^{\frac{1}{3}-\theta}e^{-i\zeta_n}$ to $n^{\frac{1}{3}-\theta}e^{-i\frac{2\pi}{3}}$, and from $n^{\frac{1}{3}-\theta}e^{i\frac{2\pi}{3}}$ to $n^{\frac{1}{3}-\theta}e^{i\zeta_n}$.

Finally, note that $|z| = n^{\frac{1}{3}-\theta}$ for all z on C_n, and that $n^{\frac{1}{3}-\theta} \to \infty$ since $\theta \in (\frac{1}{4}, \frac{1}{3})$. Therefore,

$$\left| \frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(zr + z^2u + \frac{1}{3}z^3)} \right| < \frac{\exp(|w||s| + |w|^2|v| - \frac{1}{3}|w|^3)}{\exp(\frac{1}{12}n^{1-3\theta})}, \tag{3.23}$$

uniformly for w on l_n and z on C_n. (3.18), (3.22) and (3.23) then give

$$|I_{2,n}| < 4\frac{1}{(2\pi)^2}\int_0^\infty dy_1 \int_0^{2\pi} n^{\frac{1}{3}-\theta}dy_2 \frac{4}{n^{\frac{1}{3}-\theta}}\frac{\exp(y_1|s| + y_1^2|v| - \frac{1}{3}y_1^3)}{\exp(\frac{1}{12}n^{1-3\theta})}.$$

The above integral converges, and so $I_{2,n} = O(\exp(-\frac{1}{12}n^{1-3\theta}))$. Therefore $I_{2,n} = o(1)$ since $\theta \in (\frac{1}{4}, \frac{1}{3})$. Similarly, we can show that $I_{3,n} = o(1)$ and $I_{4,n} = o(1)$. This proves (ii). $\square$

Next we examine the asymptotic behaviour of the remaining terms of (3.16):

Lemma 3.15. *Assume the conditions of Theorem 1.11. Additionally assume that one of cases (1)–(4) of Lemma 2.2 is satisfied. Fix $\theta \in (\frac{1}{4}, \frac{1}{3})$, and $\{q_n\}_{n\geq 1} \subset \mathbb{R}$ as in Definition 2.8. Define $J_n^{(l,r)}, J_n^{(r,l)}, J_n^{(r,r)}$ as in (3.16), and $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ as in Lemma 2.13. Then,*

$$\frac{n^{\frac{1}{3}}|q_n|^{-1}|t - \chi_n||t - \chi_n - \eta_n + 1|}{|A_{t,n}((u_n, r_n), (v_n, s_n))|}|J_n^{(l,r)}| = O\left(n^{\frac{1}{3}}\exp\left(-\frac{1}{12}n^{1-3\theta}\right)\right).$$

Similarly for $J_n^{(r,l)}$ and $J_n^{(r,r)}$.

Proof. We prove the result for $J_n^{(l,r)}$ for cases (1) and (2) of Lemma 2.2, and state that the remaining results follow similarly.

Assume that case (1) or (2) of Lemma 2.2 is satisfied. Lemma 2.2 and Definition 2.8 then imply that q_n converges to a positive constant as $n \to \infty$. Next note, (1.8), (1.9) and (3.16) give

$$|J_n^{(l,r)}| \leq \frac{1}{(2\pi)^2} |\gamma_n^{(l)}||\Gamma_n^{(r)}| \sup_{(w,z)\in\gamma_n^{(l)}\times\Gamma_n^{(r)}} \left| \frac{\exp(nf_n(w) - n\tilde{f}_n(z))}{w - z} \right|.$$

Recall (see (3.15)) that $\gamma_n^{(l)}$ is that section of γ_n inside $B(t, n^{-\theta}q_n)$. Therefore $|\gamma_n^{(l)}| = 2n^{-\theta}q_n$ (see part (2) of Lemma 3.5 and Definition 3.7). Thus, since q_n converges to a positive constant as $n \to \infty$, $|\gamma_n^{(l)}| = O(n^{-\theta})$. Next recall (see (3.15)) that $\Gamma_n^{(r)}$ is that part of Γ_n outside $B(t, n^{-\theta}q_n)$. Definition 3.7 and parts (4)–(6) of Lemma 3.5 thus give

$$|J_n^{(l,r)}| \leq C \sup_{w\in\gamma_n^{(l)}} |\exp(nf_n(w) - n\tilde{f}_n(\tilde{a}_{1,n}))|,$$

where $\tilde{a}_{1,n} \subset \partial B(t, n^{-\theta}q_n)$ is defined in part (2) of Lemma 3.5, and where $C > 0$ is some fixed constant. Recall that q_n converges to a positive constant as $n \to \infty$, $\gamma_n^{(l)} \subset B(t, n^{-\theta}q_n)$ and $\tilde{a}_{1,n} \in \partial B(t, n^{-\theta}q_n)$. Also recall (see previous lemma), since one of cases (1) and (2) of Lemma 2.2 is satisfied, $|\operatorname{Arg}(w - t)| = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ uniformly for w on $\gamma_n^{(l)}$, and $\operatorname{Arg}(\tilde{a}_{1,n} - t) = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$. Therefore,

$$|J_n^{(l,r)}| \leq C \sup_{w\in h_n} \left| \exp(nf_n(t + n^{-\frac{1}{3}}q_n w) - n\tilde{f}_n(t + n^{-\frac{1}{3}}q_n z_n)) \right|,$$

where $h_n \subset B(0, n^{\frac{1}{3}-\theta})$ is defined as on the left of Figure 3.4, and $z_n \subset \partial B(0, n^{\frac{1}{3}-\theta})$ is defined by $z_n := n^{\frac{1}{3}}q_n^{-1}(\tilde{a}_{1,n} - t)$. Note, $|\operatorname{Arg}(w)| = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ uniformly for w on h_n, and $\operatorname{Arg}(z_n) = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$. Also note, parts (3) and (4) of Corollary 3.3 give

$$|J_n^{(l,r)}| \leq C|\exp(nf_n(t) - n\tilde{f}_n(t) + O(n^{1-4\theta}))| \sup_{w\in h_n} \left| \frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right|.$$

Lemma 2.13 then gives

$$|J_n^{(l,r)}| \leq C \frac{|A_{t,n}((u_n, r_n),(v_n, s_n))|}{|t - \chi_n||t - \chi_n - \eta_n + 1|} \exp(O(n^{-\frac{1}{3}} + n^{1-4\theta}))$$

$$\times \sup_{w\in h_n} \left| \frac{\exp(ws + w^2v + \frac{1}{3}w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right|.$$

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Finally recall that $\theta \in (\frac{1}{4}, \frac{1}{3})$. Therefore we can choose the constant $C > 0$ such that,

$$|J_n^{(l,r)}| \leq C \frac{|A_{t,n}((u_n, r_n), (v_n, s_n))|}{|t - \chi_n||t - \chi_n - \eta_n + 1|} \sup_{w \in h_n} \left| \frac{\exp(ws + w^2 v + \frac{1}{3} w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right|. \tag{3.24}$$

Next note that,

$$\left| \frac{\exp(ws + w^2 v + \frac{1}{3} w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right| = \frac{\exp(\text{Re}(ws + w^2 v + \frac{1}{3} w^3))}{\exp(\text{Re}(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3))},$$

for all w on h_n. Recall that $|\text{Arg}(w)| = \frac{\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ uniformly for w on h_n. Therefore $\text{Re}(w^3) \leq 0$ for all w on h_n. Moreover, recall that $\text{Arg}(z_n) = \frac{2\pi}{3} + O(n^{-\frac{1}{3} + \theta})$. Therefore $\text{Re}((z_n)^3) > \frac{1}{2}|z_n|^3$. Combine the above to get,

$$\left| \frac{\exp(ws + w^2 v + \frac{1}{3} w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right| \leq \frac{\exp(|w||s| + |w|^2 |v| + 0)}{\exp(-|z_n||r| - |z_n|^2 |u| + \frac{1}{6}|z_n|^3)},$$

for all w on h_n. Finally recall that $|w| \leq n^{\frac{1}{3} - \theta}$ for all w on h_n (since $h_n \subset B(0, n^{\frac{1}{3} - \theta})$), $|z_n| = n^{\frac{1}{3} - \theta}$ (since $z_n \in \partial B(0, n^{\frac{1}{3} - \theta})$), and $n^{\frac{1}{3} - \theta} \to \infty$ (since $\theta \in (\frac{1}{4}, \frac{1}{3})$). Therefore,

$$\sup_{w \in h_n} \left| \frac{\exp(ws + w^2 v + \frac{1}{3} w^3)}{\exp(z_n r + (z_n)^2 u + \frac{1}{3}(z_n)^3)} \right| \leq \frac{\exp(O(n^{\frac{2}{3} - 2\theta}))}{\exp(O(n^{\frac{2}{3} - 2\theta}) + \frac{1}{6} n^{1 - 3\theta})}$$

$$\leq \frac{1}{\exp(\frac{1}{12} n^{1 - 3\theta})}.$$

Substitute into (3.24) to get the required result. □

(3.16), and the above two lemmas, give the asymptotic behaviour of J_n. It remains to consider the asymptotic behaviour of Φ_n (see (3.14)). Since many of the arguments of the following lemma are similar to those used in the proofs of Lemmas 3.14 and 3.15, we do not go into as much detail here:

Lemma 3.16. *Assume the conditions of Theorem 1.11. Additionally assume that one of cases* (1)–(4) *of Lemma 2.2 is satisfied. Fix* $\theta \in (\frac{1}{4}, \frac{1}{3})$, *and* $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ *as in Definition 2.8. Define* Φ_n *as in Lemma 3.8,* $\Phi : (\mathbb{R}^2)^2 \to \mathbb{R}$ *as in* (1.20), *and* $A_{t,n} : (\mathbb{Z}^2)^2 \to \mathbb{R} \setminus \{0\}$ *as in Lemma 2.13. Then, when the parameters of* (1.28) *and* (1.29) *satisfy* $(u, r) \neq (v, s)$,

$$\frac{n^{\frac{1}{3}} |q_n|^{-1}(t - \chi_n)(t - \chi_n - \eta_n + 1)}{A_{t,n}((u_n, r_n), (v_n, s_n))} \Phi_n \to 1_{(v>u)} \Phi((v, s), (u, r)).$$

Proof. First note, since $(u, r) \neq (v, s)$, part (2) of Lemma 3.8 gives

$$\Phi_n = 1_{(v>u)} \frac{1}{2\pi i} \int_{\kappa_n} dw \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(w - \frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w - \frac{j}{n})}.$$

This proves the result when $v < u$, and when $v = u$ and $s \neq r$. It thus remains to show the result when $v > u$. Assuming this, partition κ_n (see Definition 3.7 and Lemma 3.6) as $\kappa_n = \kappa_n^{(l)} + \kappa_n^{(r)}$, where $\kappa_n^{(l)}$ is that *local section* of κ_n inside $B(t, n^{-\theta}|q_n|)$, and $\kappa_n^{(l)}$ is the *remaining section* of κ_n outside $B(t, n^{-\theta}|q_n|)$. Therefore $\Phi_n = \Phi_n^{(l)} + \Phi_n^{(r)}$, where

$$\Phi_n^{(l)} := \frac{1}{2\pi i} \int_{\kappa_n^{(l)}} dw \, \frac{\prod_{j=u_n+r_n-n+1}^{u_n-1}(w - \frac{j}{n})}{\prod_{j=v_n+s_n-n}^{v_n}(w - \frac{j}{n})},$$

and $\Phi_n^{(r)}$ is defined analogously. We will show that:

(i)
$$\frac{n^{\frac{1}{3}}|q_n|^{-1}(t - \chi_n)(t - \chi_n - \eta_n + 1)}{A_{t,n}((u_n, r_n), (v_n, s_n))}\Phi_n^{(l)} \to \frac{\exp(-\frac{1}{4}\frac{(s-r)^2}{v-u})}{2\sqrt{\pi(v - u)}},$$

(ii)
$$\frac{n^{\frac{1}{3}}|q_n|^{-1}(t - \chi_n)(t - \chi_n - \eta_n + 1)}{A_{t,n}((u_n, r_n), (v_n, s_n))}\Phi_n^{(r)} \to 0.$$

Parts (i) and (ii) and (1.20) then give the required result.

Consider part (i). First note, (1.8), (1.9) and (2.25) give

$$\Phi_n^{(l)} = \frac{1}{2\pi i} \int_{\kappa_n^{(l)}} dw \, \exp(nF_n(w)).$$

Define $D_{1,n}$ as in Lemma 3.6. Also define, $\psi_n := \mathrm{Arg}(D_{1,n} - t)$, and note that Lemma 3.6 gives $\psi_n = \frac{\pi}{2} + O(n^{-\frac{1}{3}+\theta})$. Recall that $\kappa_n^{(l)}$ is that section of κ_n inside $B(t, n^{-\theta}|q_n|)$, and κ_n is counter-clockwise (see Definition 3.7). Lemma 3.6 and Figure 3.3 then imply that $\kappa_n^{(l)}$ is the lines from $t + n^{-\theta}|q_n|e^{-i\psi_n}$ to t, and from t to $t + n^{-\theta}|q_n|e^{i\psi_n}$. A change of variables thus gives

$$\Phi_n^{(l)} = \frac{n^{-\frac{1}{3}}|q_n|}{2\pi i} \int_{k_n} dw \, \exp(nF_n(t + n^{-\frac{1}{3}}|q_n|w)),$$

where k_n is the lines from $n^{\frac{1}{3}-\theta}e^{-i\psi_n}$ to 0, and from 0 to $n^{\frac{1}{3}-\theta}e^{i\psi_n}$. Thus, since $q_n > 0$ for cases (1) and (2) of Lemma 2.2, and $q_n < 0$ for cases (3) and (4),

$$\Phi_n^{(l)} = \begin{cases} \dfrac{n^{-\frac{1}{3}}|q_n|}{2\pi i} \displaystyle\int_{k_n} dw \, \exp(nF_n(t + n^{-\frac{1}{3}}q_n w)) & \text{for cases (1) and (2),} \\[3mm] \dfrac{n^{-\frac{1}{3}}|q_n|}{2\pi i} \displaystyle\int_{k_n} dw \, \exp(nF_n(t - n^{-\frac{1}{3}}q_n w)) & \text{for cases (3) and (4).} \end{cases}$$

(2.25) and parts (3) and (4) of Corollary 3.3 then give

$$\Phi_n^{(l)} = n^{-\frac{1}{3}}|q_n| \exp(nF_n(t) + O(n^{1-4\theta}))\phi_n,$$

where

$$
\phi_n := \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{k_n} dw \exp(w(s-r) + w^2(v-u)) & \text{for cases (1) and (2),} \\[2ex] \dfrac{1}{2\pi i} \displaystyle\int_{k_n} dw \exp((-w)(s-r) + (-w)^2(v-u)) & \text{for cases (3) and (4).} \end{cases}
$$

Lemma 2.13 then gives

$$
\Phi_n^{(l)} = \frac{n^{-\frac{1}{3}} |q_n| A_{t,n}((u_n, r_n), (v_n, s_n))}{(t - \chi_n)(t - \chi_n - \eta_n + 1)} \exp(O(n^{-\frac{1}{3}} + n^{1-4\theta})) \phi_n.
$$

Note that $\exp(O(n^{-\frac{1}{3}} + n^{1-4\theta})) \to 1$ since $\theta \in (\frac{1}{4}, \frac{1}{3})$. Also, since $v > u$ and $\theta \in (\frac{1}{4}, \frac{1}{3})$, we can proceed similarly to Lemma 3.14 to show that,

$$
\phi_n \to \frac{1}{2\pi i} \int_{-\infty}^{+\infty} d(iy) \exp(iy(s-r) + (iy)^2(v-u)),
$$

for cases (1) and (2). The RHS is integrable since $v > u$, and integrating gives

$$
\phi_n \to \frac{1}{2\sqrt{\pi(v-u)}} \exp\left(-\frac{1}{4} \frac{(s-r)^2}{v-u}\right).
$$

for cases (1) and (2). This proves part (i) for cases (1) and (2). Similarly for cases (3) and (4).

Consider part (ii). First note, (1.8), (1.9) and (2.25) give

$$
\Phi_n^{(r)} = \frac{1}{2\pi i} \int_{\kappa_n^{(r)}} dw \exp(n F_n(w)),
$$

where $\kappa_n^{(r)}$ is that section of κ_n outside $B(t, n^{-\theta}|q_n|)$. Therefore,

$$
|\Phi_n^{(r)}| \le \frac{1}{2\pi} |\kappa_n^{(r)}| \sup_{w \in \kappa_n^{(r)}} |\exp(n F_n(w))| \le C |\exp(n F_n(D_{1,n}))|,
$$

where $C > 0$ is some fixed constant, and the second part above follows from Lemma 3.6. We can then proceed similarly to the proof of Lemma 3.15 to prove part (ii). □

We are finally ready to prove the main result of this section:

Proof of Theorem 1.11 when $t \in R_\mu^+$. In this proof, for brevity, denote $a_n := (t - \chi_n)(t - \chi_n - \eta_n + 1)$. First recall that $t \in R_\mu^+$ if and only if one of cases (1)–(4) of Lemma 2.2 is satisfied. Then, (1.6) and (3.14), and the expression for β_n in the statement of Lemma 3.8 give

$$
\frac{n}{n - r_n} \frac{K_n((u_n, r_n), (v_n, s_n))}{B_n(r_n, s_n)} = J_n - \Phi_n.
$$

(1.5) then gives

$$\mathcal{K}_n((v_n, s_n), (u_n, r_n)) = \frac{n - r_n}{n} \frac{J_n - \Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}.$$

Then, (3.16), and Lemmas 3.14 and 3.15 give

$$\mathcal{K}_n((v_n, s_n), (u_n, r_n)) = n^{-\frac{1}{3}} |q_n| a_n^{-1} \frac{n - r_n}{n} (\widetilde{K}_{\mathrm{Ai}}((v, s), (u, r)) + o(1))$$

$$- \frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}. \quad (3.25)$$

Moreover, since one of cases (1)–(4) is satisfied, the expression for Φ_n in part (1) of Lemma 3.8 gives

$$\frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}$$

$$= \frac{1_{(v_n \geq u_n, s_n > r_n)}}{A_{t,n}((u_n, r_n), (v_n, s_n))} \frac{(v_n + s_n - u_n - r_n - 1)!}{(s_n - r_n - 1)!(v_n - u_n)!} \frac{(n - r_n)!}{(n - s_n)!} n^{r_n - s_n}. \quad (3.26)$$

Alternatively, when $(u, r) \neq (v, s)$, Lemma 3.16 gives

$$\frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}$$

$$= n^{-\frac{1}{3}} |q_n| a_n^{-1} \frac{n - r_n}{n} (1_{(v > u)} \Phi((v, s), (u, r)) + o(1)). \quad (3.27)$$

Recall, for cases (1)–(4), that $f_t'''(t) \neq 0$, $e^{C(t)} > 1$, and $C'(t) < 0$ (see Lemma 2.3). Moreover, defining $f_{t,n}'$ as in (2.18), recall (see Lemma 2.7) that $f_{t,n}'''(t) \to f_t'''(t)$ as $n \to \infty$. Definition 2.8 then gives

$$|q_n| = 2^{\frac{1}{3}} |f_{t,n}'''(t)|^{-\frac{1}{3}} \to 2^{\frac{1}{3}} |f_t'''(t)|^{-\frac{1}{3}},$$

a non-zero constant. Also, recalling that $a_n = (t - \chi_n)(t - \chi_n - \eta_n + 1)$, and $(\chi_n, \eta_n) = (\chi_n(t), \eta_n(t))$, Definition 1.8 gives

$$a_n = \frac{(e^{C_n(t)} - 1)^2}{e^{C_n(t)} C_n'(t)^2}.$$

Moreover, (1.28) and Definition 1.8 give

$$\frac{n - r_n}{n} = 1 - \eta_n + O(n^{-\frac{1}{3}}) = -\frac{(e^{C_n(t)} - 1)^2}{e^{C_n(t)} C_n'(t)} + O(n^{-\frac{1}{3}}).$$

Thus, since $e^{C_n(t)} \to e^{C(t)} \notin \{0, 1\}$, and $C_n'(t) \to C'(t) \neq 0$ (see (1.27)),

$$a_n \to \frac{(e^{C(t)} - 1)^2}{e^{C(t)} C'(t)^2} \qquad \text{and} \qquad \frac{n - r_n}{n} \to -\frac{(e^{C(t)} - 1)^2}{e^{C(t)} C'(t)},$$

non-zero constants. Combined, the above give

$$|q_n| a_n^{-1} \tfrac{n-r_n}{n} \to -2^{\frac{1}{3}} |f_t'''(t)|^{-\frac{1}{3}} C'(t) = 2^{\frac{1}{3}} |f_t'''(t)|^{-\frac{1}{3}} |C'(t)| = \beta(t), \qquad (3.28)$$

where the second part follows since $C'(t) < 0$, the last part follows from the definition of $\beta(t)$ in the statement of Theorem 1.10.

Note, (3.25) and (3.28) give

$$\mathcal{K}_n((v_n, s_n), (u_n, r_n)) = n^{-\frac{1}{3}} (\beta(t) + o(1))(\widetilde{K}_{\mathrm{Ai}}((v, s), (u, r)) + o(1))$$
$$- \frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}.$$

Recall that $t > \chi_n > \chi_n + \eta_n - 1$ for cases (1)–(4) of Lemma 2.2, and so $A_{t,n}((u_n, r_n), (v_n, s_n)) > 0$ (see Lemma 2.13). (1.6) and (3.26) then give

$$\frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))}$$
$$= \frac{1_{(v_n \geq u_n, s_n > r_n)}}{|A_{t,n}((u_n, r_n), (v_n, s_n))| B(r_n, s_n)} \frac{(v_n + s_n - u_n - r_n - 1)!}{(s_n - r_n - 1)!(v_n - u_n)!}.$$

Moreover, when $(u, r) \neq (v, s)$, (3.27) and (3.28) alternatively give

$$\frac{n - r_n}{n} \frac{\Phi_n}{A_{t,n}((u_n, r_n), (v_n, s_n))} = n^{-\frac{1}{3}} (\beta(t) + o(1))(1_{(v>u)} \Phi((v, s), (u, r)) + o(1)).$$

Swap (u_n, r_n) and (v_n, s_n) in the above expressions to get the required result for cases (1)–(4) of Lemma 2.2, i.e., when $t \in R_\mu^+$. $\qquad\square$

4. Existence of appropriate contours of descent/ascent

We finally prove Lemmas 3.5 and 3.6.

4.1. Lemma 3.5 for case (1) of Lemma 2.2

Assume the conditions of Lemma 3.5. Additionally assume that case (1) of Lemma 2.2 is satisfied. Fix $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied.

We begin by considering the roots of the functions f_t', f_n' and $\tilde{f}_n'$ in this case. We consider f_n' and state that $\tilde{f}_n'$ can be treated similarly. Recall the definitions given in (1.12), (2.3), (2.4), (2.5), (2.10) and (2.12), and the properties discussed in (2.2) and (2.9). Recall that $\underline{S_1} := \inf S_1$, $\overline{S_1} := \sup S_1$, etc. (see Section 1.6). Then:

Lemma 4.1. *Assume the above conditions. Then, $t \in J_1 = (\overline{S_1}, +\infty)$. Moreover:*

(1) $f_t'(s) > 0$ *for all* $s \in (\overline{S_1}, t)$, $f_t'(t) = f_t''(t) = 0$ *and* $f_t'''(t) > 0$, *and* $f_t'(s) > 0$ *for all* $s \in (t, +\infty)$.

(2) f_t' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_2, J_3, J_4\}$.

(3) f_t' *has at most 1 root in each of* $\bigcup_{i=1}^3 \{K_1^{(i)}, K_2^{(i)}, \ldots\}$.

Next note that $t \in J_{1,n} = (\overline{S_{1,n}}, +\infty)$. *Indeed, fixing* $\xi > 0$ *as above, we have* $t - 4\xi > \overline{S_1}$ *and* $t - 2\xi > \overline{S_{1,n}}$. *Also* f_n' *has 2 roots in* $B(t, \xi)$. *We denote these by* $\{t_{1,n}, t_{2,n}\}$ *as in Lemma 2.6, and we recall that one of the possibilities* (a)–(c) *discussed in that lemma must be satisfied. Then, whenever possibility* (a) *is satisfied:*

(a1) $t_{1,n} \in (t - \xi, t + \xi)$ *and* $t_{1,n} = t_{2,n}$. *Moreover* $f_n'(s) > 0$ *for all* $s \in (\overline{S_{1,n}}, t_{1,n})$, $f_n'(t_{1,n}) = f_n''(t_{1,n}) = 0$ *and* $f_n'''(t_{1,n}) > 0$, *and* $f_n'(s) > 0$ *for all* $s \in (t_{1,n}, +\infty)$.

(a2) f_n' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_{2,n}, J_{3,n}, J_{4,n}\}$.

(a3) f_n' *has 1 root in each of* $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$.

Moreover, whenever possibility (b) *is satisfied:*

(b1) $\{t_{1,n}, t_{2,n}\} \subset (t - \xi, t + \xi)$ *and* $t_{1,n} > t_{2,n}$. *Moreover* $f_n'(s) > 0$ *for all* $s \in (\overline{S_{1,n}}, t_{2,n})$, $f_n'(t_{2,n}) = 0$ *and* $f_n''(t_{2,n}) < 0$, $f_n'(s) < 0$ *for all* $s \in (t_{2,n}, t_{1,n})$, $f_n'(t_{1,n}) = 0$ *and* $f_n''(t_{1,n}) > 0$, *and* $f_n'(s) > 0$ *for all* $s \in (t_{1,n}, +\infty)$.

(b2) f_n' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_{2,n}, J_{3,n}, J_{4,n}\}$.

(b3) f_n' *has 1 root in each of* $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$.

Finally, whenever possibility (c) *is satisfied:*

(c1) $t_{1,n} \in B(t, \xi) \cap \mathbb{H}$ *and* $t_{2,n}$ *is the complex conjugate of* $t_{1,n}$. *Moreover,* $f_n'(s) > 0$ *for all* $s \in (\overline{S_{1,n}}, +\infty)$.

(c2) f_n' *has 0 roots in each of* $\{\mathbb{C} \setminus (\mathbb{R} \cup \{t_{1,n}, t_{2,n}\}), J_{2,n}, J_{3,n}, J_{4,n}\}$

(c3) f_n' *has 1 root in each of* $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\}$.

Proof. Consider f_t'. Note, (2.3), (2.4), and case (1) of Lemma 2.2, give $t \in L_t = J_1 = (\overline{S_1}, +\infty)$. Also, (1.18) gives $f_t'(t) = f_t''(t) = 0$, and case (1) of Lemma 2.2 gives $f_t'''(t) > 0$. Parts (1)–(3) then follow from Lemma 2.1.

FIGURE 4.1. Top: T_t, the set of roots of f_t', as described by Lemma 4.1. Note, for each $i \in \{1, 2, 3\}$, S_i does not necessarily equal $[\underline{S_i}, \overline{S_i}]$, and subintervals of $[\underline{S_i}, \overline{S_i}] \setminus S_i$ contain at most 1 root. (a), (b), (c): T_n, the set of roots of f_n', as described by Lemma 4.1 for possibilities (a)–(c). Roots of multiplicity 1 and 2 are represented by $\times$ and $*$ respectively, and elements of $S_n = S_{1,n} \cup S_{2,n} \cup S_{3,n}$ are represented by $\bullet$.

Consider f_n'. First note, since $L_t = J_1 = (\overline{S_1}, +\infty)$, (2.11) gives $t - 4\xi > \overline{S_1}$. Also, (2.12) gives $t \in L_n = J_{1,n} = (\overline{S_{1,n}}, +\infty)$, and $t - 2\xi > \overline{S_{1,n}}$. Also, part (1) of Lemma 2.6 implies that f_n' has 2 roots in $B(t, \xi)$.

Consider part (b1). First note, since possibility (b) of Lemma 2.6, is satisfied, that $\{t_{1,n}, t_{2,n}\} \subset (t - \xi, t + \xi) \subset L_n = J_{1,n} = (\overline{S_{1,n}}, +\infty)$, $t_{1,n} > t_{2,n}$, and $t_{1,n}$ and $t_{2,n}$ are roots of f_n' of multiplicity 1. Next note, (2.7) implies that $(f_n')|_{J_{1,n}}$ is real-valued and continuous, and

$$\lim_{w \in \mathbb{R}, w \downarrow \overline{S_{1,n}}} f_n'(w) = +\infty.$$

Finally note that part (1) of Lemma 2.6 implies that f_n' has 0 roots in $(t - \xi, t + \xi) \setminus \{t_{1,n}, t_{2,n}\}$, and 0 roots in $L_n \setminus (t - \xi, t + \xi) = J_{1,n} \setminus (t - \xi, t + \xi) = (\overline{S_{1,n}}, t - \xi] \cup [t + \xi, +\infty)$. (b1) follows from the above observations. Parts (b2) and (b3) similarly follow from Lemma 2.6, as do parts (a1)–(a3) and (c1)–(c3). □

Next, for convenience, define,

$$T_t := \text{Set of roots of } f_t' \quad \text{and} \quad T_n := \text{Set of roots of } f_n'. \tag{4.1}$$

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FIGURE 4.2. Top left: $R_t|_{J_1}$. Top right: The directions of steepest decent/ascent for f_t at t. "a" represents ascent and "d" represents descent. (a), (b), (c) left: $R_n|_{J_{1,n}}$ for possibilities (a)–(c) of Lemma 4.1. (a), (b), (c) right: The directions of steepest decent/ascent for f_n at $t_{1,n}/t_{2,n}$.

Then, recalling that $f_t' : \mathbb{C} \setminus S \to \mathbb{C}$ and $f_n' : \mathbb{C} \setminus S_n \to \mathbb{C}$ are analytic, T_t and T_n are discrete subsets of $\mathbb{C} \setminus S$ and $\mathbb{C} \setminus S_n$ respectively. The previous lemma discusses the

locations of the elements of these sets, and this is depicted in Figure 4.1. Note that,

$$T_t = T_{1,t} \cup T_{2,t} \cup T_{3,t} \cup \{t\} \qquad \text{and} \qquad T_n = T_{1,n} \cup T_{2,n} \cup T_{3,n} \cup \{t_{1,n}, t_{2,n}\}, \qquad (4.2)$$

where $T_{i,t}$ is the set of roots of f_t' in $[\underline{S_i}, \overline{S_i}] \setminus S_i$, and similarly for $T_{i,n}$.

Next recall (3.7) gives

$$(R_t|_{J_1})' = (f_t')|_{J_1} \qquad \text{and} \qquad (R_n|_{J_{1,n}})' = (f_n')|_{J_{1,n}},$$

and similarly for the higher order derivatives. Above, R_t is the real-part of f_t and R_n is the real-part of f_n (see (3.2) and (3.3)). The functions on the LHSs are the "real-derivatives" of $R_t|_{J_1}$ and $R_n|_{J_{1,n}}$, and the functions on the RHSs are those given in (2.1) and (2.7) restricted to J_1 and $J_{1,n}$ respectively. Part (1) of Lemma 4.1 then implies that $R_t|_{J_1}$ has the behaviour shown on the top left of Figure 4.2. Also, parts (a1), (b1) and (c1) of Lemma 4.1 imply that $R_n|_{J_{1,n}}$ have those behaviours shown on the left of Figure 4.2 for the possibilities (a)–(c). Part (3) of Lemma 3.4 also shows that f_t and f_n have those directions of steepest descent/ascent shown on the right of Figure 4.2. The following lemma examines some of the resulting contours of steepest descent/ascent:

Lemma 4.2. *There exists simple contours, D_t and A_t, as shown on the top of Figure 4.3 with the following properties:*

(1) *D_t and A_t both start at t, enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, end in the intervals shown, and are otherwise contained in $\mathbb{H}$.*

(2) *D_t and A_t are contours of steepest descent and ascent (respectively) for f_t.*

(3) *D_t and A_t do not intersect except at t.*

Also, whenever possibility (a) is satisfied, there exists simple contours, D_n and A_n, as shown in Figure 4.3 with the following properties:

(a1) *D_n and A_n both start at $t_{1,n}$, enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, end in the intervals shown, and are otherwise contained in $\mathbb{H}$.*

(a2) *D_n and A_n are contours of steepest descent and ascent (respectively) for f_n.*

(a3) *D_n and A_n do not intersect except at $t_{1,n}$.*

Next, whenever possibility (b) is satisfied, there exists simple contours, D_n and A_n, as shown in Figure 4.3 with the following properties:

(b1) *D_n and A_n start at $t_{1,n}$ and $t_{2,n}$ respectively, both enter $\mathbb{H}$ in the direction $\frac{\pi}{2}$, end in the intervals shown, and are otherwise contained in $\mathbb{H}$.*

(b2) D_n and A_n are contours of steepest descent and ascent (respectively) for f_n.

(b3) D_n and A_n do not intersect.

Next, whenever possibility (c) *is satisfied, there exists simple contours,* D_n, A_n, D'_n, A'_n, *as shown in Figure 4.3 with the following properties:*

(c1) D_n, A_n, D'_n, A'_n *all start at* $t_{1,n} \in \mathbb{H}$, *leave* $t_{1,n}$ *in orthogonal directions in the counter-clockwise order* D_n, A_n, D'_n, A'_n, *end in the intervals shown or are unbounded, and are otherwise contained in* $\mathbb{H}$.

(c2) D_n, D'_n *are contours of steepest descent for* f_n, *and* A_n, A'_n *are contours of steepest ascent for* f_n.

(c3) D_n, D'_n, A_n, A'_n *do not intersect except at* $t_{1,n}$.

Proof. Consider f_n. Recall (see (4.1)) that T_n is the set of roots of f'_n, and the behaviour of T_n is described in Lemma 4.1 and displayed in Figure 4.1. Also, the directions of steepest decent/ascent for f_n at $t_{1,n}/t_{2,n}$ are shown on the right of Figure 4.2. Define:

(i) For possibility (a), let D_n and A_n denote the contours of steepest descent and ascent (respectively) for f_n, which start at $t_{1,n} \in (\overline{S_{1,n}}, +\infty)$, and which enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively. For possibility (b), let D_n and A_n denote the contours of steepest descent and ascent (respectively), which start at $t_{1,n} \in (\overline{S_{1,n}}, +\infty)$ and $t_{2,n} \in (\overline{S_{1,n}}, +\infty)$ respectively, and which enter $\mathbb{H}$ in the direction $\frac{\pi}{2}$. For possibility (c), let $\{D_n, D'_n\}$ and $\{A_n, A'_n\}$ denote the contours of steepest descent and ascent (respectively) which start at $t_{1,n} \in \mathbb{H}$. All contours are defined to follow the unique directions of steepest descent and ascent (as appropriate) at each $w \in \mathbb{C} \setminus (S_n \cup T_n)$ (see part (3) of Lemma 3.4), and are defined to end whenever/if they first intersect a point in $S_n \cup T_n$. Finally, if no such point of intersection exists, the contours are unbounded.

Then, using the above definition, we will first show:

(ii) For possibility (a), D_n and A_n do not intersect except at $t_{1,n}$. For possibility (b), D_n and A_n never intersect. For possibility (c), D_n, A_n, D'_n, A'_n do not intersect except at $t_{1,n}$. For all possibilities, the contours are simple.

FIGURE 4.3. The contours of Lemma 4.2.

Next we investigate the possible end-points of the contours in $S_n \cup T_n$. Recall that these are depicted in Figure 4.1, and that $T_n = T_{1,n} \cup T_{2,n} \cup T_{3,n} \cup \{t_{1,n}, t_{2,n}\}$ (see (4.2)). Also, since $t_{1,n}$ and $t_{2,n}$ are the start-points of D_n, A_n, D'_n, A'_n, part (ii) implies that the contours do not intersect $\{t_{1,n}, t_{2,n}\}$ again. Part (i) thus implies that they end whenever/if they intersect a point in $S_{1,n} \cup T_{1,n} \cup S_{2,n} \cup T_{2,n} \cup S_{3,n} \cup T_{3,n} \subset \mathbb{R}$. Next, we will show:

(iii) Each of D_n, A_n, D'_n, A'_n either eventually intersect and end in $S_{1,n} \cup T_{1,n} \cup S_{2,n} \cup T_{2,n} \cup S_{3,n} \cup T_{3,n} \subset \mathbb{R}$, or they do not intersect $\mathbb{R}$ and are unbounded. Moreover, if we exclude the start-points and end-points, the contours are contained in $\mathbb{H}$.

Finally, we use the above observations to show:

(iv) Each of D_n, D'_n eventually intersect and end in $S_{1,n} \cup T_{1,n} \cup S_{3,n} \cup T_{3,n} \subset \mathbb{R}$.

(v) Each of A_n, A'_n either eventually intersect and end in $S_{2,n} \cup T_{2,n} \subset \mathbb{R}$, or they do not intersect $\mathbb{R}$ and are unbounded.

(vi) For possibilities (a) and (b), D_n ends in $S_{3,n} \cup T_{3,n}$ and A_n ends in $S_{2,n} \cup T_{2,n}$. For possibility (c), one of $\{D_n, D'_n\}$ ends in $S_{3,n} \cup T_{3,n}$, one of $\{A_n, A'_n\}$ ends in $S_{2,n} \cup T_{2,n}$, one of $\{D_n, D'_n\}$ ends in $S_{1,n} \cup T_{1,n}$, one of $\{A_n, A'_n\}$ does not intersect $\mathbb{R}$ and is unbounded. Moreover, if we label so that D_n ends in $S_{3,n} \cup T_{3,n}$ and A_n ends in $S_{2,n} \cup T_{2,n}$, then they leave $t_{1,n}$ in the counter-clockwise order D_n, A_n, D'_n, A'_n.

The required results follow from parts (i), (ii), (iii) and (vi) since $S_{i,n} \cup T_{i,n} \subset [\underline{S_{i,n}}, \overline{S_{i,n}}]$ for all $i \in \{1, 2, 3\}$ (see Figure 4.1 and (4.2)).

Consider (ii) for possibility (a). Recall (see part (i)) that D_n, A_n both start at $t_{1,n}$. Also, recall (see part (1) of Lemma 3.4) that R_n strictly decreases along D_n, where R_n is the real-part of f_n, and R_n strictly increases along A_n. A contradiction argument then proves part (ii) for possibility (a). Consider (ii) for possibility (b). Recall (see part (i)) that D_n starts at $t_{1,n} \in \mathbb{R}$ and A_n starts at $t_{2,n} \in \mathbb{R}$, and $t_{1,n} > t_{2,n}$. Also, recall (see left of Figure 4.2) that R_n strictly decreases as we move from $t_{2,n}$ to $t_{1,n}$ along $\mathbb{R}$. A similar contradiction argument then proves part (ii) for possibility (b). Part (ii) for possibility (c) also follows similarly.

Consider (iii) for possibility (a). Recall (see part (i)) that D_n and A_n both start at $t_{1,n} \in \mathbb{R}$ and immediately enter $\mathbb{H}$. Also, recall that the contours either end in $S_n \cup T_n$ or they are unbounded. Finally, recall that $S_n \cup T_n \subset \mathbb{R}$. Thus, to prove part (iii) for possibility (a), it is sufficient to show that contours of steepest descent and ascent cannot intersect $\mathbb{R} \setminus (S_n \cup T_n)$ from $\mathbb{H}$. To show this, fix $s \in \mathbb{R} \setminus (S_n \cup T_n)$. Recall that $f'_n(s) \in \mathbb{R}$ (see (2.7)), and note that $f'_n(s) \neq 0$ since T_n is the set of roots of f'_n. Therefore $f'_n(s) > 0$ or $f'_n(s) < 0$. Part (3) of Lemma 3.4 then implies that f_n has 1 direction of steepest ascent at s. Moreover, the direction of steepest ascent is along the positive real axis whenever $f'_n(s) > 0$, and along the negative real axis whenever $f'_n(s) < 0$. Thus a contour of steepest descent cannot intersect s from $\mathbb{H}$. Similarly for contours of steepest ascent. This proves part (iii) for possibility (a). We can similarly prove part (iii) for possibilities (b) and (c).

Consider (iv). First note, part (4) of Lemma 3.4 implies that D_n, D'_n are bounded. Part (iii) thus implies that each of D_n, D'_n eventually intersects and ends in $S_{1,n} \cup T_{1,n} \cup S_{2,n} \cup T_{2,n} \cup S_{3,n} \cup T_{3,n} \subset \mathbb{R}$. Next note, part (5) of Lemma 3.4 implies that D_n, D'_n do not intersect $S_{2,n}$. Thus, to prove part (iv), it remains to show that contours of steepest descent cannot intersect $T_{2,n}$ from $\mathbb{H}$. To show this, fix $s \in T_{2,n}$. Recall (see Figure 4.1 and (4.1)) that $s \in (x, y)$ where x and y are consecutive elements of $S_{2,n}$, s is a root of f'_n of multiplicity 1, and s is the unique root of f'_n in (x, y) (see parts (a3), (b3) and (c3) of Lemma 4.1). Also, (2.7) gives $f'_n(w) \in \mathbb{R}$ for all $w \in (x, y)$, and

$$\lim_{w \in \mathbb{R}, w \uparrow y} f'_n(w) = +\infty \qquad \text{and} \qquad \lim_{w \in \mathbb{R}, w \downarrow x} f'_n(w) = -\infty.$$

It thus follows that $f'_n(s) = 0$ and $f''_n(s) > 0$. Part (3) of Lemma 3.4 thus shows that there are 2 directions of steepest descent and 2 directions of steepest ascent for f_n at s. Moreover, the directions of steepest descent are given by $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, and the directions of steepest ascent are given by 0 and π. Thus a contour of steepest descent cannot intersect s from $\mathbb{H}$. This proves (iv).

Consider (v). First, recall (see part (iii)) that each of A_n, A'_n eventually intersects and ends in $S_{1,n} \cup T_{1,n} \cup S_{2,n} \cup T_{2,n} \cup S_{3,n} \cup T_{3,n} \subset \mathbb{R}$, or they do not intersect $\mathbb{R}$ and are unbounded. Next note, part (5) of Lemma 3.4 implies that A_n, A'_n do not intersect $S_{1,n} \cup S_{3,n}$. Thus, to prove part (v), it remains to show that contours of steepest ascent cannot intersect $T_{1,n} \cup T_{3,n}$ from $\mathbb{H}$. This follows from similar arguments to those used in the proof of part (iv).

Consider (vi) for possibility (a). Recall (see part (iv)) that D_n ends either in $S_{1,n} \cup T_{1,n}$ or in $S_{3,n} \cup T_{3,n}$. We argue by contradiction: Assume that D_n ends in $S_{1,n} \cup T_{1,n}$. Recall (see part (i)) that D_n and A_n leave $t_{1,n}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively. Next recall (see part (v)) that A_n ends either in $S_{2,n} \cup T_{2,n}$, or A_n does not intersect $\mathbb{R}$ and is unbounded. Thus, since $S_{1,n} \cup T_{1,n} \subset [S_{1,n}, \overline{S_{1,n}}]$ and $S_{2,n} \cup T_{2,n} \subset [S_{2,n}, \overline{S_{2,n}}]$, Figure 4.1 clearly implies that A_n must intersect $\overline{D_n}$. This contradicts part (ii). Thus D_n cannot end in $S_{1,n} \cup T_{1,n}$, and must end in $S_{3,n} \cup T_{3,n}$. Moreover, a similar argument by contradiction shows that A_n cannot be unbounded, and so A_n must end in $S_{2,n} \cup T_{2,n}$. This proves (vi) for possibility (a). (vi) for possibility (b) follows similarly.

Consider (vi) for possibility (c). Recall (see part (iv)) each of $\{D_n, D'_n\}$ end either in $S_{1,n} \cup T_{1,n}$ or in $S_{3,n} \cup T_{3,n}$. We argue by contradiction: Assume that both of $\{D_n, D'_n\}$ end in $S_{1,n} \cup T_{1,n}$. Recall (see part (i) and right of Figure 4.2) that D_n, D'_n, A_n, A'_n leave from $t_{1,n} \in \mathbb{H}$ in orthogonal directions, and we alternately encounter elements from $\{D_n, D'_n\}$ and $\{A_n, A'_n\}$ when proceeding counter-clockwise in a neighbourhood of $t_{1,n}$. Next recall (see part (v)) that each of $\{A_n, A'_n\}$ end either in $S_{2,n} \cup T_{2,n}$, or they do not intersect $\mathbb{R}$ and are unbounded. Thus, since $S_{1,n} \cup T_{1,n} \subset [S_{1,n}, \overline{S_{1,n}}]$ and $S_{2,n} \cup T_{2,n} \subset [S_{2,n}, \overline{S_{2,n}}]$, Figure 4.1 clearly implies that one of $\{A_n, A'_n\}$ must intersect at least one of $\overline{\{D_n, D'_n\}}$.

This contradicts part (ii). Thus both of $\{D_n, D_n'\}$ cannot end in $S_{1,n} \cup T_{1,n}$. Similarly, both of $\{D_n, D_n'\}$ cannot end in $S_{3,n} \cup T_{3,n}$. Thus one of $\{D_n, D_n'\}$ must end in $S_{1,n} \cup T_{1,n}$, and the other must end $S_{3,n} \cup T_{3,n}$. Also, we can similarly argue by contradiction that one of $\{A_n, A_n'\}$ must end in $S_{2,n} \cup T_{2,n}$, and the other is unbounded. Finally, if we choose the labelling as described in part (vi), it remains to show that the contours do not leave $t_{1,n}$ in the counter-clockwise order D_n, A_n', D_n', A_n. This again follows from a similar argument by contradiction. This proves (vi).

Next consider f_t. Recall (see (4.1)) that T_t is the set of roots of f_t', and the behaviour of T_t is described in Lemma 4.1 and displayed in Figure 4.1. Also, the directions of steepest decent/ascent for f_t at t are shown on the right of Figure 4.2. We define:

(vii) Let D_t and A_t denote the contours of steepest descent and ascent (respectively) for f_t which start $t \in (\overline{S_1}, +\infty)$, and which enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively. The contours are defined to follow the unique directions of steepest descent and ascent (as appropriate) at each $w \in \mathbb{C} \setminus (S \cup T)$ (see part (3) of Lemma 3.4), and are defined to end whenever/if they first intersect a point in $S \cup T$. Finally, if no such point of intersection exists, the contours are unbounded.

This definition, and similar arguments to those used to prove part (ii), above, then give:

(viii) D_t and A_t do not intersect except at t, and they are simple.

Next we investigate the possible end-points of the contours in $S \cup T_t$. Recall that these are depicted in Figure 4.1, and $T_t = T_{1,t} \cup T_{2,t} \cup T_{3,t} \cup \{t\}$ (see (4.2)). Also, since t is the start-point of D_t and A_t, part (viii) implies that the contours do not intersect $\{t\}$ again. Part (vii) thus implies that they end whenever/if they intersect a point in $S_1 \cup T_{1,t} \cup S_2 \cup T_{2,t} \cup S_3 \cup T_{3,t} \subset \mathbb{R}$. Similar arguments to those used to prove part (iii), above, then give:

(ix) D_t and A_t either eventually intersect and end in $S_1 \cup T_{1,t} \cup S_2 \cup T_{2,t} \cup S_3 \cup T_{3,t} \subset \mathbb{R}$, or they do not intersect $\mathbb{R}$ and are unbounded. Moreover, if we exclude the start-points and end-points, the contours are contained in $\mathbb{H}$.

Next, we use the above observations to show:

(x) D_t ends in $(\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2})$.

(xi) A_t ends in $(\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2})$, or it does not intersect $\mathbb{R}$ and is unbounded.

Finally, we will show:

(xii) D_t ends in $(\underline{S_3}, \overline{S_3})$.

(xiii) A_t ends in $(\underline{S_2}, \overline{S_2})$.

The required results then follows from parts (vii), (viii), (ix), (xii) and (xiii).

Consider (x). First note, part (4) of Lemma 3.4 implies that D_t is bounded. Part (ix) then implies that D_t ends in $S_1 \cup T_{1,t} \cup S_2 \cup T_{2,t} \cup S_3 \cup T_{3,t} \subset \mathbb{R}$. Thus, since $S_i \cup T_{i,t} \subset [\underline{S_i}, \overline{S_i}]$ for all $i \in \{1, 2, 3\}$ (see (4.2)), D_t ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, \overline{S_1}]$. Thus, to show part (x), it is sufficiently to show that D_t does not end in $\{\underline{S_3}, \overline{S_2}\} \cup [\underline{S_1}, \overline{S_1}]$. We argue by contradiction: First assume that D_t ends at $\overline{S_1}$. Then (see parts (vii) and (ix)) D_t starts at $t > \overline{S_1}$ in the direction $\frac{\pi}{3}$, ends at $\overline{S_1}$, and is otherwise contained in $\mathbb{H}$. Consider the simple closed contour consisting of D_t and the interval $[\overline{S_1}, t]$. Note, Figure 3.1 and part (2) of Lemma 3.4 implies that the function $w \mapsto \text{Im}(f_t(w))$ is constant on this contour (indeed, the constant valve is 0). Therefore, since this function is harmonic, it is also constant in the domain bounded by the simple closed contour. (3.4) easily shows that this is not true. Thus we have a contradiction, and so D_t does not end at $\overline{S_1}$. Next assume that D_t ends at a point $d_t \in [\underline{S_1}, \overline{S_1})$. Note, as above, $w \mapsto \text{Im}(f_t(w))$ is constant on D_t. In particular, this gives $\text{Im}(f_t(d_t)) = \text{Im}(f_t(t))$. Recall (see (2.2)) that $(\overline{S_1} - \epsilon, \overline{S_1}] \subset S_1$ for all $\epsilon > 0$ sufficiently small. (3.5) and Figure 3.1 then give $\text{Im}(f_t(s)) > 0$ for all $s \in [\underline{S_1}, \overline{S_1})$. However, these also give $\text{Im}(f_t(t)) = 0$. This contradicts $\text{Im}(f_t(d_t)) = \text{Im}(f_t(t))$, and so D_t does not end in $[\underline{S_1}, \overline{S_1})$. Next assume that D_t ends at $\overline{S_2}$ or at $\underline{S_3}$. Note, (2.2), (3.5) and Figure 3.1 give $\text{Im}(f_t(\overline{S_2})) = \pi\mu[S_1] > 0$ and $\text{Im}(f_t(\underline{S_3})) = \pi\eta > 0$. We then proceed as in the previous case to get a contradiction, and so D_t does not end at $\overline{S_2}$ or at $\underline{S_3}$. This proves (x). Part (xi) follows similarly.

Consider (xii). Part (x) implies that it is sufficient to show that D_t does not end in $\{\overline{S_3}\} \cup [\underline{S_2}, \overline{S_2})$. We argue by contradiction: First assume that D_t ends in $[\underline{S_2}, \overline{S_2})$. Then (see parts (vii) and (ix)) D_t starts at t in the direction $\frac{\pi}{3}$, ends in $[\underline{S_2}, \overline{S_2})$, and is otherwise contained in $\mathbb{H}$. Denote the end-point by d_t. Parts (vii), (viii) and (xi) then imply that A_t stars at t in the direction $\frac{2\pi}{3}$, A_t and D_t do not intersect except at t, A_t ends in $(d_t, \overline{S_2})$, and A_t is otherwise contained in $\mathbb{H}$. Denote the end-point of A_t by $a_t \in (d_t, \overline{S_2})$. Note, part (2) of Lemma 3.4 implies that the function $w \mapsto \text{Im}(f_t(w))$ is constant on D_t and A_t (indeed, the constant valve is 0). In particular this gives $\text{Im}(f_t(d_t)) = \text{Im}(f_t(a_t))$. (3.5) and Figure 3.1 imply that this can only happen when (d_t, a_t) is entirely contained in a sub-interval of $[\underline{S_2}, \overline{S_2}] \setminus S_2$, in which case $w \mapsto \text{Im}(f_t(w))$ is constant on the interval $[d_t, a_t]$ also. Therefore $w \mapsto \text{Im}(f_t(w))$ is an harmonic function which is constant on the simple closed contour consisting of D_t and A_t and interval $[d_t, a_t]$, and so it also constant in the domain bounded by the closed contour. (3.4) easily shows that this is not true. Thus

we have a contradiction, and so D_t does not end in $[S_2, \overline{S_2})$. Similarly, we can argue by contradiction that D_t does not end at $\overline{S_3}$. This proves (xii).

Consider (xiii). Part (xi) implies that it is sufficient to show that A_t does not end in $(S_3, \overline{S_3}] \cup \{S_2\}$, and that A_t is not unbounded. Using parts (vii) and (xii), this follows from arguments by contradiction similar to those used in the proof of (xii). □

Remark 4.3. Note, the nature of D_t and A_t for case (1) of Lemma 2.2, as described above, proves an interesting inequality. Recall that D_t and A_t both start at $t \in (\overline{S_1}, +\infty)$, D_t ends at a point $d_t \in (S_3, \overline{S_3})$, and A_t ends at a point $a_t \in (S_2, \overline{S_2})$. Also recall that $w \mapsto \text{Im}(f_t(w))$ is constant along D_t and A_t. Figure 3.1 implies that $\text{Im}(f_t(t)) = 0$, and so $0 = \text{Im}(f_t(t)) = \text{Im}(f_t(d_t)) = \text{Im}(f_t(a_t))$. Figure 3.1 also implies that this can only occur if $\mu[S_1] - (\lambda - \mu)[S_2] < 0$. This inequality must be satisfied whenever case (1) is satisfied. The authors are not aware of a direct proof of this inequality, or of its significance. Though we will not discuss them, analogous inequalities exist for the other cases of Lemma 2.2.

We are now in a position to define the contours, $\gamma_{1,n}^+$ and $\Gamma_{1,n}^+$, that satisfy Lemma 3.5 for case (1) of Lemma 2.2. As above, fix $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied. Next, fix $\theta \in (\frac{1}{4}, \frac{1}{3})$ as in Lemma 3.5, and $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ as in Definition 2.8. Note, since $f_t'''(t) > 0$ (see case (1) of Lemma 2.2), Lemma 2.7 and Definition 2.8 imply that $\{q_n\}_{n \geq 1}$ converges to a positive constant as $n \to \infty$. Also, part (4) of Lemma 2.9 gives $t_{1,n} = t + O(n^{-\frac{1}{3}})$ and $t_{2,n} = t + O(n^{-\frac{1}{3}})$, and so

$$\{t_{1,n}, t_{2,n}\} \subset B(t, n^{-\theta} q_n) \subset B(t, \xi).$$

Thus, D_n and A_n (see previous lemma) both start inside $B(t, n^{-\theta} q_n)$ and $B(t, \xi)$. Moreover, assuming for simplicity that $[0, 1]$ is the domain of definition of these contours, we define:

$$
\begin{aligned}
d_{1,n} &:= D_n(\sup\{y \in [0, 1] : D_n(y) \in \text{cl}(B(t, n^{-\theta} q_n))\}), \\
d_{2,n} &:= D_n(\sup\{y \in [0, 1] : D_n(y) \in \text{cl}(B(t, \xi))\}), \\
a_{1,n} &:= A_n(\sup\{y \in [0, 1] : A_n(y) \in \text{cl}(B(t, n^{-\theta} q_n))\}), \\
a_{2,n} &:= A_n(\sup\{y \in [0, 1] : A_n(y) \in \text{cl}(B(t, \xi))\}).
\end{aligned}
\tag{4.3}
$$

In words, $d_{1,n}$ and $d_{2,n}$ denote the points where D_n "exits" $B(t, n^{-\theta} q_n)$ and $B(t, \xi)$ respectively. Similarly for $a_{1,n}$, $a_{2,n}$ and A_n.

Next denote the equivalent quantities for D_t and A_t by $d_{1,t}, d_{2,t}, a_{1,t}, a_{2,t}$. Also, fixing $\xi > 0$ sufficiently small as above, note that case (1) of Lemma 2.2 and (3.1) give

$$\mathbb{C}_\xi = \{w \in \mathbb{C} : \text{Re}(w) > \overline{S_1} + 2\xi \text{ or } |\text{Im}(w)| > \xi^4\}.$$

Recall that D_t ends in $(S_3, \overline{S_3}) \subset \mathbb{C} \setminus \mathbb{C}_\xi$, and A_t ends in $(S_2, \overline{S_2}) \subset \mathbb{C} \setminus \mathbb{C}_\xi$, and let $d_{3,t}$ and $a_{3,t}$ denote the points where D_t and A_t "exit" $\mathbb{C}_\xi$ respectively. Note, it is always possible

to choose the $\xi > 0$ sufficiently small that $d_{3,t}$ and $a_{3,t}$ are on the upper boundary of $\mathbb{C}_\xi$:

$$\text{Im}(d_{3,t}) = \text{Im}(a_{3,t}) = \xi^4. \tag{4.4}$$

Also note, since D_t and A_t end in $(\underline{S_3}, \overline{S_3})$ and $(\underline{S_2}, \overline{S_2})$ respectively, it is always possible to choose the $\xi > 0$ sufficiently small such that,

$$\text{Re}(d_{3,t}) \in (\underline{S_3} + c, \overline{S_3} - c) \qquad \text{and} \qquad \text{Re}(a_{3,t}) \in (\underline{S_2} + c, \overline{S_2} - c), \tag{4.5}$$

where $c = c(t) > 0$ is some constant which is independent of ξ. Finally define the following contours, which are depicted in Figure 4.4:

Definition 4.4. Define $\gamma_{1,n}^+$ to be the simple contour which:

- starts at $t \in (\overline{S_1}, +\infty)$,

- then traverses the straight line from t to $d_{1,n} \in \partial B(t, n^{-\theta} q_n)$,

- then traverses that section of D_n from $d_{1,n} \in \partial B(t, n^{-\theta} q_n)$ to $d_{2,n} \in \partial B(t, \xi)$,

- then traverses the shortest arc of $\partial B(t, \xi)$ from $d_{2,n} \in \partial B(t, \xi)$ to $d_{2,t} \in \partial B(t, \xi)$,

- then traverses that section of D_t from $d_{2,t}$ to $d_{3,t}$,

- then traverses the straight line from $d_{3,t}$ to $\text{Re}(d_{3,t})$,

- then ends at $\text{Re}(d_{3,t}) \in (\underline{S_3} + c, \overline{S_3} - c)$.

Next, let $\tilde{D}_n$ and $\tilde{A}_n$ denote the contours of steepest descent and ascent (respectively) for $\tilde{f}_n$ which are analogous to D_n and A_n. Moreover, let $\tilde{d}_{1,n}, \tilde{d}_{2,n}, \tilde{a}_{1,n}, \tilde{a}_{2,n}$ denote analogous quantities to those defined in (4.3). Finally, define $\Gamma_{1,n}^+$ to be the simple contour which:

- starts at $t \in (\overline{S_1}, +\infty)$,

- then traverses the straight line from t to $\tilde{a}_{1,n} \in \partial B(t, n^{-\theta} q_n)$,

- then traverses that section of $\tilde{A}_n$ from $\tilde{a}_{1,n} \in \partial B(t, n^{-\theta} q_n)$ to $\tilde{a}_{2,n} \in \partial B(t, \xi)$,

- then traverses the shortest arc of $\partial B(t, \xi)$ from $\tilde{a}_{2,n} \in \partial B(t, \xi)$ to $a_{2,t} \in \partial B(t, \xi)$,

- then traverses that section of A_t from $a_{2,t}$ to $a_{3,t}$,

- then traverses the straight line from $a_{3,t}$ to $\text{Re}(a_{3,t})$,
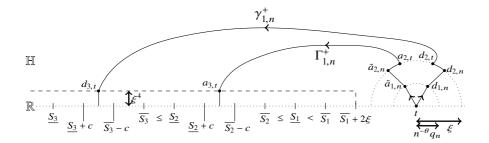
FIGURE 4.4. The contours defined in Definition 4.4. The dashed lines represent boundaries of $\mathbb{C}_\xi$. $c = c(t) > 0$ is independent of ξ.

- then ends at $\mathrm{Re}(a_{3,t}) \in (\underline{S_2} + c, \overline{S_2} - c)$.

We finally show that the above contours satisfy the requirements of Lemma 3.5:

Proof of Lemma 3.5 for case (1) *of Lemma 2.2.* Part (1) of Lemma 3.5 follows easily from Lemma 4.2 and Definition 4.4 (see also Figures 3.2 and 4.4). Consider (2). We will show that:

(i) $\mathrm{Arg}(d_{1,n} - t) = \frac{\pi}{3} + O(n^{-\frac{1}{3}+\theta})$ and $\mathrm{Arg}(a_{1,n} - t) = \frac{2\pi}{3} + O(n^{-\frac{1}{3}+\theta})$.

Similarly, we can show that $\mathrm{Arg}(\tilde{d}_{1,n}-t) = \frac{\pi}{3}+O(n^{-\frac{1}{3}+\theta})$ and $\mathrm{Arg}(\tilde{a}_{1,n}-t) = \frac{2\pi}{3}+O(n^{-\frac{1}{3}+\theta})$. This proves (2).

Consider (3). We will show, for all $\xi > 0$ sufficiently small as in the statement of this lemma, that there exists an integer $n(\xi) > 0$ such that the following are satisfied for all $n > n(\xi)$:

(ii) $\mathrm{Arg}(d_{2,n} - t) = \frac{\pi}{3} + O(\xi)$ and $\mathrm{Arg}(a_{2,n} - t) = \frac{2\pi}{3} + O(\xi)$.

Then, letting R_n denote the real-part of f_n (see (3.3)), we will use this to show that there exists a choice of the above ξ such that the following are satisfied:

(iii) $R_n(w) \le R_n(d_{1,n})$ for all w on that section of D_n from $d_{1,n}$ to $d_{2,n}$.

(iv) $R_n(w) \le R_n(d_{1,n})$ for all w on the shortest arc of $\partial B(t, \xi)$ from $d_{2,n}$ to $d_{2,t}$.

(v) $R_n(w) \le R_n(d_{1,n})$ for all w on that section of D_t from $d_{2,t}$ to $d_{3,t}$.

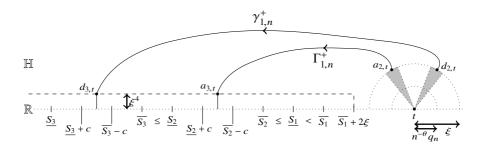(vi) $R_n(w) \le R_n(d_{1,n})$ for all w on the straight line from $d_{3,t}$ to $\mathrm{Re}(d_{3,t})$.

FIGURE 4.5. The contours defined in Definition 4.4 and depicted in Figure 4.4. Here, we do not depict those sections of the contours in $cl(B(t, \xi))$. Instead, we depict two cones (the shaded areas). For all $\xi > 0$ sufficiently small and $n > n(\xi)$, $Arg(w - t) = \frac{\pi}{3} + O(\xi)$ and $Arg(z - t) = \frac{2\pi}{3} + O(\xi)$ uniformly for w and z in the right and left cones respectively, $\gamma_{1,n}^+ \cap B(t, \xi)$ and $\Gamma_{1,n}^+ \cap B(t, \xi)$ are contained in the right and left cones respectively, and $\gamma_{1,n}^+ \setminus B(t, \xi)$ and $\Gamma_{1,n}^+ \setminus B(t, \xi)$ are independent of n.

Definition 4.4 then implies that $R_n(w) \le R_n(d_{1,n})$ for all $w \in \gamma_{1,n}^+ \setminus B(t, n^{-\theta}q_n)$. This proves (3). (4) follows similarly.

Consider (5). First note, we can proceed similarly to the proofs of parts (i) and (ii) to show the following: For all $\xi > 0$ sufficiently small, there exists an integer $n(\xi) > 0$ such that $Arg(w - t) = \frac{\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for $w \in \gamma_{1,n}^+ \cap cl(B(t, \xi))$, and $Arg(z - t) = \frac{2\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for $z \in \Gamma_{1,n}^+ \cap cl(B(t, \xi))$. These contour sections are thus contained in the cones shown in Figure 4.5. Next note, Definition 4.4 implies that $\gamma_{1,n}^+$ and $\Gamma_{1,n}^+$ depend on n inside the cones, are independent of n outside the cones, and the parts outside the cones never intersect. Then, for $\xi > 0$ fixed sufficiently small, Figure 4.5 and a simple geometric argument proves part (5).

Consider (6). Recall (see the proof of part (5)), that for all $\xi > 0$ sufficiently small, there exists an integer $n(\xi) > 0$ such that $Arg(w - t) = \frac{\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for w on that section of D_n from $d_{1,n}$ to $d_{2,n}$. Looking at Figure 4.5, this contour is contained in that section of the right cone in $B(t, \xi) \setminus B(t, n^{-\theta}q_n)$. We will show the following:

(vii) Consider that section of the right cone discussed above. Then, we can choose the above ξ and $n(\xi)$ such that the direction of steepest descent for f_n at w equals $\frac{\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for w in that section.

Finally recall (see Lemma 4.2 and Definition 4.4) that D_n follows the uniquely defined directions of steepest descent for f_n. The above observations imply that the length of that section of D_n from $d_{1,n}$ to $d_{2,n}$ is of order $O(\xi)$ for all $n > n(\xi)$. Moreover, Definition 4.4 trivially implies that the remaining sections of $\gamma_{1,n}^+$ are of order at most $O(1)$ for all n sufficiently large. Therefore we can fix $\xi > 0$ sufficiently small such that $|\gamma_{1,n}^+| = O(1)$. Similarly, we can show that $|\Gamma_{1,n}^+| = O(1)$. This proves (6).

Consider (i). First recall (see Lemma 4.2) that D_n is a contour of steepest descent for f_n which starts at $t_{1,n}$, and (see (4.3)) $d_{1,n} \in \partial B(t, n^{-\theta} q_n)$ denotes that point at which D_n "exits" $B(t, n^{-\theta} q_n)$. Thus, letting R_n and I_n respectively denote the real and imaginary-parts of f_n, parts (1) and (2) of Lemma 3.4 give

$$R_n(d_{1,n}) < R_n(t_{1,n}) \qquad \text{and} \qquad I_n(d_{1,n}) = I_n(t_{1,n}).$$

Thus, since $t_{1,n} = t + O(n^{-\frac{1}{3}})$ (see part (4) of Lemma 2.9), and since $f_n(t) \in \mathbb{R}$, part (1) of Corollary 3.3 gives

$$R_n(d_{1,n}) < f_n(t) + O(n^{-1}) \qquad \text{and} \qquad I_n(d_{1,n}) = O(n^{-1}). \tag{4.6}$$

Similar considerations for the contour, A_n, of steepest ascent for f_n give

$$R_n(a_{1,n}) > f_n(t) + O(n^{-1}) \qquad \text{and} \qquad I_n(a_{1,n}) = O(n^{-1}). \tag{4.7}$$

Next note, since $\theta \in (\frac{1}{4}, \frac{1}{3})$, part (2) of Lemma 3.2 (take $\xi_n = n^{-\theta}$) gives

$$f_n(t + n^{-\theta} q_n e^{i\alpha}) = f_n(t) + \frac{1}{3} n^{-3\theta} e^{3i\alpha} + O(n^{-\frac{1}{3} - 2\theta}),$$

uniformly for $\alpha \subset (-\pi, \pi]$. Therefore, since $f_n(t) \in \mathbb{R}$,

$$R_n(t + n^{-\theta} q_n e^{i\alpha}) = f_n(t) + \frac{1}{3} n^{-3\theta} \cos(3\alpha) + O(n^{-\frac{1}{3} - 2\theta}), \tag{4.8}$$

$$I_n(t + n^{-\theta} q_n e^{i\alpha}) = \frac{1}{3} n^{-3\theta} \sin(3\alpha) + O(n^{-\frac{1}{3} - 2\theta}), \tag{4.9}$$

uniformly for $\alpha \subset (-\pi, \pi]$.

(4.9), and the second parts of (4.6) and (4.7) give

$$\sin(3 \operatorname{Arg}(d_{1,n} - t)) = O(n^{-\frac{1}{3} + \theta}) \quad \text{and} \quad \sin(3 \operatorname{Arg}(a_{1,n} - t)) = O(n^{-\frac{1}{3} + \theta}). \tag{4.10}$$

Then, since $\theta \in (\frac{1}{4}, \frac{1}{3})$, (4.8) and (4.10), and the first parts of (4.6) and (4.7) imply the following:

- Either $d_{1,n} = d'_{1,n}$ or $d_{1,n} = d''_{1,n}$ where $\operatorname{Arg}(d'_{1,n} - t) = \frac{\pi}{3} + O(n^{-\frac{1}{3} + \theta})$ and $\operatorname{Arg}(d''_{1,n} - t) = \pi + O(n^{-\frac{1}{3} + \theta})$. In either case, $R_n(d_{1,n}) = f_n(t) - \frac{1}{3} n^{-3\theta} + O(n^{-\frac{1}{3} - 2\theta})$.

- Either $a_{1,n} = a'_{1,n}$ or $a_{1,n} = a''_{1,n}$ where $\operatorname{Arg}(a'_{1,n} - t) = O(n^{-\frac{1}{3} + \theta})$ and $\operatorname{Arg}(a''_{1,n} - t) = \frac{2\pi}{3} + O(n^{-\frac{1}{3} + \theta})$. In either case, $R_n(a_{1,n}) = f_n(t) + \frac{1}{3} n^{-3\theta} + O(n^{-\frac{1}{3} - 2\theta})$.
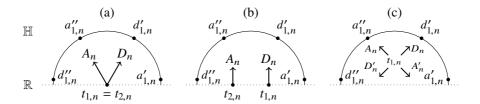
FIGURE 4.6. The contours A_n, D_n, A'_n, D'_n of Lemma 4.2 in $B(t, n^{-\theta} q_n) \cap$ $\mathbb{H}$ for each of the possibilities (a)–(c) of Lemmas 4.1 and 4.2.

These exit points, and each of the possibilities (a), (b) and (c) of Lemmas 4.1 and 4.2, are shown in Figure 4.6. We will show, for each possibility, that $d_{1,n} = d'_{1,n}$ and $a_{1,n} = a''_{1,n}$. This proves (i).

For possibilities (a) and (b), recall that D_n and A_n have the behaviours described in Lemma 4.2 and shown in Figure 4.3. Note that the contours do not intersect outside $B(t, n^{-\theta} q_n)$, and it is easy to see that these behaviours are only possible if D_n "exits" $B(t, n^{-\theta} q_n)$ at $d'_{1,n}$ and A_n "exits" at $a''_{1,n}$. Thus $d_{1,n} = d'_{1,n}$ and $a_{1,n} = a''_{1,n}$, as required, for possibilities (a) and (b). For possibility (c), proceeding as above, we can show that $a'_{1,n}$ and $a''_{1,n}$ are the possible "exit" points of both A_n and A'_n, and $d'_{1,n}$ and $d''_{1,n}$ are the possible "exit" points of both D_n and D'_n. Also, the end-points of the contours, and the fact that they do not intersect outside $B(t, n^{-\theta} q_n)$, implies that one of $\{A_n, A'_n\}$ "exits" at $a'_{1,n}$, one of $\{A_n, A'_n\}$ "exits" at $a''_{1,n}$, one of $\{D_n, D'_n\}$ "exits" at $d'_{1,n}$, and one of $\{D_n, D'_n\}$ "exits" at $d''_{1,n}$. Finally, we can proceed similarly to the proof of part (vi) of the proof of Lemma 4.2 to show that A'_n, D_n, A_n, D'_n respectively "exit" at $a'_{1,n}$, $d'_{1,n}$, $a''_{1,n}$, $d''_{1,n}$. Thus $d_{1,n} = d'_{1,n}$ and $a_{1,n} = a''_{1,n}$, as required, for possibility (c).

Consider (ii). First recall (see (4.3)) that $d_{2,n}$ and $a_{2,n}$ denote the points at which D_n and A_n respectively "exit" $B(t, \xi)$. Then, proceed as in (i) to get,

$$R_n(d_{2,n}) < f_n(t) + O(n^{-1}) \quad \text{and} \quad I_n(d_{2,n}) = O(n^{-1}),$$
$$R_n(a_{2,n}) > f_n(t) + O(n^{-1}) \quad \text{and} \quad I_n(a_{2,n}) = O(n^{-1}).$$

Thus, since $f_n(t) \to f_t(t)$,

$$R_n(d_{2,n}) < f_t(t) + o(1) \quad \text{and} \quad I_n(d_{2,n}) = o(1),$$
$$R_n(a_{2,n}) > f_t(t) + o(1) \quad \text{and} \quad I_n(a_{2,n}) = o(1).$$

Next, recall that ξ is any positive number for which (2.11), (2.14), (2.17) and (2.31) hold. Note that this can be fixed arbitrarily small. Also, for all such $\xi > 0$ chosen sufficiently small, part (2) of Lemma 3.2 implies (take $\xi_n = \xi q_n^{-1}$) that there exists a positive integer,

$n(\xi)$, such that,

$$f_n(t + \xi e^{i\alpha}) = f_n(t) + \frac{1}{3}\xi^3 q_n^{-3} e^{3i\alpha} + O(\xi^4),$$

for all $n > n(\xi)$ and uniformly for $\alpha \subset (-\pi, \pi]$. Next recall that $f_n(t) \to f_t(t)$ and $q_n \to q_t$, where q_t is some positive number. It thus follows that there exists a choice of $n(\xi)$ such that,

$$f_n(t + \xi e^{i\alpha}) = f_t(t) + \frac{1}{3}\xi^3 q_t^{-3} e^{3i\alpha} + O(\xi^4),$$

for all $n > n(\xi)$ and uniformly for $\alpha \subset (-\pi, \pi]$. Therefore, since $f_t(t) \in \mathbb{R}$,

$$R_n(t + \xi e^{i\alpha}) = f_t(t) + \frac{1}{3}\xi^3 q_t^{-3} \cos(3\alpha) + O(\xi^4),$$

$$I_n(t + \xi e^{i\alpha}) = \frac{1}{3}\xi^3 q_t^{-3} \sin(3\alpha) + O(\xi^4),$$

for all $n > n(\xi)$ and uniformly for $\alpha \subset (-\pi, \pi]$. Then, for all $\xi > 0$ sufficiently small, it follows from similar arguments to those used in part (i) that there exists a choice of $n(\xi)$ such that we have the following possibilities for $d_{2,n}$ and $a_{2,n}$ for all $n > n(\xi)$:

- Either $d_{2,n} = d'_{2,n}$ or $d_{2,n} = d''_{2,n}$ where $\mathrm{Arg}(d'_{2,n} - t) = \frac{\pi}{3} + O(\xi)$ and $\mathrm{Arg}(d''_{2,n} - t) = \pi + O(\xi)$. In either case, $R_n(d_{2,n}) = f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4)$.

- Either $a_{2,n} = a'_{2,n}$ or $a_{2,n} = a''_{2,n}$ where $\mathrm{Arg}(a'_{2,n} - t) = O(\xi)$ and $\mathrm{Arg}(a''_{2,n} - t) = \frac{2\pi}{3} + O(\xi)$. In either case, $R_n(a_{2,n}) = f_t(t) + \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4)$.

(ii) then follows from similar arguments to those used in part (i), above.

Consider (iii). This follows trivially from part (1) of Lemma 3.4, since D_n is a contour of steepest descent for f_n. Consider (iv). First note, proceeding similarly to the proof of part (ii) above, we can show that we can choose ξ and $n(\xi)$ such that,

$$\mathrm{Arg}(w - t) = \frac{\pi}{3} + O(\xi) \qquad \text{and} \qquad R_n(w) = f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4),$$

for all $n > n(\xi)$ and uniformly for w on the shortest arc of $\partial B(t, \xi)$ from $d_{2,n}$ to $d_{2,t}$. Also recall (see proof of part (i)) that $R_n(d_{1,n}) = f_n(t) - \frac{1}{3}n^{-3\theta} + O(n^{-\frac{1}{3}-2\theta})$. Therefore we can choose ξ and $n(\xi)$ such that $R_n(w) \le R_n(d_{1,n})$ for all $n > n(\xi)$ and w on this arc. This proves (iv).

Consider (v). Note, that section of D_t from $d_{2,t}$ to $d_{3,t}$ is independent of n and is contained in $\mathbb{C}_\xi$, and so part (2) of Lemma 3.1 implies that $R_n(w) = R_t(w) + o(1)$ uniformly for all w on this section. Thus, for all $\xi > 0$, there exists an integer $n(\xi) > 0$ such that $R_n(w) = R_t(w) + O(\xi^4)$ for all $n > n(\xi)$ and uniformly for w on this section. Next recall that D_t is a contour of steepest descent for f_t. Part (1) of Lemma 3.4 thus gives $R_t(w) \le R_t(d_{2,t})$ for all w on this section. Finally, proceeding similarly to part (ii)

above, we can show that for all $\xi > 0$ sufficiently small, we can choose the above $n(\xi) > 0$ such that the following is satisfied for all $n > n(\xi)$: $R_t(d_{2,t}) = f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4)$. Combined, these observations give

$$R_n(w) = R_t(w) + O(\xi^4) \leq R_t(d_{2,t}) + O(\xi^4) = f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4), \qquad (4.11)$$

for all $n > n(\xi)$ and uniformly for w on that section of D_t from $d_{2,t}$ to $d_{3,t}$. Finally recall (see proof of part (i)) that $R_n(d_{1,n}) = f_n(t) - \frac{1}{3}n^{-3\theta} + O(n^{-\frac{1}{3}-2\theta})$. Therefore we can choose ξ and $n(\xi)$ such that $R_n(w) \leq R_n(d_{1,n})$ for all $n > n(\xi)$ and w on this section.

Consider (vi). Note that we trivially have $|w - x| \leq |d_{3,t} - x|$ for all w on the vertical straight line from $d_{3,t}$ to $\mathrm{Re}(d_{3,t})$ and $x \in \mathbb{R}$. (3.3) then gives

$$R_n(w) \leq \frac{1}{n}\sum_{x \in S_{1,n}} \log|d_{3,t} - x| - \frac{1}{n}\sum_{x \in S_{2,n}} \log|w - x| + \frac{1}{n}\sum_{x \in S_{3,n}} \log|d_{3,t} - x|,$$

for all w on the vertical line. Next recall that $\underline{S_{2,n}} \geq \chi + \eta - 1 + o(1)$ (see (1.18) and (2.5)) and $\chi + \eta - 1 \geq \overline{S_3}$ (see (2.2)). (4.5) thus implies that $\underline{S_{2,n}} > \mathrm{Re}(d_{3,t}) + \frac{1}{2}c$ for all $\xi > 0$ sufficiently small and n sufficiently large chosen independently, where $c = c(t) > 0$ is some constant independent of ξ and n. (4.4) and the above expression thus give

$$R_n(w) \leq \frac{1}{n}\sum_{x \in S_{1,n}} \log|d_{3,t} - x| - \frac{1}{n}\sum_{x \in S_{2,n}} (\log|d_{3,t} - x| + O(\xi^4)) + \frac{1}{n}\sum_{x \in S_{3,n}} \log|d_{3,t} - x|,$$

uniformly for w on the vertical line. Then, (3.3) gives $R_n(w) \leq R_n(d_{3,t}) + O(\xi^4)$ for all such n, w. Next note, substituting $d_{3,t}$ in (4.11) gives $R_n(d_{3,t}) \leq f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4)$ for all $n > n(\xi)$. Combined, these give $R_n(w) \leq f_t(t) - \frac{1}{3}\xi^3 q_t^{-3} + O(\xi^4)$ for all $n > n(\xi)$ and uniformly for w on the vertical line. Finally recall (see proof of part (i)) that $R_n(d_{1,n}) = f_n(t) - \frac{1}{3}n^{-3\theta} + O(n^{-\frac{1}{3}-2\theta})$. Therefore we can choose ξ and $n(\xi)$ such that $R_n(w) \leq R_n(d_{1,n})$ for all $n > n(\xi)$ and w on the vertical line. This proves (vi).

Consider (vii). Fixing a constant $c > 0$, and recalling that $\{q_n\}_{n \geq 1}$ is a convergent sequence with a positive limit, note that it is sufficient to show the following:

- We can fix the ξ sufficiently small, and choose the $n(\xi)$, such that the direction of steepest descent for f_n at $t + rq_n e^{i\alpha}$ equals $\frac{\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for $r \in [n^{-\theta}, \xi q_n^{-1}]$ and $\alpha \in (\frac{\pi}{3} - c\xi, \frac{\pi}{3} + c\xi)$.

To see this first note, for all such r, α, part (3) of Lemma 3.4 implies that the unique direction of steepest descent for f_n at $t + rq_n e^{i\alpha}$ is $\pi - \mathrm{Arg}(f_n'(t + rq_n e^{i\alpha}))$. Then, we can

proceed as in part (2) of Lemma 3.2 to show that we can choose the above $n(\xi) > 0$ such that,

$$f_n'(t + rq_n e^{i\alpha}) = r^2 q_n^{-1} e^{2i\alpha} + O(n^{-\frac{2}{3}} + n^{-\frac{1}{3}}r + n^{-\frac{1}{3}}r^2 + r^3)$$

$$= r^2 q_n^{-1} e^{2i\alpha} \left(1 + O(n^{-\frac{2}{3}}r^{-2} + n^{-\frac{1}{3}}r^{-1} + n^{-\frac{1}{3}} + r)\right),$$

for all $n > n(\xi)$, and uniformly for all such r, α. Also note, since $\theta < \frac{1}{3}$ and $r \in [n^{-\theta}, \xi q_n^{-1}]$, that $n^{-\frac{2}{3}}r^{-2} + n^{-\frac{1}{3}}r^{-1} + n^{-\frac{1}{3}} = o(1)$, and $r = O(\xi)$. Therefore we can choose the above ξ sufficiently small, and we can choose the $n(\xi) > 0$, such that $f_n'(t + rq_n e^{i\alpha}) = r^2 q_n^{-1} e^{2i\alpha}(1 + O(\xi))$ for all $n > n(\xi)$ and uniformly for all such r, α. Finally recall that $\alpha \in (\frac{\pi}{3} - c\xi, \frac{\pi}{3} + c\xi)$ for some $c > 0$. Thus we can choose the above ξ sufficiently small, and we can choose the $n(\xi) > 0$, such that

$$\mathrm{Arg}(f_n'(t + rq_n e^{i\alpha})) = 2\alpha + O(\xi) = \frac{2\pi}{3} + O(\xi),$$

for all $n > n(\xi)$, and uniformly for all such r, α. Thus the direction of steepest descent for f_n at $t + rq_n e^{i\alpha}$ equals $\pi - \mathrm{Arg}(f_n'(t + rq_n e^{i\alpha})) = \frac{\pi}{3} + O(\xi)$ for all $n > n(\xi)$ and uniformly for all such r, α, as required. $\qquad\square$

Remark 4.5. Note, it may happen that $\mathrm{Re}(d_{3,t}) \in P_n$ (see (1.22)). If this happens, for simplicity, we deform $\gamma_{1,n}^+$ (see Definition 4.4) in neighbourhoods of $\mathrm{Re}(d_{3,t})$ to avoid this. Note, since such deformations are arbitrarily small, they do not significantly affect the proof of the previous lemma.

4.2. **Lemma 3.5 for case (2) of Lemma 2.2**

Assume the conditions of Lemma 3.5. Additionally assume that case (2) of Lemma 2.2 is satisfied. Fix $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) and (2.31) are satisfied. Note, many of the arguments in this section are similar to those used in Section 4.1 for case (1). Therefore, we shall not go into as much detail here, but we shall highlight the differences where necessary.

We begin by consider the roots of the functions f_t', f_n' and $\tilde{f}_n'$ in this case. We consider f_n' and state that $\tilde{f}_n'$ can be treated similarly. Recall the definitions given in (1.12), (2.3), (2.4), (2.5), (2.10) and (2.12), and the properties discussed in (2.2) and (2.9). These, and case (2) of Lemma 2.2, give the following:

- L_t is an open interval with $t \in L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, $\{\underline{L_t}, \overline{L_t}\} \subset \mathrm{Supp}(\mu)$, and $\overline{S_1} > \overline{L_t} > t > \underline{L_t} > \underline{S_1} \geq \chi$. Moreover, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) > 0$.

- L_n is an open interval with $t \in L_n \in \{K_{1,n}^{(1)}, K_{2,n}^{(1)}, \ldots\}$, $\underline{L_n}$ and $\overline{L_n}$ are two consecutive elements of $S_{1,n}$, and $\underline{L_n} \to \underline{L_t}$ and $\overline{L_n} \to \overline{L_t}$.

Moreover:

Lemma 4.6. *Assume the above conditions. Then, one of the following is satisfied:*

(A) f_t' *has 1 root in* $(t, \overline{L_t})$, *denoted by* s_t.

(B) f_t' *has 0 roots in* $(t, \overline{L_t})$.

Moreover, whenever possibility (A) *is satisfied:*

(A1) $f_t'(s) > 0$ *for all* $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ *and* $f_t'''(t) > 0$, $f_t'(s) > 0$ *for all* $s \in (t, s_t)$, $f_t'(s_t) = 0$ *and* $f_t''(s_t) < 0$, *and* $f_t'(s) < 0$ *for all* $s \in (s_t, \overline{L_t})$.

(A2) f_t' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$.

(A3) f_t' *has at most 1 root in each of* $\bigcup_{i=1}^3 \{K_1^{(i)}, K_2^{(i)}, \ldots\} \setminus \{L_t\}$.

Also, whenever possibility (B) *is satisfied:*

(B1) $f_t'(s) > 0$ *for all* $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ *and* $f_t'''(t) > 0$, $f_t'(s) > 0$ *for all* $s \in (t, \overline{L_t})$.

(B2) f_t' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_1, J_2, J_3, J_4\}$.

(B3) f_t' *has at most 1 root in each of* $\bigcup_{i=1}^3 \{K_1^{(i)}, K_2^{(i)}, \ldots\} \setminus \{L_t\}$.

Next, fixing $\xi > 0$ *as above, then* $(t - 4\xi, t + 4\xi) \subset L_t$, *and* $(t - 2\xi, t + 2\xi) \subset L_n$. *Also,* f_n' *has 2 roots in* $B(t, \xi)$ (*denoted by* $\{t_{1,n}, t_{2,n}\}$), *0 roots in* $(\underline{L_n}, t - \xi]$, *and 1 root in* $[t + \xi, \overline{L_n})$ (*denoted* s_n). *Moreover, whenever possibility* (a) *of Lemma 2.6 is satisfied:*

(a1) $t_{1,n} \in (t - \xi, t + \xi)$ *and* $t_{1,n} = t_{2,n}$. *Moreover* $f_n'(s) > 0$ *for all* $s \in (\underline{L_n}, t_{1,n})$, $f_n'(t_{1,n}) = f_n''(t_{1,n}) = 0$ *and* $f_n'''(t_{1,n}) > 0$, $f_n'(s) > 0$ *for all* $s \in (t_{1,n}, s_n)$, $f_n'(s_n) = 0$ *and* $f_n''(s_n) < 0$, *and* $f_n'(s) < 0$ *for all* $s \in (s_n, \overline{L_n})$.

(a2) f_n' *has 0 roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$.

(a3) f_n' *has 1 root in each of* $\bigcup_{i=1}^3 \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \setminus \{L_n\}$.

Moreover, whenever possibility (b) *is satisfied:*

(b1) $\{t_{1,n}, t_{2,n}\} \subset (t - \xi, t + \xi)$ *and* $t_{1,n} > t_{2,n}$. *Moreover* $f_n'(s) > 0$ *for all* $s \in (\underline{L_n}, t_{2,n})$, $f_n'(t_{2,n}) = 0$ *and* $f_n''(t_{2,n}) < 0$, $f_n'(s) < 0$ *for all* $s \in (t_{2,n}, t_{1,n})$, $f_n'(t_{1,n}) = 0$ *and* $f_n''(t_{1,n}) > 0$, $f_n'(s) > 0$ *for all* $s \in (t_{1,n}, s_n)$, $f_n'(s_n) = 0$ *and* $f_n''(s_n) < 0$, *and* $f_n'(s) < 0$ *for all* $s \in (s_n, \overline{L_n})$.

(b2) f_n' *has* 0 *roots in each of* $\{\mathbb{C} \setminus \mathbb{R}, J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$.

(b3) f_n' *has* 1 *root in each of* $\bigcup_{i=1}^{3} \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \setminus \{L_n\}$.

Finally, whenever possibility (c) *is satisfied:*

(c1) $t_{1,n} \in B(t, \xi) \cap \mathbb{H}$ *and* $t_{2,n}$ *is the complex conjugate of* $t_{1,n}$. *Moreover,* $f_n'(s) > 0$ *for all* $s \in (\underline{L_n}, s_n)$, $f_n'(s_n) = 0$ *and* $f_n''(s_n) < 0$, $f_n'(s) < 0$ *for all* $s \in (s_n, \overline{L_n})$.

(c2) f_n' *has* 0 *roots in each of* $\{\mathbb{C} \setminus (\mathbb{R} \cup \{t_{1,n}, t_{2,n}\}), J_{1,n}, J_{2,n}, J_{3,n}, J_{4,n}\}$

(c3) f_n' *has* 1 *root in each of* $\bigcup_{i=1}^{3} \{K_{1,n}^{(i)}, K_{2,n}^{(i)}, \ldots\} \setminus \{L_n\}$.

Proof. Consider f_t'. First recall, since case (2) of Lemma 2.2 is satisfied, that $t \in L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, and $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) > 0$. Part (1) of Lemma 2.1 then implies that f_t' has at most 1 root in $L_t \setminus \{t\}$. Thus we have three possibilities:

- f_t' has 0 roots in $(\underline{L_t}, t)$, and 1 root in $(t, \overline{L_t})$ (denoted s_t).

- f_t' has 0 roots in $(\underline{L_t}, t)$, and 0 roots in $(t, \overline{L_t})$.

- f_t' has 1 root in $(\underline{L_t}, t)$ (denoted s_t), and 0 roots in $(t, \overline{L_t})$.

Whenever the first possibility is satisfied, we will show that possibility (A) and parts (A1)–(A3) are also satisfied. Moreover, whenever the second possibility is satisfied, we can similarly show that possibility (B) and parts (B1)–(B3) are also satisfied. Finally, we will show that the third possibility is never satisfied.

Suppose that the first possibility is satisfied. Possibility (A) is then trivially satisfied. Moreover, f_t' has 0 roots in $(\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) > 0$, f_t' has 0 roots in (t, s_t), $f_t'(s_t) = 0$ and $f_t''(s_t) \neq 0$, and f_t' has 0 roots in $(s_t, \overline{L_t})$. Thus, since $(f_t')|_{L_t}$ is real-valued and continuous (see (2.1)), the above observations are only possible if part (A1) is satisfied. Moreover, since $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, parts (2) and (3) of Lemma 2.1 imply parts (A2) and (A3). Thus, whenever the first possibility is satisfied, possibility (A) and parts (A1)–(A3) are also satisfied.
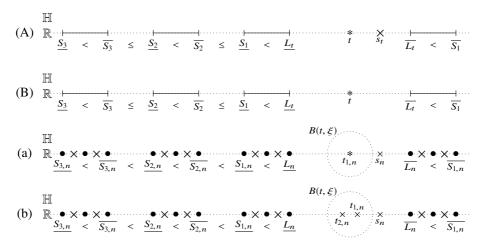
FIGURE 4.7. (A), (B): T_t, the set of roots of f_t', as described by Lemma 4.1, for possibilities (A) and (B). Note, for each $i \in \{1, 2, 3\}$, the subintervals of $[\underline{S_i}, \overline{S_i}] \setminus S_i$ contain at most 1 root (except L_t). (a), (b), (c): T_n, the set of roots of f_t', as described by Lemma 4.1 for possibilities (a), (b) and (c). Roots of multiplicity 1 and 2 are represented by $\times$ and $*$ respectively, and elements of $S_n = S_{1,n} \cup S_{2,n} \cup S_{3,n}$ are represented by $\bullet$.

Suppose that the third possibility is satisfied. First, recall that $s_t \in (\underline{L_t}, t)$ is a root of f_t' of multiplicity 1. Thus, we can fix an $\epsilon > 0$ sufficiently small such that $\overline{L_t} + \epsilon < s_t < t - \epsilon$, and s_t is the unique root of f_t' in $B(s_t, \epsilon)$. Next, we can proceed similarly to part (iii) in the proof of Lemma 2.6 to show that f_n' has 1 root in $B(s_t, \epsilon)$. Also, since roots of f_n' occur in complex conjugate pairs (see (2.7)), this root must be contained in $(s_t - \epsilon, s_t + \epsilon)$. Finally, note that (2.12) implies that we can choose ϵ and ξ such that $(s_t - \epsilon, s_t + \epsilon) \subset (\underline{L_n}, t - \xi]$. Therefore f_n' has a root in $(\underline{L_n}, t - \xi]$. However, as we will shortly show below, such a root does not exist. Thus we have a contradiction, and so the third possibility is never satisfied.

Consider f_n'. First recall that $(t - 4\xi, t + 4\xi) \subset L_t$, and $(t - 2\xi, t + 2\xi) \subset L_n$ comes from (2.11) and (2.12). Also, part (1) of Lemma 2.6 implies that f_n' has 2 roots in $B(t, \xi)$.

Suppose that possibility (a) of Lemma 2.6 is satisfied, i.e., that $t_{1,n} \in (t - \xi, t + \xi) \subset L_n$, and $t_{1,n} = t_{2,n}$ is root of f_n' of multiplicity 2. Thus $f_n'(t_{1,n}) = f_n''(t_{1,n}) = 0$ and $f_n'''(t_{1,n}) \neq 0$.

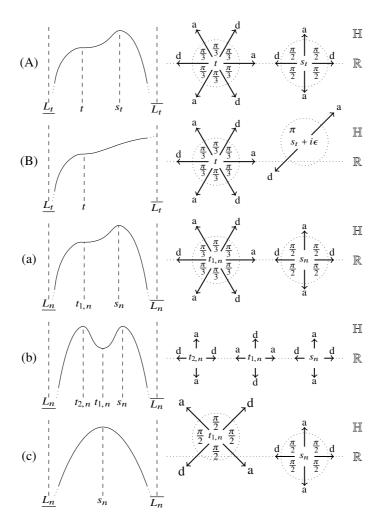FIGURE 4.8. (A), (B), left: $R_t|_{L_t}$ for possibilities (A) and (B) of Lemma 4.6. (a), (b), (c), left: $R_n|_{L_n}$ for possibilities (a)–(c) of Lemma 4.6. (A), (B), right: The associated directions of steepest decent/ascent in $\mathbb{C}$ for f_t for possibilities (A) and (B). (a), (b), (c), right: The associated directions of steepest decent/ascent in $\mathbb{C}$ for f_n for possibilities (a)–(c).

Indeed, $f_n'''(t_{1,n}) \to f_t'''(t) > 0$ (see part (4) of Lemma 2.9, and Lemma 3.1). Next note, (2.7) implies that $(f_n')|_{L_n}$ is real-valued and continuous. Moreover, recalling that $\underline{L_n}$ and $\overline{L_n}$ are two consecutive elements of $S_{1,n}$, this equation gives

$$\lim_{w \in \mathbb{R}, w \downarrow \underline{L_n}} f_n'(w) = +\infty \qquad \text{and} \qquad \lim_{w \in \mathbb{R}, w \uparrow \overline{L_n}} f_n'(w) = -\infty.$$

Also, recalling that $L_n \in \{K_{1,n}^{(1)}, K_{2,n}^{(1)}, \ldots\}$, part (1) of Lemma 2.6 implies that f_n' has 0 roots in $(t - \xi, t + \xi) \setminus \{t_{1,n}\}$, and 1 root (denoted s_n) in $L_n \setminus (t - \xi, t + \xi) = (\underline{L_n}, t - \xi] \cup [t + \xi, \overline{L_n})$. The above observations are only possible if $s_n \in [t + \xi, \overline{L_n})$ and part (a1) is satisfied. Finally, since $t_{1,n} \in (t - \xi, t + \xi)$ and $t_{1,n} = t_{2,n}$ (see possibility (a) of Lemma 2.6), and since $L_n \in \{K_{1,n}^{(1)}, K_{2,n}^{(1)}, \ldots\}$, parts (1)–(3) of Lemma 2.6 imply parts (a2) and (a3). Thus, whenever possibility (a) of Lemma 2.6 is satisfied, we have parts (a1)–(a3). Similarly, whenever possibilities (b) and (c) are satisfied, we have parts (b1)–(b3) and (c1)–(c3) respectively. $\qquad\square$

As in Section 4.1, let $T_t \subset \mathbb{C} \setminus S$ and $T_n \subset \mathbb{C} \setminus S_n$ respectively denote the set of roots of f_t' and the set of roots of f_n'. The previous lemma discusses the locations of the elements of these discrete sets, and this is depicted in Figure 4.7. Next recall, similarly to Section 4.1, (3.7) gives $(R_t|_{L_t})' = (f_t')|_{L_t}$ and $(R_n|_{L_n})' = (f_n')|_{L_n}$, and similarly for the higher order derivatives. Parts (A1), (B1), (a1), (b1) and (c1) of Lemma 4.6 then imply that $R_t|_{L_t}$ and $R_n|_{L_t}$ have those behaviours shown on the left of Figure 4.8 for the various possibilities. Part (3) of Lemma 3.4 also shows that f_t and f_n have those directions of steepest descent/ascent shown on the right of Figure 4.8. For possibility (B), in anticipation of the following lemma, we also display the directions of steepest descent/ascent in the neighbourhood of an arbitrary point $s_t + i\epsilon \in \mathbb{H}$, where $s_t \in (t, \overline{L_t})$ and $\epsilon > 0$. Note, part (B2) of Lemma 4.6 implies that $f_t'(s_t + i\epsilon) \neq 0$ for any such $s_t + i\epsilon$, and so part (3) of Lemma 3.4 implies that there is a unique direction of steepest descent and a unique direction of steepest ascent as shown. The following lemma examines the resulting contours of steepest descent/ascent:

Lemma 4.7. *Whenever possibility* (A) *of Lemma 4.6 is satisfied, $s_t \in [t + \xi, \overline{L_t})$ (see (2.11) and part* (A1) *of Lemma 4.6). Moreover, there exists simple contours, D_t, A_t, A_t'', as shown in Figure 4.9 with the following properties:*

(A1) *D_t, A_t, A_t'' start at t, t, s_t respectively, enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{2}$ respectively, and end in the intervals shown or are unbounded.*

(A2) *D_t is a contour of steepest descent for f_t, and A_t, A_t'' are contours of steepest ascent for f_t.*

(A3) D_t and A_t intersect at t, and D_t, A_t, A_t'' do not otherwise intersect.

When possibility (B) is satisfied, fix an arbitrary $s_t \in (t, \overline{L_t})$. Moreover, fix the $\xi > 0$ sufficiently small such that $s_t \in [t + \xi, \overline{L_t})$, and fix an arbitrary $\epsilon > 0$. Then, when the $\epsilon > 0$ is fixed sufficiently small, there exists simple contours, D_t, A_t, D_t'', A_t'', as shown in Figure 4.9 with the following properties:

(B1) D_t and A_t both start at t, enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$ and $\frac{2\pi}{3}$ respectively, and end in the intervals shown. D_t'' and A_t'' both start at $s_t + i\epsilon \in \mathbb{H}$, leave $s_t + i\epsilon$ in opposite directions, and end in the interior of the intervals shown or are unbounded.

(B2) D_t, D_t'' are contours of steepest descent for f_t, and A_t, A_t'' are contours of steepest ascent for f_t.

(B3) D_t and A_t intersect at t, D_t'' and A_t'' intersect at $s_t + i\epsilon$, and D_t, A_t, D_t'', A_t'' do not otherwise intersect.

Also, whenever possibility (a) is satisfied, there exists simple contours, D_n, A_n, A_n'', as shown in Figure 4.9 with the following properties:

(a1) D_n, A_n, A_n'' start at $t_{1,n}$, $t_{1,n}$, s_n respectively, enter $\mathbb{H}$ in the directions $\frac{\pi}{3}$, $\frac{2\pi}{3}$, $\frac{\pi}{2}$ respectively, and end in the intervals shown or are unbounded.

(a2) D_n is a contour of steepest descent for f_n, and A_n, A_n'' are contours of steepest ascent for f_n.

(a3) D_n and A_n intersect at $t_{1,n}$, and D_n, A_n, A_n'' do not otherwise intersect.

Next, whenever possibility (b) is satisfied, there exists simple contours, D_n, A_n, A_n'', as shown in Figure 4.9 with the following properties:

(b1) D_n, A_n, A_n'' start at $t_{1,n}$, $t_{2,n}$, s_n respectively, all enter $\mathbb{H}$ in the direction $\frac{\pi}{2}$, and end in the intervals shown or are unbounded.

(b2) D_n is a contour of steepest descent for f_n, and A_n, A_n'' are contours of steepest ascent for f_n.

(b3) D_n, A_n, A_n'' do not intersect.

Next, whenever possibility (c) is satisfied, there exists simple contours, D_n, A_n, D_n', A_n', A_n'', as shown in Figure 4.9 with the following properties:

(c1) D_n, A_n, D'_n, A'_n all start at $t_{1,n} \in \mathbb{H}$, leave $t_{1,n}$ in orthogonal directions in the counter-clockwise order D_n, A_n, D'_n, A'_n, and end in the intervals shown or are unbounded. A''_n starts at s_n, enters $\mathbb{H}$ in the direction $\frac{\pi}{2}$, and is unbounded.

(c2) D_n, D'_n are contours of steepest descent for f_n, and A_n, A'_n, A''_n are contours of steepest ascent for f_n.

(c3) D_n, A_n, D'_n, A'_n intersect at $t_{1,n}$, and $D_n, A_n, D'_n, A'_n, A''_n$ does not otherwise intersect.

Proof. Many of the arguments here are similar to those used in Lemma 4.2, so we do not go into as much detail here.

Consider f_n. First, for possibilities (a) and (b), define D_n and A_n as in Lemma 4.2. Also, for possibility (c), define D_n, A_n, D'_n, A'_n as in Lemma 4.2. Moreover, for possibilities (a)–(c), let A''_n denote the contour of steepest ascent for f_n which starts at $s_n \in L_n$, and which enters $\mathbb{H}$ in the direction $\frac{\pi}{2}$. Then, proceeding similarly to the proof of Lemma 4.2, we can show that each of D_n, D'_n end in $S_n \cup T_n$, each of A_n, A'_n, A''_n either end in $S_n \cup T_n$ or are unbounded, and the contours are otherwise in $\mathbb{H}$. We will show:

(i) There exists a $\delta \in (0, \xi)$ for which A''_n does not intersect cl$(B(t, \delta))$.

(ii) For possibility (a), D_n and A_n do not intersect except at $t_{1,n}$, and D_n and A''_n never intersect. For possibility (b), D_n and A_n never intersect, and D_n and A''_n never intersect. For possibility (c), D_n, A_n, D'_n, A'_n do not intersect except at $t_{1,n}$, D_n and A''_n never intersect, and D'_n and A''_n never intersect. For all possibilities, the contours are simple.

Then, we can investigate the possible end-points of $D_n, A_n, D'_n, A'_n, A''_n$ using arguments by contradiction similar to those used in the proof of Lemma 4.2. We omit the details, and simply state that this investigation gives the required results.

Consider (i). First recall that A''_n starts at s_n, and R_n strictly increases along A''_n. It is thus sufficient to show that there exists a $\delta \in (0, \xi)$ for which the following is satisfied:

$$R_n(w) < R_n(s_n) \text{ for all } w \in \text{cl}(B(t, \delta)). \tag{4.12}$$

To see this first note, part (2) of Lemma 3.1 implies that $R_n(w) = R_t(w) + o(1)$ uniformly for $w \in B(t, \xi)$. Also, since R_t is continuous, $R_t(w) = R_t(t) + O(\delta) = f_t(t) + O(\delta)$ for all $\delta > 0$ sufficiently small and uniformly for $w \in \text{cl}(B(t, \delta))$. Next recall (see statement of this lemma) that $s_t \in [t + \xi, \overline{L_t})$ for possibilities (A) and (B), and so parts (A1) and (B1) of Lemma 4.6 imply that $f_t(t) < f_t(t + \xi)$. Moreover, part (2) of Lemma 3.1 implies that
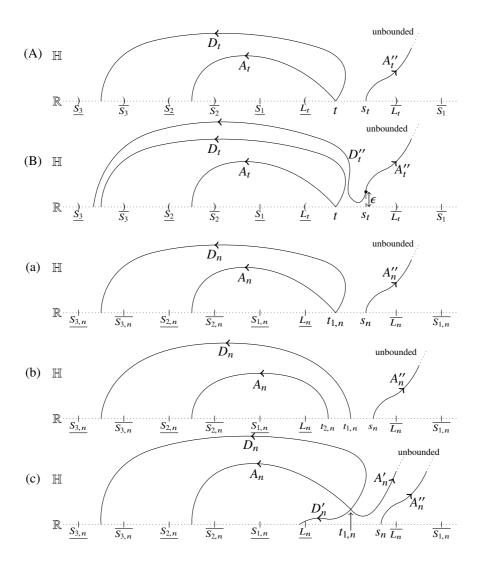
FIGURE 4.9. The contours of Lemma 4.7.

$f_t(t + \xi) = f_n(t + \xi) + o(1)$. Finally, parts (a1), (b1) and (c1) of Lemma 4.6 imply that $f_n(t + \xi) < f_n(s_n) = R_n(s_n)$. Combined, the above imply (4.12). This proves part (i).

Consider (ii) for possibility (a). Recall that D_n and A_n start at $t_{1,n}$, R_n strictly decreases along D_n, and R_n strictly increases along A_n. Also recall that D_n starts at $t_{1,n} \in L_n$ and A_n''

starts at $s_n \in L_n$, R_n strictly decreases along D_n, R_n strictly increases along A_n'', $s_n > t_{1,n}$, and R_n strictly increases as we move from $t_{1,n}$ to s_n along L_n (see left of Figure 4.8). A contradiction argument then proves part (ii) for possibility (a). Part (ii) for possibility (b) follows similarly.

Consider (ii) for possibility (c). This follows from similar arguments to those used in the proof of Lemma 4.2, except we need to additionally argue that D_n never intersects A_n'', and D_n' never intersects A_n''. Recall that D_n and D_n' start at $t_{1,n}$, A_n'' starts at s_n, R_n strictly decreases along D_n and D_n', R_n strictly increases along A_n'', and $R_n(t_{1,n}) < R_n(s_n)$ (recall that $t_{1,n} \to t$, and apply part (i) above). A contradiction argument then proves that D_n and D_n' never intersect A_n''. This proves part (ii) for possibility (c).

Consider f_t. First, define D_t and A_t as in Lemma 4.2. Next, for possibility (A) of Lemma 4.6, let A_t'' denote the contour of steepest ascent for f_t which starts at $s_t \in L_t$, and which enters $\mathbb{H}$ in the direction $\frac{\pi}{2}$. For possibility (B), let D_t'' and A_t'' denote the contours of steepest descent and ascent (respectively) which start at $s_t + i\epsilon \in \mathbb{H}$. Then, proceeding similarly to the proof of Lemma 4.2, we can show that each of D_t, D_t'' end in $S \cup T_t$, each of A_t, A_t'' either end in $S \cup T_t$ or are unbounded, and the contours are otherwise in $\mathbb{H}$. We will show:

(iii) For possibility (A), D_t and A_t do not intersect except at t, and D_t and A_t'' never intersect. For possibility (B), D_t and A_t do not intersect except at t, D_t'' and A_t'' do not intersect except at $s_t + i\epsilon$, and we can fix the $\epsilon > 0$ sufficiently small such that D_t'', A_t'' never intersect D_t, A_t. For both possibilities, the contours are simple.

(iv) For possibilities (A) and (B), D_t ends in $[\underline{S_3}, \overline{S_3})$, and A_t ends in $(\underline{S_2}, \overline{S_2})$.

(v) For possibility (A), A_t'' is unbounded. For possibility (B), fixing the $\epsilon > 0$ sufficiently small as above, A_t'' is unbounded and D_t'' ends in $[\underline{S_3}, d_t)$, where d_t denotes the end-point of D_t (thus, necessarily, we must have $d_t \in (\underline{S_3}, \overline{S_3})$).

These prove the required results.

Consider (iii) for possibility (A). This follows from similar arguments to those used to show part (ii) for possibility (a), above. Consider (iii) for possibility (B). Similar arguments show that D_t and A_t do not intersect except at t, D_t'' and A_t'' do not intersect except at $s_t + i\epsilon$, and the contours are simple. It thus remains to show that we can fix the $\epsilon > 0$ sufficiently small such that D_t'', A_t'' never intersect D_t, A_t. Recalling that $\mathrm{Im}(f_t(w)) = \mathrm{Im}(f_t(t))$ and $\mathrm{Im}(f_t(z)) = \mathrm{Im}(f_t(s_t + i\epsilon))$ for all w on $D_t \cup A_t$ and z on $D_t'' \cup A_t''$ (see part (2) of Lemma 3.4), and $\mathrm{Im}(f_t(t)) = \mathrm{Im}(f_t(s_t))$ (recall that t and s are both in $L_t \subset \mathbb{R} \setminus S$, and see Figure 3.1), it is thus sufficient to show that $\mathrm{Im}(f_t(s_t + i\epsilon)) \neq \mathrm{Im}(f_t(s_t))$ for all $\epsilon > 0$ sufficiently small. To see this recall that $s_t \in (t, \overline{L_t})$, and so part (B1) of Lemma 4.6 implies

that $f_t'(s_t) > 0$. Also, a Taylor expansion gives $f_t(s_t + i\epsilon) = f_t(s_t) + i\epsilon f_t'(s_t) + O(\epsilon^2)$ for all $\epsilon > 0$ sufficiently small. Therefore, for all $\epsilon > 0$ sufficiently small,

$$\mathrm{Im}(f_t(s_t + i\epsilon)) > \mathrm{Im}(f_t(s_t)) + \tfrac{1}{2}\epsilon f_t'(s_t). \qquad (4.13)$$

This proves part (iii) for possibility (B).

Consider (iv). Recall that D_t ends in $S \cup T_t$, and A_t either ends $S \cup T_t$ or is unbounded, where T_t is the set of roots of f_t'. The behaviours of T_t for possibilities (A) and (B) are discussed in Lemma 4.6, and these behaviours are displayed in Figure 4.7. It follows that $S \cup T_t \subset [\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup \{t, s_t\} \cup [\overline{L_t}, \overline{S_1}]$ for possibility (A), and $S \cup T_t \subset [\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup \{t\} \cup [\overline{L_t}, \overline{S_1}]$ for possibility (B). Next recall that D_t and A_t start at t, and A_t'' starts at s_t for possibility (A). The above observations, and part (iii), thus imply the following for both possibilities: D_t ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$, and A_t either ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$ or is unbounded. Then, since D_t and A_t start at $t \in L_t$, part (iv) follows from similar arguments to those used to show parts (xii) and (xiii) in the proof of Lemma 4.2.

Consider (v) for possibility (A). Recall that A_t'' starts at $s_t \in (t, \overline{L_t})$, and ends either in $S \cup T_t$ or is unbounded. Then, similar considerations to those used in part (iv), above, show that A_t'' either ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$ or is unbounded. We must thus show that A_t'' does not end in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$. We argue by contradiction: First assume that A_t'' ends in $[d_t, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t]$, where $d_t \in [\underline{S_3}, \overline{S_3})$ denotes the end point of D_t (see part (iv)). Then D_t starts at $t \in L_t$ and ends at d_t, A_t'' starts at $s_t \in (t, \overline{L_t})$ and ends in $[d_t, t)$, and D_t and A_t'' are otherwise contained in $\mathbb{H}$. A_t'' must therefore intersect D_t, which contradicts part (iii). Next assume that A_t'' ends at a point $a_t'' \in [\underline{S_3}, d_t]$. Then D_t starts at $t \in L_t$ and ends at $d_t \in [\underline{S_3}, \overline{S_3})$, A_t'' starts at $s_t \in (t, \overline{L_t})$ and ends at $a_t'' \in [\underline{S_3}, d_t]$, and D_t and A_t'' are otherwise contained in $\mathbb{H}$ and do not intersect (see part (iii)). Consider the simple closed contour consisting of D_t and A_t'' and $[a_t'', d_t]$ and $[t, s_t]$. We can proceed as in part (xii) in the proof of Lemma 4.2 to show that $w \mapsto \mathrm{Im}(f_t(w))$ is a constant harmonic function on the above simple closed contour, and so it also constant in the domain bounded by the closed contour. (3.4) easily shows that this is not true. Thus we have a contradiction, and so A_t'' does not end in $[\underline{S_3}, d_t]$. Finally suppose that A_t'' ends in $[\overline{L_t}, \overline{S_1}]$. Then A_t'' starts at $s_t \in (t, \overline{L_t})$, ends in $[\overline{L_t}, \overline{S_1}]$, and is otherwise contained in $\mathbb{H}$. We can then proceed as in part (xii) in the proof of Lemma 4.2 to show that this gives a contradiction, and so A_t'' does not end in $[\overline{L_t}, \overline{S_1}]$. This proves (v) for possibility (A).

Consider (v) for possibility (B). Recall that D_t'' and A_t'' start at $s_t + i\epsilon$ where $s_t \in (t, \overline{L_t})$, D_t'' ends in $S \cup T_t$, and A_t'' ends either in $S \cup T_t$ or is unbounded. Then, similar considerations to those used in part (iv), above, show that D_t'' ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$,

and A_t'' either ends in $[\underline{S_3}, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t] \cup [\overline{L_t}, \overline{S_1}]$ or is unbounded. Also, letting $d_t \in [\underline{S_3}, \overline{S_3})$ denote the end point of D_t (see part (iv)), similar considerations to those used above (in the proof of part (v) for possibility (A)) show that both D_t'' and A_t'' do not end in $[d_t, \overline{S_3}] \cup [\underline{S_2}, \overline{S_2}] \cup [\underline{S_1}, L_t]$. It thus remains to show that D_t'' does not end in $[\overline{L_t}, \overline{S_1}]$, and A_t'' does not end in $[\underline{S_3}, d_t) \cup [\overline{L_t}, \overline{S_1}]$. We argue by contradiction: First assume that D_t'' and A_t'' both end in $[\underline{S_3}, d_t)$, at d_t'' and a_t'' respectively. Then D_t'' starts at $s_t + i\epsilon$ and ends at $d_t'' \in [\underline{S_3}, d_t)$, A_t'' starts at $s_t + i\epsilon$ and ends at $a_t'' \in [\underline{S_3}, d_t)$, and D_t'' and A_t'' are otherwise contained in $\mathbb{H}$ and do not intersect except at $s_t + i\epsilon$ (see part (iii)). Consider the simple closed contour consisting of D_t'' and A_t'' and $[a_t'' \wedge d_t'', a_t'' \vee d_t'']$. We can proceed as in part (xii) in the proof of Lemma 4.2 to show that $w \mapsto \mathrm{Im}(f_t(w))$ is a constant harmonic function on the above simple closed contour, and so it also constant in the domain bounded by the closed contour. (3.4) easily show that this is not true. Thus we have a contradiction, and so D_t'' and A_t'' cannot both end in $[\underline{S_3}, d_t)$. Next assume that D_t'' ends at a point $d_t'' \in [\overline{L_t}, \overline{S_1}]$. Recall (see part (2) of Lemma 3.4) that $w \mapsto \mathrm{Im}(f_t(w))$ is constant on D_t''. In particular this gives $\mathrm{Im}(f_t(d_t'')) = \mathrm{Im}(f_t(s_t + i\epsilon))$. Also recall (see case (2) of Lemma 2.2 and (2.4)) that $s_t \in L_t \subset [\underline{S_1}, \overline{S_1}] \setminus S_1$ and $[\overline{L_t}, \overline{L_t} + \delta) \subset S_1$ for all $\delta > 0$ sufficiently small. (3.5) and Figure 3.1 then give $\mathrm{Im}(f_t(s_t)) \geq \mathrm{Im}(f_t(s))$ for all $s \in [\overline{L_t}, \overline{S_1}]$. (4.13) thus gives $\mathrm{Im}(f_t(s_t + i\epsilon)) > \mathrm{Im}(f_t(s))$ for all $s \in [\overline{L_t}, \overline{S_1}]$. This contradicts $\mathrm{Im}(f_t(d_t'')) = \mathrm{Im}(f_t(s_t + i\epsilon))$, and so D_t'' does not end in $[\overline{L_t}, \overline{S_1}]$. We can similarly show that A_t'' does not end in $[\overline{L_t}, \overline{S_1}]$. This proves (v) for possibility (B). $\qquad\square$

As in the Section 4.1 for case (1) of Lemma 2.2, we are now in a position to define the contours, $\gamma_{1,n}^+$ and $\Gamma_{1,n}^+$ and $\Gamma_{2,n}^+$, that satisfy Lemma 3.5 for case (2) of Lemma 2.2. We do not go into as much detail here as the Section 4.1, as the construction is quite similar to that used for case (1), but we will highlight the differences.

Fix $\xi > 0$, $\theta \in (\frac{1}{4}, \frac{1}{3})$ and $\{q_n\}_{n \geq 1} \subset \mathbb{R}$ as in the Section 4.1. Note, since $f_t'''(t) > 0$ (see case (2) of Lemma 2.2), Lemma 2.7 and Definition 2.8 imply that $\{q_n\}_{n \geq 1}$ converges to a positive constant. Then, part (4) of Lemma 2.9 implies that $\{t_{1,n}, t_{2,n}\} \subset B(t, n^{-\theta}q_n) \subset B(t, \xi)$. Thus, D_n and A_n both start inside $B(t, n^{-\theta}q_n)$ and $B(t, \xi)$. Define $d_{1,n}, d_{2,n}, a_{1,n}, a_{2,n}$ as in (4.3). Denote the equivalent quantities for D_t and A_t by $d_{1,t}, d_{2,t}, a_{1,t}, a_{2,t}$. Also, fixing $\xi > 0$ sufficiently small as above, define $\mathbb{C}_\xi$ is in (3.1). Recall that D_t ends in $[\underline{S_3}, \overline{S_3}) \subset \mathbb{C} \setminus \mathbb{C}_\xi$, and A_t ends in $(\underline{S_2}, \overline{S_2}) \subset \mathbb{C} \setminus \mathbb{C}_\xi$, and let $d_{3,t}$ and $a_{3,t}$ denote the points where D_t and A_t "exit" $\mathbb{C}_\xi$ respectively. We choose the $\xi > 0$ sufficiently small such that (4.4) is satisfied in this case also: $\mathrm{Im}(d_{3,t}) = \mathrm{Im}(a_{3,t}) = \xi^4$. Moreover, we choose the $\xi > 0$ sufficiently small such that $\mathrm{Re}(d_{3,t}) \in (-\infty, \overline{S_3} - c)$ and $\mathrm{Re}(a_{3,t}) \in (\underline{S_2} + c, \overline{S_2} - c)$, where $c = c(t) > 0$ is some constant which is independent of ξ. Next, define $\epsilon := 0$ whenever possibility (A) of Lemmas 4.6 and 4.7 is satisfied, and fix $\epsilon > 0$ as in Lemma 4.7 whenever possibility (B) is satisfied. Finally, fix $E > 0$ such that
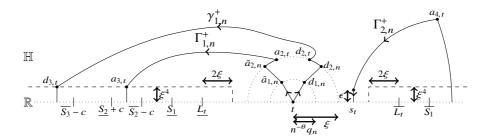
FIGURE 4.10. The contours defined in Definition 4.8. The dashed lines represent boundaries of $\mathbb{C}_\xi$. $c = c(t) > 0$ is independent of ξ.

$B(0, E)$ contains $S_1 \cup S_2 \cup S_3$ and the point $s_t + i\epsilon$, and let $a_{4,t} \in \partial B(0, E)$ denote that point where A_t'' "exits" $B(0, E)$ for the first time. We then define the following contours, which are depicted in Figure 4.10:

Definition 4.8. Define $\gamma_{1,n}^+$ and $\Gamma_{1,n}^+$ similarly as in Definition 4.4 (they both start at $t \in L_t = (\underline{L_t}, \overline{L_t})$, $\gamma_{1,n}^+$ ends at $\mathrm{Re}(d_{3,t}) \in (-\infty, \overline{S_3} - c)$, $\Gamma_{1,n}^+$ ends at $\mathrm{Re}(a_{3,t}) \in (\underline{S_2} + c, \overline{S_2} - c)$ etc.). Moreover, define $\Gamma_{2,n}^+$ to be the opposite (traversed in the opposite direction) of the simple contour which:

- is independent of n,

- starts at $s_t \in (t, \overline{L_t})$,

- then traverses the straight line from s_t to $s_t + i\epsilon$,

- then traverses that section of A_t'' from $s_t + i\epsilon$ to $a_{4,t} \in \partial B(0, E)$,

- then traverses (clockwise) the arc of $\partial B(t, \xi)$ from $a_{4,t} \in \partial B(0, E)$ to E,

- then ends at $E \in (\overline{S_1}, +\infty)$.

We finally show that the above contours satisfy the requirements of Lemma 3.5:

Proof of Lemma 3.5 for case (2) *of Lemma 2.2.* The proof for case (2) of Lemma 2.2 is very similar to the proof for case (1) (see the end of Section 4.1). Parts (1), (2), (3), (5) and (6) follow from similar arguments. Similarly for part (4), except that we need to

additionally show that $\tilde{R}_n(z) \geq \tilde{R}_n(\tilde{a}_{1,n})$ for all $z \in \Gamma_{2,n}^+$, where $\tilde{R}_n$ denote the real-part of $\tilde{f}_n$. We will show that there exists a sufficiently small choice ϵ, ξ in Definition 4.8, and a sufficiently large choice of the $E > 0$, such that the following are satisfied:

(i) $\tilde{R}_n(z) > \tilde{R}_n(\tilde{a}_{1,n})$ for all $z \in \mathrm{cl}(B(s_t, \epsilon))$.

(ii) $\tilde{R}_n(z) > \tilde{R}_n(\tilde{a}_{1,n})$ for all z on that section of A_t'' from $s_t + i\epsilon$ to $a_{4,t} \in \partial B(0, E)$.

(iii) $\tilde{R}_n(z) > \tilde{R}_n(\tilde{a}_{1,n})$ for all $z \in \partial B(0, E)$.

Definition 4.8 then implies that $\tilde{R}_n(z) > \tilde{R}_n(\tilde{a}_{1,n})$ for all $z \in \Gamma_{2,n}^+$. This proves (4).

Consider (i) for possibility (B). First, we fix the $\xi, \epsilon > 0$ sufficiently small such that $\mathrm{cl}(B(s_t, \epsilon)) \subset \mathbb{C}_\xi$ (see Figure 4.10 to see that this can be done). Then, part (2) of Lemma 3.1 implies that $\tilde{R}_n(z) = R_t(z) + o(1)$ uniformly for $z \in \mathrm{cl}(B(s_t, \epsilon))$. Next note, since R_t is continuous at s_t, that $R_t(z) = R_t(s_t) + O(\epsilon)$ for all $\epsilon > 0$ sufficiently small and uniformly for $z \in \mathrm{cl}(B(s_t, \epsilon))$. Next recall that $s_t \in (t, \overline{L_t})$, and so $R_t(s_t) > R_t(t)$ (see part (B1) of Lemma 4.6 and Figure 4.8). Finally note, since $\tilde{a}_{1,n} = t + o(1)$, (2.8), (3.2) and (3.3) give $R_t(t) = \tilde{R}_n(\tilde{a}_{1,n}) + o(1)$. Combined, the above proves part (i) for possibility (B). Part (i) for possibility (A), where $\epsilon = 0$ by the remarks preceding Definition 4.8, can be shown similarly.

Consider (ii). First, we fix ξ, ϵ sufficiently small such that $A_t'' \subset \mathbb{C}_\xi$ (see Figure 4.10 to see that this can be done). Then, part (2) of Lemma 3.1 implies that $\tilde{R}_n(z) = R_t(z) + o(1)$ uniformly for z on that section of A_t'' from $s_t + i\epsilon$ to $a_{4,t} \in \partial B(0, E)$. Next note, part (1) of Lemma 3.4 gives $R_t(z) \geq R_t(s_t + i\epsilon)$ for z on A_t''. Moreover, since R_t is continuous at s_t, $R_t(s_t + i\epsilon) = R_t(s_t) + O(\epsilon)$ for all $\epsilon > 0$ sufficiently small. We can then proceed as in the proof of part (i), above, to prove part (ii).

Consider (iii). First note, letting log be natural logarithm, (3.3) gives the following when $z \neq 0$:

$$\tilde{R}_n(z) = \frac{1}{n} \left(|\tilde{S}_{1,n}| - |\tilde{S}_{2,n}| + |\tilde{S}_{3,n}| \right) \log |z|$$
$$+ \frac{1}{n} \sum_{x \in \tilde{S}_{1,n}} \log \left| 1 - \frac{x}{z} \right| - \frac{1}{n} \sum_{x \in \tilde{S}_{2,n}} \log \left| 1 - \frac{x}{z} \right| + \frac{1}{n} \sum_{x \in \tilde{S}_{3,n}} \log \left| 1 - \frac{x}{z} \right|.$$

Recall (see (2.2) and (2.9)) that $\frac{1}{n}(|\tilde{S}_{1,n}| - |\tilde{S}_{2,n}| + |\tilde{S}_{3,n}|) \to \eta \in (0, 1)$. Also recall (see (2.2)) that $\sup_{x \in S_{1,n} \cup S_{2,n} \cup S_{3,n}} |x| = O(1)$. Thus $\tilde{R}_n(z) > \frac{1}{2}\eta \log |z|$ for all n and $|z|$ sufficiently large, chosen independently. Finally recall (see the proof of part (i), above) that $\tilde{R}_n(\tilde{a}_{1,n}) \to R_t(t)$. Combined, the above prove part (iii). $\square$

4.3. **Lemma 3.5 for cases (3-12) of Lemma 2.2**

In Sections 4.1 and 4.2 we proved Lemma 3.5 for cases (1) and (2) of Lemma 2.2. In this section we discuss cases (3)–(6). We will be very brief here, as no new ideas are needed. Instead we seek to highlight some differences which may cause confusion. Indeed, we only recall the possible behaviours of the roots of f_t' in L_t for cases (1) and (2), and state the possible behaviours of the roots of f_t' in L_t for cases (3)–(6). For cases (3)–(6), the behaviour of the remaining roots of f_t', the behaviour of the roots of f_n'', the contours of steepest descent/ascent, and the definitions of the contours $\gamma_{1,n}^+$, etc., then follow from similar considerations to those used for cases (1) and (2). Similar considerations also apply for cases (7)–(12).

First recall the situation for case (1) of Lemma 2.2 (see Section 4.1). Here, $t > \chi$, $t \in L_t$, $L_t = J_1 = (\overline{S_1}, +\infty)$, and $f_t'''(t) > 0$. Moreover $f_t'(s) > 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t''(t) > 0$, and $f_t'(s) > 0$ for all $s \in (t, \overline{L_t})$.

Next recall the situation for case (2) of Lemma 2.2 (see Section 4.2). Here, $t > \chi$, $t \in L_t$, $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, and $f_t'''(t) > 0$. Moreover, one of the following is satisfied:

(A) There exists an $s_t \in (t, \overline{L_t})$ for which: $f_t'(s) > 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) > 0$, $f_t'(s) > 0$ for all $s \in (t, s_t)$, $f_t'(s_t) = 0$ and $f_t''(s_t) < 0$, and $f_t'(s) < 0$ for all $s \in (s_t, \overline{L_t})$.

(B) $f_t'(s) > 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) > 0$, $f_t'(s) > 0$ for all $s \in (t, \overline{L_t})$.

Consider case (3). Lemma 2.2 gives $t > \chi$, $t \in L_t$, $L_t \in \{K_1^{(1)}, K_2^{(1)}, \ldots\}$, and $f_t'''(t) < 0$. Moreover, we state that one of the following is satisfied:

(A) There exists an $s_t \in (\underline{L_t}, t)$ for which: $f_t'(s) > 0$ for all $s \in (\underline{L_t}, s_t)$, $f_t'(s_t) = 0$ and $f_t''(s_t) < 0$, $f_t'(s) < 0$ for all $s \in (s_t, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) < 0$, and $f_t'(s) < 0$ for all $s \in (t, \overline{L_t})$.

(B) $f_t'(s) < 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) < 0$, $f_t'(s) < 0$ for all $s \in (t, \overline{L_t})$.

Consider cases (4) and (5). Lemma 2.2 gives $t > \chi$ for case (4), and $t \in (\chi + \eta - 1, \chi)$ for case (5). Moreover, for both cases, $t \in L_t$, $\chi \in L_t$, $L_t = J_3 = (\overline{S_2}, S_1)$ and $f_t'''(t) < 0$. Moreover, we state that $f_t'(s) < 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) < 0$, $f_t'(s) < 0$ for all $s \in (t, \overline{L_t})$.

Consider case (6). Lemma 2.2 gives $t \in (\chi + \eta - 1, \chi)$, $t \in L_t$, $L_t \in \{K_1^{(2)}, K_2^{(2)}, \ldots\}$, and $f_t'''(t) < 0$. Moreover, we state that one of the following is satisfied:

(A) There exists an $s_t \in (t, \overline{L_t})$ for which: $f_t'(s) < 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) < 0$, $f_t'(s) < 0$ for all $s \in (t, s_t)$, $f_t'(s_t) = 0$ and $f_t''(s_t) > 0$, and $f_t'(s) > 0$ for all $s \in (s_t, \overline{L_t})$.

(B) $f_t'(s) < 0$ for all $s \in (\underline{L_t}, t)$, $f_t'(t) = f_t''(t) = 0$ and $f_t'''(t) < 0$, and $f_t'(s) < 0$ for all $s \in (t, \overline{L_t})$.

4.4. Lemma 3.6 for cases (1)–(4) of Lemma 2.2

Note, many of the arguments in this section are similar to those used in the previous sections. Therefore, we shall not go into as much detail here. Assume the conditions of Lemma 3.6. Additionally assume that one of cases (1)–(4) of Lemma 2.2 is satisfied. Fix $\xi > 0$ sufficiently small such that (2.11), (2.14), (2.17) are (2.31) are satisfied. Define U_n and V_n as in (2.29), and recall (2.30).

First, since one of case (1)–(4) of Lemma 2.2 is satisfied, and since $v > u$, we can proceed as in the proof of Lemma 3.13 to show the following:

$$v_n > u_n \quad \text{and} \quad v_n + s_n - 1 > u_n + r_n + 1 \quad \text{and} \quad VU^{(n)} \neq \emptyset \quad \text{and} \quad UV_{(n)} \neq \emptyset,$$

and

$$t - 2\xi > \chi + 2\xi > \max(VU^{(n)}) > \min(VU^{(n)}) > \chi - 2\xi$$
$$> \chi + \eta - 1 + 2\xi > \max(UV_{(n)}) > \min(UV_{(n)}) > \chi + \eta - 1 - 2\xi. \quad (4.14)$$

Also, (2.32) gives

$$F_n(w) = -\frac{1}{n} \sum_{x \in VU^{(n)}} \log(w - x) + \frac{1}{n} \sum_{x \in UV_{(n)}} \log(w - x),$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$, where the branch cuts are all $(-\infty, 0]$. Therefore,

$$F_n'(w) = -\frac{1}{n} \sum_{x \in VU^{(n)}} \frac{1}{w - x} + \frac{1}{n} \sum_{x \in UV_{(n)}} \frac{1}{w - x},$$

and F_n' extends analytically to $\mathbb{C} \setminus ((VU^{(n)}) \cup (UV_{(n)}))$. Finally define G_t as in (2.26). Recall that this is also analytic in $(\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$, and recall (see (2.27)) that G_t' extends analytically to $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$.

We now consider the roots of G_t' and F_n':

Lemma 4.9. *Assume the above conditions. Then, $t \in (\chi, +\infty)$. Indeed, $t - 4\xi > \chi$. Moreover:*

(1) $G_t'(s) < 0$ *for all* $s \in (\chi, t)$, $G_t'(t) = 0$ *and* $G_t''(t) > 0$, *and* $G_t'(s) > 0$ *for all* $s \in (t, +\infty)$.

(2) G_t' *has no other roots in* $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$.

Next, note $t \in (\max(VU^{(n)}), +\infty)$. *Indeed,* $t - 2\xi > \chi + 2\xi > \max(VU^{(n)})$. *Also* F_n' *has* 1 *root in* $(t - \xi, t + \xi)$. *We denote this by* w_n *as in Lemma 2.12. Finally:*

(3) $F_n'(s) < 0$ *for all* $s \in (\max(VU^{(n)}), w_n)$, $F_n'(w_n) = 0$ *and* $F_n''(w_n) > 0$, *and* $F_n'(s) > 0$ *for all* $s \in (w_n, +\infty)$.

(4) F_n' *has* 1 *root in each interval of the form* (x, y), *when* x *and* y *are any two consecutive elements of either* $VU^{(n)}$ *or* $UV_{(n)}$.

(5) F_n' *has no other roots, other than those listed above.*

Proof. Consider G_t'. (4.14) gives $t - 4\xi > \chi$. Also, Lemma 2.10 implies that G_t' has a root of multiplicity 1 at t, has no other roots in $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$, and that,

$$G_t''(t) = \frac{e^{C(t)} C'(t)^2}{e^{C(t)} - 1}.$$

Thus, since one of cases (1)–(4) of Lemma 2.2 is satisfied, Lemma 2.3 gives $G_t''(t) > 0$. This proves parts (1) and (2).

Consider F_n'. First note, (4.14) implies that $t - 2\xi > \chi + 2\xi > \max(VU^{(n)})$. Next note, part (1) of Lemma 2.12 implies that F_n' has 1 root in $(t - \xi, t + \xi)$. We denote this by w_n, and we recall that $w_n \to t$ (see part (5) of Lemma 2.12). Moreover, part (2) of Lemma 2.11 gives

$$n^{\frac{1}{3}} F_n''(w) = m_n(v - u) G_{t,n}''(w) + O(n^{-\frac{1}{3}}),$$

uniformly for $w \in B(t, \xi)$. Recall that $v - u > 0$ by assumption. Also recall that $\{m_n\}_{n \geq 1}$ is a convergent sequence of real-numbers with a non-zero limit (see Definition 2.8), $m_n(e^{C_n(t)} - 1)/e^{C_n(t)} > 0$ (see Definition 2.8), and $e^{C_n(t)} - 1 \to e^{C(t)} - 1 > 0$ (see cases (1)–(4) of Lemma 2.2, Lemma 2.3, and (1.27)). Therefore $m_n \to m_t$ for some $m_t > 0$. Moreover (see Lemma 2.10), $G_{t,n}''(w) = G_t''(w) + o(1)$ uniformly for $w \in B(t, \xi)$. Finally (see part (5) of Lemma 2.12), $w_n \to t$. Combined, the above observations give $n^{\frac{1}{3}} F_n''(w_n) \to m_t(v - u) G_t''(t) > 0$. Therefore $F_n''(w_n) > 0$. Parts (3)–(5) then follow from parts (2)–(4) of Lemma 2.12. $\qquad\square$
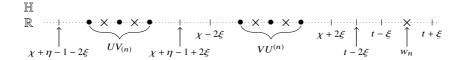
FIGURE 4.11. Top: The roots of G_t'. Bottom: The roots of F_n'. Roots of multiplicity 1 are represented by $\times$, and elements of $VU^{(n)}$ and $UV_{(n)}$ are represented by $\bullet$.
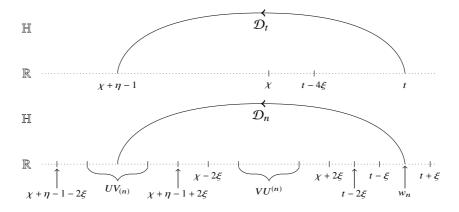

FIGURE 4.12. The contours of Lemma 4.10.

A depiction of (4.14) and the root behaviours described in Lemma 4.9 is given in Figure 4.11. Next we state (without proof) a result which follows from similar arguments to those used to show Lemma 4.2:

Lemma 4.10. *There exists simple contours, $\mathcal{D}_t$ and $\mathcal{D}_n$, as shown in Figure 4.12 with the following properties:*

- *$\mathcal{D}_t$ starts at t, enter $\mathbb{H}$ in the direction $\frac{\pi}{2}$, ends at $\chi + \eta - 1$, and is otherwise contained in $\mathbb{H}$. Moreover, $\mathcal{D}_t$ is a contour of steepest descent for G_t.*

- *$\mathcal{D}_n$ starts at w_n, enters $\mathbb{H}$ in the direction $\frac{\pi}{2}$, ends in the interval shown, and is otherwise contained in $\mathbb{H}$. Moreover, $\mathcal{D}_n$ is a contour of steepest descent for F_n.*

Similarly to the previous sections, we are now in a position to define the contour, κ_n^+, that satisfies Lemma 3.6 for cases (1)–(4) of Lemma 2.2. As in those sections, recalling (see part (5) of Lemma 2.12) that $w_n = t + O(n^{-\frac{1}{3}})$, then $w_n \in B(t, n^{-\theta}|q_n|) \subset B(t, \xi)$. Also, similarly to the previous section, we let $D_{1,n}$ and $D_{2,n}$ denote the points where $\mathcal{D}_n$ "exits" $B(t, n^{-\theta}|q_n|)$ and $B(t, \xi)$ respectively. We denote the equivalent quantities for $\mathcal{D}_t$ by $D_{1,t}, D_{2,t}$. Finally, define $\mathbb{C}_\xi$ as in (3.1), and let $D_{3,t}$ denote the point where $\mathcal{D}_t$ "exits" $\mathbb{C}_\xi$. Similarly to the previous sections, note that it is always possible to choose the $\xi > 0$ sufficiently small such that,

$$\mathrm{Im}(D_{3,t}) = \xi^4 \qquad \text{and} \qquad \mathrm{Re}(D_{3,t}) \in (-\infty, \chi - c),$$

where $c = c(t) > 0$ is some constant which is independent of ξ. Finally define:

Definition 4.11. Define κ_n^+ to be the simple contour which:

- starts at $t \in (\chi, +\infty)$,

- then traverses the straight line from t to $D_{1,n} \in \partial B(t, n^{-\theta}|q_n|)$,

- then traverses that section of $\mathcal{D}_n$ from $D_{1,n} \in \partial B(t, n^{-\theta}|q_n|)$ to $D_{2,n} \in \partial B(t, \xi)$,

- then traverses the shortest arc of $\partial B(t, \xi)$ from $D_{2,n} \in \partial B(t, \xi)$ to $D_{2,t} \in \partial B(t, \xi)$,

- then traverses that section of $\mathcal{D}_t$ from $D_{2,t}$ to $D_{3,t}$,

- then traverses the straight line from $D_{3,t}$ to $\mathrm{Re}(D_{3,t})$,

- then ends at $\mathrm{Re}(D_{3,t}) \in (-\infty, \chi - c)$,

where $c = c(t) > 0$ is some constant which is independent of ξ.

We finally show that the above contour satisfies the requirements of Lemma 3.6:

Proof of Lemma 3.6 for case (1)–(4) *of Lemma 2.2.* Note, Lemma 4.10 and Figure 4.12 and Definition 4.11 imply that κ_n^+ starts at t, ends in the interior of the intervals shown in Figure 3.3, is otherwise contained in $\mathbb{H}$, and is independent of n outside $\mathrm{cl}(B(t, \xi))$. Moreover, using similar arguments to those used in the proof of Lemma 3.5 (see end of Section 4.1), we can show that $\mathrm{Re}(F_n(w)) \leq \mathrm{Re}(F_n(D_{1,n}))$ for all $w \in \kappa_n^+ \setminus B(t, n^{-\theta}|q_n|)$, and that $|\kappa_n^+| = O(1)$. It thus remains to show that $\mathrm{Arg}(D_{1,n} - t) = \frac{\pi}{2} + O(n^{-\frac{1}{3}+\theta})$.

To see the above, first recall (see Lemma 4.10), that $\mathcal{D}_n$ is a contour of steepest descent for F_n which starts at $w_n \in (t - \xi, t + \xi)$, and which "exits" $B(t, n^{-\theta}|q_n|)$ at $D_{1,n}$. Thus parts (1) and (2) of Lemma 3.4 give

$$\mathrm{Re}(F_n(D_{1,n})) < \mathrm{Re}(F_n(w_n)) \qquad \text{and} \qquad \mathrm{Im}(F_n(D_{1,n})) = \mathrm{Im}(F_n(w_n)).$$

Thus, since $w_n = t + O(n^{-\frac{1}{3}})$ (see part (5) of Lemma 2.12), and since $F_n(t) \in \mathbb{R}$, and since $F_n = f_n - \tilde{f}_n$ (see (2.25)), parts (1) and (2) of Corollary 3.3 give

$$\text{Re}(F_n(D_{1,n})) < F_n(t) + O(n^{-1}) \qquad \text{and} \qquad \text{Im}(F_n(D_{1,n})) = O(n^{-1}).$$

Next note, since $\theta \in (\frac{1}{4}, \frac{1}{3})$, and since $F_n = f_n - \tilde{f}_n$, parts (2) and (3) of Lemma 3.2 (take $\xi_n = n^{-\theta}$) give

$$F_n(t + n^{-\theta} q_n e^{i\alpha}) = F_n(t) + n^{-\frac{1}{3}-2\theta}(v - u)e^{2i\alpha} + O(n^{-\frac{2}{3}-\theta}),$$

uniformly for $\alpha \subset (-\pi, \pi]$. Therefore, since $F_n(t) \in \mathbb{R}$,

$$\text{Re}(F_n(t + n^{-\theta} q_n e^{i\alpha})) = F_n(t) + n^{-\frac{1}{3}-2\theta}(v - u)\cos(2\alpha) + O(n^{-\frac{2}{3}-\theta}),$$

$$\text{Im}(F_n(t + n^{-\theta} q_n e^{i\alpha})) = n^{-\frac{1}{3}-2\theta}(v - u)\sin(2\alpha) + O(n^{-\frac{2}{3}-\theta}),$$

uniformly for $\alpha \subset (-\pi, \pi]$. Finally recall that $v > u$. We can then proceed similarly to the proof of part (i) of Lemma 3.5 (see end of Section 4.1) to show that $\text{Arg}(D_{1,n} - t) = \frac{\pi}{2} + O(n^{-\frac{1}{3}+\theta})$, as required. $\qquad\qquad\square$

4.5. Lemma 3.6 for cases (5)–(12) of Lemma 2.2

In Section 4.4 we proved Lemma 3.6 for cases (1)–(4) of Lemma 2.2. In this section we discuss cases (5) and (6). As in Section 4.3, we will be very brief, as the considerations are very similar. We only seek to highlight the differences here. Similar considerations also apply for cases (7)–(12).

Assume the conditions of Lemma 3.6, and that one of cases (5) and (6) are satisfied. Note, since $v > u$, we can proceed similarly to the proof of Lemma 3.13 to get:

$$v_n > u_n \quad \text{and} \quad v_n + s_n < u_n + r_n \quad \text{and} \quad VU^{(n)} \neq \emptyset \quad \text{and} \quad VU_{(n)} \neq \emptyset,$$

and

$$\chi + 2\xi > \max(VU^{(n)}) > \min(VU^{(n)}) > \chi - 2\xi > t + 2\xi > t - 2\xi$$
$$> \chi + \eta - 1 + 2\xi > \max(VU_{(n)}) > \min(VU_{(n)}) > \chi + \eta - 1 - 2\xi.$$

Also, (2.32) gives

$$F_n(w) = -\frac{1}{n} \sum_{x \in VU^{(n)}} \log(w - x) - \frac{1}{n} \sum_{x \in VU_{(n)}} \log(w - x),$$

for all $w \in (\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$, where the branch cuts in the 1st sum on the RHS are all $[0, +\infty)$, and the branch cuts in the 2nd sum are all $(-\infty, 0]$. Therefore,

$$F_n'(w) = -\frac{1}{n} \sum_{x \in VU^{(n)}} \frac{1}{w - x} - \frac{1}{n} \sum_{x \in VU_{(n)}} \frac{1}{w - x},$$

and F_n' extends analytically to $\mathbb{C} \setminus ((VU^{(n)}) \cup (VU_{(n)}))$. Finally, define G_t as in (2.26). Recall that this is also analytic in $(\mathbb{C} \setminus \mathbb{R}) \cup (t - 2\xi, t + 2\xi)$, and recall (see (2.27) that G_t' extends analytically to $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$. Then $t \in (\chi + \eta - 1, \chi)$, and we can proceed as in the proof of Lemma 4.9 to show that $G_t'(s) < 0$ for all $s \in (\chi + \eta - 1, t)$, $G_t'(t) = 0$ and $G_t''(t) > 0$, and $G_t'(s) > 0$ for all $s \in (t, \chi)$. Moreover, G_t' has no other roots in $\mathbb{C} \setminus \{\chi, \chi + \eta - 1\}$. The behaviour of the roots of F_n', the contours of steepest descent, and the definitions of the contour κ_n^+, then follow from similar considerations in the previous section.

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