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Volume 27, n° 2 (2020), p. 159-180.

http://ambp.centre-mersenne.org/item?id=AMBP_2020__27_2_159_0



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*Publication éditée par le laboratoire de mathématiques Blaise Pascal
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Publication membre du
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Pseudo-Riemannian metrics on bicovariant bimodules

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Abstract

We study pseudo-Riemannian invariant metrics on bicovariant bimodules over Hopf algebras. We clarify some properties of such metrics and prove that pseudo-Riemannian invariant metrics on a bicovariant bimodule and its cocycle deformations are in one to one correspondence.

1. Introduction

The notion of metrics on covariant bimodules on Hopf algebras have been studied by a number of authors including Heckenberger and Schmüdgen ([6, 7, 8]) as well as Beggs, Majid and their collaborators ([11] and references therein). The goal of this article is to characterize bicovariant pseudo-Riemannian metrics on a cocycle-twisted bicovariant bimodule. As in [8], the symmetry of the metric comes from Woronowicz's braiding map σ on the bicovariant bimodule. However, since our notion of non-degeneracy of the metric is slightly weaker than that in [8], we consider a slightly larger class of metrics than those in [8]. The positive-definiteness of the metric does not play any role in what we do.

We refer to the later sections for the definitions of pseudo-Riemannian metrics and cocycle deformations. Our strategy is to exploit the covariance of the various maps between bicovariant bimodules to view them as maps between the finite-dimensional vector spaces of left-invariant elements of the respective bimodules. This was already observed and used crucially by Heckenberger and Schmüdgen in the paper [8]. We prove that bi-invariant pseudo-Riemannian metrics are automatically bicovariant maps and compare our definition of pseudo-Riemannian metric with some of the other definitions available in the literature. Finally, we prove that the pseudo-Riemannian bi-invariant metrics on a bicovariant bimodule and its cocycle deformation are in one to one correspondence. These results will be used in the companion article [2].

In Section 2, we discuss some generalities on bicovariant bimodules. In Section 3, we define and study pseudo-Riemannian left metrics on a bicovariant differential calculus. Finally, in Section 4, we prove our main result on bi-invariant metrics on cocycle-deformations.

Let us set up some notations and conventions that we are going to follow. All vector spaces will be assumed to be over the complex field. For vector spaces V_1 and V_2 ,

Keywords: bicovariant bimodules, pseudo-Riemannian metric, cocycle deformations.

$\sigma^{\text{can}} : V_1 \otimes_{\mathbb{C}} V_2 \rightarrow V_2 \otimes_{\mathbb{C}} V_1$ will denote the canonical flip map, i.e, $\sigma^{\text{can}}(v_1 \otimes_{\mathbb{C}} v_2) = v_2 \otimes_{\mathbb{C}} v_1$. For the rest of the article, (\mathcal{A}, Δ) will denote a Hopf algebra. We will use the Sweedler notation for the coproduct Δ . Thus, we will write

$$\Delta(a) = a_{(1)} \otimes_{\mathbb{C}} a_{(2)}. \quad (1.1)$$

For a right \mathcal{A} -module V , the notation V^* will stand for the set $\text{Hom}_{\mathcal{A}}(V, \mathcal{A})$.

Following [16], the comodule coaction on a left \mathcal{A} -comodule V will be denoted by the symbol Δ_V . Thus, Δ_V is a \mathbb{C} -linear map $\Delta_V : V \rightarrow \mathcal{A} \otimes_{\mathbb{C}} V$ such that

$$(\Delta \otimes_{\mathbb{C}} \text{id})\Delta_V = (\text{id} \otimes_{\mathbb{C}} \Delta_V)\Delta_V, \quad (\epsilon \otimes_{\mathbb{C}} \text{id})\Delta_V(v) = v$$

for all v in V (here ϵ is the counit of \mathcal{A}). We will use the notation

$$\Delta_V(v) = v_{(-1)} \otimes_{\mathbb{C}} v_{(0)}. \quad (1.2)$$

Similarly, the comodule coaction on a right \mathcal{A} -comodule will be denoted by ${}_V\Delta$ and we will write

$${}_V\Delta(v) = v_{(0)} \otimes_{\mathbb{C}} v_{(1)}. \quad (1.3)$$

Finally, for a Hopf algebra \mathcal{A} , ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ will denote the categories of various types of mixed Hopf-bimodules as in [13, Subsection 1.9].

2. Covariant bimodules on quantum groups

In this section we recall and prove some basic facts on covariant bimodules. These objects were studied by many Hopf-algebraists (as Hopf-bimodules) including Abe ([1]) and Sweedler ([15]). During the 1980's, they were re-introduced by Woronowicz ([16]) for studying differential calculi over Hopf algebras. Schauenburg ([14]) proved a categorical equivalence between bicovariant bimodules and Yetter–Drinfeld modules over a Hopf algebra, the latter being introduced by Yetter in [17].

We start by recalling the notions on covariant bimodules from Section 2 of [16]. Suppose M is a bimodule over \mathcal{A} such that (M, Δ_M) is a left \mathcal{A} -comodule. Then (M, Δ_M) is called a left-covariant bimodule if this tuple is an object of the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, i.e, for all a in \mathcal{A} and m in M , the following equation holds.

$$\Delta_M(am) = \Delta(a)\Delta_M(m), \quad \Delta_M(ma) = \Delta_M(m)\Delta(a).$$

Similarly, if ${}_M\Delta$ is a right comodule coaction on M , then $(M, {}_M\Delta)$ is called a right covariant bimodule if it is an object of the category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, i.e, for any a in \mathcal{A} and m in M ,

$${}_M\Delta(am) = \Delta(a){}_M\Delta(m), \quad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a).$$

Finally, let M be a bimodule over \mathcal{A} and $\Delta_M : M \rightarrow \mathcal{A} \otimes_{\mathbb{C}} M$ and ${}_M\Delta : M \rightarrow M \otimes_{\mathbb{C}} \mathcal{A}$ be \mathbb{C} -linear maps. Then we say that $(M, \Delta_M, {}_M\Delta)$ is a bicovariant bimodule if this triplet is an object of ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Thus,

- (i) (M, Δ_M) is left-covariant bimodule,
- (ii) $(M, {}_M\Delta)$ is a right-covariant bimodule,
- (iii) $(\text{id} \otimes_{\mathbb{C}} M\Delta)\Delta_M = (\Delta_M \otimes_{\mathbb{C}} \text{id}){}_M\Delta$.

The vector space of left (respectively, right) invariant elements of a left (respectively, right) covariant bimodules will play a crucial role in the sequel and we introduce notations for them here.

Definition 2.1. Let (M, Δ_M) be a left-covariant bimodule over \mathcal{A} . The subspace of left-invariant elements of M is defined to be the vector space

$${}_0M := \{m \in M : \Delta_M(m) = 1 \otimes_{\mathbb{C}} m\}.$$

Similarly, if $(M, {}_M\Delta)$ is a right-covariant bimodule over \mathcal{A} , the subspace of right-invariant elements of M is the vector space

$$M_0 := \{m \in M : {}_M\Delta(m) = m \otimes_{\mathbb{C}} 1\}.$$

Remark 2.2. We will say that a bicovariant bimodule $(M, \Delta_M, {}_M\Delta)$ is finite if ${}_0M$ is a finite dimensional vector space. Throughout this article, we will only work with bicovariant bimodules which are finite.

Let us note the immediate consequences of the above definitions.

Lemma 2.3 ([16, Theorem 2.4]). *Suppose M is a bicovariant \mathcal{A} - \mathcal{A} -bimodule. Then*

$${}_M\Delta({}_0M) \subseteq {}_0M \otimes_{\mathbb{C}} \mathcal{A}, \quad \Delta_M(M_0) \subseteq \mathcal{A} \otimes_{\mathbb{C}} M_0.$$

More precisely, if $\{m_i\}_i$ is a basis of ${}_0M$, then there exist elements $\{a_{ji}\}_{i,j}$ in \mathcal{A} such that

$${}_M\Delta(m_i) = \sum_j m_j \otimes_{\mathbb{C}} a_{ji}. \tag{2.1}$$

Proof. This is a simple consequence of the fact that ${}_M\Delta$ commutes with Δ_M . □

The category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ has a natural monoidal structure. Indeed, if (M, Δ_M) and (N, Δ_N) are left-covariant bimodules over \mathcal{A} , then we have a left coaction $\Delta_{M \otimes_{\mathcal{A}} N}$ of \mathcal{A} on $M \otimes_{\mathcal{A}} N$ defined by the following formula:

$$\Delta_{M \otimes_{\mathcal{A}} N}(m \otimes_{\mathcal{A}} n) = m_{(-1)}n_{(-1)} \otimes_{\mathbb{C}} m_{(0)} \otimes_{\mathcal{A}} n_{(0)}.$$

Here, we have made use of the Sweedler notation introduced in (1.2). This makes $M \otimes_{\mathcal{A}} N$ into a left covariant \mathcal{A} - \mathcal{A} -bimodule. Similarly, there is a right coaction $M \otimes_{\mathcal{A}} N \Delta$ on $M \otimes_{\mathcal{A}} N$ if $(M, {}_M \Delta)$ and $(N, {}_N \Delta)$ are right covariant bimodules.

The fundamental theorem of Hopf modules (Theorem 1.9.4 of [13]) states that if V is a left-covariant bimodule over \mathcal{A} , then V is free as a left (as well as a right) \mathcal{A} -module. This was reproved by Woronowicz in [16]. In fact, one has the following result:

Proposition 2.4 ([16, Theorem 2.1 and Theorem 2.3]). *Let (M, Δ_M) be a bicovariant bimodule over \mathcal{A} . Then the multiplication maps ${}_0M \otimes_{\mathbb{C}} \mathcal{A} \rightarrow M$, $\mathcal{A} \otimes_{\mathbb{C}} {}_0M \rightarrow M$, $M_0 \otimes_{\mathbb{C}} \mathcal{A} \rightarrow M$ and $\mathcal{A} \otimes_{\mathbb{C}} M_0 \rightarrow M$ are isomorphisms.*

Corollary 2.5. *Let (M, Δ_M) and (N, Δ_N) be left-covariant bimodules over \mathcal{A} and $\{m_i\}_i$ and $\{n_j\}_j$ be vector space bases of ${}_0M$ and ${}_0N$ respectively. Then each element of $M \otimes_{\mathcal{A}} N$ can be written as $\sum_{ij} a_{ij} m_i \otimes_{\mathcal{A}} n_j$ and $\sum_{ij} m_i \otimes_{\mathcal{A}} n_j b_{ij}$, where a_{ij} and b_{ij} are uniquely determined.*

A similar result holds for right-covariant bimodules $(M, {}_M \Delta)$ and $(N, {}_N \Delta)$ over \mathcal{A} . Finally, if (M, Δ_M) is a left-covariant bimodule over \mathcal{A} with basis $\{m_i\}_i$ of ${}_0M$, and $(N, {}_N \Delta)$ is a right-covariant bimodule over \mathcal{A} with basis $\{n_j\}_j$ of N_0 , then any element of $M \otimes_{\mathcal{A}} N$ can be written uniquely as $\sum_{ij} a_{ij} m_i \otimes_{\mathcal{A}} n_j$ as well as $\sum_{ij} m_i \otimes_{\mathcal{A}} n_j b_{ij}$.

Proof. The proof of this result is an adaptation of [16, Lemma 3.2] and we omit it. \square

The next proposition will require the definition of right Yetter–Drinfeld modules for which we refer to [17] and [14, Definition 4.1].

Proposition 2.6 ([14, Theorem 5.7]). *The functor $M \mapsto {}_0M$ induces a monoidal equivalence of categories ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ and the category of right Yetter–Drinfeld modules. Therefore, if (M, Δ_M) and (N, Δ_M) be left-covariant bimodules over \mathcal{A} , then*

$${}_0(M \otimes_{\mathcal{A}} N) = \text{span}_{\mathbb{C}}\{m \otimes_{\mathcal{A}} n : m \in {}_0M, n \in {}_0N\}. \quad (2.2)$$

Similarly, if $(M, {}_M \Delta)$ and $(N, {}_N \Delta)$ are right-covariant bimodules over \mathcal{A} , then we have that

$$(M \otimes_{\mathcal{A}} N)_0 = \text{span}_{\mathbb{C}}\{m \otimes_{\mathcal{A}} n : m \in M_0, n \in N_0\}.$$

Thus, ${}_0(M \otimes_{\mathcal{A}} N) = {}_0M \otimes_{\mathbb{C}} {}_0N$ and $(M \otimes_{\mathcal{A}} N)_0 = M_0 \otimes_{\mathbb{C}} N_0$.

Remark 2.7. In the light of Proposition 2.6, we are allowed to use the notations ${}_0M \otimes_{\mathbb{C}} {}_0N$ and ${}_0(M \otimes_{\mathcal{A}} N)$ interchangeably.

We recall now the definition of covariant maps between bimodules.

Definition 2.8. Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant \mathcal{A} -bimodules and T be a \mathbb{C} -linear map from M to N .

T is called left-covariant if T is a morphism in the category ${}^{\mathcal{A}}\mathcal{M}$, i.e, for all $m \in M$,

$$(\text{id} \otimes_{\mathbb{C}} T)(\Delta_M(m)) = \Delta_N(T(m)).$$

T is called right-covariant if T is a morphism in the category $\mathcal{M}^{\mathcal{A}}$. Thus, for all $m \in M$,

$$(T \otimes_{\mathbb{C}} \text{id})_M \Delta(m) = {}_N \Delta(T(m)).$$

Finally, a map which is both left and right covariant will be called a bicovariant map. In other words, a bicovariant map is a morphism in the category ${}^{\mathcal{A}}\mathcal{M}^{\mathcal{A}}$.

We end this section by recalling the following fundamental result of Woronowicz.

Proposition 2.9 ([16, Proposition 3.1]). *Given a bicovariant bimodule \mathcal{E} there exists a unique bimodule homomorphism*

$$\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \quad \text{such that } \sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega \quad (2.3)$$

for any left-invariant element ω and right-invariant element η in \mathcal{E} . σ is invertible and is a bicovariant \mathcal{A} -bimodule map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself. Moreover, σ satisfies the following braid equation on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$:

$$(\text{id} \otimes_{\mathcal{A}} \sigma)(\sigma \otimes_{\mathcal{A}} \text{id})(\text{id} \otimes_{\mathcal{A}} \sigma) = (\sigma \otimes_{\mathcal{A}} \text{id})(\text{id} \otimes_{\mathcal{A}} \sigma)(\sigma \otimes_{\mathcal{A}} \text{id}).$$

3. Pseudo-Riemannian metrics on bicovariant bimodules

In this section, we study pseudo-Riemannian metrics on bicovariant differential calculus on Hopf algebras. After defining pseudo-Riemannian metrics, we recall the definitions of left and right invariance of a pseudo-Riemannian metrics from [8]. We prove that a pseudo-Riemannian metric is left (respectively, right) invariant if and only if it is left (respectively, right) covariant. The coefficients of a left-invariant pseudo-Riemannian metric with respect to a left-invariant basis of \mathcal{E} are scalars. We use this fact to clarify some properties of pseudo-Riemannian invariant metrics. We end the section by comparing our definition with those by Heckenberger and Schmüdgen ([8]) as well as by Beggs and Majid.

Definition 3.1 ([8]). Suppose \mathcal{E} is a bicovariant \mathcal{A} -bimodule \mathcal{E} and $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ be the map as in Proposition 2.9. A pseudo-Riemannian metric for the pair (\mathcal{E}, σ) is a right \mathcal{A} -linear map $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A}$ such that the following conditions hold:

- (i) $g \circ \sigma = g$.
- (ii) If $g(\rho \otimes_{\mathcal{A}} \nu) = 0$ for all ν in \mathcal{E} , then $\rho = 0$.

For other notions of metrics on covariant differential calculus, we refer to [11] and references therein.

Definition 3.2 ([8]). A pseudo-Riemannian metric g on a bicovariant \mathcal{A} -bimodule \mathcal{E} is said to be left-invariant if for all ρ, ν in \mathcal{E} ,

$$(\text{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)) = g(\rho \otimes_{\mathcal{A}} \nu).$$

Similarly, a pseudo-Riemannian metric g on a bicovariant \mathcal{A} -bimodule \mathcal{E} is said to be right-invariant if for all ρ, ν in \mathcal{E} ,

$$(\epsilon g \otimes_{\mathbb{C}} \text{id})(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)) = g(\rho \otimes_{\mathcal{A}} \nu).$$

Finally, a pseudo-Riemannian metric g on a bicovariant \mathcal{A} - \mathcal{A} bimodule \mathcal{E} is said to be bi-invariant if it is both left-invariant as well as right-invariant.

We observe that a pseudo-Riemannian metric is invariant if and only if it is covariant.

Proposition 3.3. *Let g be a pseudo-Riemannian metric on the bicovariant bimodule \mathcal{E} . Then g is left-invariant if and only if $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A}$ is a left-covariant map. Similarly, g is right-invariant if and only if $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A}$ is a right-covariant map.*

Proof. Let g be a left-invariant metric on \mathcal{E} , and ρ, ν be elements of \mathcal{E} . Then the following computation shows that g is a left-covariant map.

$$\begin{aligned} \Delta g(\rho \otimes_{\mathcal{A}} \nu) &= \Delta((\text{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu))) \\ &= \Delta(\text{id} \otimes_{\mathbb{C}} \epsilon g)(\rho_{(-1)} \nu_{(-1)} \otimes_{\mathbb{C}} \rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}) \\ &= \Delta(\rho_{(-1)} \nu_{(-1)}) \epsilon g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}) \\ &= (\rho_{(-1)})_{(1)} (\nu_{(-1)})_{(1)} \otimes_{\mathbb{C}} (\rho_{(-1)})_{(2)} (\nu_{(-1)})_{(2)} \epsilon g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}) \\ &= (\rho_{(-1)})_{(1)} (\nu_{(-1)})_{(1)} \otimes_{\mathbb{C}} ((\text{id} \otimes_{\mathbb{C}} \epsilon g)((\rho_{(-1)})_{(2)} (\nu_{(-1)})_{(2)} \otimes_{\mathbb{C}} \rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)})) \\ &= \rho_{(-1)} \nu_{(-1)} \otimes_{\mathbb{C}} ((\text{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}))) \\ &\quad (\text{where we have used co associativity of comodule coactions}) \\ &= \rho_{(-1)} \nu_{(-1)} \otimes_{\mathbb{C}} g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}) \\ &= (\text{id} \otimes_{\mathbb{C}} g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)). \end{aligned}$$

On the other hand, suppose $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A}$ is a left-covariant map. Then we have

$$\begin{aligned} (\text{id} \otimes_{\mathbb{C}} \epsilon g) \Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu) &= (\text{id} \otimes_{\mathbb{C}} \epsilon)(\text{id} \otimes_{\mathbb{C}} g) \Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu) \\ &= (\text{id} \otimes_{\mathbb{C}} \epsilon) \Delta g(\rho \otimes_{\mathcal{A}} \nu) = g(\rho \otimes_{\mathcal{A}} \nu). \end{aligned}$$

The proof of the right-covariant case is similar. □

The following key result will be used throughout the article.

Lemma 3.4 ([8]). *If g is a pseudo-Riemannian metric which is left-invariant on a left-covariant \mathcal{A} -bimodule \mathcal{E} , then $g(\omega_1 \otimes_{\mathcal{A}} \omega_2) \in \mathbb{C} \cdot 1$ for all ω_1, ω_2 in ${}_0\mathcal{E}$. Similarly, if g is a right-invariant pseudo-Riemannian metric on a right-covariant \mathcal{A} -bimodule, then $g(\eta_1 \otimes_{\mathcal{A}} \eta_2) \in \mathbb{C} \cdot 1$ for all η_1, η_2 in \mathcal{E}_0 .*

Let us clarify some of the properties of a left-invariant and right-invariant pseudo-Riemannian metrics. To that end, we make the next definition which makes sense as we always work with finite bicovariant bimodules (see Remark 2.2). The notations used in the next definition will be used throughout the article.

Definition 3.5. Let \mathcal{E} and g be as above. For a fixed basis $\{\omega_1, \dots, \omega_n\}$ of ${}_0\mathcal{E}$, we define $g_{ij} = g(\omega_i \otimes_{\mathcal{A}} \omega_j)$. Moreover, we define $V_g : \mathcal{E} \rightarrow \mathcal{E}^* = \text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ to be the map defined by the formula

$$V_g(e)(f) = g(e \otimes_{\mathcal{A}} f).$$

Proposition 3.6. *Let g be a left-invariant pseudo-Riemannian metric for \mathcal{E} as in Definition 3.1. Then the following statements hold:*

- (i) *The map V_g is a one-one right \mathcal{A} -linear map from \mathcal{E} to \mathcal{E}^* .*
- (ii) *If $e \in \mathcal{E}$ is such that $g(e \otimes_{\mathcal{A}} f) = 0$ for all $f \in {}_0\mathcal{E}$, then $e = 0$. In particular, the map $V_g|_{{}_0\mathcal{E}}$ is one-one and hence an isomorphism from ${}_0\mathcal{E}$ to $({}_0\mathcal{E})^*$.*
- (iii) *The matrix $((g_{ij}))_{ij}$ is invertible.*
- (iv) *Let g^{ij} denote the (i, j) -th entry of the inverse of the matrix $((g_{ij}))_{ij}$. Then g^{ij} is an element of $\mathbb{C} \cdot 1$ for all i, j .*
- (v) *If $g(e \otimes_{\mathcal{A}} f) = 0$ for all e in ${}_0\mathcal{E}$, then $f = 0$.*

Proof. The right \mathcal{A} -linearity of V_g follows from the fact that g is a well-defined map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to \mathcal{A} . The condition (ii) of Definition 3.1 forces V_g to be one-one. This proves (i).

For proving (ii), note that $V_g|_{{}_0\mathcal{E}}$ is the restriction of a one-one map to a subspace. Hence, it is a one-one \mathbb{C} -linear map. Since g is left-invariant, by Lemma 3.4, for any e in ${}_0\mathcal{E}$, $V_g(e)({}_0\mathcal{E})$ is contained in \mathbb{C} . Therefore, V_g maps ${}_0\mathcal{E}$ into $({}_0\mathcal{E})^*$. Since, ${}_0\mathcal{E}$ and $({}_0\mathcal{E})^*$ have the same finite dimension as vector spaces, $V_g|_{{}_0\mathcal{E}} : {}_0\mathcal{E} \rightarrow ({}_0\mathcal{E})^*$ is an isomorphism. This proves (ii).

Now we prove (iii). Let $\{\omega_i\}_i$ be a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ be a dual basis, i.e, $\omega_i^*(\omega_j) = \delta_{ij}$. Since $V_g|_{{}_0\mathcal{E}}$ is a vector space isomorphism from ${}_0\mathcal{E}$ to $({}_0\mathcal{E})^*$ by part (ii),

there exist complex numbers a_{ij} such that

$$(V_g)^{-1}(\omega_i^*) = \sum_j a_{ij}\omega_j.$$

This yields

$$\delta_{ik} = \omega_i^*(\omega_k) = g \left(\sum_j a_{ij}\omega_j \otimes_{\mathcal{A}} \omega_k \right) = \sum_j a_{ij}g_{jk}.$$

Therefore, $((a_{ij}))_{ij}$ is the left-inverse and hence the inverse of the matrix $((g_{ij}))_{ij}$. This proves (iii).

For proving (iv), we use the fact that g_{ij} is an element of $\mathbb{C}.1$ for all i, j . Since

$$\sum_k g(\omega_i \otimes_{\mathcal{A}} \omega_k)g^{kj} = \delta_{ij}.1 = \sum_k g^{ik}g(\omega_k \otimes_{\mathcal{A}} \omega_j) = \delta_{ij},$$

we have

$$\sum_k g(\omega_i \otimes_{\mathcal{A}} \omega_k)\epsilon(g^{kj}) = \delta_{ij} = \sum_k \epsilon(g^{ik})g(\omega_k \otimes_{\mathcal{A}} \omega_j).$$

So, the matrix $((\epsilon(g^{ij})))_{ij}$ is also an inverse to the matrix $((g(\omega_i \otimes_{\mathcal{A}} \omega_j)))_{ij}$ and hence $g^{ij} = \epsilon(g^{ij})$ and g^{ij} is in $\mathbb{C}.1$.

Finally, we prove (v) using (iv). Suppose f be an element in \mathcal{E} such that $g(e \otimes_{\mathcal{A}} f) = 0$ for all e in ${}_0\mathcal{E}$. Let $f = \sum_k \omega_k a_k$ for some elements a_k in \mathcal{A} . Then for any fixed index i_0 , we obtain

$$0 = g \left(\sum_j g^{i_0j}\omega_j \otimes_{\mathcal{A}} \sum_k \omega_k a_k \right) = \sum_k \sum_j g^{i_0j}g_{jk}a_k = \sum_k \delta_{i_0k}a_k = a_{i_0}.$$

Hence, we have that $f = 0$. This finishes the proof. \square

We apply the results in Proposition 3.6 to exhibit a basis of the free right \mathcal{A} -module $V_g(\mathcal{E})$. This will be used in making Definition 4.11 which is needed to prove our main Theorem 4.15.

Lemma 3.7. *Suppose $\{\omega_i\}_i$ is a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ be the dual basis as in the proof of Proposition 3.6. If g is a pseudo-Riemannian left-invariant metric on \mathcal{E} , then $V_g(\mathcal{E})$ is a free right \mathcal{A} -module with basis $\{\omega_i^*\}_i$.*

Proof. We will use the notations $(g_{ij})_{ij}$ and g^{ij} from of Proposition 3.6. Since V_g is a right \mathcal{A} -linear map, $V_g(\mathcal{E})$ is a right \mathcal{A} -module. Since

$$V_g(\omega_i) = \sum_j g_{ij}\omega_j^* \tag{3.1}$$

and the inverse matrix $(g^{ij})_{ij}$ has scalar entries (Proposition 3.6), we get

$$\omega_k^* = \sum_i g^{ki} V_g(\omega_i)$$

and so ω_k^* belongs to $V_g(\mathcal{E})$ for all k . By the right \mathcal{A} -linearity of the map V_g , we conclude that the set $\{\omega_i^*\}_i$ is right \mathcal{A} -total in $V_g(\mathcal{E})$.

Finally, if a_i are elements in \mathcal{A} such that $\sum_k \omega_k^* a_k = 0$, then by (3.1), we have

$$0 = \sum_{i,k} g^{ki} V_g(\omega_i) a_k = V_g \left(\sum_i \omega_i \left(\sum_k g^{ki} a_k \right) \right).$$

As V_g is one-one and $\{\omega_i\}_i$ is a basis of the free module \mathcal{E} , we get

$$\sum_k g^{ki} a_k = 0 \quad \forall i.$$

Multiplying by g_{ij} and summing over i yields $a_j = 0$. This proves that $\{\omega_i^*\}_i$ is a basis of $V_g(\mathcal{E})$ and finishes the proof. \square

Remark 3.8. Let us note that for all $e \in \mathcal{E}$, the following equation holds:

$$e = \sum_i \omega_i \omega_i^*(e). \tag{3.2}$$

The following proposition was kindly pointed out to us by the referee for which we will need the notion of a left dual of an object in a monoidal category. We refer to Definition 2.10.1 of [5] or Definition XIV.2.1 of [9] for the definition.

Proposition 3.9. *Suppose g is a pseudo-Riemannian \mathcal{A} -bilinear pseudo-Riemannian metric on a finite bicovariant \mathcal{A} -bimodule. Let $\widetilde{\mathcal{E}}$ denote the left dual of the object \mathcal{E} in the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Then $\widetilde{\mathcal{E}}$ is isomorphic to \mathcal{E} as objects in the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ via the morphism V_g .*

Proof. It is well-known that $\widetilde{\mathcal{E}}$ and \mathcal{E}^* are isomorphic objects in the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. This follows by using the bicovariant \mathcal{A} -bilinear maps

$$\begin{aligned} \text{ev} : \widetilde{\mathcal{E}} \otimes_{\mathcal{A}} \mathcal{E} &\longrightarrow \mathcal{A}; & \text{coev} : \mathcal{A} &\longrightarrow \mathcal{E} \otimes_{\mathcal{A}} \widetilde{\mathcal{E}}; \\ \phi \otimes_{\mathcal{A}} e &\longmapsto \phi(e), & 1 &\longmapsto \sum_i \omega_i \otimes_{\mathcal{A}} \omega_i^* \end{aligned}$$

We define $\text{ev}_g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \mathcal{A}$ and $\text{coev}_g : \mathcal{A} \rightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ by the following formulas:

$$\text{ev}_g(e \otimes_{\mathcal{A}} f) = g(e \otimes_{\mathcal{A}} f), \quad \text{coev}_g(1) = \sum_i \omega_i \otimes_{\mathcal{A}} V_g^{-1}(\omega_i^*).$$

Then since g is both left and right \mathcal{A} -linear, ev_g and coev_g are \mathcal{A} - \mathcal{A} -bilinear maps. The bicovariance of g implies the bicovariance of ev_g while the bicovariance of $\text{coev}_g = (\text{id} \otimes_{\mathcal{A}} V_g^{-1}) \circ \text{coev}$ follows from the bicovariance of V_g and coev .

Since the left dual of an object is unique upto isomorphism, we need to check the following identities for all e in \mathcal{E} :

$$(\text{ev}_g \otimes_{\mathcal{A}} \text{id})(\text{id} \otimes_{\mathcal{A}} \text{coev}_g)(e) = e, \quad (\text{id} \otimes_{\mathcal{A}} \text{ev}_g)(\text{coev}_g \otimes_{\mathcal{A}} \text{id})(e) = e.$$

But these follow by a simple computation using the fact that ${}_0\mathcal{E}$ is right \mathcal{A} -total in \mathcal{E} and the identity (3.2).

From the above discussion, we have that \mathcal{E} and \mathcal{E}^* are two left duals of the object \mathcal{E} in the category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Then by the proof of Proposition 2.10.5 of [5], we know that $(\text{ev}_g \otimes_{\mathcal{A}} \text{id}_{\mathcal{E}^*})(\text{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \text{coev})$ is an isomorphism from \mathcal{E} to \mathcal{E}^* . But it can be easily checked that $(\text{ev}_g \otimes_{\mathcal{A}} \text{id}_{\mathcal{E}^*})(\text{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \text{coev}) = V_g$. This completes the proof. \square

Now we state a result on bi-invariant (i.e both left and right-invariant) pseudo-Riemannian metric.

Proposition 3.10. *Let g be a pseudo-Riemannian metric on \mathcal{E} and the symbols $\{\omega_i\}_i, \{g_{ij}\}_{ij}$ be as above. If*

$${}_{\mathcal{E}}\Delta(\omega_i) = \sum_j \omega_j \otimes_{\mathbb{C}} R_{ji} \tag{3.3}$$

(see (2.1)), then g is bi-invariant if and only if the elements g_{ij} are scalar and

$$g_{ij} = \sum_{kl} g_{kl} R_{ki} R_{lj}. \tag{3.4}$$

Proof. Since g is left-invariant, the elements g_{ij} are in \mathbb{C} . Moreover, the right-invariance of g implies that g is right-covariant (Proposition 3.3), i.e.

$$\begin{aligned} 1 \otimes_{\mathbb{C}} g_{ij} &= \Delta(g_{ij}) = (g \otimes_{\mathcal{A}} \text{id})_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} \Delta(\omega_i \otimes_{\mathbb{C}} \omega_j) \\ &= (g \otimes_{\mathcal{A}} \text{id}) \left(\sum_{kl} \omega_k \otimes_{\mathcal{A}} \omega_l \otimes_{\mathbb{C}} R_{ki} R_{lj} \right) = 1 \otimes_{\mathbb{C}} \sum_{kl} g_{kl} R_{ki} R_{lj}, \end{aligned}$$

so that

$$g_{ij} = \sum_{kl} g_{kl} R_{ki} R_{lj}. \tag{3.5}$$

Conversely, if $g_{ij} = g(\omega_i \otimes_{\mathcal{A}} \omega_j)$ are scalars and (3.4) is satisfied, then g is left-invariant and right-covariant. By Proposition 3.3, g is right-invariant. \square

We end this section by comparing our definition of pseudo-Riemannian metrics with some of the other definitions available in the literature.

Proposition 3.6 shows that our notion of pseudo-Riemannian metric coincides with the right \mathcal{A} -linear version of a “symmetric metric” introduced in Definition 2.1 of [8] if we impose the condition of left-invariance.

Next, we compare our definition with the one used by Beggs and Majid in Proposition 4.2 of [10] (also see [11] and references therein). To that end, we need to recall the construction of the two forms by Woronowicz ([16]).

If \mathcal{E} is a bicovariant \mathcal{A} -bimodule and σ be the map as in Proposition 2.9, Woronowicz defined the space of two forms as:

$$\Omega^2(\mathcal{A}) := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) / \text{Ker}(\sigma - 1).$$

The symbol \wedge will denote the quotient map

$$\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \rightarrow \Omega^2(\mathcal{A}).$$

Thus,

$$\text{Ker}(\wedge) = \text{Ker}(\sigma - 1). \tag{3.6}$$

In Proposition 4.2 of [10], the authors define a metric on a bimodule \mathcal{E} over a (possibly) noncommutative algebra \mathcal{A} as an element h of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that $\wedge(h) = 0$. We claim that metrics in the sense of Beggs and Majid are in one to one correspondence with elements $g \in \text{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ (not necessarily left-invariant) such that $g \circ \sigma = g$. Thus, modulo the nondegeneracy condition (ii) of Definition 3.1, our notion of pseudo-Riemannian metric matches with the definition of metric by Beggs and Majid.

Indeed, if $g \in \text{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ as above and $\{\omega_i\}_i$, is a basis of ${}_0\mathcal{E}$, then the equation $g \circ \sigma = g$ implies that

$$g \circ \sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = g(\omega_i \otimes_{\mathcal{A}} \omega_j).$$

However, by equation (3.15) of [16], we know that

$$\sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = \sum_{k,l} \sigma_{ij}^{kl} \omega_k \otimes_{\mathcal{A}} \omega_l$$

for some scalars σ_{ij}^{kl} . Therefore, we have

$$\sum_{k,l} \sigma_{ij}^{kl} g(\omega_k \otimes_{\mathcal{A}} \omega_l) = g(\omega_i \otimes_{\mathcal{A}} \omega_j). \tag{3.7}$$

We claim that the element $h = \sum_{i,j} g(\omega_i \otimes_{\mathcal{A}} \omega_j) \omega_i \otimes_{\mathcal{A}} \omega_j$ satisfies $\wedge(h) = 0$. Indeed, by virtue of (3.6), it is enough to prove that $(\sigma - 1)(h) = 0$. But this directly follows from (3.7) using the left \mathcal{A} -linearity of σ .

This argument is reversible and hence starting from $h \in \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ satisfying $\wedge(h) = 0$, we get an element $g \in \text{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ such that for all i, j ,

$$g \circ \sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = g(\omega_i \otimes_{\mathcal{A}} \omega_j).$$

Since $\{\omega_i \otimes_{\mathcal{A}} \omega_j : i, j\}$ is right \mathcal{A} -total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ (Corollary 2.5) and the maps g, σ are right \mathcal{A} -linear, we get that $g \circ \sigma = g$. This proves our claim. Let us note that since we did not assume g to be left invariant, the quantities $g(\omega_i \otimes_{\mathcal{A}} \omega_j)$ need not be scalars. However, the proof goes through since the elements σ_{kl}^{ij} are scalars.

4. Pseudo-Riemannian metrics for cocycle deformations

This section concerns the braiding map and pseudo-Riemannian metrics of bicovariant bimodules under cocycle deformations of Hopf algebras. This section contains two main results. We start by recalling that a bicovariant bimodule \mathcal{E} over a Hopf algebra \mathcal{A} can be deformed in the presence of a 2-cocycle γ on \mathcal{A} to a bicovariant \mathcal{A}_γ -bimodule \mathcal{E}_γ . We prove that the canonical braiding map of the bicovariant bimodule \mathcal{E}_γ (Proposition 2.9) is a cocycle deformation of the canonical braiding map of \mathcal{E} . Finally, we prove that pseudo-Riemannian bi-invariant metrics on \mathcal{E} and \mathcal{E}_γ are in one to one correspondence.

Throughout this section, we will make heavy use of the Sweedler notations as spelled out in (1.1), (1.2) and (1.3). The coassociativity of Δ will be expressed by the following equation:

$$(\Delta \otimes_{\mathbb{C}} \text{id})\Delta(a) = (\text{id} \otimes_{\mathbb{C}} \Delta)\Delta(a) = a_{(1)} \otimes_{\mathbb{C}} a_{(2)} \otimes_{\mathbb{C}} a_{(3)}.$$

Also, when m is an element of a bicovariant bimodule, we will use the notation

$$(\text{id} \otimes_{\mathbb{C}M} \Delta)\Delta_M(m) = (\Delta_M \otimes_{\mathbb{C}} \text{id})_M \Delta(m) = m_{(-1)} \otimes_{\mathbb{C}} m_{(0)} \otimes_{\mathbb{C}} m_{(1)}. \quad (4.1)$$

Definition 4.1. A cocycle γ on a Hopf algebra (\mathcal{A}, Δ) is a \mathbb{C} -linear map $\gamma : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \rightarrow \mathbb{C}$ such that it is convolution invertible, unital, i.e.,

$$\gamma(a \otimes_{\mathbb{C}} 1) = \epsilon(a) = \gamma(1 \otimes_{\mathbb{C}} a)$$

and for all a, b, c in \mathcal{A} ,

$$\gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)})\gamma(a_{(2)}b_{(2)} \otimes_{\mathbb{C}} c) = \gamma(b_{(1)} \otimes_{\mathbb{C}} c_{(1)})\gamma(a \otimes_{\mathbb{C}} b_{(2)}c_{(2)}). \quad (4.2)$$

Given a Hopf algebra (\mathcal{A}, Δ) and such a cocycle γ as above, we have a new Hopf algebra $(\mathcal{A}_\gamma, \Delta_\gamma)$ which is equal to \mathcal{A} as a vector space, the coproduct Δ_γ is equal to Δ while the algebra structure $*_\gamma$ on \mathcal{A}_γ is defined by the following equation:

$$a *_\gamma b = \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)})a_{(2)}b_{(2)}\bar{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)}). \quad (4.3)$$

Here, $\bar{\gamma}$ is the convolution inverse to γ which is unital and satisfies the following equation:

$$\bar{\gamma}(a_{(1)}b_{(1)} \otimes_{\mathbb{C}} c)\bar{\gamma}(a_{(2)} \otimes_{\mathbb{C}} b_{(2)}) = \bar{\gamma}(a \otimes_{\mathbb{C}} b_{(1)}c_{(1)})\bar{\gamma}(b_{(2)} \otimes_{\mathbb{C}} c_{(2)}). \quad (4.4)$$

We refer to [4, Theorem 1.6] for more details.

Suppose M is a bicovariant \mathcal{A} - \mathcal{A} -bimodule. Then M can also be deformed in the presence of a cocycle. This is the content of the next proposition.

Proposition 4.2 ([12, Theorem 2.5]). *Suppose M is a bicovariant \mathcal{A} -bimodule and γ is a 2-cocycle on \mathcal{A} . Then we have a bicovariant \mathcal{A}_γ -bimodule M_γ which is equal to M as a vector space but the left and right \mathcal{A}_γ -module structures are defined by the following formulas:*

$$a *_\gamma m = \gamma(a_{(1)} \otimes_{\mathbb{C}} m_{(-1)})a_{(2)}.m_{(0)}\bar{\gamma}(a_{(3)} \otimes_{\mathbb{C}} m_{(1)}) \quad (4.5)$$

$$m *_\gamma a = \gamma(m_{(-1)} \otimes_{\mathbb{C}} a_{(1)})m_{(0)}.a_{(2)}\bar{\gamma}(m_{(1)} \otimes_{\mathbb{C}} a_{(3)}), \quad (4.6)$$

for all elements m of M and for all elements a of \mathcal{A} . Here, $*_\gamma$ denotes the right and left \mathcal{A}_γ -module actions, and $.$ denotes the right and left \mathcal{A} -module actions.

The \mathcal{A}_γ -bicovariant structures are given by

$$\Delta_{M_\gamma} := \Delta_M : M_\gamma \rightarrow \mathcal{A}_\gamma \otimes_{\mathbb{C}} M_\gamma \text{ and } {}_{M_\gamma}\Delta := {}_M\Delta : M_\gamma \rightarrow M_\gamma \otimes_{\mathbb{C}} \mathcal{A}_\gamma. \quad (4.7)$$

Remark 4.3. From Proposition 4.2, it is clear that if M is a finite bicovariant bimodule (see Remark 2.2), then M_γ is also a finite bicovariant bimodule.

We end this subsection by recalling the following result on the deformation of bicovariant maps.

Proposition 4.4 ([12, Theorem 2.5]). *Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant \mathcal{A} -bimodules, $T : M \rightarrow N$ be a \mathbb{C} -linear bicovariant map and γ be a cocycle as above. Then there exists a map $T_\gamma : M_\gamma \rightarrow N_\gamma$ defined by $T_\gamma(m) = T(m)$ for all m in M . Thus, $T_\gamma = T$ as \mathbb{C} -linear maps. Moreover, we have the following:*

- (i) *the deformed map $T_\gamma : M_\gamma \rightarrow N_\gamma$ is an \mathcal{A}_γ bicovariant map,*
- (ii) *if T is a bicovariant right (respectively left) \mathcal{A} -linear map, then T_γ is a bicovariant right (respectively left) \mathcal{A}_γ -linear map,*
- (iii) *if $(P, \Delta_P, {}_P\Delta)$ is another bicovariant \mathcal{A} -bimodule, and $S : N \rightarrow P$ is a bicovariant map, then $(S \circ T)_\gamma : M_\gamma \rightarrow P_\gamma$ is a bicovariant map and $S_\gamma \circ T_\gamma = (S \circ T)_\gamma$.*

4.1. Deformation of the braiding map

Suppose \mathcal{E} is a bicovariant \mathcal{A} -bimodule, σ be the bicovariant braiding map of Proposition 2.9 and g be a bi-invariant metric. Then Proposition 4.4 implies that we have deformed maps σ_γ and g_γ . In this subsection, we study the map σ_γ . The map g_γ will be discussed in the next subsection. We will need the following result:

Proposition 4.5 ([12, Theorem 2.5]). *Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant bimodules over a Hopf algebra \mathcal{A} and γ be a cocycle as above. Then there exists a bicovariant \mathcal{A}_γ -bimodule isomorphism*

$$\xi : M_\gamma \otimes_{\mathcal{A}_\gamma} N_\gamma \rightarrow (M \otimes_{\mathcal{A}} N)_\gamma.$$

The isomorphism ξ and its inverse are respectively given by

$$\begin{aligned} \xi(m \otimes_{\mathcal{A}_\gamma} n) &= \gamma(m_{(-1)} \otimes_{\mathbb{C}} n_{(-1)})m_{(0)} \otimes_{\mathcal{A}} n_{(0)}\bar{\gamma}(m_{(1)} \otimes_{\mathbb{C}} n_{(1)}) \\ \xi^{-1}(m \otimes_{\mathcal{A}} n) &= \bar{\gamma}(m_{(-1)} \otimes_{\mathbb{C}} n_{(-1)})m_{(0)} \otimes_{\mathcal{A}_\gamma} n_{(0)}\gamma(m_{(1)} \otimes_{\mathbb{C}} n_{(1)}) \end{aligned}$$

As an illustration, we make the following computation which will be needed later in this subsection:

Lemma 4.6. *Suppose $\omega \in {}_0\mathcal{E}, \eta \in \mathcal{E}_0$. Then the following equation holds:*

$$\xi^{-1}(\gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)})) = \eta \otimes_{\mathcal{A}_\gamma} \omega.$$

Proof. Let us first clarify that we view $\gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)})$ as an element in $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. Then the equation holds because of the following computation:

$$\begin{aligned} &\xi^{-1}(\gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)})) \\ &= \gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\xi^{-1}(\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)})\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)}) \\ &= \gamma(\eta_{(-2)} \otimes_{\mathbb{C}} 1)\bar{\gamma}(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}_\gamma} \omega_{(0)}\gamma(1 \otimes_{\mathbb{C}} \omega_{(1)})\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(2)}) \\ &\quad (\text{since } \omega \in {}_0\mathcal{E}, \eta \in \mathcal{E}_0) \\ &= \epsilon(\eta_{(-2)})\epsilon(\eta_{(-1)})\eta_{(0)} \otimes_{\mathcal{A}_\gamma} \omega_{(0)}\epsilon(\omega_{(1)})\epsilon(\omega_{(2)}) \quad (\text{since } \bar{\gamma} \text{ and } \gamma \text{ are normalised}) \\ &= \eta \otimes_{\mathcal{A}_\gamma} \omega. \quad \square \end{aligned}$$

Now, we are in a position to study the map σ_γ . By Proposition 4.2, \mathcal{E}_γ is a bicovariant \mathcal{A}_γ -bimodule. Then Proposition 2.9 guarantees the existence of a canonical braiding from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself. We show that this map is nothing but the deformation σ_γ of the map σ associated with the bicovariant \mathcal{A} -bimodule \mathcal{E} . By the definition of σ_γ , it is a map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. However, by virtue of Proposition 4.5, the map ξ defines an isomorphism from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. By an abuse of notation, we will denote the map

$$\xi^{-1}\sigma_\gamma\xi : \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma \rightarrow \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$$

by the symbol σ_γ again.

Theorem 4.7 ([12, Theorem 2.5]). *Let \mathcal{E} be a bicovariant \mathcal{A} -bimodule and γ be a cocycle as above. Then the deformation σ_γ of σ is the unique bicovariant \mathcal{A}_γ -bimodule braiding map on \mathcal{E}_γ given by Proposition 2.9.*

Proof. Since σ is a bicovariant \mathcal{A} -bimodule map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself, part (ii) of Proposition 4.4 implies that σ_{γ} is a bicovariant \mathcal{A}_{γ} -bimodule map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_{\gamma} \cong \mathcal{E}_{\gamma} \otimes_{\mathcal{A}_{\gamma}} \mathcal{E}_{\gamma}$ to itself. By Proposition 2.9, there exists a unique \mathcal{A}_{γ} -bimodule map σ' from $\mathcal{E}_{\gamma} \otimes_{\mathcal{A}_{\gamma}} \mathcal{E}_{\gamma}$ to itself such that $\sigma'(\omega \otimes_{\mathcal{A}_{\gamma}} \eta) = \eta \otimes_{\mathcal{A}_{\gamma}} \omega$ for all ω in ${}_0(\mathcal{E}_{\gamma})$, η in $(\mathcal{E}_{\gamma})_0$.

Since ${}_0(\mathcal{E}_{\gamma}) = {}_0\mathcal{E}$ and $(\mathcal{E}_{\gamma})_0 = \mathcal{E}_0$, it is enough to prove that $\sigma_{\gamma}(\omega \otimes_{\mathcal{A}_{\gamma}} \eta) = \eta \otimes_{\mathcal{A}_{\gamma}} \omega$ for all ω in ${}_0\mathcal{E}$, η in \mathcal{E}_0 .

We will need the concrete isomorphism between $\mathcal{E}_{\gamma} \otimes_{\mathcal{A}_{\gamma}} \mathcal{E}_{\gamma}$ and $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_{\gamma}$ defined in Proposition 4.5. Since ω is in ${}_0\mathcal{E}$ and η is in \mathcal{E}_0 , this isomorphism maps the element $\omega \otimes_{\mathcal{A}_{\gamma}} \eta$ to $\gamma(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\bar{\gamma}(\omega_{(1)} \otimes_{\mathbb{C}} 1)$. Then, by the definition of σ_{γ} , we compute the following:

$$\begin{aligned} \sigma_{\gamma}(\omega \otimes_{\mathcal{A}_{\gamma}} \eta) &= \sigma(\gamma(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\bar{\gamma}(\omega_{(1)} \otimes_{\mathbb{C}} 1)) \\ &= \sigma(\epsilon(\eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\epsilon(\omega_{(1)})) = \epsilon(\eta_{(-1)})\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\epsilon(\omega_{(1)}) \\ &= \gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\bar{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)}) = \eta \otimes_{\mathcal{A}_{\gamma}} \omega, \end{aligned}$$

where, in the last step we have used Lemma 4.6. \square

Remark 4.8. Proposition 4.2, Proposition 4.4, Proposition 4.5 and Theorem 4.7 together imply that the categories ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}^{\mathcal{A}}$ and ${}^{\mathcal{A}_{\gamma}}\mathcal{M}_{\mathcal{A}_{\gamma}}^{\mathcal{A}_{\gamma}}$ are isomorphic as braided monoidal categories. This was the content of Theorem 2.5 of [12]. The referee has pointed out that this is a special case of a much more generalized result of Bichon ([3, Theorem 6.1]) which says that if two Hopf algebras are monoidally equivalent, then the corresponding categories of right-right Yetter Drinfeld modules are also monoidally equivalent.

However, in Theorem 4.7, we have proved in addition that the braiding on ${}^{\mathcal{A}_{\gamma}}\mathcal{M}_{\mathcal{A}_{\gamma}}^{\mathcal{A}_{\gamma}}$ is precisely the Woronowicz braiding of Proposition 2.9.

Corollary 4.9. *If the unique bicovariant \mathcal{A} -bimodule braiding map σ for a bicovariant \mathcal{A} -bimodule \mathcal{E} satisfies the equation $\sigma^2 = 1$, then the bicovariant \mathcal{A}_{γ} -bimodule braiding map σ_{γ} for the bicovariant \mathcal{A}_{γ} -bimodule \mathcal{E}_{γ} also satisfies $\sigma_{\gamma}^2 = 1$.*

In particular, if \mathcal{A} is the commutative Hopf algebra of regular functions on a compact semisimple Lie group G and \mathcal{E} is its canonical space of one-forms, then the braiding map σ_{γ} for \mathcal{E}_{γ} satisfies $\sigma_{\gamma}^2 = 1$.

Proof. By Theorem 4.7, σ_{γ} is the unique braiding map for the bicovariant \mathcal{A}_{γ} -bimodule \mathcal{E}_{γ} . Since, by our hypothesis, $\sigma^2 = 1$, the deformed map σ_{γ} also satisfies $\sigma_{\gamma}^2 = 1$ by part (ii) of Proposition 4.4.

Next, if \mathcal{A} is a commutative Hopf algebra as in the statement of the corollary and \mathcal{E} is its canonical space of one-forms, then we know that the braiding map σ is just the flip map, i.e. for all e_1, e_2 in \mathcal{E} ,

$$\sigma(e_1 \otimes_{\mathcal{A}} e_2) = e_2 \otimes_{\mathcal{A}} e_1,$$

and hence it satisfies $\sigma^2 = 1$. Therefore, for every cocycle deformation \mathcal{E}_γ of \mathcal{E} , the corresponding braiding map satisfies $\sigma_\gamma^2 = 1$. \square

4.2. Pseudo-Riemannian bi-invariant metrics on \mathcal{E}_γ

Suppose \mathcal{E} is a bicovariant \mathcal{A} -bimodule and \mathcal{E}_γ be its cocycle deformation as above. The goal of this subsection is to prove that a pseudo-Riemannian bi-invariant metric on \mathcal{E} naturally deforms to a pseudo-Riemannian bi-invariant metric on \mathcal{E}_γ . Since g is a bicovariant (i.e, both left and right covariant) map from the bicovariant bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself, then by Proposition 4.4, we have a right \mathcal{A}_γ -linear bicovariant map g_γ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself. We need to check the conditions (i) and (ii) of Definition 3.1 for the map g_γ .

The proof of the equality $g_\gamma = g_\gamma \circ \sigma_\gamma$ is straightforward. However, checking condition (ii), i.e, verifying that the map V_{g_γ} is an isomorphism onto its image needs some work. The root of the problem is that we do not yet know whether $\mathcal{E}^* = V_g(\mathcal{E})$. Our strategy to verify condition (ii) is the following: we show that the right \mathcal{A} -module $V_g(\mathcal{E})$ is a bicovariant right \mathcal{A} -module (see Definition 4.10) in a natural way. Let us remark that since the map g (hence V_g) is not left \mathcal{A} -linear, $V_g(\mathcal{E})$ need not be a left \mathcal{A} -module. Since bicovariant right \mathcal{A} -modules and bicovariant maps between them can be deformed (Proposition 4.13), the map V_g deforms to a right \mathcal{A}_γ -linear isomorphism from \mathcal{E}_γ to $(V_g(\mathcal{E}))_\gamma$. Then in Theorem 4.15, we show that $(V_g)_\gamma$ coincides with the map V_{g_γ} and the latter is an isomorphism onto its image. This is the only subsection where we use the theory of bicovariant right modules (as opposed to bicovariant bimodules).

Definition 4.10. Let M be a right \mathcal{A} -module, and $\Delta_M : M \rightarrow \mathcal{A} \otimes_{\mathbb{C}} M$ and ${}_M\Delta : M \rightarrow M \otimes_{\mathbb{C}} \mathcal{A}$ be \mathbb{C} -linear maps. We say that $(M, \Delta_M, {}_M\Delta)$ is a bicovariant right \mathcal{A} -module if the triplet is an object of the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}^{\mathcal{A}}$, i.e,

- (i) (M, Δ_M) is a left \mathcal{A} -comodule,
- (ii) $(M, {}_M\Delta)$ is a right \mathcal{A} -comodule,
- (iii) $(\text{id} \otimes_{\mathbb{C}} M\Delta)\Delta_M = (\Delta_M \otimes_{\mathbb{C}} \text{id}){}_M\Delta$,
- (iv) For any a in \mathcal{A} and m in M ,

$$\Delta_M(ma) = \Delta_M(m)\Delta(a), \quad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a).$$

For the rest of the subsection, \mathcal{E} will denote a bicovariant \mathcal{A} -bimodule. Moreover, $\{\omega_i\}_i$ will denote a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ the dual basis, i.e, $\omega_i^*(\omega_j) = \delta_{ij}$.

Let us recall that (2.1) implies the existence of elements R_{ij} in \mathcal{A} such that

$$\varepsilon\Delta(\omega_i) = \sum_{ij} \omega_j \otimes_{\mathbb{C}} R_{ji}. \quad (4.8)$$

We want to show that $V_g(\mathcal{E})$ is a bicovariant right \mathcal{A} -module. To this end, we recall that (Lemma 3.7) $V_g(\mathcal{E})$ is a free right \mathcal{A} -module with basis $\{\omega_i^*\}_i$. This allows us to make the following definition.

Definition 4.11. Let $\{\omega_i\}_i$ and $\{\omega_i^*\}_i$ be as above and g a bi-invariant pseudo-Riemannian metric on \mathcal{E} . Then we can endow $V_g(\mathcal{E})$ with a left-coaction $\Delta_{V_g(\mathcal{E})} : V_g(\mathcal{E}) \rightarrow \mathcal{A} \otimes_{\mathbb{C}} V_g(\mathcal{E})$ and a right-coaction $v_{g(\mathcal{E})}\Delta : V_g(\mathcal{E}) \rightarrow V_g(\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A}$, defined by the formulas

$$\begin{aligned} \Delta_{V_g(\mathcal{E})} \left(\sum_i \omega_i^* a_i \right) &= \sum_i (1 \otimes_{\mathbb{C}} \omega_i^*) \Delta(a_i), \quad v_{g(\mathcal{E})}\Delta \left(\sum_i \omega_i^* a_i \right) \\ &= \sum_{ij} (\omega_j^* \otimes_{\mathbb{C}} S(R_{ij})) \Delta(a_i), \end{aligned} \quad (4.9)$$

where the elements R_{ij} are as in (4.8).

Then we have the following result.

Proposition 4.12. *The triplet $(V_g(\mathcal{E}), \Delta_{V_g(\mathcal{E})}, v_{g(\mathcal{E})}\Delta)$ is a bicovariant right \mathcal{A} -module. Moreover, the map $V_g : \mathcal{E} \rightarrow V_g(\mathcal{E})$ is bicovariant, i.e, we have*

$$\Delta_{V_g(\mathcal{E})}(V_g(e)) = (\text{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(e), \quad v_{g(\mathcal{E})}\Delta(V_g(e)) = (V_g \otimes_{\mathbb{C}} \text{id})_{\mathcal{E}}\Delta(e). \quad (4.10)$$

Proof. The fact that $(V_g(\mathcal{E}), \Delta_{V_g(\mathcal{E})}, v_{g(\mathcal{E})}\Delta)$ is a bicovariant right \mathcal{A} -module follows immediately from the definition of the maps $\Delta_{V_g(\mathcal{E})}$ and $v_{g(\mathcal{E})}\Delta$. So we are left with proving (4.10). Let $e \in \mathcal{E}$. Then there exist elements a_i in \mathcal{A} such that $e = \sum_i \omega_i a_i$. Hence, by (3.1), we obtain

$$\begin{aligned} \Delta_{V_g(\mathcal{E})}(V_g(e)) &= \Delta_{V_g(\mathcal{E})} \left(V_g \left(\sum_i \omega_i a_i \right) \right) = \Delta_{V_g(\mathcal{E})} \left(\sum_{ij} g_{ij} \omega_j^* a_i \right) \\ &= \sum_{ij} (1 \otimes_{\mathbb{C}} g_{ij} \omega_j^*) \Delta(a_i) = \sum_i ((\text{id} \otimes_{\mathbb{C}} V_g)(1 \otimes_{\mathbb{C}} \omega_i)) \Delta(a_i) \\ &= \sum_i (\text{id} \otimes_{\mathbb{C}} V_g)(\Delta_{\mathcal{E}}(\omega_i)) \Delta(a_i) \\ &= \sum_i (\text{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(\omega_i a_i) = (\text{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(e). \end{aligned}$$

This proves the first equation of (4.10).

For the second equation, we begin by making an observation. Since $\varepsilon\Delta(\omega_i) = \sum_j \omega_j \otimes_{\mathbb{C}} R_{ji}$, we have

$$\delta_{ij} = \varepsilon(R_{ij}) = m(S \otimes_{\mathbb{C}} \text{id})\Delta(R_{ij}) = \sum_k S(R_{ik})R_{kj}.$$

Therefore, multiplying (3.5) by $S(R_{jm})$ and summing over j , we obtain

$$\sum_j g_{ij}S(R_{jm}) = \sum_j g_{jm}R_{ji}. \quad (4.11)$$

Now by using (3.1), we compute

$$\begin{aligned} v_g(\varepsilon)\Delta(V_g(e)) &= v_g(\varepsilon)\Delta\left(V_g\left(\sum_i \omega_i a_i\right)\right) = v_g(\varepsilon)\Delta\left(\sum_{ij} g_{ij}\omega_j^* a_i\right) \\ &= \sum_{ij} v_g(\varepsilon)\Delta(g_{ij}\omega_j^*)\Delta(a_i) = \sum_{ijk} g_{ij}\omega_k^* \otimes_{\mathbb{C}} S(R_{jk})\Delta(a_i) \\ &= \sum_{ik} \omega_k^* \otimes_{\mathbb{C}} \sum_j g_{ij}S(R_{jk})\Delta(a_i) \\ &= \sum_{ik} \omega_k^* \otimes_{\mathbb{C}} \sum_j g_{jk}R_{ji}\Delta(a_i) \quad (\text{by (4.11)}) \\ &= \sum_{ijk} g_{jk}\omega_k^* \otimes_{\mathbb{C}} R_{ji}\Delta(a_i) = \sum_{ij} V_g(\omega_j) \otimes_{\mathbb{C}} R_{ji}\Delta(a_i) \\ &= \sum_i (V_g \otimes_{\mathbb{C}} \text{id})\varepsilon\Delta(\omega_i)\Delta(a_i) \quad (\text{by (4.8)}) \\ &= \sum_i (V_g \otimes_{\mathbb{C}} \text{id})\varepsilon\Delta(\omega_i a_i) = (V_g \otimes_{\mathbb{C}} \text{id})\varepsilon\Delta(e). \end{aligned}$$

This finishes the proof. \square

Now we recall that bicovariant right \mathcal{A} -modules (i.e, objects of the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$) can be deformed too.

Proposition 4.13 ([14, Theorem 5.7]). *Let $(M, \Delta_M, {}_M\Delta)$ be a bicovariant right \mathcal{A} -module and γ be a 2-cocycle on \mathcal{A} . Then*

- (i) M deforms to a bicovariant right \mathcal{A}_γ -module, denoted by M_γ ,
- (ii) if $(N, \Delta_N, {}_N\Delta)$ is another bicovariant right \mathcal{A} -module and $T : M \rightarrow N$ is a bicovariant right \mathcal{A} -linear map, then the deformation $T_\gamma : M_\gamma \rightarrow N_\gamma$ is a bicovariant right \mathcal{A}_γ -linear map,
- (iii) T_γ , as in (ii), is an isomorphism if and only if T is an isomorphism.

Proof. Parts (i) and (ii) follow from the equivalence of categories ${}^{\mathcal{A}}\mathcal{M}$ and ${}^{\mathcal{A}_\gamma}\mathcal{M}$ combined with the ${}^{\mathcal{A}}\mathcal{M}_{{}^{\mathcal{A}}}$ analogue of (non-monoidal part of) the second assertion of Proposition 5.7 of [14]. Part (iii) follows by noting that since the map T is a bicovariant right \mathcal{A} -linear map, its inverse T^{-1} is also a bicovariant right \mathcal{A} -linear map. Thus, the deformation $(T^{-1})_\gamma$ of T^{-1} exists and is the inverse of the map T_γ . \square

As an immediate corollary, we make the following observation.

Corollary 4.14. *Let g be a bi-invariant pseudo-Riemannian metric on a bicovariant \mathcal{A} -bimodule \mathcal{E} . Then the following map is a well-defined isomorphism.*

$$(V_g)_\gamma : \mathcal{E}_\gamma \rightarrow (V_g(\mathcal{E}))_\gamma = (V_g)_\gamma(\mathcal{E}_\gamma)$$

Proof. Since both \mathcal{E} and $V_g(\mathcal{E})$ are bicovariant right \mathcal{A} -modules, and V_g is a right \mathcal{A} -linear bicovariant map (Proposition 4.12), Proposition 4.13 guarantees the existence of $(V_g)_\gamma$. Since g is a pseudo-Riemannian metric, by (ii) of Definition 3.1, $V_g : \mathcal{E} \rightarrow V_g(\mathcal{E})$ is an isomorphism. Then, by (iii) of Proposition 4.13, $(V_g)_\gamma$ is also an isomorphism. \square

Now we are in a position to state and prove the main result of this section which shows that there is an abundant supply of bi-invariant pseudo-Riemannian metrics on \mathcal{E}_γ . Since g is a map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to \mathcal{A} , g_γ is a map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ to \mathcal{A}_γ . But we have the isomorphism ξ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ (Proposition 4.5). As in Subsection 4.1, we will make an abuse of notation to denote the map $g_\gamma \xi^{-1}$ by the symbol g_γ .

Theorem 4.15. *If g is a bi-invariant pseudo-Riemannian metric on a finite bicovariant \mathcal{A} -bimodule \mathcal{E} (as in Remark 2.2) and γ is a 2-cocycle on \mathcal{A} , then g deforms to a right \mathcal{A}_γ -linear map g_γ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself. Moreover, g_γ is a bi-invariant pseudo-Riemannian metric on \mathcal{E}_γ . Finally, any bi-invariant pseudo-Riemannian metric on \mathcal{E}_γ is a deformation (in the above sense) of some bi-invariant pseudo-Riemannian metric on \mathcal{E} .*

Proof. Since g is a right \mathcal{A} -linear bicovariant map (Proposition 3.3), g indeed deforms to a right \mathcal{A}_γ -linear map g_γ from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma \cong \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ (see Proposition 4.5) to \mathcal{A}_γ . The second assertion of Proposition 4.4 implies that g_γ is bicovariant. Then Proposition 3.3 implies that g_γ is bi-invariant. Since $g \circ \sigma = g$, part (iii) of Proposition 4.4 implies that

$$g_\gamma = (g \circ \sigma)_\gamma = g_\gamma \circ \sigma_\gamma.$$

This verifies condition (i) of Definition 3.1

Next, we prove that g_γ satisfies (ii) of Definition 3.1. Let ω be an element of ${}_0\mathcal{E} = {}_0(\mathcal{E}_\gamma)$ and η be an element of $\mathcal{E}_0 = (\mathcal{E}_\gamma)_0$. Then we have

$$\begin{aligned} (V_g)_\gamma(\omega)(\eta) &= (V_g(\omega))_\gamma(\eta) = V_g(\omega)(\eta) \\ &= g(\omega \otimes_{\mathcal{A}} \eta) = g_\gamma(\bar{\gamma}(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}_\gamma} \eta_{(0)}\gamma(\omega_{(1)} \otimes_{\mathbb{C}} 1)) \\ &\quad \text{(by the definition of } \xi^{-1} \text{ in Proposition 4.5)} \\ &= g_\gamma(\epsilon(\eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}_\gamma} \eta_{(0)}\epsilon(\omega_{(1)})) = g_\gamma(\omega \otimes_{\mathcal{A}_\gamma} \eta) = V_{g_\gamma}(\omega)(\eta). \end{aligned}$$

Then, by the right- \mathcal{A}_γ linearity of $(V_g)_\gamma(\omega)$ and $V_{g_\gamma}(\omega)$, we get, for all a in \mathcal{A} ,

$$V_{g_\gamma}(\omega)(\eta *_\gamma a) = V_{g_\gamma}(\omega)(\eta) *_\gamma a = (V_g)_\gamma(\omega)(\eta) *_\gamma a = (V_g)_\gamma(\omega)(\eta *_\gamma a).$$

Therefore, by the right \mathcal{A} -totality of $(\mathcal{E}_\gamma)_0 = \mathcal{E}_0$ in \mathcal{E}_γ , we conclude that the maps $(V_g)_\gamma$ and V_{g_γ} agree on ${}_0(\mathcal{E}_\gamma)$. But since ${}_0(\mathcal{E}_\gamma) = {}_0\mathcal{E}$ is right \mathcal{A}_γ -total in \mathcal{E}_γ and both V_{g_γ} and $(V_g)_\gamma$ are right- \mathcal{A}_γ linear, $(V_g)_\gamma = V_{g_\gamma}$ on the whole of \mathcal{E}_γ .

Next, since V_g is a right \mathcal{A} -linear isomorphism from \mathcal{E} to $V_g(\mathcal{E})$, hence by Corollary 4.14, $(V_g)_\gamma$ is an isomorphism onto $(V_g(\mathcal{E}))_\gamma = (V_g)_\gamma(\mathcal{E}_\gamma) = V_{g_\gamma}(\mathcal{E}_\gamma)$. Therefore V_{g_γ} is an isomorphism from \mathcal{E}_γ to $V_{g_\gamma}(\mathcal{E}_\gamma)$. Hence g_γ satisfies (ii) of Definition 3.1.

To show that every pseudo-Riemannian metric on \mathcal{E}_γ is obtained as a deformation of a pseudo-Riemannian metric on \mathcal{E} , we view \mathcal{E} as a cocycle deformation of \mathcal{E}_γ under the cocycle γ^{-1} . Then given a pseudo-Riemannian metric g' on \mathcal{E}_γ , by the first part of this proof, $(g')_{\gamma^{-1}}$ is a bi-invariant pseudo-Riemannian metric on \mathcal{E} . Hence, $g' = ((g')_{\gamma^{-1}})_\gamma$ is indeed a deformation of the bi-invariant pseudo-Riemannian metric $(g')_{\gamma^{-1}}$ on \mathcal{E} . \square

Remark 4.16. We have actually used the fact that \mathcal{E} is finite in order to prove Theorem 4.15. Indeed, since \mathcal{E} is finite, we can use the results of Section 3 to derive Proposition 4.12 which is then used to prove Corollary 4.14. Finally, Corollary 4.14 is used to prove Theorem 4.15.

Also note that the proof of Theorem 4.15 also implies that the maps $(V_g)_\gamma$ and V_{g_γ} are equal.

When g is a pseudo-Riemannian bicovariant bilinear metric on \mathcal{E} , then we have a much shorter proof of the fact that g_γ is a pseudo-Riemannian metric on \mathcal{E}_γ which avoids bicovariant right \mathcal{A} -modules. We learnt the proof of this fact from communications with the referee and is as follows: We will work in the categories ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ and ${}^{\mathcal{A}_\gamma}\mathcal{M}_{\mathcal{A}_\gamma}$. Firstly, as g is bilinear, V_g is a morphism of the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ and can be deformed to a bicovariant \mathcal{A}_γ -bilinear map $(V_g)_\gamma$ from \mathcal{E}_γ to $(\mathcal{E}^*)_\gamma$. Similarly, g deforms to a \mathcal{A}_γ -bilinear map from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to \mathcal{A}_γ . Then as in the proof of Theorem 4.15, we can easily check that $(V_g)_\gamma = V_{g_\gamma}$.

On the other hand, it is well-known that the left dual $\widetilde{\mathcal{E}}$ of \mathcal{E} is isomorphic to \mathcal{E}^* . Since g is bilinear, Proposition 3.9 implies that the morphism V_g (in the category ${}^{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$) is an isomorphism from \mathcal{E} to \mathcal{E}^* .

Therefore, we have an isomorphism $(V_g)_\gamma$ is an isomorphism from \mathcal{E}_γ to $(\mathcal{E}^*)_\gamma \cong (\mathcal{E}_\gamma)^*$ by Exercise 2.10.6 of [5]. As $(V_g)_\gamma = V_{g_\gamma}$, we deduce that V_{g_γ} is an isomorphism from \mathcal{E}_γ to $(\mathcal{E}_\gamma)^*$. Since the equation $g_\gamma \circ \sigma_\gamma = g_\gamma$, this completes the proof.

Acknowledgement

We are immensely grateful to the referee for clarifying the connections with monoidal categories and their deformations. Their enlightening remarks has greatly improved our understanding. We would like to thank the referee for pointing out the relevant results in [3, 12, 14]. Proposition 3.6, Remark 4.8, Remark 4.16 and the discussion at the very end of the article are due to the referee.

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