ANNALES MATHÉMATIQUES

BLAISE PASCAL

V. P. Anoop & Sanjay Parui

Hardy–Littlewood–Sobolev Inequality for Upper Half Space

Volume 28, nº 2 (2021), p. 117-140.

<http://ambp.centre-mersenne.org/item?id=AMBP_2021__28_2_117_0>

Publication éditée par le laboratoire de mathématiques Blaise Pascal
de l'université Clermont Auvergne, UMR 6620 du CNRS
Clermont-Ferrand — France

CENTRE MERSENNE

Publication membre du
Centre Mersenne pour l'édition scientifique ouverte
http://www.centre-mersenne.org/

Annales mathématiques Blaise Pascal **28**, 117-140 (2021)

Hardy–Littlewood–Sobolev Inequality for Upper Half Space

V. P. Anoop
Sanjay Parui

Abstract

We define an extension operator and study (L^p, L^q) boundedness of Hardy–Littlewood–Sobolev inequality and weighted Hardy–Littlewood–Sobolev inequality on upper Half space for the Dunkl transform.

1. Introduction and Main Theorems

Weighted inequalities have applications in problems of Harmonic analysis and partial differential equations. Hardy–Littlewood–Sobolev and weighted Hardy–Littlewood–Sobolev inequality have attracted a great attention to many people and it has been extended from Euclidean space to other manifolds. One of the simplest manifolds with boundary is upper-half space $\mathbb{R}^N \times (0,\infty)$. The Hardy–Littlewood–Sobolev (HLS) inequalities are equivalent to (L^p, L^q) boundedness of the convolution with the Riesz potential. Parallel to the potential equation $\triangle u = f$ on $\mathbb{R}^N$, one can consider the Laplacian on $\mathbb{R}^N \times (0,\infty)$ with Neumann boundary condition. For $f \in C_0^\infty(\mathbb{R}^N)$ $N \geq 2$ the pointwise solution of the equation

$$\begin{aligned} -\triangle u(x, x_{N+1}) &= 0 &&\text{for } x_{N+1} > 0 \text{ and } x \in \mathbb{R}^N \\ u_{x_{N+1}}(x,0) &= -f(x) &&\text{for } x \in \mathbb{R}^N \end{aligned}$$

is equivalent to up to a constant multiplier and harmonic function, the following integral equation

$$u(x) = \int_{\mathbb{R}^N} \frac{f(y)}{\left(|x-y|^2 + x_{N+1}^2\right)^{\frac{N-1}{2}}} \mathrm{d}y. \tag{1.1}$$

This equation can be viewed of another type of harmonic extension of f. With this motivation Dou and Zhu [2] introduced the extension operator

$$\mathcal{E}_\alpha f(x, x_{N+1}) = \int_{\mathbb{R}^N} \frac{f(y)}{\left(|x-y|^2 + x_{N+1}^2\right)^{\frac{N+1-\alpha}{2}}} \mathrm{d}y$$

First and second author are supported by national postdoctoral fellowship - NPDF (PDF/2019/001963) and MATRICS grant MTR/2019/001119 respectively from the Science and Engineering Research Board (SERB), India.
Keywords: Dunkl transform, Hardy–Littlewood–Sobolev inequality, Weighted Hardy inequality.
2020 *Mathematics Subject Classification*: 42B10, 42B35, 42B37.

for $\alpha \in (1, N+1)$ and studied the sharp Hardy-Littlewood -Sobolev inequalities for $\mathcal{E}_\alpha$ and it's dual operator. Dou [4] also proved double weighted HLS inequality on the upper half space and study the existence of extremal functions for the sharp inequality. Dunkl transform is one of the generalizations of classical Fourier transform and many important classical results have been extended to the $\mathbb{R}^N$ with Dunkl weight. Boundedness of Riesz potential for Dunkl transform was first proved by Thangavelu and Xu in [10] and a different proof was given by Gorbachev et al. [5]. Same authors of [5] proved sharped weighted HLS inequality (Stein–Weiss inequality) with Dunkl weight and found the sharp constant for $p = q$ in [6]. Our main goal in this paper is to define the extension operator $\mathcal{E}_\alpha^k$ analogous to $\mathcal{E}_\alpha$ and prove the HLS inequality and weighted HLS inequality associated with the Dunkl transform on the manifold $\mathbb{R}^N \times (0, \infty)$.

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $\mathbb{R}^N$ and $|\cdot| := \sqrt{\langle \cdot, \cdot \rangle}$. Let $R \subset \mathbb{R}^N \setminus \{0\}$ be a finite set. Then R is called a root system, if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ for all $\alpha \in R$ and $\sigma_\alpha(R) = R$ for all $\alpha \in R$. A root system can be written as the disjoint union of $R_+ \cup (-R_+)$ and these R_+ and $(-R_+)$ are separated by a hyper plane passing through the origin. This R_+ is called as the positive roots of the root system. The subgroup $G = G(R) \subseteq O(N, \mathbb{R})$ which is generated by reflections $\{\sigma_\alpha : \alpha \in R\}$ is called reflection group (or Coxeter-group) associated with R. A G-invariant function k defined on R, is called as a multiplicity function. We fix a reflection group G and a multiplicity function k. We can define the G-invariant homogeneous weight function $h_k^2(x) = \prod_{\alpha \in R_+} |\langle x, \alpha \rangle|^{2k(\alpha)}$ of degree $2\gamma_k$, where $\gamma_k := \sum_{\alpha \in R_+} k(\alpha)$. Let $d_k := N + 2\gamma_k$ and $\mathrm{d}\mu_k(x)$ be the weighted measure $c_k h_k^2(x)\mathrm{d}x$, where

$$c_k^{-1} = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{2}} h_k^2(x)\mathrm{d}x = 2^{\frac{d_k-2}{2}} \Gamma\left(\frac{d_k}{2}\right).$$

For $j \in \{1, 2, \ldots, N\}$ the differential-difference operators T_j(the Dunkl operators) are defined by

$$T_j f(x) := \partial_j f(x) + E_j f(x), \; f \in C^1(\mathbb{R}^N)$$

where $E_j f(x) = \sum_{\alpha \in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$. The Dunkl operators T_j is a generalization of the partial differential operator in the classical analysis. For a fixed $y \in \mathbb{R}^N$, it is known that there exists a unique real analytic solution $f(x) = E_k(x, y)$ for the system $T_i f = y_i f;\; 1 \le i \le N$ satisfying $f(0) = 1$. The kernel $E_k(x, y)$ is called the Dunkl kernel and it is clearly a generalization of the exponential functions $e^{\langle x, y \rangle}$. Dunkl transform is defined as a generalization of Fourier transform. For $u \in L^1(\mathbb{R}^N, \mathrm{d}\mu_k(x))$, its Dunkl Fourier transform is defined by

$$\mathcal{F}_k u(\xi) = \int_{\mathbb{R}^N} u(x) E_k(-i\xi, x)\mathrm{d}\mu_k(x).$$

Dunkl translation operator is defined through the Dunkl transform. The Dunkl translation $\tau_y^k f$ is defined by $\mathcal{F}_k(\tau_y^k f)(\xi) = E_k(iy,\xi)\mathcal{F}_k f(\xi)$ and it makes sense for all $f \in L^2(\mathbb{R}^N, \mathrm{d}\mu_k(x))$ as $E_k(iy,\xi)$ is a bounded function. Dunkl translation has the property; $\tau_y^k f(x) = \tau_{-x}^k f(-y)$. We refer [7, 9, 10] and the references there in for further reading on Dunkl kernel and Dunkl transform. Let $L^p(\mathbb{R}^N, \mathrm{d}\mu_k)$ be the space of complex valued measurable functions f such that

$$\|f\|_{p,\mathrm{d}\mu_k} = \left(\int_{\mathbb{R}^N} |f(x)|^p \mathrm{d}\mu_k(x)\right)^{\frac{1}{p}} < \infty$$

and $L^p(\mathbb{R}^N \times (0,\infty), \mathrm{d}\mu_k \mathrm{d}x_{N+1})$ be the space of complex valued measurable functions g such that

$$\|g\|_{p,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} = \left(\int_{\mathbb{R}^N} |g(x,x_{N+1})|^p \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\right)^{\frac{1}{p}} < \infty.$$

Thangavelu and Xu [10] defined the Riesz potential I_α^k associated to Dunkl operator on Schwartz space by

$$I_\alpha^k f(x) = (\gamma_\alpha^k)^{-1} \int_{\mathbb{R}^N} \tau_{-y}^k f(x)|y|^{\alpha-d_k} \mathrm{d}\mu_k(y),$$

where $0 < \alpha < d_k$ and $\gamma_\alpha^k = 2^{\alpha-d_k}\Gamma(\alpha/2)/\Gamma((d_k-\alpha)/2)$. Maximal function $M_k f$ defined for $f \in \mathcal{S}(\mathbb{R}^N)$ was introduced and studied by Thangavelu and Xu [10] as follows

$$M_k f(x) = \sup_{r>0} \frac{|(f *_k \chi_{B_r})(x)|}{\int_{\mathbb{R}^N} \chi_{B_r}(x)\mathrm{d}\mu_k(x)},$$

where χ_{B_r} is the characteristic function of the Euclidean ball B_r of radius r centered at the origin. Later Deleaval [3] proved a refined scalar maximal theorem. With these notations we can state the results of Gorbachev et al. [6] as follows.

Theorem 1.1. *Let $N \in \mathbb{N}$, $1 \le p \le q < \infty$, $\gamma < \frac{d_k}{q}$, $\gamma + \beta \ge 0$, $0 < \alpha < d_k$ and $\alpha - \gamma - \beta = d_k(\frac{1}{p} - \frac{1}{q})$.*

(a) *If $1 < p \le q < \infty$ and $\beta < \frac{d_k}{p'}$ then*

$$\||x|^{-\gamma} I_\alpha^k f\|_{q,\mathrm{d}\mu_k} \le C(\alpha,\beta,\gamma,p,q,d_k)\||x|^\beta f\|_{p,\mathrm{d}\mu_k} \tag{1.2}$$

with the sharp constant $C(\alpha,\beta,\gamma,p,q,d_k)$. Moreover for $p = q$ the sharp constant is given by

$$C(\alpha,\beta,\gamma,p,p,d_k) = 2^{-\alpha} \frac{\Gamma(\frac{1}{2}(\frac{d_k}{p}-\gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p'}-\beta))}{\Gamma(\frac{1}{2}(\frac{d_k}{p'}+\gamma))\Gamma(\frac{1}{2}(\frac{d_k}{p}+\beta))}.$$

(b) *If* $p = 1$, $1 < q < \infty$, $\beta \le 0$ *and* $\lambda > 0$, *then for* $f \in \mathcal{S}(\mathbb{R}^N)$

$$\int_{\{x\in\mathbb{R}^N : |x|^{-\gamma}|I_\alpha^k f(x)|>\lambda\}} \mathrm{d}\mu_k(x) \lesssim \left(\frac{\||\cdot|^\beta f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^q .$$

Dou and Zhu [2] established the following HLS inequality for the extension operator $\mathcal{E}_\alpha$ on the upper half space.

Theorem 1.2. *For any* $1 < \alpha < N+1$, $1 < p < \frac{N}{\alpha-1}$ *and*

$$\frac{1}{q} = \frac{N}{N+1}\left(\frac{1}{p} - \frac{\alpha-1}{N}\right)$$

there is a best constant $C(N,\alpha,p) > 0$ *such that*

$$\left(\int_{\mathbb{R}^N\times(0,\infty)} |\mathcal{E}_\alpha f(x,x_{N+1})|^q \,\mathrm{d}x\mathrm{d}x_{N+1}\right)^{\frac{1}{q}} \le C(N,\alpha,p)\left(\int_{\mathbb{R}^N} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$.

Using the weighted Hardy type inequality on the upper half space Dou [4] proved the following weighted HLS inequality for the extension operator $\mathcal{E}_\alpha$.

Theorem 1.3. *Let* $1 \le p \le q < \infty$, $\gamma < \frac{N+1}{q}$, $\beta < \frac{N}{p'}$, $\gamma+\beta \ge 0$, $1 < \alpha < N+1$, *and*

$$\frac{1}{q} = \frac{N}{N+1}(\frac{1}{p} - \frac{\alpha-\gamma-\beta-1}{N})$$

then for $1 < p \le q < \infty$ *there is a best constant* $C(N,\alpha,p) > 0$ *such that*

$$\left(\int_{\mathbb{R}^N\times(0,\infty)} ||x|^{-\gamma}\mathcal{E}_\alpha f(x,x_{N+1})|^q \,\mathrm{d}x\mathrm{d}x_{N+1}\right)^{\frac{1}{q}} \le C(N,\alpha,p)\left(\int_{\mathbb{R}^N} ||x|^\beta f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}}$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$.

One can ask for the natural extension operator analogous to $\mathcal{E}_\alpha$ in Dunkl setting and expect to extend the Theorem 1.1 on the upper half space.

We introduce an extension operator $\mathcal{E}_\alpha^k$ for $\alpha \in (1, d_k+1)$ as the following integral operator which acts on $f \in \mathcal{S}(\mathbb{R}^N)$ as

$$\begin{aligned}\mathcal{E}_\alpha^k f(x,x_{N+1}) &= (\gamma_{\alpha-1}^k)^{-1}\int_{\partial\mathbb{R}_+^{N+1}} f(y)\tau_{-y}^k\left(\frac{1}{(x_N^2+|\cdot|^2)^{\frac{d_k+1-\alpha}{2}}}\right)(x)\mathrm{d}\mu_k(y)\\ &= (\gamma_{\alpha-1}^k)^{-1}\int_{\partial\mathbb{R}_+^{N+1}} \tau_{-y}^k f(x)\frac{1}{(x_N^2+|y|^2)^{\frac{d_k+1-\alpha}{2}}}\mathrm{d}\mu_k(y),\end{aligned}$$

where $(\gamma_{\alpha-1}^k) = \frac{2^{\alpha-1-\frac{d_k}{2}}\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{d_k-\alpha+1}{2})}$. Our first main theorem is the following Hardy–Littlewood–Sobolev inequality on the upper half space.

Theorem 1.4. *For* $1 < \alpha < d_k + 1$, $1 < p < \frac{d_k}{\alpha-1}$ *and*

$$\frac{1}{q} = \frac{d_k}{d_k+1}\left(\frac{1}{p} - \frac{\alpha-1}{d_k}\right) \tag{1.3}$$

there exist an optimal constant $C(d_k, p, \alpha) > 0$ *such that*

$$\left(\int_{\mathbb{R}^N\times(0,\infty)} |\mathcal{E}_\alpha^k f(x, x_{N+1})|^q \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\right)^{\frac{1}{q}} \le C(d_k, p, \alpha)\left(\int_{\mathbb{R}^N} |f(x)|^p \mathrm{d}\mu_k(x)\right)^{\frac{1}{p}} \tag{1.4}$$

for all $f \in \mathcal{S}(\mathbb{R}^N)$.

Also the map $f \mapsto \mathcal{E}_\alpha^k f$ *is of weak type* $(1, q)$.

A simple proof of HLS inequality for Dunkl operator was given by Gorbachev et.al. [5, Theorem 4.1] using the L^p boundedness of the spherical mean operator in the Dunkl setting. We will follow this approach with some modification to prove the Theorem 1.4. The key point to prove the inequality (1.2) was to rewrite the inequality as a convolution inequality in the multiplicative group $\mathbb{R}_+$ with the Haar measure $\mathrm{d}r/r$ and then apply Hausdorff Young inequality for the multiplicative group $\mathbb{R}_+$. Our proof of the following weighted HLS inequality is inspired by the approach of Gorbachev et al. in proving the inequality (1.2).

Theorem 1.5. *Let* $1 \le p \le q < \infty, \gamma < \frac{d_k+1}{q}, \beta < \frac{d_k}{p'}, \gamma+\beta \ge 0, 1 < \alpha < d_k+1$, *and*

$$\frac{1}{q} = \frac{d_k}{d_k+1}\left(\frac{1}{p} - \frac{\alpha-\gamma-\beta-1}{d_k}\right). \tag{1.5}$$

If $1 < p \le q < \infty$, *then*

(1) $$\left(\int_{\mathbb{R}^N\times(0,\infty)} ||(x, x_{N+1})|^{-\gamma}\mathcal{E}_\alpha^k f(x, x_{N+1})|^q \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\right)^{\frac{1}{q}}$$
$$\le C(\alpha, \beta, \gamma, p, q, d_k)\left(\int_{\mathbb{R}^N} ||x|^\beta f(x)|^p \mathrm{d}\mu_k(x)\right)^{\frac{1}{p}}.$$

(2) *If* $p = 1$, $1 < q < \infty$, $\beta \le 0$, *then for* $f \in \mathcal{S}(\mathbb{R}^N)$ *and* $\lambda > 0$

$$\int_{\{x\in\mathbb{R}^N : |x|^{-\gamma}|\mathcal{E}_\alpha^k f(x, x_{N+1})| > \lambda\}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \le C\left(\frac{\||\cdot|^\beta f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^q.$$

We will first represent the extension operator $\mathcal{E}^k_\alpha$ as a kernel operator and study basic properties of the kernel. Proof of the HLS inequality (1.4) is given in the Section 1. In Section 2 we stated an auxiliary lemma and proved sharp Hardy type inequalities which will be used to give a proof of the weighted HLS inequality in Section 3. We added an Appendix where the best constant for weighted HLS inequality in the case of $p = q$ is expressed as an integral involved with hypergeometric function.

For each $\alpha \in (1, d_k+1)$ we consider the extension operator which acts on $f \in \mathcal{S}(\mathbb{R}^N)$ as

$$\mathcal{E}^k_\alpha f(x) = (\gamma^k_{\alpha-1})^{-1} \int_{\partial\mathbb{R}^N_+} f(y)\tau^k_{-y}\left(\frac{1}{(x_N^2 + |\cdot|^2)^{\frac{d_k+1-\alpha}{2}}}\right)(x)\mathrm{d}\mu_k(y),$$

where $(\gamma^k_{\alpha-1}) = \frac{2^{\alpha-1-\frac{d_k}{2}}\Gamma(\frac{\alpha-1}{2})}{\Gamma(\frac{d_k-\alpha+1)}{2})}$. If we denote

$$\Phi^k_\alpha((x, x_{N+1}), y) = \tau^k_{-y}\left(\frac{1}{(x_N^2 + |\cdot|^2)^{\frac{d_k+1-\alpha}{2}}}\right)(x)$$

then the kernel of the operator $\mathcal{E}^\alpha_k$ is $(\gamma^k_{\alpha-1})^{-1}\Phi^k_\alpha((x, x_{N+1}), y)$. Therefore, using the integral identity

$$\frac{1}{|x|^{d_k+1-\alpha}} = \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})}\int_0^\infty s^{\frac{d_k+1-\alpha}{2}-1}e^{-s|x|^2}\mathrm{d}s \tag{1.6}$$

and the formula $\tau^k_y e^{-s|\cdot|^2}(x) = e^{-s(|x|^2+|y|^2)}E_k(\sqrt{2s}x, \sqrt{2s}y)$ ([9])

$$\begin{aligned}\Phi^k_\alpha((x, x_{N+1}), y) &= \tau^k_{-y}\left(\frac{1}{(x_N^2 + |\cdot|^2)^{\frac{d_k+1-\alpha}{2}}}\right)(x)\\ &= \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})}\int_0^\infty e^{-sx_{N+1}^2}s^{\frac{d_k-1-\alpha}{2}}e^{-s(|x|^2+|y|^2)}E_k(\sqrt{2s}x, \sqrt{2s}y)\mathrm{d}s.\end{aligned}$$

Now writing $x = rx'$, $y = ty'$ and changing the variable $s \to \frac{u}{r^2}$ we find

$$\begin{aligned}&\Phi^k_\alpha((x, x_{N+1}), y)\\ &\quad= \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})}r^{-d_k-1+\alpha}\int_0^\infty e^{-\frac{u}{r^2}x_{N+1}^2}u^{\frac{d_k-1-\alpha}{2}}e^{-u(1+\frac{t^2}{r^2})}E_k\left(\sqrt{2u}x', \sqrt{2u}\frac{t}{r}y'\right)\mathrm{d}u\\ &\quad= \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})}r^{-d_k-1+\alpha}\Phi^k_\alpha\left(\left(x', \frac{x_{N+1}}{r}\right), \frac{t}{r}y'\right).\end{aligned}$$

If we use the change of variable $s \to \frac{u}{t^2}$ then we obtain

$$\Phi_\alpha^k((x, x_{N+1}), y) = \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})} t^{-d_k-1+\alpha} \Phi_\alpha^k\left(\left(x', \frac{x_{N+1}}{t}\right), \frac{r}{t} y'\right).$$

Let $\mathrm{d}\sigma_k(x') = a_k h_k^2(x')\mathrm{d}\sigma(x')$ be the normalized surface measure on S^{N-1}. Then we find $\mathrm{d}\mu_k(x) = b_k r^{d_k-1}\mathrm{d}r\mathrm{d}\sigma_k(x')$, where $b_k^{-1} = 2^{\frac{d_k}{2}-1}\Gamma(\frac{d_k}{2})$. From [6, p. 12] we note down the formula

$$\int_{S^{N-1}} \tau_{-ty'}^k(e^{-s|\cdot|^2})(rx')\mathrm{d}\sigma_k(x') = \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\pi e^{-s(r^2+t^2-2rt\cos\phi)} \sin^{d_k-2}\phi \mathrm{d}\phi$$

and

$$\int_{S^{N-1}} \tau_{-ty'}^k(e^{-s|\cdot|^2})(rx')\mathrm{d}\sigma_k(y') = \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\pi e^{-s(r^2+t^2-2rt\cos\phi)} \sin^{d_k-2}\phi \mathrm{d}\phi.$$

Hence from the definition of $\Phi_\alpha^k((x, x_{N+1}), y)$ we get

$$\begin{aligned}
&\int_{S^{N-1}} \Phi_\alpha^k((x', tx_{N+1}), ty')\mathrm{d}\sigma_k(x') \\
&\quad = \int_{S^{N-1}} \Phi_\alpha^k((x', tx_{N+1}), ty')\mathrm{d}\sigma_k(y') \\
&\quad = \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\pi (t^2x_{N+1}^2 + 1 + t^2 - 2t\cos\phi)^{\frac{\alpha-d_k-1}{2}} \sin^{d_k-2}\phi \mathrm{d}\phi \\
&\quad := \Phi_{\alpha,0}^k((1, x_{N+1}), t). \qquad (1.7)
\end{aligned}$$

For radial functions $f \in \mathcal{S}(\mathbb{R}^N)$ we have the following formula [7]

$$\tau_y^k f(x) = \int_{\mathbb{R}^N} f_0(\sqrt{|x|^2 + |y|^2 - 2\langle y, \eta\rangle})\mathrm{d}\mu_x^k(\eta),$$

where $f_0(|x|) = f(x)$ and μ_x^k is the probability measure with supp $\mu_x^k \subset \{\eta : |\eta| \leq |x|\}$. Therefore using the identity (1.6) we find another expression for the kernel

$$\Phi_\alpha^k((x, x_{N+1}), y) = \frac{1}{\Gamma(\frac{d_k+1-\alpha}{2})} \int_{\mathbb{R}^N} (|x|^2 + x_{N+1}^2 + |y|^2 - 2 < y, \eta >)^{\frac{\alpha+1-d_k}{2}} \mathrm{d}\mu_x^k(\eta).$$

Proof of Theorem 1.4

From the definition of the extension operator $\mathcal{E}_\alpha^k$ we write using the polar coordinates

$$\begin{aligned}\gamma_{\alpha-1}^k \mathcal{E}_\alpha^k f(x, x_{N+1}) &= \int_0^\infty \int_{S^{N-1}} \tau_{-ty'}^k f(x) \frac{1}{(x_N^2+t^2)^{\frac{d_k+1-\alpha}{2}}} \mathrm{d}\sigma_k(y') t^{d_k-1} \mathrm{d}t \\ &= \int_0^\infty T_t f(x) \frac{1}{(x_N^2+t^2)^{\frac{d_k+1-\alpha}{2}}} t^{d_k-1} \mathrm{d}t\end{aligned}$$

where

$$T_t f(x) = \int_{S^{N-1}} \tau_{-ty'}^k f(x) \mathrm{d}\sigma_k(y').$$

Using the integral Minkowski's inequality we get

$$\begin{aligned}&\gamma_{\alpha-1}^k \left(\int_{\mathbb{R}^N} |\mathcal{E}_\alpha^k f(x, x_{N+1})|^q \ \mathrm{d}x_{N+1} \right)^{\frac{1}{q}} \\ &\qquad = \int_0^\infty |T_t f(x)| \left(\int_0^\infty \frac{1}{(x_{N+1}^2+t^2)^{\frac{q(d_k+1-\alpha)}{2}}} \mathrm{d}x_{N+1} \right)^{\frac{1}{q}} t^{d_k-1} \mathrm{d}t \\ &\qquad \lesssim \int_0^\infty |T_t f(x)| t^{-(d_k-(\alpha-1+\frac{1}{q}))} t^{d_k-1} \mathrm{d}t.\end{aligned}$$

We write for $R > 0$

$$\begin{aligned}I &= \int_0^\infty |T_t f(x)| t^{-(d_k-(\alpha-1+\frac{1}{q}))} t^{d_k-1} \mathrm{d}t \\ &= \int_0^R |T_t f(x)| t^{-(d_k-(\alpha-1+\frac{1}{q}))} t^{d_k-1} \mathrm{d}t + \int_R^\infty |T_t f(x)| t^{-(d_k-(\alpha-1+\frac{1}{q}))} t^{d_k-1} \mathrm{d}t \\ &= I_1 + I_2.\end{aligned}$$

Rösler [7] proved that T^t is a positive operator on $C^\infty(\mathbb{R}^N)$ and hence

$$I_1 \leq \int_0^R T_t |f|(x) t^{-(d_k-(\alpha-1+\frac{1}{q}))} t^{d_k-1} \mathrm{d}t.$$

Now we can proceed as [5] and keeping in mind the relation (1.3) to get the estimate

$$I_1 \leq R^{(\frac{d_k}{p} - \frac{d_k}{q})} M_k |f|(x) \quad \text{and} \quad I_2 \leq R^{-\frac{d_k}{q}} \|f\|_{p,\mathrm{d}\mu_k}$$

Choosing $R^{-\frac{d_k}{p}} = (M_k|f|(x)/\|f\|_{p,\mathrm{d}\mu_k})$ we find

$$I \leq (M_k|f|(x))^{p/q} (\|f\|_{p,\mathrm{d}\mu_k})^{1-\frac{p}{q}}. \tag{1.8}$$

for $1 \leq p < q$. Using the fact that for $p > 1$, $\|M_k|f|\|_{p,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}$ and the inequality (1.8) we have for $1 < p < q$

$$\left(\int_{\mathbb{R}^N\times(0,\infty)} |\mathcal{E}^k_\alpha f(x,x_{N+1})|^q \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\right)^{\frac{1}{q}} \lesssim \|M_k|f|\|^{p/q}_{p,\mathrm{d}\mu_k}\|f\|^{1-\frac{p}{q}}_{p,\mathrm{d}\mu_k} \lesssim \|f\|_{p,\mathrm{d}\mu_k}.$$

Since the maximal function M_k is of weak type $(1,1)$, putting $p = 1$ in (1.8) we conclude the map $f \mapsto \mathcal{E}^k_\alpha f$ is weak $(1,q)$. □

2. **Auxiliary Lemma and Hardy type Inequalities**

We define the operators V and W analogous to the Hardy operator and Bellman operator defined in [6]

$$Wf(x,x_{N+1}) = \int_{|y|\leq|(x,x_{N+1})|} f(y)\mathrm{d}\mu_k(y)$$

and

$$Vg(y) = \int_{|(x,x_{N+1})|\leq|y|} g(x,x_{N+1})\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}.$$

Our interest to prove the following weighted Hardy type inequalities

$$\||(x,x_{N+1})|^{-a}Wf\|_{p,\mathrm{d}\mu_k\mathrm{d}x_{N+1}} \leq c^W_k(a,b,p)\||y|^b f\|_{p,\mathrm{d}\mu_k}$$

and

$$\||y|^{-a}Vg\|_{p,\mathrm{d}\mu_k} \leq c^V_k(a,b,p)\||(x,x_{N+1})|^b g\|_{p,\mathrm{d}\mu_k\mathrm{d}x_{N+1}}.$$

with the sharp constant $c^W_k(a,b,p)$, $c^V_k(a,b,p)$.

We state here the following lemma which will be used to prove Hardy type inequalities and our main theorem.

Lemma 2.1. *Let* $1 \leq p \leq \infty$ *and* $f \in L^p(\mathbb{R}_+\frac{\mathrm{d}r}{r})$, $g \in L^1(\mathbb{R}_+\frac{\mathrm{d}r}{r})$, $h \in L^{p'}(\mathbb{R}_+\frac{\mathrm{d}r}{r})$. *Then*

$$\left|\int_0^\infty\int_0^\infty h(r)f(t)g(r/t)\frac{\mathrm{d}t}{t}\frac{\mathrm{d}r}{r}\right| \leq \left(\int_0^\infty |h(t)|^{p'}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p'}}\left(\int_0^\infty |f(t)|^p\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}\int_0^\infty |g(r)|\frac{\mathrm{d}r}{r}.$$

Moreover if we define the operator A_g *by*

$$A_g f(r) = \int_0^\infty f(r/t)g(t)\frac{\mathrm{d}t}{t}$$

and $g \geq 0$ *then* $\|A_g\|_{p\to p} = \|g\|_1$.

For a proof of this lemma we refer to [6].

Theorem 2.2. *Let* $1 \le p \le \infty$. *Then there exists a sharp constant* $c_k^W(a,b,p)$ *such that*

$$\||(x,x_{N+1})|^{-a} Wf\|_{p,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \le c_k^W(a,b,p)\||x|^b f\|_{p,\mathrm{d}\mu_k}. \tag{2.1}$$

if and only if $b < \frac{d_k}{p'}$ *and* $a+b = d_k + \frac{1}{p}$.

Moreover $c_k^W(a,b,p) = \frac{b_k}{\frac{d_k}{p'}-b}\left(\frac{1}{2}\frac{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})}{\Gamma(\frac{d_k+1}{2})}\right)^{\frac{1}{p}}$.

Proof. Consider the modified operator

$$\widetilde{H}f(x,x_{N+1}) = \int_{|y|\le(|x|^2+x_{N+1}^2)^{1/2}} |y|^{-b} f(y)\,\mathrm{d}\mu_k(y).$$

Using the polar coordinates the operator $\widetilde{H}$ can be written as

$$\widetilde{H}(f)(rx', rx_{N+1}) = b_k \int_0^\infty \int_{S^{N-1}} f(ty') g(r/t(1,x_{N+1})) t^{-b+d_k} \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t},$$

where $g(t(1,x_{N+1})) = 1$ if $t^2 > (1+x_{N+1}^2)^{-1}$, otherwise it is zero. We express $\||(x,x_{N+1})|^{-ap}\widetilde{H}f\|_p^p$ as

$$\begin{aligned}
&\int_{\mathbb{R}^N\times(0,\infty)} |(x,x_{N+1})|^{-ap}|\widetilde{H}f(x,x_{N+1})|^p \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\\
&= b_k \int_0^\infty \int_0^\infty (r^2+x_{N+1}^2)^{-\frac{ap}{2}} \int_{S^{N-1}} r^{d_k} |\widetilde{H}f(rx', x_{N+1})|^p \mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}\\
&= b_k \int_0^\infty \int_0^\infty (1+x_{N+1}^2)^{-\frac{ap}{2}} \int_{S^{N-1}} r^{-ap+d_k+1} |\widetilde{H}f(rx', rx_{N+1})|^p \mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}.
\end{aligned}$$

On the other hand

$$\int_{\mathbb{R}^N} |f(x)|^p \mathrm{d}\mu_k(x) = b_k \int_0^\infty \int_{S^{N-1}} \left|t^{\frac{d_k}{p}} f(ty')\right|^p \mathrm{d}\sigma_k(y')\frac{\mathrm{d}t}{t}.$$

Replacing f by $|x|^{-\frac{d_k}{p}} f$ we have to show

$$\begin{aligned}
\int_0^\infty \int_0^\infty (1+x_{N+1}^2)^{-\frac{ap}{2}} \int_{S^{N-1}} r^{-ap+d_k+1}\left|\widetilde{H}\left(t^{-\frac{d_k}{p}} f\right)(rx', rx_{N+1})\right|^p \mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}\\
\le c_k^W(a,b,p)\int_0^\infty \int_{S^{N-1}} |f(ty')|^p \mathrm{d}\sigma_k(y')\frac{\mathrm{d}t}{t}.
\end{aligned}$$

We set

$$\begin{aligned}
J = \int_0^\infty \int_0^\infty \int_{S^{N-1}} (1+x_{N+1}^2)^{-\frac{a}{2}} r^{-a+\frac{d_k+1}{p}}\\
\left|\widetilde{H}\left(t^{-\frac{d_k}{p}} f\right)(rx', rx_{N+1})\right| h(rx', x_{N+1}) \mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}.
\end{aligned}$$

Now

$$\widetilde{H}\left(|\cdot|^{-\frac{d_k}{p}} f\right)(rx', rx_{N+1})$$
$$= b_k r^{-b+\frac{d_k}{p'}} \int_0^\infty \int_{S^{N-1}} f(ty')g(r/t(1, x_{N+1})) \left(\frac{r}{t}\right)^{b-\frac{d_k}{p'}} \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t}.$$

Hence by Lemma 2.1 we obtain

$$J \le b_k \int_0^\infty \int_{S^{N-1}} \int_{S^{N-1}} (1 + x_{N+1}^2)^{-\frac{a}{2}} \left(|h(rx', x_{N+1})|^{p'} \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p'}} \left(\int_0^\infty |f(ty')|^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}$$
$$\int_0^\infty |g(t(1, x_{N+1}))| t^{b-\frac{d_k}{p'}} \frac{\mathrm{d}t}{t} \mathrm{d}\sigma_k(x')\mathrm{d}\sigma_k(y')\mathrm{d}x_{N+1}$$
$$\le C_k^W(a, b, p) \left(\int_0^\infty \int_0^\infty \int_{S^{N-1}} |h(rx', x_{N+1})|^{p'} \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \mathrm{d}x_{N+1}\right)^{\frac{1}{p'}}$$
$$\left(\int_0^\infty \int_{S^{N-1}} |f(ty')|^p \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}},$$

where $C_k^W(a, b, p) = b_k \left(\int_0^\infty (1 + x_{N+1}^2)^{-a\frac{p}{2}} (\int_0^\infty |g(t(1, x_{N+1}))| t^{b-\frac{d_k}{p'}} \frac{\mathrm{d}t}{t})^p \mathrm{d}x_{N+1}\right)^{\frac{1}{p}}$.
Therefore we find

$$C_k^W(a, b, p) = b_k \left(\frac{1}{\frac{d_k}{p'} - b}\right) \left(\int_0^\infty (1 + x_{N+1}^2)^{-\frac{(d_k+1)}{2}} \mathrm{d}x_{N+1}\right)^{\frac{1}{p}}$$
$$= \frac{b_k}{\frac{d_k}{p'} - b} \left(\frac{1}{2} \frac{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})}{\Gamma(\frac{d_k+1}{2})}\right)^{\frac{1}{p}}.$$

One can show the constant $C_k^W(a, b, p)$ is optimal by taking a radial function f and then following the method used in [6]. If the inequality (2.1) holds then considering $f_t(x) = f(tx)$ we can find the condiditon $a + b = d_k + \frac{1}{p}$. □

Theorem 2.3. *Let $1 \le p \le \infty$. Then there exists a sharp constant $c_k^V(a, b, p)$ such that*

$$\||y|^{-a} V g\|_{p, \mathrm{d}\mu_k} \le c_k^V(a, b, p) \||(x, x_{N+1})|^b g\|_{p, \mathrm{d}\mu_k \mathrm{d}x_{N+1}} \tag{2.2}$$

if and only if $b < \frac{d_k+1}{p'}$, $a + b = d_k + \frac{1}{p'}$ and

$$c_k^V(a, b, p) = \frac{b_k}{\frac{d_k+1}{p'} - b} \left(\frac{1}{2} \frac{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})}{\Gamma(\frac{d_k+1}{2})}\right)^{\frac{1}{p'}}.$$

Proof. We consider the modified operator

$$\widetilde{B}g(y) = \int_{\{(x,x_{N+1}):|(x,x_{N+1})|\le|y|\}} |(x,x_{N+1})|^{-b} g(x,x_{N+1}) \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}.$$

Using the polar coordinates we can write

$$\begin{aligned}
&\widetilde{B}g(ty') \\
&= b_k \int_0^\infty\int_0^\infty\int_{S^{N-1}} r^{-b+d_k+1}(1+x_{N+1}^2)^{-\frac{b}{2}} g(rx',rx_{N+1}) f\Big(\frac{t}{r}(x'.x_{N+1})\Big)\mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1} \\
&= b_k t^{-b+(d_k+1)} \int_0^\infty \int_0^\infty \int_{S^{N-1}} (1+x_{N+1}^2)^{-\frac{b}{2}} g(rx',rx_{N+1}) \left(\frac{t}{r}\right)^{b-(d_k+1)} \\
&\qquad f\Big(\frac{t}{r}(x'.x_{N+1})\Big)\mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1},
\end{aligned}$$

where $f(t(x',x_{N+1})) = 1$ if $t^2 > (1+x_{N+1}^2)$. Since

$$\int_{\mathbb{R}^N} |y|^{-ap}|\widetilde{B}g(y)|^p \mathrm{d}\mu_k(y) = b_k \int_0^\infty \int_{S^{N-1}} t^{-ap+d_k}|\widetilde{B}g(ty')|^p\mathrm{d}\sigma_k(y')\frac{\mathrm{d}t}{t}$$

and

$$\begin{aligned}
&\int_0^\infty \int_{\mathbb{R}^N} |g(x,x_{N+1})|^p \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \\
&\qquad\qquad = b_k \int_0^\infty \int_0^\infty \int_{S^{N-1}} |g(rx',rx_{N+1})|^p\mathrm{d}\sigma_k(x') r^{d_k+1}\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}
\end{aligned}$$

replacing g by $|(x,x_{N+1})|^{-\frac{d_k+1}{p}} g$ we need to prove

$$\begin{aligned}
&\int_0^\infty \int_{S^{N-1}} t^{-ap+d_k}\Big|\widetilde{B}\Big(r^{-\frac{d_k+1}{p}}g\Big)(rx',rx_{N+1})\Big|^p\mathrm{d}\sigma_k(y')\frac{\mathrm{d}t}{t} \\
&\qquad\qquad \le C_k^V(a,b,p)\int_0^\infty\int_0^\infty\int_{S^{N-1}} |g(rx',rx_{N+1})|^p\mathrm{d}\sigma_k(x')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}.
\end{aligned}$$

We write

$$J = \int_0^\infty \int_{S^{N-1}} t^{-a+\frac{d_k}{p}}\widetilde{B}\Big(r^{-\frac{d_k+1}{p}}g\Big)(ty')h(ty')\mathrm{d}\sigma_k(y')\frac{\mathrm{d}t}{t}.$$

Using the relation $a+b = d_k + \frac{1}{p'}$

$$\begin{aligned}
J = b_k &\int_0^\infty\int_0^\infty\int_0^\infty\int_{S^{N-1}}\int_{S^{N-1}} (1+x_{N+1}^2)^{-\frac{b}{2}} g(rx',rx_{N+1})\left(\frac{t}{r}\right)^{b-\frac{d_k+1}{p'}} \\
&\qquad f\left(\frac{t}{r}(x'.x_{N+1})\right) h(ty')\mathrm{d}\sigma_k(x')\mathrm{d}\sigma_k(y')\frac{\mathrm{d}r}{r}\mathrm{d}x_{N+1}\frac{\mathrm{d}t}{t}.
\end{aligned}$$

Using the Lemma 2.1

$$
\begin{aligned}
J \leq b_k & \int_0^\infty (1+x_{N+1}^2)^{-\frac{b}{2}} \int_{S^{N-1}} \int_{S^{N-1}} \left(\int_0^\infty |h(ty')|^{p'} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p'}} \left(\int_0^\infty |g(rx', rx_{N+1})|^p \frac{\mathrm{d}r}{r} \right)^{\frac{1}{p}} \\
& \int_0^\infty |f(r(x', x_{N+1}))| r^{b-\frac{d_k+1)}{p'}} \frac{\mathrm{d}r}{r} \mathrm{d}\sigma_k(x') \mathrm{d}\sigma_k(y') \mathrm{d}x_{N+1} \\
\leq & \frac{b_k}{\frac{d_k+1}{p'} - b} \int_0^\infty (1+x_N^2)^{-\frac{d_k+1}{2p'}} \int_{S^{N-1}} \int_{S^{N-1}} \left(\int_0^\infty |g(rx', rx_{N+1})|^p \frac{\mathrm{d}r}{r} \right)^{\frac{1}{p}} \\
& \left(\int_0^\infty |h(ty')|^{p'} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p'}} \mathrm{d}\sigma_k(x') \mathrm{d}\sigma_k(y') \\
\leq & C_k^V(a,b,p) \left(\int_0^\infty \int_0^\infty \int_{S^{N-1}} |g(rx', rx_{N+1})|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \right)^{\frac{1}{p}} \\
& \left(\int_0^\infty \int_{S^{N-1}} |h(ty')|^{p'} \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p'}},
\end{aligned}
$$

where

$$
\begin{aligned}
C_k^V(a,b,p) &= \frac{b_k}{\frac{d_k+1}{p'} - b} \left(\int_0^\infty (1+x_N^2)^{-\frac{d_k+1}{2})} \right)^{\frac{1}{p'}} \\
&= \frac{b_k}{\frac{d_k+1}{p'} - b} \left(\frac{1}{2} \frac{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})}{\Gamma(\frac{d_k+1}{2})} \right)^{\frac{1}{p'}}.
\end{aligned}
$$

One can show the constant $C_k^V(a,b,p)$ is optimal by considering a function $g(x, x_{N+1})$, radial in x and then following the method used in [6]. If the inequality (2.2) holds then considering $g_t(x, x_{N+1}) = g(t(x, x_{N+1)})$ we can find the conditon $a + b = d_k + \frac{1}{p'}$. □

3. Weighted Hardy–Littlewood–Sobolev Inequality

In this section we will give a proof of the Theorem 1.5. In the classical case it was proved by Dou [4] using the weighted Hardy type inequality for upper half space. However we will reduce the inequality to a convolution inequality on $\mathbb{R}_+$ with Haar measure $\frac{\mathrm{d}r}{r}$ and apply the Lemma 2.1. For the case $p = q$ we are able to find the sharp constant in terms of integral involving hypergeometric function. We will closely follow the idea from [6]. First we will consider the case $p = q$ and then it will be used together with (2.1) and 2.2 to prove the case $1 < p < q$ by suitably decomposing the space $\mathbb{R}^N \times (0,\infty) \times \mathbb{R}^N$ into three regions.

Proof of Theorem 1.5

Proof of (1). First we will consider the case $1 < p = q$. In this case from the equation (1.5) we have $\frac{1}{p} = \gamma + \beta + (1 - \alpha)$. Let us consider the modified operator

$$\begin{aligned}\widetilde{\mathcal{E}_\alpha^k} f(rx', x_{n+1}) &= \int_{\mathbb{R}^N} f(y)|y|^{-\beta} \Phi_\alpha^k((x, x_{N+1}), y) \mathrm{d}\mu_k(y) \\ &= \int_0^\infty \int_{S^{N-1}} f(ty')t^{-\beta+d_k} \Phi_\alpha^k((rx', x_{N+1}), ty') \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t}.\end{aligned}$$

Then it is equivalent to prove the inequality of the form

$$\begin{aligned}&\left(\int_0^\infty \int_0^\infty (r^2 + x_{N+1}^2)^{-\frac{\gamma p}{2}} r^{d_k} \int_{S^{N-1}} |\widetilde{\mathcal{E}_\alpha^k} f(rx', x_{N+1})|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \mathrm{d}x_{N+1}\right)^{\frac{1}{p}} \\ &\qquad \leq C'(\alpha, \beta, \gamma, p, q, d_k) \left(\int_0^\infty \int_{S^{N-1}} |r^{d_k/p} f(rx')|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p}}.\end{aligned}$$

By a change variable x_{N+1} to rx_{N+1} and replacing f by $|\cdot|^{-\frac{d_k}{p}} f$, it is enough to show

$$\begin{aligned}&\left(\int_0^\infty \int_0^\infty (1 + x_{N+1}^2)^{-\frac{\gamma p}{2}} r^{-\gamma p + d_k + 1} \int_{S^{N-1}} \left|\widetilde{\mathcal{E}_\alpha^k}\left(r^{-\frac{d_k}{p}} f\right)(rx', rx_{N+1})\right|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \mathrm{d}x_{N+1}\right)^{\frac{1}{p}} \\ &\qquad \leq C'(\alpha, \beta, \gamma, p, q, d_k) \left(\int_0^\infty \int_{S^{N-1}} |f(rx')|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p}}.\end{aligned}$$

Recalling the fact that $\Phi_\alpha^k((rx', rx_{N+1}, ty') = t^{-d_k - 1 + \alpha} \Phi_\alpha^k((x', \frac{r}{t} x_{N+1}), \frac{r}{t} y')$ together with the relation $\gamma + \beta + 1 - \alpha = \frac{1}{p}$ we can express $\widetilde{\mathcal{E}_\alpha^k}(r^{-\frac{d_k}{p}} f)(rx', rx_{N+1})$ as an integral operator

$$\begin{aligned}&\widetilde{\mathcal{E}_\alpha^k}\left(r^{-\frac{d_k}{p}} f\right)(rx', rx_{N+1}) \\ &\qquad = b_k \int_0^\infty \int_{S^{N-1}} f(ty') t^{-\frac{d_k+1}{p} + \gamma} \Phi_\alpha^k\left(\left(x', \frac{r}{t} x_{N+1}\right), \frac{r}{t} y'\right) \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t} \\ &\qquad = b_k r^{\gamma - \frac{d_k+1}{p}} \int_0^\infty \int_{S^{N-1}} f(ty') \Phi_{\alpha,1}^k(r/t, (x', x_{N+1}), y') \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t},\end{aligned}$$

where

$$\Phi_{\alpha,1}^k(t, (x', x_{N+1}), y') = t^{-\gamma + \frac{d_k+1}{p}} \Phi_\alpha^k((x', tx_{N+1}), ty').$$

We set

$$J = b_k \int_0^\infty \int_0^\infty \int_{S^{N-1}} (1+x_{N+1}^2)^{-\frac{\gamma}{2}} r^{-\gamma+\frac{d_k+1}{p}} \widetilde{\mathcal{E}_\alpha^k}(r^{-\frac{d_k}{p}} f)(rx', rx_{N+1})$$

$$h(rx', x_{N+1}) \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \mathrm{d}x_{N+1}.$$

Our aim is to show

$$|J| \le C'(\alpha,\beta,\gamma,p,p,d_k) \left(\int_0^\infty \int_{S^{N-1}} |f(rx')|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p}}$$

$$\left(\int_0^\infty \int_0^\infty \int_{S^{N-1}} |h(rx', x_{N+1})|^p \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p'}},$$

where the constant $C'(\alpha,\beta,\gamma,p,p,d_k)$ will be specified latter. Using the Lemma 2.1 we obtain

$$|J| \le \int_0^\infty \int_{S^{N-1}} \int_{S^{N-1}} (1+x_{N+1}^2)^{-\frac{\gamma}{2}} \left(\int_0^\infty |h(rx', x_{N+1})|^{p'} \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p'}} \left(\int_0^\infty |f(ty')|^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}$$

$$\int_0^\infty \Phi_{\alpha,1}^k((t,(x',x_{N+1})),y') \frac{\mathrm{d}t}{t} \mathrm{d}\sigma_k(x') \mathrm{d}\sigma_k(y') \mathrm{d}x_{N+1}.$$

Let

$$\begin{aligned}
\widetilde{\Phi}_{\alpha,0}^k(1,x_{N+1}) &= \int_{S^{N-1}} \int_0^\infty \Phi_{\alpha,1}^k((t,(x',x_{N+1})),y') \frac{\mathrm{d}t}{t} \mathrm{d}\sigma_k(x') \\
&= \int_{S^{N-1}} \int_0^\infty \Phi_{\alpha,1}^k((t,(x',x_{N+1})),y') \frac{\mathrm{d}t}{t} \mathrm{d}\sigma_k(y') \\
&= \int_0^\infty \Phi_{\alpha,0}^k((1,x_{N+1}),t) t^{-\gamma+\frac{d_k+1}{p}} \frac{\mathrm{d}t}{t}. \qquad (3.1)
\end{aligned}$$

From the equation (1.7) and the fact $\frac{1}{p'} + \frac{1}{p} = 1$ we can write

$$|J| \le b_k \int_0^\infty (1+x_{N+1}^2)^{-\frac{\gamma}{2}} \widetilde{\Phi}_{\alpha,0}^k(1,x_{N+1}) \left(\int_0^\infty \int_{S^{N-1}} |h(rx', x_{N+1})|^{p'} \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r}\right)^{\frac{1}{p'}} \mathrm{d}x_{N+1}$$

$$\left(\int_0^\infty \int_{S^{N-1}} |f(ty')|^p \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}$$

$$\le C'(\alpha,\beta,\gamma,p,p,d_k) \left(\int_0^\infty \int_0^\infty \int_{S^{N-1}} |h(rx', x_{N+1})|^{p'} \mathrm{d}\sigma_k(x') \frac{\mathrm{d}r}{r} \mathrm{d}x_{N+1}\right)^{\frac{1}{p'}}$$

$$\left(\int_0^\infty \int_{S^{N-1}} |f(ty')|^p \mathrm{d}\sigma_k(y') \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}},$$

where $C'(\alpha,\beta,\gamma,p,p,d_k) = b_k\left(\int_0^\infty (1+x_{N+1}^2)^{-\gamma\frac{p}{2}}(\widetilde{\Phi}^k_{\alpha,0}(1,x_{N+1}))^p \mathrm{d}x_{N+1}\right)^{\frac{1}{p}} < \infty$ which follows from the Remark 4.2 in the appendix. Thus it follows from Lemma 4.1 in the Appendix and the definition of $\mathcal{E}^k_\alpha$

$$\||\cdot|^{-\gamma}\mathcal{E}^k_\alpha f\|_{p,\mathrm{d}\mu_k\mathrm{d}x_{N+1}} \leq C(\alpha,\beta,\gamma,p,p,d_k)\||\cdot|^\beta f\|_{p,\mathrm{d}\mu_k},$$

with

$$\begin{aligned} &C(\alpha,\beta,\gamma,p,p,d_k)\\ &\quad= \frac{2^{1-\alpha}}{\Gamma(\frac{\alpha-1}{2})}\left(\int_0^\infty (1+x_{N+1}^2)^{-\frac{d_k+1}{2}}|F(a',b',c';(1+x_{N+1}^2)^{-1})|^p \mathrm{d}x_{N+1}\right)^{\frac{1}{p}}. \end{aligned}$$

One can show that the constant $C(\alpha,\beta,\gamma,p,p,d_k)$ is optimal by considering a function $f(x)$ which is radial and then following the argument used in [6].

Case: $1 < p < q < \infty$. We recall that

$$\widetilde{\mathcal{E}^k_\alpha} f(x) = \int_{\mathbb{R}^N} f(y)\Phi^k_\alpha((x,x_{N+1}),y)\mathrm{d}\mu_k(y).$$

Here the kernel $\Phi^k_\alpha((x,x_{N+1}),y)$ has the following expression

$$\Phi^k_\alpha((x,x_{N+1}),y) = \frac{1}{\Gamma(\frac{d_k+\alpha-1}{2})}\int_{\mathbb{R}^N}(|x|^2+x_N^2+|y|^2-2\langle y,\eta\rangle)^{\frac{\alpha-1-d_k}{2}}\mathrm{d}\mu^k_x(\eta),$$

and μ^k_x is the probability measure with $\operatorname{supp}\mu^k_x \subset \{\eta : |\eta| \leq |x|\}$. We define

$$J^k_\alpha = \int_0^\infty\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{f(y)g(x,x_{N+1})\Phi^k_\alpha((x,x_{N+1}),y)}{|(x,x_{N+1})|^\gamma|y|^\beta}\mathrm{d}\mu_k(y)\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}.$$

Then it is sufficient to show

$$J^k_\alpha \lesssim \|f\|_p\|g\|_{q'} \tag{3.2}$$

for $f \in \mathcal{S}(\mathbb{R}^N), g \in \mathcal{S}(\mathbb{R}^N\times(0,\infty))$ with $f,g \geq 0$. Note that for $\gamma = 0, \beta = 0$ and α satisfying $\frac{1}{q} = \frac{d_k}{d_k+1}(\frac{1}{p} - \frac{(\alpha-1)}{d_k})$ the inequality (3.2) holds by Theorem 1.4. Essentially following the idea of Stein [8], the domain $\mathbb{R}^N\times(0,\infty)\times\mathbb{R}^N$ is decomposed into three domains F_1, F_2, F_3 and the integral on each domain is estimated to apply the Theorem 1.4. Let $\mathbb{R}^N\times(0,\infty)\times\mathbb{R}^N = F_1\cup F_2\cup F_3$, where

$$F_1 = \left\{((x,x_{N+1}),y) : \frac{1}{2}|y| < |(x,x_{N+1})| < 2|y|\right\},$$

$$F_2 = \left\{((x,x_{N+1}),y) : |(x,x_{N+1})| < \frac{1}{2}|y|\right\}$$

$$F_3 = \left\{((x,x_{N+1}),y) : |y| < \frac{1}{2}|(x,x_{N+1})|\right\}.$$

Then $J_\alpha^k = \sum_{j=1}^3 J_{\alpha,i}^k$, where

$$J_{\alpha,i}^k = \int_{F_i} \frac{f(y)g(x,x_{N+1})\Phi_\alpha^k((x,x_{N+1}),y)}{|(x,x_{N+1})|^\gamma |y|^\beta} \mathrm{d}\mu_k(y)\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}.$$

Estimate for $J_{\alpha,1}^k$. For $(x, x_{N+1}) \in F_1$ and $|\eta| \leq |x|$

$$(|x|^2 + x_{N+1}^2 + |y|^2 - 2\langle y,\eta\rangle)^{\frac{\gamma+\beta}{2}} \lesssim |(x,x_{N+1})|^\gamma |y|^\beta$$

and hence

$$\frac{\Phi_\alpha^k((x,x_{N+1}),y)}{|(x,x_{N+1})|^\gamma |y|^\beta} \lesssim \Phi_{\tilde{\alpha}}^k((x,x_{N+1}),y),$$

where $\widetilde{\alpha} = \alpha - \gamma - \beta$. From the relation (1.5) $1 < \widetilde{\alpha} < d_k + 1$ and $\frac{1}{q} = \frac{d_k}{d_k+1}(\frac{1}{p} - \frac{\tilde{\alpha}-1}{d_k})$. Therefore, we find the desired inequality

$$J_{\alpha,1}^k \lesssim \|f\|_{p,\mathrm{d}\mu_k} \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}.$$

by applying the Theorem 1.4.

Estimate for $J_{\alpha,2}^k$. For $((x,x_{N+1}),y) \in F_2$

$$\Phi_\alpha^k((x,x_{N+1}),y) \lesssim |y|^{\alpha-1-d_k}.$$

Since $F_2 \subset \{((x,x_{N+1}),y) : |(x,x_{N+1})| \leq |y|\}$

$$\begin{aligned}
J_{\alpha_2}^k &\lesssim \int_{\mathbb{R}^N} \int_{|(x,x_{N+1})|\leq|y|} \frac{f(y)g(x,x_{N+1})}{|(x,x_{N+1})|^\gamma |y|^{\beta-\alpha+1+d_k}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\mathrm{d}\mu_k(y) \\
&\lesssim \int_{\mathbb{R}^N} f(y)|y|^{\alpha-1-\beta-\gamma+\frac{1}{q}} \widetilde{V}g(y)\mathrm{d}\mu_k(y) \\
&\lesssim \|f\|_p \left(\int_{\mathbb{R}^N} |y|^{p'(\alpha-1-\beta-\gamma+\frac{1}{q})} |\widetilde{V}g(y)|^{p'-q'} |\widetilde{V}g(y)|^{q'}\right)^{\frac{1}{p'}},
\end{aligned}$$

where

$$\widetilde{V}g(y) = |y|^{\gamma-(d_k+\frac{1}{q})} \int_{|(x,x_{N+1})|\leq|y|} \frac{g(x,x_{N+1})}{|(x,x_{N+1})|^\gamma} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}.$$

A straight forward calculation shows that

$$\int_{|(x,x_{N+1})|\leq|y|} |(x,x_{N+1})|^{-\gamma q} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \lesssim |y|^{-\gamma q+(d_k+1)}$$

and hence

$$|\widetilde{V}g(y)| \lesssim |y|^{-\frac{d_k}{q'}} \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}. \tag{3.3}$$

Taking $b = \gamma$, $a = d + k + \frac{1}{q} - \gamma$ in the Theorem 2.3 we deduce

$$\|\widetilde{V}g\|_{q',\mathrm{d}\mu_k} \lesssim \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}. \tag{3.4}$$

From the relation (1.5), inequality (3.3) and (3.4) we have for $p > 1$

$$
\begin{aligned}
J^k_{\alpha,2} &\lesssim \|f\|_{p,\mathrm{d}\mu_k} \|g\|^{1-\frac{q'}{p'}}_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \|\widetilde{V}g\|^{\frac{q'}{p'}}_{q',\mathrm{d}\mu_k} \\
&\lesssim \|f\|_{p,\mathrm{d}\mu_k} \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}.
\end{aligned}
$$

For $p = 1$, $\gamma < \frac{d_k+1}{q}$

$$
\begin{aligned}
J^k_{\alpha,2} &\lesssim \int_{\mathbb{R}^N} |y|^{\alpha-1-\beta-d_k} f(y) \int_{|(x,x_{N+1})|\le|y|} \frac{g(x,x_{N+1})}{|(x,x_{N+1})|^{\gamma}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1}\mathrm{d}\mu_k(y) \\
&\lesssim \int_{\mathbb{R}^N} f(y)|y|^{\alpha-1-\beta-d_k-\gamma+\frac{d_k+1}{q}} \mathrm{d}\mu_k(y)\|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} = \|f\|_{1,\mathrm{d}\mu_k}\|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}.
\end{aligned}
$$

Estimate for $J^k_{\alpha,3}$. If $((x,x_{N+1}),y) \in F_3$ we have

$$
\sqrt{|x|^2+|y|^2-2\langle y,\eta\rangle} \ge \frac{1}{2}|(x,x_{N+1})|
$$

and hence

$$
\Phi^k_\alpha((x,x_{N+1}),y) \lesssim |(x,x_{N+1})|^{\alpha-1-d_k}.
$$

Therefore,

$$
\begin{aligned}
J^k_{\alpha,3} &\lesssim \int_0^\infty \int_{\mathbb{R}^N} \int_{|y|\le|(x,x_{N+1})|} \frac{f(y)g(x,x_{N+1})}{|(x,x_{N+1})|^{\gamma-\alpha+1+d_k}|y|^{\beta}} \mathrm{d}\mu_k(y)\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \\
&= \int_0^\infty \int_{\mathbb{R}^N} g(x,x_{N+1})|(x,x_{N+1})|^{\alpha-1-\beta-\gamma+\frac{1}{p}} \widetilde{W}f(x,x_{N+1})\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1},
\end{aligned}
\tag{3.5}
$$

where

$$
\widetilde{W}f(x,x_{N+1}) = |(x,x_{N+1})|^{\beta-d_k-\frac{1}{p}} \int_{|y|\le|(x,x_{N+1})|} |y|^{-\beta} f(y)\mathrm{d}\mu_k(y).
$$

Applying Hölder's inequality and the fact that

$$
\int_{|y|\le|(x,x_{N+1})|} |y|^{-\beta p'}\mathrm{d}\mu_k(y) = |(x,x_{n+1})|^{-\beta p'+d_k}
$$

we obtain

$$
\widetilde{W}f(x,x_{N+1}) \lesssim |(x,x_{N+1})|^{-\frac{d_k+1}{p}} \|f\|_{p,\mathrm{d}\mu_k}. \tag{3.6}
$$

Applying the Theorem 2.1 with $b = \beta$ and $a = d_k + \frac{1}{p} - \beta$

$$
\|\widetilde{W}f\|_{p,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \lesssim \|f\|_{p,\mathrm{d}\mu_k}. \tag{3.7}
$$

Taking into account the relation (1.5) and the inequality (3.6) we get

$$
\begin{aligned}
|(x,x_{N+1})|^{q(\alpha-1-\beta-\gamma+\frac{1}{p})}|\widetilde{W}f|^q &= |(x,x_{N+1})|^{q(\alpha-1-\beta-\gamma+\frac{1}{p})}|\widetilde{W}f|^{q-p}|\widetilde{W}f|^p \\
&\lesssim \|f\|_p^{q-p}|\widetilde{W}f|^p.
\end{aligned}
\tag{3.8}
$$

Now combining (3.7) and (3.8) and applying Hölder's inequality to (3.5) for $p > 1$ we obtain

$$J^k_{\alpha,3} \lesssim \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \|f\|_{p,\mathrm{d}\mu_k}.$$

For $p = 1$, we find $\beta \le 0$. First consider the case $\beta < 0$. Using the fact $\beta < 0$ from the definition of $\widetilde{W} f$ for $p = 1$ we obtain

$$\widetilde{W} f(x, x_{N+1}) \lesssim |(x, x_{N+1})|^{-(d_k+1)} \|f\|_{1,\mathrm{d}\mu_k} \tag{3.9}$$

and

$$\|\widetilde{W} f\|_{1,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \lesssim \|f\|_{1,\mathrm{d}\mu_k}. \tag{3.10}$$

Taking into account the relation (1.5) for $p = 1$ and the inequality (3.9) we get

$$\begin{aligned} |(x, x_{N+1})|^{q(\alpha-\beta-\gamma)} |\widetilde{W} f|^q &= |(x, x_{N+1})|^{q(\alpha-\beta-\gamma)} |\widetilde{W} f|^{q-1} |\widetilde{W} f| \\ &\lesssim \|f\|^{q-1}_{1,\mathrm{d}\mu_k} |\widetilde{W} f|. \end{aligned} \tag{3.11}$$

Therefore by Hölder's inequality,

$$J^k_{\alpha,3} \lesssim \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \|f\|^{1-\frac{1}{q}}_{1,\mathrm{d}\mu_k} \|W f\|_{1,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \lesssim \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \|f\|_{1,\mathrm{d}\mu_k}.$$

For $\beta = 0$ from the relation (1.5) $\alpha - \gamma = \frac{d_k+1}{q'}$ and proceeding as before we can show (3.11) holds with $\beta = 0$ and hence the desired inequality $J^k_{\alpha,3} \lesssim \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \|f\|_{1,\mathrm{d}\mu_k}$.

Proof of (2). It is sufficient to prove that for any $\lambda > 0$

$$S_\alpha = \int_{\{(x,x_{N_1}):|(x,x_{N+1})|^{-\gamma} \tilde{\mathcal{E}}^k_\alpha f\,(x,x_{N+1}) \ge \lambda\}} \mathrm{d}\mu_k(x) \mathrm{d}x_{N+1} \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{\lambda} \right)^q$$

for $f \in \mathcal{S}(\mathbb{R}^N)$. We define the operators

$$A^i_\alpha f(x, x_{N+1}) = \int_{\mathbb{R}^N} f(y)|y|^{-\beta} \Phi^k_\alpha((x, x_{N+1}), y) \chi_{F_i}((x, x_{N+1}), y) \mathrm{d}\mu_k(y)$$

for $i = 1, 2, 3$. Noting the fact $\widetilde{\mathcal{E}^k_\alpha} = \sum_{i=1}^3 A^i_\alpha$ we get
$S_\alpha = \sum_{i=1}^3 S_{\alpha,i}$, where

$$S_{\alpha,i} = \int_{\{(x,x_{N_1}):|(x,x_{N+1})|^{-\gamma} A^i_\alpha f\,(x,x_{N+1}) \ge \lambda/3\}} \mathrm{d}\mu_k(x) \mathrm{d}x_{N+1}.$$

Now we will estimate each of $S_{\alpha,i}$ for $i = 1, 2, 3$.

Estimate of $S_{\alpha,1}$. Proceeding as in the estimate of $J^k_{\alpha,1}$, we deduce

$$\begin{aligned} &|(x, x_{N+1})|^{-\gamma} A_{\alpha,1} f(x, x_{N+1}) \\ &\qquad \lesssim \int_{\mathbb{R}^N} f(y)|y|^{-\beta} \Phi^k_{\tilde{\alpha}}((x, x_{N+1}), y) \mathrm{d}\mu_k(y) = E^k_{\tilde{\alpha}} f(x, x_{N+1}) \end{aligned}$$

with $\widetilde{\alpha}$ satisfying $\frac{1}{q} = \frac{d_k}{d_k+1}(1 - \frac{\tilde{\alpha}-1}{d_k})$. Now it follows $S_{\alpha,1} \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^{\frac{1}{q}}$ from the fact that $f \mapsto E^k_\alpha f$ is weak $(1, q)$ (Theorem 1.4).

Estimate of $S_{\alpha,2}$. Applying the estimate obtained in $J^k_{\alpha,2}$ we obtain

$$A_{\alpha,2} f(x, x_{N+1}) \lesssim \int_{|y| \geq |(x,x_{N+1})|} |y|^{\alpha-1-\beta-d_k} f(y)\mathrm{d}\mu_k(y) := V_1 f(x, x_{N+1}).$$

From the calculations performed in $J^k_{\alpha,2}$ with $p = 1$, $\beta \leq 0$, $\gamma < \frac{d_k+1}{q}$ we obtain

$$\int_0^\infty \int_{\mathbb{R}^N} |(x, x_{N+1})|^{-\gamma} V_1 f(x, x_{N+1}) g(x, x_{N+1})\mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \lesssim \|f\|_{1,\mathrm{d}\mu_k} \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}$$

and hence

$$\||(x, x_{N+1})|^{-\gamma} V_1 f\|_{q,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \lesssim \|f\|_{1,\mathrm{d}\mu_k}.$$

Therefore,

$$S_{\alpha,2} \lesssim \int_{\{(x,x_{N+1}):|(x,x_{N+1})|^{-\gamma} V_1 f\,(x,x_{N+1}) \geq \lambda\}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^{\frac{1}{q}}.$$

Estimate of $S_{\alpha,3}$. Applying the estimate obtained $J_{\alpha,3}$ we deduce

$$A_{\alpha,3} f(x, x_{N+1}) \lesssim |(x, x_{N+1})|^{\alpha-1-d_k} \int_{|y| \leq |(x,x_{N+1})|} |y|^{-\beta} f(y)\mathrm{d}\mu_k(y) := W_1 f(x, x_{N+1}).$$

For $p = 1$ we note that $\gamma < \frac{d_k}{q}$ and $\beta < 0$. Again from the calculation in obtaining the estimate $J_{\alpha,3}$ for the case $p = 1$

$$\int_{\mathbb{R}^N} \int_0^\infty |(x, x_{N+1})|^{-\gamma} |W_1 f(x, x_{N+1})| g(x, x_{N+1}) \leq \|f\|_{1,\mathrm{d}\mu_k} \|g\|_{q',\mathrm{d}\mu_k \mathrm{d}x_{N+1}}$$

and hence we obtain $\||(x, x_{N+1})|^{-\gamma} W_1 f\|_{q,\mathrm{d}\mu_k \mathrm{d}x_{N+1}} \leq \|f\|_{1,\mathrm{d}\mu_k}$. Therefore,

$$S_{\alpha,3} \lesssim \int_{\{(x,x_{N+1}):|(x,x_{N+1})|^{-\gamma} W_1 f\,(x,x_{N+1}) \geq \lambda\}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^{\frac{1}{q}}.$$

If $\beta = 0$ then from the relation (1.5) $\alpha - \gamma = \frac{d_k+1}{q'}$ and

$$\begin{aligned} |(x, x_{N+1})|^{-\gamma} W_1 f(x, x_{N+1}) &= |(x, x_{N+1})|^{-\frac{d_k+1}{q}} \int_{|y| \leq |(x,x_{N+1})|} f(y)\mathrm{d}\mu_k(y) \\ &\leq |(x, x_{N+1})|^{-\frac{d_k+1}{q}} \|f\|_{1\mathrm{d}\mu_k}. \end{aligned}$$

Therefore,

$$S_{\alpha,3} \lesssim \int_{\{(x,x_{N+1}):|(x,x_{N+1})|^{-\frac{d_k+1}{q}} \|f\|_{1,\mathrm{d}\mu_k} \geq \lambda\}} \mathrm{d}\mu_k(x)\mathrm{d}x_{N+1} \lesssim \left(\frac{\|f\|_{1,\mathrm{d}\mu_k}}{\lambda}\right)^{q}.$$

Hence the proof is completed. □

4. Appendix

We will express the function $\widetilde{\Phi}^k_{\alpha,0}(1, x_{N+1})$ in terms of hypergeometric function [1, Chapter II] defined by

$$F(a', b', c'; z) = \sum_{m=0}^{\infty} \frac{\Gamma(a'+m)\Gamma(b'+m)}{\Gamma(1+m)\Gamma(c'+m)} z^k.$$

Lemma 4.1.

$$\widetilde{\Phi}^k_{\alpha,0}(1, x_{N+1}) = \frac{1}{2} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k+1-\alpha}{2})} (1 + x_{N+1}^2)^{\frac{1}{2}(\gamma - \frac{d_k+1}{p})} F(a', b', c', (1 + x_{N+1}^2)^{-1}),$$

where F is a hypergeometric function with $a' = -\frac{\gamma}{2} + \frac{d_k+1}{2p}$, $b' = \frac{\gamma}{2} - \frac{\alpha}{2} + \frac{d_k+1}{2p}$, $c' = \frac{d_k}{2}$.

Proof. Write $b = \frac{d_k+1-\alpha}{2}$, $l = d_k - 2$, then we can write

$$\begin{aligned}
&\widetilde{\Phi}^k_{\alpha,0}(1, x_{N+1}) \\
&= \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\infty \int_0^\pi t^{-\gamma+\frac{d_k+1}{p}} (t^2 a^2 + 1 - 2t\cos\phi)^{-b} \sin^l \phi \mathrm{d}\phi \frac{\mathrm{d}t}{t} \\
&= \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\infty \int_0^\pi t^{-\gamma+\frac{d_k+1}{p}} \left(t^2 + 1 - 2\frac{t}{a}\cos\phi\right)^{-b} \sin^l \phi \mathrm{d}\phi \frac{\mathrm{d}t}{t} \\
&= a^{\gamma - \frac{d_k+1}{p}} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \int_0^\infty \int_0^\pi (1+t^2)^{-b} t^{-\gamma+\frac{d_k+1}{p}} \left(1 - \frac{2t}{a(1+t^2)}\cos\phi\right)^{-b} \sin^l \phi \mathrm{d}\phi \frac{\mathrm{d}t}{t}.
\end{aligned}$$

The integral with respect to t has singularity at $t = 0, 1, \infty$. The integral converges at the origin and ∞ if and only if $\gamma < \frac{d_k+1}{p}$ and $\gamma + \frac{d_k+1}{p} - \alpha > 0$ which is equivalent to $\beta < \frac{d_k}{p'}$ by the relation (1.5). When $a = 1$ that is $x_{N+1} = 0$ the integral has singularity at $t = 1$ and is integrable at the singular point $t = 1$ (see [6]).

We observe that if m is odd and $l > 0$, $\int_0^\pi \cos^m\theta \sin^l \theta d\theta = 0$ where as for m even and $l > 0$

$$\begin{aligned}
\int_0^\pi \cos^m\phi \sin^l \phi \mathrm{d}\phi &= 2\int_0^{\pi/2} \cos^m\phi \sin^l \phi \mathrm{d}\phi \\
&= \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(m+\frac{l}{2}+1)}.
\end{aligned}$$

Now using the expansion of $(1 - r\cos\phi)^{-b}$ for $r < 1$

$$\int_0^\pi (1 - r\cos\phi)^{-b} \sin^l \phi \mathrm{d}\phi = \frac{1}{\Gamma(b)} \sum_{m=0}^{\infty} \frac{\Gamma(b+2m)}{\Gamma(2m+1)} \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(m+\frac{l}{2}+1)} r^{2m}.$$

This identity leads to conclude

$$\begin{aligned}
&\widetilde{\Phi}^k_{\alpha,0}(1, x_{N+1})\\
&\quad = a^{\gamma - \frac{d_k+1}{p}} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \frac{1}{\Gamma(b)} \int_0^\infty \sum_0^\infty 2^{2m} a^{-2m} \frac{\Gamma(b+2m)}{\Gamma(2m+1)} \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(m+\frac{l}{2}+1)}\\
&\qquad \int_0^\infty t^{-\gamma+\frac{d_k+1}{p}+2m-1}(1+t^2)^{-(b+2m)}\mathrm{d}t\\
&\quad = a^{\gamma - \frac{d_k+1}{p}} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k-1}{2})} \frac{\Gamma(\frac{l+1}{2})}{\Gamma(b)} \sum_0^\infty 2^{2m} a^{-2m} \frac{\Gamma(b+2m)}{\Gamma(2m+1)} \frac{\Gamma(m+\frac{1}{2})\Gamma(\frac{l+1}{2})}{\Gamma(m+\frac{l}{2}+1)}\\
&\qquad \frac{\Gamma(m - \frac{\gamma}{2} + \frac{d_k+1}{2p})\Gamma(m + \frac{\gamma}{2} + b - \frac{d_k+1}{2p})}{\Gamma(b+2m)}\\
&\quad = \frac{1}{2} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k+1-\alpha}{2})} a^{\gamma - \frac{d_k+1}{p}} \sum_0^\infty \frac{\Gamma(m + \frac{d_k+1}{2p} - \frac{\gamma}{2})\Gamma(m + \frac{d_k+1}{2p'} + \frac{\gamma}{2} - \frac{\alpha}{2})}{\Gamma(m+1)\Gamma(m+\frac{d_k}{2})} a^{-2m}\\
&\quad = \frac{1}{2} \frac{\Gamma(\frac{d_k}{2})}{\Gamma(1/2)\Gamma(\frac{d_k+1-\alpha}{2})} (1 + x_{N+1}^2)^{\frac{1}{2}(\gamma - \frac{d_k+1}{p})} F(a', b', c'; (1 + x_{N+1}^2)^{-1}),
\end{aligned}$$

where $a' = \frac{d_k+1}{2p} - \frac{\gamma}{2}$, $b' = \frac{d_k+1}{2p} + \frac{\gamma}{2} - \frac{\alpha}{2}$, $c' = \frac{d_k}{2}$.

Thus we obtain

$$\begin{aligned}
&b_k(\gamma_k^{\alpha-1})^{-1}\widetilde{\Phi}^k_{\alpha,0}(1, x_{N+1})\\
&\qquad = \frac{2^{1-\alpha}}{\Gamma(\frac{\alpha-1}{2})} (1 + x_{N+1}^2)^{\frac{1}{2}(\gamma - \frac{d_k+1}{p})} F(a', b', c'; (1 + x_{N+1}^2)^{-1}). \qquad \square
\end{aligned}$$

Remark 4.2. We remark that $a' + b' - c' = \frac{1-\alpha}{2} < 0$, and hence $F(a', b', c'; (1+x_{N+1}^2)^{-1})$ is a bounded function and $\int_0^\infty \left((1+x_{N+1}^2)^{-\gamma/2}\widetilde{\Phi}_{\alpha,0}^k(1, x_{N+1})\right)^p \, \mathrm{d}x_{N+1} < \infty$.

Acknowledgements

We would like to thank the referee for their remarks and comments which helped improving the quality of the article.

References

[1] Harry Bateman, Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher Transcendental Functions*. Macgraw Hill Book Company, 1953.

[2] Jingbo DDou and Meijun Zhu. Sharp Hardy-Littlewood-Sobolev inequality on the upper half space. *Int. Math. Res. Not.*, 2015(3):651–687, 2015.

[3] Luc Deleaval. Two results on the Dunkl maximal operator. *Stud. Math.*, 203(1):47–68, 2011.

[4] Jingbo Dou. Weighted Hardy–Littlewood–Sobolev inequalities on the upper half space. *Commun. Contemp. Math.*, 18(5): article no. 1550067 (20 pages), 2016.

[5] Dmitriĭ V. Gorbachev, Valeriĭ I. Ivanov, and Sergey Yu. Tikhonov. Positive L^p-bounded Dunkl-type generalized translation operator and its applications. *Constr. Approx.*, 49:555–605, 2019.

[6] Dmitriĭ V. Gorbachev, Valeriĭ I. Ivanov, and Sergey Yu. Tikhonov. Riesz potential and maximal function for Dunkl transform. *Potential Anal.*, 55:513–538, 2021.

[7] Margit Rösler. Dunkl operators: theory and applications. In *Orthogonal Polynomials and Special Functions*, volume 1817 of *Lecture Notes in Mathematics*. Springer, 2003.

[8] Elias M. Stein and Guido Weiss. Fractional integrals on $n-$dimensional Euclidean space. *J. Math. Mech.*, 7:503–514, 1958.

[9] Sundaram Thangavelu and Yuan Xu. Convolution operator and maximal function for the Dunkl transform. *J. Anal. Math.*, 97:25–55, 2005.

[10] Sundaram Thangavelu and Yuan Xu. Riesz transform and Riesz potentials for Dunkl transform. *J. Comput. Appl. Math.*, 199:181–195, 2007.

V. P. Anoop
Department of Mathematics
Indian Institute of Science, Bangalore
India, 560012.
anoopvp89@gmail.com

Sanjay Parui
School of Mathematical Sciences
National Institute of Science Education
and Research, Bhubaneswar
India, 752050.
and
Homi Bhabha National Institute,
Training School Complex, Anushakti Nagar,
Mumbai, India, 400094.
parui@niser.ac.in