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Localized calculus for the Hecke category

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Abstract

We construct a functor from the Hecke category to a groupoid built from the underlying Coxeter group. This fixes a gap in an earlier work of the authors. This functor provides an abstract realization of the localization of the Hecke category at the field of fractions. Knowing explicit formulas for the localization is a key technical tool in software for computations with Soergel bimodules.

Calcul localisé pour la catégorie de Hecke

Résumé

Nous construisons un foncteur de la catégorie de Hecke vers un groupoïde construit à partir du groupe de Coxeter sous-jacent. Cette construction corrige une lacune dans un travail antérieur des auteurs. Ce foncteur fournit une réalisation abstraite de la localisation de la catégorie de Hecke en le corps des fractions. Connaître des formules explicites pour la localisation est un outil technique clé pour le calcul algorithmique avec les bimodules de Soergel.

1. Introduction

Soergel bimodules provide a concrete incarnation of the Hecke category associated to any Coxeter group. This category is a fundamental object in geometric representation theory. In [17] the authors provided a diagrammatic (or “generators and relations”) description of Soergel bimodules, building on earlier work of Libedinsky [24, 25], Elias–Khovanov [11] and the first author [8]. This description has provided new impetus for the algebraic study of the Hecke category, and has had significant applications in modular and higher representation theory (see e.g. [5, 7, 10, 14, 20, 22, 30, 33]). Abe has also introduced another more algebraic incarnation of the Hecke category [3], which is equivalent to the diagrammatic category in many situations, and has already found important applications in representation theory [2, 6]. The reader is referred to the introduction to [17] and [34] for a discussion of the origins of the Hecke algebra and category, as well as its significance in geometric and higher representation theory.

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Soergel bimodules are a full monoidal subcategory of graded bimodules over a polynomial ring R . A key tool in the theory is provided by localization, which embeds Soergel bimodules inside a much simpler monoidal category, a subcategory of bimodules over the quotient field Q of R which we call the “ Q -groupoid” associated to the underlying Coxeter group. (In geometric settings, this technique is related to the localization theorem in equivariant cohomology.) After base change from R to Q , any Soergel bimodule splits a direct sum of (grading shifts of) objects in the Q -groupoid.

In [17] this idea was imported into the diagrammatic study, and provided a key tool in the proofs. First, in [18] we provided a diagrammatic description of this Q -groupoid by generators and relations, using the topology of the Coxeter complex to prove the correctness of our presentation. In [17] we provided a diagrammatic description of a “mixed calculus” which combined the diagrammatics of the Hecke category and the Q -groupoid. This mixed category is tailor-made to be simultaneously equivalent to both

- the Karoubi envelope of the diagrammatic Hecke category, after base change from R to Q , and
- the additive envelope of the diagrammatic Q -groupoid.

Combining these equivalences, we obtain an implicit construction of a functor, which we denote Λ in this paper, from the diagrammatic Hecke category to the additive envelope of the Q -groupoid. The main goal of this paper is to make this “localization functor” Λ explicit.

As kindly pointed out to us by Simon Riche, the treatment of localization in [17] contains a gap as written. We constructed a full and essentially surjective functor, denoted Std in [17], from the additive closure of the Q -groupoid to our mixed calculus, but the faithfulness of this functor was not properly justified. The faithfulness of Std is what demonstrates that the mixed calculus, and hence the diagrammatic Hecke category, is nonzero. We gave two separate arguments in [17] that the functor Std is faithful. The first (see the end of [17, §5.4]) relies on the existence of a functor \mathcal{F} from the diagrammatic Hecke category to Soergel bimodules. We made a separate error here (in [17, Claim 5.13]), claiming to construct the functor \mathcal{F} in more generality than we actually did. Regardless, our intention was to study the diagrammatic Hecke category in additional generality, where such a functor \mathcal{F} might not exist. Consequently we wrote [17, §5.5, in particular Proposition 5.25], where we sketched a method for constructing the localization functor Λ explicitly. In turn, the localization functor would induce a functor from the mixed calculus which would be quasi-inverse to Std , thus proving its faithfulness. Computer calculations of the second author verified this method for (most realizations of) those Coxeter groups which satisfy $m_{st} \leq 4$ for all simple reflections s, t . However, it was a major oversight to

think that the general case would be trivial! This entire paper is a construction of Λ , and thereby completes the sketch of [17, Proposition 5.25].

Remark 1.1. Between the argument using \mathcal{F} and the computer calculations made, the proofs in [17] are sufficient to deal with many versions of the Hecke category (e.g. coming from different realizations of the Coxeter group), and in particular, are sufficient for all of the applications to date. However, our intention in [17] was to work in the broadest reasonable generality, and this paper allows us to. For more information on the errors in [17], and a summary of the assumptions now required to quote various theorems, see Section 5.

Remark 1.2. The target of the localization functor of [17] was to be the additive envelope of the diagrammatic version of the Q -groupoid, as constructed in [18] using the topology of the Coxeter complex. In this paper, the target of our localization functor Λ is the algebraic Q -groupoid, defined below in Section 1.1. Originally, it was the topological proof of [18] that formed the bedrock upon which the well-definedness of the Hecke category was established. This paper removes the dependency on topology.

Remark 1.3. Throughout this paper, we assume our Coxeter system has no parabolic subgroup of type H_3 . The diagrammatic Hecke category is not defined in type H_3 , because one crucial relation (the “Zamolodchikov relation”) is missing. Further discussion on type H_3 can be found in Section 3.6.

A secondary purpose of this paper is the following: when writing [17] the authors viewed localization primarily as a theoretical, rather than computational, tool. However subsequently there have been several instances where it has been necessary to understand the localization of the diagrammatic category in concrete terms. An important example is software developed by the second author and Thorge Jensen for efficient computation of the p -canonical basis of Hecke algebras (see [19, 23] for details, and [27] for an application of this technology). A second instance is recent work of Libedinsky and the second author [26] which uses localization to prove the positivity of anti-spherical Kazhdan–Lusztig polynomials.

Initially, ad hoc formulas were written down for the localization (and these were used to do various computer checks in [17]). Subsequently, much cleaner and general formulations were found (see (2.7), (2.27), Lemma 2.11). In this paper we provide these formulas.

A third goal of this paper is to improve the state of the literature relating to (two-colored) Jones–Wenzl projectors, certain idempotents in the (two-colored) Temperley–Lieb category. The two-colored Jones–Wenzl projectors were studied heavily in [8], where they were used to construct idempotents projecting to indecomposable Soergel bimodules for

the dihedral groups¹. In particular, for the diagrammatic Hecke category to be well-defined, it is extremely important whether or not a given two-colored Jones–Wenzl projector is *rotatable*, i.e. it is rotation-invariant up to scalar. Negligibility of the Jones–Wenzl projector is a necessary but not a sufficient condition. Another important property (that we need for our construction of Λ) is that the “polynomial evaluation” of a particular Jones–Wenzl projector is the product of the positive roots for the corresponding dihedral group. The discussion of these properties in [8] was not done in sufficient generality for our needs in this paper, and new ideas were needed to generalize them. Our results here are new and should have independent interest even in the study of ordinary Jones–Wenzl projectors. We postpone further discussion of these aspects to the end of the introduction.

We now provide a more technical introduction to the results of this paper.

1.1. The Q -groupoid

Let W be a group acting by homomorphisms on a commutative ring Q . From this data one can construct a monoidal category $\Omega_Q W$ (the “ Q -groupoid of W ”), which provides a categorification of W . Its definition is as follows. It has objects $\{r_x \mid x \in W\}$ with

$$\mathrm{Hom}(r_x, r_y) = \begin{cases} Q & \text{if } x = y, \\ 0 & \text{otherwise.} \end{cases} \quad (1.1)$$

The monoidal structure on objects is given by $r_x r_y := r_{xy}$. The tensor product of morphisms is as follows: if $f, g \in Q$ are regarded as elements of $\mathrm{End}(r_x)$ and $\mathrm{End}(r_y)$ respectively, then

$$f \otimes g := fx(g) \quad (1.2)$$

as an element of $Q = \mathrm{End}(r_{xy})$.

If the W -action on Q is faithful, $\Omega_Q W$ may be understood concretely as follows. For $x \in W$ one can define a Q -bimodule Q_x , which is isomorphic to Q as a left Q -module, and has right Q -action twisted by x : $q \cdot q' := x(q') \cdot q$ for $q \in Q_x$ and $q' \in Q$. Such bimodules are called *standard bimodules*. There exists a canonical isomorphism $Q_x \otimes_Q Q_y = Q_{xy}$, so standard bimodules form a monoidal category inside Q -bimodules. It is easy to construct a functor $\mathcal{F}_{\mathrm{std}}$ from $\Omega_Q W$ to the category of standard bimodules, sending r_x to Q_x . When the W -action on Q is faithful this functor is an equivalence.

However, when the W -action on Q is not faithful, there are isomorphisms $Q_x \cong Q_y$ for $x \neq y$; such an isomorphism does not exist in the abstractly-defined category $\Omega_Q W$. In this case $\mathcal{F}_{\mathrm{std}}$ is faithful but not full; it induces an isomorphism from all *nonzero* morphism spaces in $\Omega_Q W$ to the corresponding morphism spaces between bimodules,

¹The relationship between Soergel bimodules for dihedral groups and the two-colored Temperley–Lieb 2-category is an instance of “quantum geometric Satake at a root of unity” in type A_1 , see [9].

but entirely misses some other morphism spaces between bimodules. One can think of $\Omega_Q W$ as a model for the category of standard bimodules, but a model which does not detect degenerate behavior. The isomorphism classes in $\Omega_Q W$ are always in bijection with W , regardless of any surprises that the bimodule category may have in store.

Remark 1.4. We will view morphism spaces in $\Omega_Q W$ as being Q -bimodules, just as morphism spaces between standard bimodules are. The identification of $\text{End}(r_x)$ with Q intertwines the left action of Q on morphism spaces. Meanwhile, the right action of Q on $\text{End}(r_x)$ is twisted by x , so that $\text{End}(r_x)$ is actually isomorphic to Q_x as a Q -bimodule, just as $\text{End}(Q_x)$ is. With this convention, the functor \mathcal{F}_{std} preserves the Q -bimodule structure on morphism spaces.

1.2. The Hecke category and our main theorem

Now fix a commutative domain \mathbb{k} , and choose a *realization* of (W, S) over \mathbb{k} , see [17, Definition 3.1]. Roughly speaking, a realization is the data of a representation of W on a free \mathbb{k} -module \mathfrak{h}^* for which W acts by “reflections,” together with data on the reflecting hyperplanes (i.e. roots and coroots). Associated to this realization one has a polynomial ring R , whose linear terms are given by \mathfrak{h}^* . Let Q denote the fraction field of R . By slight abuse of language, we still refer to elements of Q as polynomials. Note that W acts on R and Q by algebra homomorphisms. In a general realization the action of W on \mathfrak{h}^* (and hence on R and Q) need not be faithful.

Let \mathcal{H} denote the (diagrammatic) Hecke category associated to this realization, as defined in [17]. Morphism spaces in \mathcal{H} are R -bimodules. The Hecke category \mathcal{H} is meant to encode the morphisms between certain R -bimodules known as Bott–Samelson bimodules (whose summands are called Soergel bimodules): for certain realizations there is a functor \mathcal{F} from \mathcal{H} to the category of Bott–Samelson bimodules. Just like $\Omega_Q W$, \mathcal{H} is a model which does not detect any degenerate behavior which may appear in the bimodule setting for special realizations.

Remark 1.5. In order to define the diagrammatic Hecke category associated to a realization, one requires the existence and rotatability of certain two-colored Jones–Wenzl projectors. This requirement is a (somewhat mysterious) condition on the Cartan matrix of the realization (i.e. the pairing of roots and coroots), and on the base ring \mathbb{k} . We assume that these projectors exist until Section 5, and discuss the conditions for them to exist in more detail in Section 6.

Let B be a Bott–Samelson bimodule. After base change from R to Q on one side, it becomes possible to equip $B \otimes_R Q$ with a left action of Q , making it into a Q -bimodule. Then one can prove that $B \otimes_R Q$ splits into a direct sum of standard bimodules Q_x . In fact,

this splitting lifts to the diagrammatic models as well. Let $(\Omega_Q W)_\oplus$ denote the additive envelope of $\Omega_Q W$. The main result of this paper is the following theorem:

Theorem 1.6. *There exists a monoidal functor $\Lambda : \mathcal{H} \rightarrow (\Omega_Q W)_\oplus$.*

Remark 1.7. In the literature, the symbol \mathcal{H}_{BS} is often used to refer to the pre-additive category defined by generators and relations in [17], while the symbol \mathcal{H} refers to the Karoubi envelope of its additive, graded closure. We only care about the category \mathcal{H}_{BS} in this paper, so we refer to it as \mathcal{H} for simplicity. Since $(\Omega_Q W)_\oplus$ is Karoubian, the functor Λ automatically lifts from \mathcal{H} to its Karoubi envelope.

The functor $\Lambda : \mathcal{H} \rightarrow (\Omega_Q W)_\oplus$ is meant to describe what happens to a morphism between Bott–Samelson bimodules after passage to the localization, where this morphism becomes a matrix of morphisms between the standard summands. For some realizations, such as when the W -action is not faithful, neither the functor \mathcal{F} from \mathcal{H} to the category of Bott–Samelson bimodules, nor the functor \mathcal{F}_{std} from $\Omega_Q W$ to the category of standard bimodules, is an equivalence. Nonetheless, one can show abstractly that after base changing the morphisms from R to Q on one side, the objects in \mathcal{H} split as direct sums of objects in $\Omega_Q W$. The functor Λ still records what happens to morphisms after localization.

Remark 1.8. Recently, Abe has proved that the formulas here define morphisms in his bimodule theoretic incarnation of the Hecke category, under a natural assumption on the realization [1]. (The key difficulty is to show that a morphism between localizations of bimodules is in fact the localization of a morphism.) This constructs a version of the functor \mathcal{F} from the Hecke category to Soergel bimodules in broad generality. We discuss the Abe’s results in more detail in Section 5.3. Even more recently, Hazi [21] has established strong results on the existence and rotability of Jones–Wenzl projectors. His work completely settles Conjecture 6.27.

Morphisms between objects in $\Omega_Q W$ are relatively easy to understand, thanks to (1.1): they are polynomials in Q , or are zero. Morphisms in $(\Omega_Q W)_\oplus$ between direct sums of objects in $\Omega_Q W$ are also easy to understand: they are matrices of morphisms in $\Omega_Q W$, and hence matrices whose entries are either polynomials or zero. This is unlike morphisms in \mathcal{H} , which are relatively complicated and difficult to compose.

One use of the functor Λ is to prove that the double leaves basis (see [17, §6]) of \mathcal{H} is, in fact, a basis. One proves that these morphisms span using a complicated diagrammatic argument. To prove linear independence, one applies the functor Λ , and proves that their image is still linearly independent using an upper triangularity argument. Consequently, double leaves are a basis, and the functor Λ is therefore faithful. For this reason, the well-definedness of Λ was crucial to the results of [17]. See [17, Propositions 6.6 and 6.9] for more details.

Another use of the functor Λ is to enable computer calculations. With current software, computers are not good at diagrammatic algebra. On the other hand, computers are excellent at multiplying sparse matrices of polynomials. Since one can check that two morphisms are equal after applying Λ , this transforms a complicated diagrammatic problem into matrix multiplication. To program this, one must know the matrix of polynomials associated to any generating morphism of \mathcal{H} . For each pair of simple reflections forming a finite dihedral subgroup of order $2m$, there is a generating morphism called the $2m$ -valent vertex. For small values of m , the matrix associated to the $2m$ -valent vertex was already in use by the authors for their computer calculations, but was not in the literature. Here, we present a precise formula for all m .

1.3. Organization of the paper

In Section 2 we define the functor Λ by defining it on generators. In Section 3 we check that the functor is well-defined by checking the relations. Thus we prove Theorem 1.6.

In Section 2 and Section 3 we provide and manipulate a whole lot of matrices with values in Q (which describe the functor Λ on morphisms), without explaining where these matrices come from. The goal of Section 4 is to explain how we computed these matrices. Much of the authors' intuition for the functor Λ is gained from a diagrammatic presentation of $\Omega_Q W$, which we developed in [17, §4]. This led to a combined diagrammatic calculus for localized Soergel bimodules, found in [18, §6], in which one can describe the heuristic method for computing Λ . We recall and use this technology in Section 4, and we encourage the reader to briefly acquaint themselves with [17, §4] and [18, §6]. Because these heuristics are not needed to follow the rest of the paper, and because we wished to emphasize the logical independence of Theorem 1.6 from this additional diagrammatic technology, we chose to put the cart in front of the horse and prove the theorem before motivating the construction.

In Section 5 we describe and rectify the error from our previous work, and elaborate on which assumptions one needs to use various results and constructions.

Our proofs in Section 3 rely on several technical properties of (two-colored) Jones–Wenzl projectors. These properties all appear in our earlier works, but in varying degrees of explicitness and with varying levels of proof. Jones–Wenzl projectors are traditionally studied using recursive formulas, but these recursive formulas might not apply in general (e.g. when we need to study the n -th Jones–Wenzl projector, but the k -th Jones–Wenzl projector is not defined for $k < n$). Some of the proofs² in [8] tacitly relied on these recursive formulas, and so they do not apply in all cases (e.g. for non-faithful realizations of dihedral groups). Altogether, we felt it was necessary to provide a complete discussion

²In particular, see Lemma 6.26 for a result where the proof in [8] is insufficiently general.

of Jones–Wenzl projectors and to carefully prove the required properties even when recursive formulas do not apply, which we do in Section 6. We give a general argument that, whenever a Jones–Wenzl projector exists over a (sufficiently nice) base ring, then its coefficients “agree” with the coefficients over a generic field. Over the generic field we can use the standard recursive formulas. This argument is new and general, and may be of independent interest. In Section 6.11 we describe Jones–Wenzl projectors in some special cases of note, including types B_2 and G_2 in small characteristic.

The diagrammatic calculus for \mathcal{H} has some additional complications when the realization is *odd-unbalanced*. We also describe the localization functor Λ in this setting. The additional technicalities are distracting, so we postpone this case until the final section. Again, we also feel like the accounts given of the unbalanced case in previous work are hasty and not easy for the reader to piece together. Thus we have aimed to present a self-contained exposition in Section 7.

Acknowledgements

We would like to thank Simon Riche for initially pointing out the gap in [17], which gave rise to this paper, as well as several very interesting questions and observations (particularly concerning Jones–Wenzl projectors in small characteristic) which gave rise to Section 6.11. We would also like to thank the referee for a detailed reading, and suggesting changes which have clarified the exposition. Both authors wish to thank the IAS for their hospitality.

2. Definition of the localization functor

Both \mathcal{H} and $(\Omega_Q W)_\oplus$ are monoidal and \mathcal{H} is presented by generators and relations. The strategy to construct the functor $\Lambda: \mathcal{H} \rightarrow (\Omega_Q W)_\oplus$ is simple: we define the images of the generating objects and morphisms, and then check the relations. In this section we define the functor on objects and morphisms, and in the subsequent sections we check the relations. We assume throughout that our realization is such that \mathcal{H} is well-defined, and elaborate further on what this entails when it becomes relevant. Starting in Section 2.5 we assume our realization is balanced for simplicity; the unbalanced case is treated in Section 7.

2.1. Definition on objects

The Hecke category \mathcal{H} is generated as a monoidal category by the set $\{B_s \mid s \in S\}$. On objects we set

$$\Lambda : B_s \longmapsto r_{\text{id}} \oplus r_s \tag{2.1}$$

If we think of $\text{id} = s^0$ and $s = s^1$ inside W , then

$$\Lambda(B_s) = \bigoplus_{e \in \{0,1\}} r_{s^e}. \quad (2.2)$$

By definition, we have canonical inclusions and projections from $\Lambda(B_s)$ to its direct summands r_{id} and r_s .

A sequence $\underline{w} = (s_1, \dots, s_d)$ of simple reflections ($s_i \in S$) is called an *expression*, and it has *length* d . We remove the underline and write w for the corresponding element in W , i.e. $w = s_1 \dots s_d$. In $\Omega_Q W$ we have

$$r_w = r_{s_1} \otimes r_{s_2} \otimes \dots \otimes r_{s_d}. \quad (2.3)$$

If $\underline{w} = (s_1, \dots, s_d)$ is an expression, a *subexpression* of \underline{w} is a sequence $\underline{e} = (e_1, \dots, e_d)$ of zeroes and ones of the same length as \underline{w} . We write $\underline{e} \subset \underline{w}$. This subexpression is thought of as representing the subword $s_1^{e_1} s_2^{e_2} \dots s_d^{e_d}$ of \underline{w} , and we call the corresponding element of W the *endpoint* of the subexpression. We employ the notation

$$\underline{w}^{\underline{e}} := s_1^{e_1} \dots s_d^{e_d} \in W \quad (2.4)$$

and we have

$$r_{\underline{w}^{\underline{e}}} = r_{s_1^{e_1}} \otimes \dots \otimes r_{s_d^{e_d}}. \quad (2.5)$$

The objects of \mathcal{H} are tensor products of various objects B_s , and thus can be identified with expressions. For an expression $\underline{w} = (s_1, \dots, s_d)$ we write $B_{\underline{w}} := B_{s_1} \otimes \dots \otimes B_{s_d}$. Because Λ is monoidal, it follows that we have a canonical isomorphism

$$\Lambda(B_{\underline{w}}) = (r_{\text{id}} \oplus r_{s_1}) \otimes \dots \otimes (r_{\text{id}} \oplus r_{s_d}) = \bigoplus_{\underline{e} \subset \underline{w}} r_{\underline{w}^{\underline{e}}}. \quad (2.6)$$

In this way, subexpressions of \underline{w} identify the 2^d summands inside $\Lambda(B_{\underline{w}})$. Given a subexpression $\underline{e} \subset \underline{w}$ with $\underline{w}^{\underline{e}} = x \in W$ we denote by

$$i(\underline{e}) : r_x \longrightarrow \Lambda(B_{\underline{w}}) \quad \text{and} \quad p(\underline{e}) : \Lambda(B_{\underline{w}}) \longrightarrow r_x$$

the inclusion of and projection to the canonical summand of (2.6) indexed by \underline{e} .

2.2. Definition on one-color morphisms

Morphisms in additive categories can be described as matrices of morphisms with respect to a fixed direct sum decomposition. Thus, we can specify a morphism

$$\psi : \Lambda(B_{\underline{w}}) \longrightarrow \Lambda(B_{\underline{x}})$$

by encoding it as a matrix with elements in Q , whose columns are indexed by subexpressions $\underline{e} \subset \underline{w}$ and whose rows are indexed by subexpressions $\underline{f} \subset \underline{x}$. The entry in column \underline{e}

and row \underline{f} is the composition $p(\underline{f}) \circ \psi \circ i(\underline{e})$, which is a map $r_{\underline{w}^e} \rightarrow r_{\underline{x}^f}$, viewed as an element of Q .

The images of the one-color morphisms are as follows:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} f \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|c} \emptyset & \emptyset \\ \hline \emptyset & f \end{array} \quad (2.7a)$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|cc} & 0 & 1 \\ \hline \emptyset & \alpha & 0 \end{array} \quad (2.7b)$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|c} \emptyset & \emptyset \\ \hline 0 & 1 \\ 1 & 0 \end{array} \quad (2.7c)$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|cc} & 0 & 1 \\ \hline 00 & 1/\alpha & 0 \\ 01 & 0 & 1/\alpha \\ 10 & 0 & -1/\alpha \\ 11 & -1/\alpha & 0 \end{array} \quad (2.7d)$$

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|cccc} & 00 & 01 & 10 & 11 \\ \hline 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{array} \quad (2.7e)$$

If α represents the simple reflection $s \in S$, then α represents the corresponding simple root α_s .

Remark 2.1. As a reminder of the diagrammatic conventions, morphisms are composed from bottom to top, and their corresponding matrices are composed right to left (in the usual way).

Remark 2.2. Note that the degrees of the matrix entries can be determined as follows. The degree of any given matrix coefficient in $\Lambda(\phi)$ is the degree of ϕ , plus the length of the source expression, minus the length of the target expression. Recall that $\deg(\alpha) = 2$.

Recall that in \mathcal{H} , the cups and caps are obtained as compositions of trivalent vertices and dots:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} := \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} := \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ \text{---} \\ \text{---} \\ \text{---} \end{array} \quad (2.8)$$

Composing the corresponding matrices above gives the images of the cups and caps under Λ :

$$\begin{array}{c}
 \text{---} \\
 \cup \\
 \text{---} \\
 \text{---}
 \end{array}
 \mapsto
 \begin{array}{c|c}
 & \emptyset \\
 \hline
 00 & 1/\alpha \\
 01 & 0 \\
 10 & 0 \\
 11 & -1/\alpha
 \end{array}
 \quad (2.9)$$

$$\begin{array}{c}
 \text{---} \\
 \cap \\
 \text{---} \\
 \text{---}
 \end{array}
 \mapsto
 \begin{array}{c|cccc}
 & 00 & 01 & 10 & 11 \\
 \hline
 \emptyset & \alpha & 0 & 0 & \alpha
 \end{array}
 \quad (2.10)$$

Note that all the zero entries in the matrices above are forced to be zero, simply because $\text{Hom}(r_x, r_y) = 0$ if $x \neq y$. When it saves space, we ignore the rows and columns in our matrices which are forced to be zero, and only write the interesting part. For example, instead of (2.7b), we could have simply written

$$\begin{array}{c}
 \text{---} \\
 \uparrow \\
 \text{---}
 \end{array}
 \mapsto
 \begin{array}{c|c}
 & 0 \\
 \hline
 \emptyset & \alpha
 \end{array}.
 \quad (2.11)$$

2.3. Monoidal structure

For the sake of explicitness, we give some more detail and examples concerning the monoidal structure. Recall from Remark 1.4 that the identification of $\text{End}(r_x)$ with Q uses the left action of Q on morphism spaces. Tensoring with (the image under Λ of) B_s on the right does not affect this left action, but tensoring with (the image under Λ of) B_s on the left does.

Remark 2.3. In the diagrammatic language (see Section 4), one can think that the matrices are recording polynomials which should be placed on the left of a diagram. When morphisms are concatenated horizontally, the left of a subdiagram is no longer the left of the entire diagram, and the polynomial must be pushed to the left using the monoidal structure in $\Omega_Q W$, which was defined in (1.2).

Example 2.4. Multiplication by the polynomial f is an endomorphism of the monoidal identity, and has the 1×1 matrix (see (2.7a))

$$\begin{array}{c}
 \text{---} \\
 f \\
 \text{---}
 \end{array}
 \mapsto
 \begin{array}{c|c}
 & \emptyset \\
 \hline
 \emptyset & f
 \end{array}.
 \quad (2.12)$$

Since $B_s \cong r_{\text{id}} \oplus r_s$, we have

$$f \otimes \text{id}_{B_s} =
 \begin{array}{c}
 \text{---} \\
 f \\
 \text{---}
 \end{array}
 \mapsto
 \begin{array}{c|cc}
 & 0 & 1 \\
 \hline
 0 & f & 0 \\
 1 & 0 & f
 \end{array}
 \quad (2.13)$$

and

$$\text{id}_{B_s} \otimes f = \begin{array}{c} \text{---} \\ \text{---} \end{array} \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] f \begin{array}{c} \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|cc} & 0 & 1 \\ 0 & f & 0 \\ \hline 1 & 0 & s(f) \end{array}. \quad (2.14)$$

On the second row of the matrix of $\text{id}_{B_s} \otimes f$, the polynomial f must be pushed left through r_s , so it is acted on by s .

Similarly, if we know the $m \times n$ matrix of $\Lambda(\phi)$ for some morphism ϕ , then the matrix of $\Lambda(\phi \otimes \text{id}_{B_s})$ (resp. $\Lambda(\text{id}_{B_s} \otimes f)$) is a $2m \times 2n$ matrix, where each entry $f \in Q$ of the original matrix is replaced by a 2×2 matrix (indexed by subexpressions of (s)) according to the matrix of (2.13) (resp. (2.14)).

Example 2.5. The reader may use these rules to check that

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \right] \begin{array}{c} \text{---} \\ \text{---} \end{array} \mapsto \begin{array}{c|cc} & 0 & 1 \\ \hline 000 & 1/\alpha & 0 \\ 011 & -1/\alpha & 0 \\ 100 & 0 & s(1/\alpha) \\ 111 & 0 & s(-1/\alpha) \end{array} = \begin{array}{c|cc} & 0 & 1 \\ \hline 000 & 1/\alpha & 0 \\ 011 & -1/\alpha & 0 \\ 100 & 0 & -1/\alpha \\ 111 & 0 & 1/\alpha \end{array}. \quad (2.15)$$

We have omitted four rows (indexed by the subexpressions 001, 010, 101 and 110) which are zero.

Remark 2.6. Sometimes the decision to push all polynomials to the left of the diagram can obscure a nicer formula. For example, in a product $r_x r_y$, the fraction field Q acts on the left and on the right, but also in the middle, between the two tensor factors. The formulas for (2.7d) and (2.9) can both be rewritten so that the polynomial $\frac{1}{\alpha}$, with no sign, always appears in the middle of the target $\Lambda(B_s \otimes B_s)$. When $\frac{1}{\alpha}$ is pushed left, it picks up a sign based on whether the first B_s is replaced with r_{id} or with r_s .

2.4. Notation for positive roots

Our next goal is to provide the image of the $2m_{st}$ -valent vertex under Λ . This will require a little more notation first.

Let $\underline{w} = (s_1, \dots, s_d)$ be an expression of length $d \geq 1$. We define the corresponding element $\beta_{\underline{w}} \in \mathfrak{h}^*$ by the formula

$$\beta_{\underline{w}} = s_1 s_2 \dots s_{d-1} (\alpha_{s_d}). \quad (2.16)$$

Similarly, if $\underline{e} \subset \underline{w}$ is a subexpression, we write

$$\beta_{\underline{e}} = s_1^{e_1} s_2^{e_2} \dots s_{d-1}^{e_{d-1}} (\alpha_{s_d}). \quad (2.17)$$

Note that the value of $e_d \in \{0, 1\}$ does not affect $\beta_{\underline{e}}$.

Let $\underline{x} = (s_1, \dots, s_d)$ be any expression. We write $X_{\underline{x}}$ for the set of *leading subexpressions* of \underline{x} . That is,

$$X_{\underline{x}} = \{(s_1), (s_1, s_2), \dots, (s_1, s_2, \dots, s_{d-1}), \underline{x}\}. \quad (2.18)$$

For leading subexpressions we do not use the zero-one notation $\underline{e} \subset \underline{x}$, as we did for arbitrary subexpressions; this is because we will be interested later in subexpressions of leading subexpressions!

For the geometric realization, it is well-known that one can enumerate the positive roots Φ^+ of a finite Coxeter group using leading subexpressions of any given reduced expression for the longest element. If \underline{x} is a reduced expression for the longest element of W , then

$$\Phi^+ = \{\beta_{\underline{w}} \mid \underline{w} \in X_{\underline{x}}\}. \quad (2.19)$$

Both Φ^+ and $X_{\underline{x}}$ have size equal to the number of reflections in W , so this description is irredundant. What happens for other realizations will be discussed in Remark 2.10 below. Regardless, this discussion of positive roots is only for motivational purposes.

Remark 2.7. It is worth noting that one can *not* use tailing subexpressions to enumerate positive roots. For example, if $Y_{\underline{x}} = \{(s_d), (s_{d-1}, s_d), \dots, (s_2, \dots, s_d), \underline{x}\}$ then Φ^+ need not equal $\{\beta_{\underline{w}} \mid \underline{w} \in Y_{\underline{x}}\}$. Every such $\beta_{\underline{w}}$ will be in the W -orbit of α_{s_d} , and not all positive roots need be in the same W -orbit (e.g. long and short roots are in two different orbits).

Specifically for finite dihedral groups, we can enumerate the positive roots of the geometric representation (without redundancy) in two standard ways. Let X_s be the set of leading expressions of (s, t, s, t, \dots)

$$X_s = \{s, st, sts, \dots, stst \dots\} \quad (2.20)$$

where the final expression has length m , where m denotes the order of st . Similarly, let

$$X_t = \{t, ts, tst, \dots, tsts \dots\}. \quad (2.21)$$

We define polynomials $\pi_{s,t}$ and $\pi_{t,s}$ by the following formulas:

$$\pi_{s,t} = \alpha_s \cdot s(\alpha_t) \cdot st(\alpha_s) \cdots = \prod_{\underline{w} \in X_s} \beta_{\underline{w}} \quad (2.22)$$

and

$$\pi_{t,s} = \alpha_t \cdot t(\alpha_s) \cdot ts(\alpha_t) \cdots = \prod_{\underline{w} \in X_t} \beta_{\underline{w}}. \quad (2.23)$$

This definition makes sense in any realization, even when the realization is not faithful or balanced.

For the geometric realization we have

$$\{\beta_{\underline{w}} \mid \underline{w} \in X_s\} = \{\beta_{\underline{w}} \mid \underline{w} \in X_t\}, \quad (2.24)$$

and this is the set of positive roots $\Phi_{s,t}^+$. In particular, if $w_t \in X_t$ is the longest expression, then $\beta_{w_t} = \alpha_s$, and vice versa. Now $\pi_{s,t}$ is the product of the positive roots in $\Phi_{s,t}^+$, and (2.24) gives us

$$\pi_{s,t} = \pi_{t,s}. \quad (2.25)$$

Moreover, s preserves the set $\Phi_{s,t}^+ \setminus \{\alpha_s\}$, and sends $\alpha_s \mapsto -\alpha_s$. Thus $\pi_{s,t}$ is s -antiinvariant:

$$s(\pi_{s,t}) = -\pi_{s,t}. \quad (2.26)$$

When the realization is not faithful, the sets in (2.24) might not have size m , instead giving a redundant description of a smaller set of roots. When the realization is not balanced, the sets in (2.24) may be non-equal. Indeed, if $w_t \in X_t$ is the longest expression, then β_{w_t} is a non-trivial scalar multiple of α_s , and all the roots coming from X_t are rescalings of roots coming from X_s . See Section 7.1 for more details. In these cases it is a technical question what the appropriate definition of ‘‘positive roots’’ should be, but this is not of importance for us. The definitions of $\pi_{s,t}$ and $\pi_{t,s}$ make sense as above, and it is only important what properties they satisfy.

Example 2.8. To follow this example one should be familiar with two-colored quantum numbers, see Section 6.1. In the geometric realization of the symmetric group S_3 with simple reflections s and t , we have

$$ts(\alpha_t) = \alpha_s, \quad t(\alpha_s) = s(\alpha_t), \quad \alpha_t = st(\alpha_s), \\ \pi_{s,t} = \pi_{t,s} = \alpha_s \alpha_t s(\alpha_t).$$

Meanwhile, for an arbitrary realization, one has $st(\alpha_s) = [3]\alpha_s + [2]_t\alpha_t$. For any realization $[3] = 0$ and $[2]_s[2]_t = 1$, but $[2]_t = 1$ if and only if the realization is balanced. For example, if $[2]_t = -5$ and $[2]_s = \frac{-1}{5}$ we have

$$-5ts(\alpha_t) = \alpha_s, \quad t(\alpha_s) = -5s(\alpha_t), \quad -5\alpha_t = st(\alpha_s), \quad \pi_{s,t} = -5 \cdot \pi_{t,s}.$$

For the remainder of this section we stick with the case where the realization is balanced (but not necessarily faithful). This is sufficient to imply that (2.25) and (2.26) hold, and these are the properties of $\pi_{s,t}$ that we use. We identify $\pi_{s,t}$ and $\pi_{t,s}$ in our formulas for simplicity. We deal with the unbalanced case in Section 7. The proofs found in Section 7 also suffice to prove the desired statements (2.25) and (2.26) in the balanced case, regardless of faithfulness.

Remark 2.9. The equality (2.25) is not really essential for our proofs, so long as one carefully distinguishes between $\pi_{s,t}$ and $\pi_{t,s}$ in the formulas, as we do in Section 7. The anti-invariance of $\pi_{s,t}$, namely (2.26), is essential for our proofs, and holds only when the realization is even-balanced (see Definition 6.4). This is one reason why even-balancedness is essential for the well-definedness of \mathcal{H} .

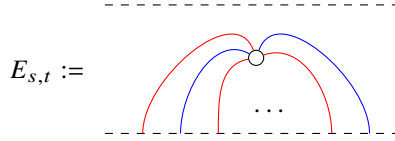
Remark 2.10. For arbitrary realizations of finite Coxeter groups, one could fix a reduced expression \underline{w} for the longest element, and define positive roots (as a multiset) using (2.19). For faithful realizations, this multiset has no nontrivial multiplicities, a well-known fact for the geometric realization. If the realization is balanced, then the multiset of positive roots does not depend on the choice of reduced expression. Otherwise, the positive roots will depend on the choice of reduced expression, though only up to rescaling: the lines spanned by these roots will be independent of the choice of reduced expression. We do not know where to find proofs of these observations in the literature. Briefly, one should prove these claims using the observation that $\beta_{\underline{w}} = \beta_{\underline{w}'}$ in the geometric realization if the two expressions are related by a sequence of transformations of the form

- Coxeter relations applied to all but the last index in the expression, and
- the equality $\beta_{\underline{w}_i} = \alpha_s$, which we can use to transform the end of the expression.

These transformations continue to apply verbatim for any balanced realization, and apply up to scalar for unbalanced realizations.

2.5. Definition on two-color morphisms

Suppose that $s, t \in S$ are two distinct reflections, such that st has order $m < \infty$ in W . We need to specify the image of the $2m$ -valent vertex in $(\Omega_Q W)_\oplus$. First, we specify the image of the morphism



obtained by “twisting all arms of the $2m$ -valent vertex down.” Given that we already know the images of the cups and caps under Λ , we can recover knowledge of the $2m_{s,t}$ -valent vertex from the knowledge of the image of $E_{s,t}$ under Λ . We do this in Section 2.6 below.

Let $\underline{w} := (s, t, s, t, \dots)$ have length $2m$, so that $B_{\underline{w}}$ is the source of $E_{s,t}$. Define

$$f : \Lambda(B_{\underline{w}}) \longrightarrow \Lambda(\mathbb{K})$$

by

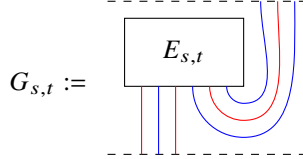
$$f_{\underline{e}}^\emptyset := \begin{cases} \pi_{s,t} & \text{if } \underline{w}^{\underline{e}} = \text{id}, \\ 0 & \text{otherwise.} \end{cases} \quad (2.27)$$

(Thus, in matrix notation, f consists of a single row in which every possible non-zero entry is $\pi_{s,t}$). We set:

$$\Lambda : E_{s,t} \longmapsto f. \quad (2.28)$$

2.6. An alternative formula

Here we give a formula for the image of the $2m$ -valent vertex before “twisting.” Consider the following explicit realization of the $2m$ -valent vertex:



(We draw the $m = 3$ case, but the reader can guess what $G_{s,t}$ looks like in general.)

When we express $\Lambda(G_{s,t})$ as a matrix, it will have a potential nonzero entry for each $\underline{e} \subset (s, t, s, \dots)$ and each $\underline{e}' \subset (t, s, t, \dots)$ with the same endpoint. Let the corresponding matrix entry be denoted $(G_{s,t})_{\underline{e}}^{\underline{e}'} \in Q$.

Recall that X_t is the set of leading subexpressions of the reduced expression (t, s, t, \dots) . If $\underline{w} \in X_t$ has length i and $\underline{e}' \subset (t, s, t, \dots)$ is a subexpression, then $\underline{e}'_{\subset \underline{w}} := \underline{e}'_{\leq i} = (e'_1, e'_2, \dots, e'_i)$ is a subexpression of \underline{w} . Let

$$\zeta(\underline{e}') = \prod_{\underline{w} \in X_t} \beta_{\underline{e}'_{\subset \underline{w}}} = \prod_{i=1}^m s_1^{e'_1} s_2^{e'_2} \dots s_{i-1}^{e'_{i-1}}(\alpha_{s_i}). \quad (2.29)$$

Note that $\zeta(\underline{e}')$ is a product of m roots, and has the same degree as $\pi_{s,t}$, regardless of the choice of \underline{e}' .

Lemma 2.11. *If $\underline{e} \subset (s, t, s, \dots)$ and $\underline{e}' \subset (t, s, t, \dots)$ have the same endpoint, then*

$$(G_{s,t})_{\underline{e}}^{\underline{e}'} = \frac{\pi_{s,t}}{\zeta(\underline{e}')} \quad (2.30)$$

We delay the proof of this lemma until the end of this section.

Remark 2.12. In the matrix of $\Lambda(G_{s,t})$, all potential nonzero entries are nonzero, and the nonzero entries in each row are equal. The lemma also shows that the formula for the image of the $2m$ -valent vertex $G_{s,t}$ is more complicated than its twisted cousin $E_{s,t}$.

Example 2.13. Let us consider the special case when $m = 2$, so s and t commute. Then $\pi_{s,t} = \alpha_s \alpha_t$, and $\pi_{s,t} = \zeta(\underline{e}')$ for all subexpressions $\underline{e}' \subset (t, s)$. Thus the matrix of $\Lambda(G_{s,t})$ is easy: all entries which could be nonzero are equal to 1.

Example 2.14. Suppose that $m = 3$. Then

$$(G_{s,t})_{(000)}^{(000)} = (G_{s,t})_{(101)}^{(000)} = \frac{\alpha_s \alpha_t s(\alpha_t)}{\alpha_t \alpha_s \alpha_t} = \frac{s(\alpha_t)}{\alpha_t}.$$

Let us deduce two consequences of this lemma. If $\underline{e} = (1, 1, \dots, 1)$ and $\underline{e}' = (1, 1, \dots, 1, 1)$ then

$$\zeta(\underline{e}') = \prod_{i=1}^m s_1 s_2 \dots s_{i-1} (\alpha_{s_i}) = \pi_{s,t}$$

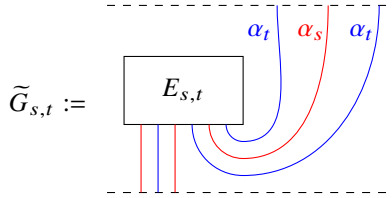
and hence

$$(G_{s,t})_{\underline{e}}^{\underline{e}'} = 1. \tag{2.31}$$

Similarly, if $\underline{e} = (0, 1, \dots, 1, 1)$ and $\underline{e}' = (1, 1, \dots, 1, 0)$ then

$$(G_{s,t})_{\underline{e}}^{\underline{e}'} = 1. \tag{2.32}$$

Proof of Lemma 2.11. Consider the following morphism (again we draw the case $m = 3$, and leave it to the reader to imagine the general case):



We compute the image of $\tilde{G}_{s,t}$ both via (2.28) and via the lemma, and see that they agree.

If we compute the image of $\tilde{G}_{s,t}$ using (2.28) and (2.9) then each of the simple roots appearing in the formulas for the cup cancels those in the diagram for $\tilde{G}_{s,t}$, and we deduce that all potentially non-zero entries are equal to $\pi_{s,t}$ (see Remark 2.6).

On the other hand, if we compute image of $\tilde{G}_{s,t}$ using (2.30) then, for each subexpression $\underline{e}' \subset (t, s, t, \dots)$, the simple roots along the top of $\tilde{G}_{s,t}$ will pull to the left to become $\zeta(\underline{e}')$, and hence all potentially non-zero entries are equal to $\pi_{s,t}$. \square

3. Checking the relations

3.1. One-color

Checking the one-color relations is routine. We give some sample computations.

The relation

$$\begin{array}{c} \text{---} \\ \bullet \\ | \\ \bullet \\ \text{---} \end{array} = \alpha$$

follows from the identity

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha \\ 0 \end{pmatrix}$$

the first matrix on the left is (2.7b), the second (2.7c).

The relation



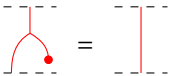
$$(3.1)$$

follows from the identity

$$\begin{pmatrix} \alpha & 0 & 0 & 0 \\ 0 & 0 & -\alpha & 0 \end{pmatrix} \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \\ 0 & -1/\alpha \\ -1/\alpha & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The second matrix on the left is (2.7d). The first matrix is obtained from (2.7b), by applying (2.14).

The relation



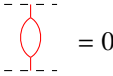
$$$$

follows from the identity

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The first matrix on the left is (2.7e). The second matrix is obtained from (2.7c), by applying (2.14).

The relation



$$= 0$$

follows from the identity

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \\ 0 & -1/\alpha \\ -1/\alpha & 0 \end{pmatrix} = 0.$$

The first matrix on the left is (2.7e), the second (2.7d).

We leave it to the reader to check the other one-color relations.

3.2. Cyclicity

We begin to check the two-color relations, starting with the cyclicity of the $2m$ -valent vertex.

When doing these computations below, if $\underline{e} \subset \underline{w}$, we write $[\underline{e}]$ for the identity endomorphism of the corresponding summand of $\Lambda(B_{\underline{w}})$.

Example 3.1. For example, if we write

$$\Lambda(\phi): [\underline{e}] \mapsto \alpha_s[\underline{f}] + \alpha_t[\underline{g}]$$

this means that the \underline{e} column of the matrix for $\Lambda(\phi)$ has only two nonzero entries, α_s in row \underline{f} and α_t in row \underline{g} .

Cyclicity of the $2m$ -valent vertex is equivalent to the following relation (together with its analogue after swapping s and t):

$$(3.2)$$

Lemma 3.2. Λ preserves the relation (3.2).

Proof. Let \underline{e} be a subexpression of the source, and consider the image of $[\underline{e}]$ under the morphisms on the left and right hand side. Because the image of these morphisms is $\Lambda(B_s) = r_{\text{id}} \oplus r_s$, $[\underline{e}]$ is mapped to zero unless it has endpoint id or s .

Case 1: \underline{e} is a subexpression for the identity. Then on the left-hand side, $[\underline{e}]$ is mapped as follows:

$$[\underline{e}] \mapsto \frac{1}{\alpha_s} [00\underline{e}] - \frac{1}{\alpha_s} [11\underline{e}] \mapsto \frac{\pi_{s,t}}{\alpha_s} [0].$$

(This is a composition of two maps, the red cup followed by $\text{id}_{B_s} \otimes E_{s,t}$. Applying the red cup yields the expression in the middle by (2.9). Afterwards, only the first term survives $\text{id}_{B_s} \otimes E_{s,t}$, since the second term applies $E_{s,t}$ to a subexpression for s rather than for id .) Similarly, on the right-hand side we find:

$$[\underline{e}] \mapsto \frac{1}{\alpha_s} [\underline{e}00] - \frac{1}{\alpha_s} [\underline{e}11] \mapsto \frac{\pi_{t,s}}{\alpha_s} [0].$$

Thus $[\underline{e}]$ has the same image on both sides, as $\pi_{s,t} = \pi_{t,s}$ (see (2.25)).

Case 2: \underline{e} is a subexpression for s . On the left hand side, $[\underline{e}]$ is mapped as follows:

$$[\underline{e}] \mapsto \frac{1}{\alpha_s} [00\underline{e}] - \frac{1}{\alpha_s} [11\underline{e}] \mapsto -\frac{s(\pi_{s,t})}{\alpha_s} [1].$$

The s in the numerator came from (2.14). On the right-hand side we find:

$$[\underline{e}] \mapsto \frac{1}{s(\alpha_s)} [\underline{e}00] - \frac{1}{s(\alpha_s)} [\underline{e}11] = -\frac{1}{\alpha_s} [\underline{e}00] + \frac{1}{\alpha_s} [\underline{e}11] \mapsto \frac{\pi_{t,s}}{\alpha_s} [1].$$

Thus $[\underline{e}]$ has the same image on both sides since $-s(\pi_{s,t}) = \pi_{t,s}$. □

3.3. Properties of the Jones–Wenzl projector

We now turn to the Jones–Wenzl relation:

$$\begin{array}{c} \text{---} \\ \boxed{E_{s,t}} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \boxed{JW_{s,t}} \\ \text{---} \end{array} \tag{3.3}$$

Here, $JW_{s,t}$ is a particular linear combination of degree 2 diagrams consisting entirely of univalent and trivalent vertices. We will need a number of basic facts about $JW_{s,t}$. Let us recall the construction of $JW_{s,t}$ so we may justify these facts.

For this discussion we assume the reader is familiar with the two-colored Temperley–Lieb category $2\mathcal{TL}$ and the deformation retract functor $\Sigma: 2\mathcal{TL} \rightarrow \mathcal{H}$. The best exposition for this can be found in the recent book [15, §9.2]; an older exposition can be found in [8, §4.3 and §5.3.2].

The functor Σ sends a cup in $2\mathcal{TL}$ to a morphism in \mathcal{H} which we call a *pitchfork*:

$$\begin{array}{c} \text{---} \\ \text{red cap} \\ \text{---} \end{array} \mapsto \begin{array}{c} \text{---} \\ \text{pitchfork} \\ \text{---} \end{array} \tag{3.4}$$

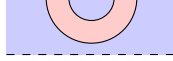
The image of a cap we call an *inverted pitchfork*:

$$\begin{array}{c} \text{---} \\ \text{blue cap} \\ \text{---} \end{array} \mapsto \begin{array}{c} \text{---} \\ \text{inverted pitchfork} \\ \text{---} \end{array} \tag{3.5}$$

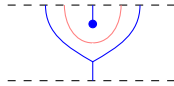
All morphisms in the image of Σ have degree 0. The functor Σ is well-defined as long as the blue circle inside a red region in $2\mathcal{TL}$ evaluates to $\alpha_s^\vee(\alpha_t)$, and the red circle inside a blue region evaluates to $\alpha_t^\vee(\alpha_s)$.

There is exactly one (n, n) crossingless matching with no cups or caps: the identity matching. This is the unique crossingless matching which matches each of the n bottom boundary points to a top boundary point. Any other matching pairs two bottom points and two top points, and has both a cap on bottom and a cup on top. Consequently, the image under Σ of any non-identity two-colored crossingless matching has a pitchfork on top and an inverted pitchfork on bottom.

Remark 3.3. Cups on top and caps on the bottom are just subdiagrams of a larger crossingless matching, and after applying Σ , the same can be said of pitchforks. When choosing a deformation retract of a crossingless matching to define Σ , one often chooses a retract where the pitchfork does not appear precisely as a subdiagram, but some slight modification thereof. For example, when taking the deformation retract of



one obtains the diagram



(3.6)

The cap on top no longer gives rise to a subdiagram identical to the red pitchfork from (3.4), but instead one finds a red pitchfork without a handle.

Of course, the red pitchfork without a handle is just the composition of a red pitchfork with a red univalent vertex, thanks to the unit relation (3.1), so this diagram factors through the pitchfork. Similarly, another deformation retract might have several pitchforks which are grafted to the same handle, which can be accounted for by the one-color associativity relation. Below, when we say that a diagram representing a morphism in \mathcal{H} has a *pitchfork on top*, we mean that it factors as the composition of a pitchfork and another diagram. For example, (3.6) has a pitchfork on top in this sense.

Let us consider two-colored crossingless matchings where the leftmost region is colored s . The two-colored Jones–Wenzl projector $JW_{s,n}$, if it exists, is a particular linear combination of (n, n) crossingless matchings. It is uniquely determined by the following properties in $2\mathcal{T}\mathcal{L}$:

- It becomes zero when any cap is placed on top (we say it is *killed by caps*).
- It becomes zero when any cup is placed on bottom (we say it is *killed by cups*).
- The coefficient of the identity crossingless matching is 1.

The existence of such a linear combination is not guaranteed, and relies upon properties of the scalars $\alpha_s^\vee(\alpha_t)$ and $\alpha_t^\vee(\alpha_s)$ in the base ring \mathbb{k} . Similarly, one can discuss $JW_{n,t}$. Jones–Wenzl projectors are discussed at great length in Section 6.

We let $\mathcal{JW}_{n,s,t}$ denote the image of $JW_{s,n}$ (if it exists) under Σ . Then $\mathcal{JW}_{n,s,t}$ is an endomorphism of the alternating length $n + 1$ object $B_s B_t B_s \dots$ in \mathcal{H} . The fact that

Jones–Wenzl projectors are killed by cups implies after applying Σ that $\mathcal{JW}_{n,s,t}$ is killed by pitchforks on bottom. That is

$$\begin{array}{c} \boxed{\mathcal{JW}_{n,s,t}} \\ \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \\ \boxed{P} \end{array} = 0 \tag{3.7}$$

whenever P has a pitchfork on top. Similarly, $\mathcal{JW}_{n,s,t}$ is killed by (inverted) pitchforks on top; when postcomposing with a morphism that has an inverted pitchfork on bottom, you get zero. We call these properties *death by pitchfork*.

The fact that Jones–Wenzl projectors have coefficient 1 of the identity, and all other diagrams have cups and caps, implies that

$$\boxed{\mathcal{JW}_{n,s,t}} = \text{---} \text{---} \text{---} \text{---} + \boxed{Q} \tag{3.8}$$

where Q represents a linear combination of diagrams, and every diagram in Q has both a pitchfork on top and an inverted pitchfork on bottom. This, combined with death by pitchfork, is the easiest way to observe that $\mathcal{JW}_{n,s,t}$ is an idempotent.

Remark 3.4. The pictures above are drawn as though n were even. By definition, the source and target of $\mathcal{JW}_{n,s,t}$ should alternate red, blue, etc. Whether the final strand is red or blue depends on the parity of n .

Remark 3.5. Any Jones–Wenzl projector is fixed by the contravariant equivalence which applies a vertical flip to all diagrams. After all, the defining properties of the Jones–Wenzl projector in $2\mathcal{T}\mathcal{L}$ are invariant under applying the vertical flip.

As for any morphism in the image of the functor Σ , the Jones–Wenzl projector commutes with trivalent vertices on the side, in the sense that

$$\begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \boxed{\mathcal{JW}_{n,s,t}} \\ \text{---} \text{---} \text{---} \text{---} \end{array} = \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \\ \boxed{\mathcal{JW}_{n,s,t}} \\ \text{---} \text{---} \text{---} \text{---} \end{array} \tag{3.9}$$

As a consequence of (3.9) and (3.1) we have

$$\mathcal{JW}_{n,s,t} = \mathcal{JW}_{n,s,t} \quad (3.10)$$

In this paper we are only interested in a particular Jones–Wenzl morphism, where $n = m - 1$ and $m = m_{s,t} < \infty$. One of the required assumptions for the category \mathcal{H} to be well-defined is that $JW_{s(m-1)}$ and $JW_{t(m-1)}$ both exist. A second assumption is that these Jones–Wenzl projectors are *rotatable*. We wish to spell out the rotatable property in the context of \mathcal{H} , for the morphism

$$\mathcal{JW}_{s,t} := \mathcal{JW}_{m_{s,t}-1,s,t}.$$

However, Σ is not a monoidal functor, and consequently rotation in $2\mathcal{TL}$ does not immediately translate to rotation in \mathcal{H} . To discuss rotation more easily, we need to focus our attention on morphisms of degree +2 rather than degree 0.

Define

$${}^2\mathcal{JW}_{s,t} = \mathcal{JW}_{s,t} \quad (3.11)$$

Then ${}^2\mathcal{JW}_{s,t}$ is a degree 2 map from $B_s B_t \dots$ to $B_t B_s \dots$, where these are the two reduced expressions of length $m - 1$. Using (3.10) we also have

$$\mathcal{JW}_{s,t} = {}^2\mathcal{JW}_{s,t} \quad (3.12)$$

Remark 3.6. One should think that there are two different ways to apply a deformation retract to the same linear combination of crossingless matchings. One produces a degree zero morphism on the planar strip, while the other produces a degree +2 morphism on the planar disk with a shorter boundary. The first process is functorial, hence the functor Σ , while the second process is rotationally equivariant. See [15, Equations (9.14) and (9.15)] for more details. Hence $\mathcal{JW}_{s,t}$ and ${}^2\mathcal{JW}_{s,t}$ are just two different kinds of deformation retract of the Jones–Wenzl projector.

The Jones–Wenzl projector JW_{m-1} in $2\mathcal{TL}$ is *rotatable* if its rotation is still orthogonal to all cups and caps, and thus is a scalar multiple of the Jones–Wenzl projector. If so, applying the degree 2 deformation retract, we directly obtain

$$= \lambda_{s,t} \quad (3.13)$$

for some scalar $\lambda_{s,t}$. One can prove that $\lambda_{s,t} = \lambda_{t,s}^{-1}$, so that

$$= \quad (3.14)$$

In this case, we say that the Jones–Wenzl projector is *invariant under rotation by two strands*. See Section 6.8 for the proofs.

One can compute that whenever (3.13) holds for some scalar $\lambda_{s,t}$, that scalar must be a particular invertible two-colored quantum number, see Lemma 6.25 for details. By definition, $\lambda_{s,t} = 1$ when the realization is *balanced*, c.f. Definition 6.4. In the balanced case, we say that the Jones–Wenzl projector is *invariant under rotation by one strand with color swap*. Rotation invariance of the Jones–Wenzl projector is a crucial compatibility between (3.3) and the cyclicity of the $2m$ -valent vertex. For the rest of this section we assume the realization is balanced. We treat the unbalanced case in Section 7.

Combining rotation invariance with death by pitchfork, we also see that ${}^2\mathcal{JW}_{s,t}$ is killed by placing pitchforks on the “side,” for example

$$= 0 \quad (3.15)$$

Another crucial property of this special Jones–Wenzl projector is that

$$\begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} \boxed{\mathcal{JW}_{s,t}} \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} \boxed{{}^2\mathcal{JW}_{s,t}} \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} = \pi_{s,t} \tag{3.16}$$

(the notation $\pi_{s,t}$ was introduced in Section 2.4). This was proven in [8, Corollary 4.15] under certain assumptions, and we prove it more generally in Lemma 6.26 below.

Remark 3.7. The proof of this result comes from counting the red and the blue regions in each crossingless matching. After applying Σ and composing with many dots as above, each region is transformed to a barbell, and evaluates to the polynomial α_s or α_t . In particular, the relation (3.16) is a consequence only of the one color relations. Death by pitchfork is also a consequence only of the one-color relations.

Finally, to apply the functor Λ most efficiently to Jones–Wenzl projectors, we will twist them using adjunction to consider maps whose target is the monoidal identity. Let

$$\begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} JW_{s,t} = \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} \boxed{{}^2\mathcal{JW}_{s,t}} \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} \tag{3.17}$$

and

$$\begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} JW'_{s,t} := \begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} \boxed{JW_{s,t}} \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} \boxed{\mathcal{JW}_{s,t}} \begin{array}{c} \dots \\ \text{---} \\ \dots \\ \text{---} \end{array} \tag{3.18}$$

(We leave it as an exercise to the reader to check the equality of the second two diagrams, using (3.10).) Various features of $JW_{s,t}$ and $JW'_{s,t}$ follow immediately from the properties of $\mathcal{JW}_{s,t}$ and ${}^2\mathcal{JW}_{s,t}$ listed above. For example, they are both killed by all pitchforks on bottom, and

$$\begin{array}{c} \text{---} \\ \dots \\ \text{---} \end{array} JW'_{s,t} = \pi_{s,t}. \tag{3.19}$$

As another example, $JW_{s,t}$ is invariant under rotation by two strands.

$$(3.20)$$

Remark 3.8. One nice feature of $JW_{s,t}$ and $JW'_{s,t}$ is that their coloration does not depend on the parity of m_{st} (as opposed to $\mathcal{J}W_{s,t}$, see Remark 3.4).

3.4. Checking the two-color relations

There are a number of versions of the Jones–Wenzl relation (3.3) we will use, which are all equivalent only using cyclicity. Rotating one red strand of (3.3) around (analogously to (3.20)) we obtain

$$(3.21)$$

which is the version we check in the following lemma.

Lemma 3.9. Λ preserves the relation (3.21).

Proof. For any subexpression \underline{e} of (s, t, \dots, s) , $[e]$ is killed by both sides of the equation unless \underline{e} is a subexpression for the identity. It follows from the definition of Λ (more precisely, from (2.7c) and (2.28)) that for any such subexpression, under the morphism of the left-hand side of the lemma we have

$$[e] \mapsto \pi_{s,t}[\emptyset].$$

So we are done if we can verify the same statement on the right hand side.

Under the RHS of (3.21), let us write

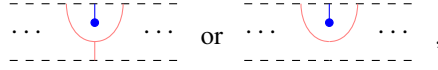
$$[e] \mapsto g_e[\emptyset]$$

for some $g_e \in Q$. We wish to show that $g_e = \pi_{s,t}$ for all subexpressions \underline{e} for the identity.

First we check this when $\underline{e} = (0, 0, \dots, 0)$. Using (2.7c), the LHS of (3.19) will (after applying Λ) send $[\emptyset]$ to $g_e[\emptyset]$ as well. Clearly the RHS of (3.19) will send $[\emptyset]$ to $\pi_{s,t}[\emptyset]$. Thus $g_e = \pi_{s,t}$ if (3.19) is preserved by Λ . Now $JW'_{s,t}$ is built from univalent and trivalent

vertices. Moreover, (3.19) is the consequence of the one-color relations, see Remark 3.7. Since we have already checked that Λ preserves the one-color relations, (3.19) is already proven to hold after applying Λ .

Now, let P (for pitchfork) denote a morphism of the form



perhaps with colors interchanged, and with additional vertical strands (identity maps) to the left and right. Death by pitchfork implies that

$$\begin{array}{c} \boxed{JW'_{s,t}} \\ \text{---} \\ \boxed{P} \end{array} = 0. \quad (3.22)$$

Death by pitchfork is also a consequence of the one-color relations, so we already know that Λ kills the LHS of (3.22).

Let $\underline{e} = (\underline{e}_1, 1, 0, 0, \underline{e}_2)$ and $\underline{e}' = (\underline{e}_1, 0, 0, 1, \underline{e}_2)$ be subexpressions for the identity which differ only in the three strands meeting the pitchfork P , which is a pitchfork of the first kind pictured above. Let $\underline{f} = (\underline{e}_1, 1, \underline{e}_2)$ be the corresponding subexpression of the source of P . Let w be the endpoint of the partial subexpression \underline{e}_1 . Then using (2.7c) and (2.7d) we see that (3.7) sends

$$[\underline{f}] \mapsto w\left(\frac{1}{\alpha_s}\right)[\underline{e}'] - w\left(\frac{1}{\alpha_s}\right)[\underline{e}] \mapsto \frac{1}{w(\alpha_s)}(g_{\underline{e}'} - g_{\underline{e}})[\emptyset].$$

The first arrow is the action of P , and the second arrow the action of $JW_{s,t}$. Since the result must be zero, and R is a domain, we deduce that

$$g_{\underline{e}} = g_{\underline{e}'}.$$

Using very similar arguments, we deduce that if two subexpressions \underline{e} and \underline{e}' are related by one of the following moves

$$\begin{aligned} (\dots, 1, 0, 0, \dots) &\longleftrightarrow (\dots, 0, 0, 1, \dots) \\ (\dots, 1, 0, 1, \dots) &\longleftrightarrow (\dots, 0, 0, 0, \dots) \end{aligned}$$

then their images $g_{\underline{e}}[\emptyset]$ and $g_{\underline{e}'}[\emptyset]$ under $\Lambda(JW_{s,t})$ agree. One checks easily that any subexpression of (s, t, \dots, s) for the identity may be transformed via a sequence of these moves to the subexpression $(0, 0, \dots, 0)$, whence the result. \square

Remark 3.10. The proof above used the “pre-composition trick”: to understand the images of various subexpressions \underline{e} , we pre-compose with a map which isolates these

subexpressions. For example, the image of $[00 \dots 0]$ can be isolated using precomposition with the all-dots map, as in (3.19). Precomposing with pitchforks isolates pairs of subexpressions.

The following is 2-color associativity:

$$(3.23)$$

(We draw the relation for $m_{st} = 3$. The following lemma and its proof holds in general.)

Lemma 3.11. Λ preserves (3.23).

The proof the lemma will be via the “precomposition trick”. However in order to apply it we need the following:

Lemma 3.12. *Precomposing (3.23) with a dot on either the first, second or last strand yields a relation which is already a consequence of the other relations: the one-color, cyclicity, and Jones–Wenzl relations.*

Remark 3.13. In fact, the lemma remains true for any dot placed on the lower strands. However the proof is more involved, and we will only need the easier version stated here.

Proof. First we claim that placing a dot below a $2m$ -valent vertex gives one term where “the dot pulls straight through the $2m$ -valent vertex,” and all the remaining terms have pitchforks. That is,

$$(3.24)$$

Here P denotes a linear combination of diagrams, and each diagram has a pitchfork on top. Moreover, if the rightmost blue strand is twisted down like so,

$$(3.25)$$

then each diagram in P' has a pitchfork on the bottom. The formulas (3.24) and (3.25) are very useful in combination with “death by pitchfork” for diagrams with multiple $2m$ -valent vertices.

In fact, (3.24) and (3.25) can both be deduced from (3.8); the term where the dot pulls through corresponds to the identity diagram in the original Jones–Wenzl projector. To get from (3.8) to (3.24) and (3.25) is a relatively straightforward exercise in diagrammatics and adjunction. First, absorb the dot into the $2m$ -valent vertex using (3.21), and define $JW'_{s,t}$ using the last diagram in (3.18). We leave the rest to the reader.

Now precompose the RHS of (3.23) with a dot morphism on the first, second, or last strand. Resolve the $2m$ -valent vertex which meets this dot by (3.24). Because $2m$ -valent vertices satisfy death by pitchfork, every diagram in P is orthogonal to the other $2m$ -valent vertex, so only the term where the dot pulls straight through will survive. It is now relatively easy to verify that precomposing (3.23) with a dot morphism on the first, second, or last strand yields a relation which follows from the previous relations; we leave this as an exercise to the reader. \square

Remark 3.14. The coloration in (3.24) and (3.25) depends on the parity of m_{st} , and they have been drawn above as though m_{st} were odd. If m_{st} were even then the dot on the RHS of (3.24) would be red rather than blue.

Proof of Lemma 3.11. Suppose that \underline{e} is a subexpression of (s, s, t, s, \dots) . We check as usual that $[\underline{e}]$ has the same image under the morphisms presented on the left-hand and right-hand sides, for all choices of \underline{e} .

First we check the case $\underline{e} = (1, 1, \dots, 1)$. We will use (2.32) and (2.31). On the left hand side its image is

$$[111 \dots 11] \mapsto [01 \dots 11] \mapsto [11 \dots 10].$$

On the right hand side its image is

$$[11 \dots 111] \mapsto [11 \dots 111] \mapsto [11 \dots 111] \mapsto [111 \dots 10].$$

We claim that all other cases can be solved using the precomposition trick, and Lemma 3.12. Indeed, by Lemma 3.12 and the fact that we have already checked that Λ satisfies the one-color, cyclicity, and Jones–Wenzl relations, we deduce that if \underline{e} has a zero in its first, second or last position, then the images under both sides of (3.23) agree.

Finally, note that both sides of (3.23) are killed by most pitchforks on bottom. Each pitchfork has a dot morphism in the middle of its three strands, and this central dot can appear on any strand but the first, second, or last. As in the proof of Lemma 3.9, this allows us to equate the images of various subexpressions under the transformations

$$\begin{aligned} (\dots, 1, 0, 0, \dots) &\longleftrightarrow (\dots, 0, 0, 1, \dots) \\ (\dots, 1, 0, 1, \dots) &\longleftrightarrow (\dots, 0, 0, 0, \dots) \end{aligned}$$

of the source, where the central zero is on one of the strands except the first, second, or last. Using these transformations, we can get from any subexpression \underline{e} with a zero

anywhere to some subexpression with a zero in the first, second, or last position. The lemma follows. \square

3.5. Zamolodchikov (or 3-color) relations

The remaining relations in \mathcal{H} are associated with finitary rank 3 parabolic subgroups. The possible types of relations are: $I_2(m) \times A_1$, A_3 , B_3 , and (the forbidden) H_3 . These relations can be found in [17, §1.4.3].

That the relation of type $I_2(m) \times A_1$ is preserved by Λ is immediate. Let $\{s, t\}$ be the simple reflections in $I_2(m)$, and u be the commuting simple reflection in A_1 . The relation is an equality of two morphisms from (u, s, t, s, \dots) to (t, s, t, \dots, u) which only use vertices $G_{s,t}$, $G_{u,s}$, and $G_{u,t}$. If $e \in \{0, 1\}$ and $\underline{f} \subset (s, t, s, \dots)$ and $\underline{f}' \subset (t, s, t, \dots)$, then we claim that after applying Λ both sides send

$$[e \underline{f}] \mapsto \sum_{\underline{f}'} (G_{s,t})_{\underline{f}}^{\underline{f}'} [e' \underline{f}'].$$

The proof is relatively straightforward using Example 2.13 and the fact that u fixes $\pi_{s,t}$ and $\zeta(\underline{f}')$ for any \underline{f}' .

The relations of type A_3 and B_3 are complicated, but (unlike the two-color relations) they do not come in families, so one need not provide a general argument. Both relations have been checked by computer. For example, the B_3 relation says that two diagrams are equal; both diagrams have 9 strands on bottom and top, and consist of thirteen $2m$ -valent vertices. Checking that this relation is preserved by Λ amounts to checking that two different thirteen-fold products of sparse $2^9 \times 2^9$ matrices are equal.

Computer computation with the formulas in this paper is possible with the Magma package ASLoc <https://github.com/joelgibson/ASLoc>. The reader is referred to [19] for more detail on the implementation. The reader wishing to check the Zamolodchikov relations in types A_3 and B_3 should go to the above link, install the package, and read the section on “Verifying relations”.

Remark 3.15. The reader may be consoled that the relations of type A_3 and B_3 can also be checked with the precomposition trick, but the details are technical and not worth repeating here. One can prove that placing a dot below these relations yields an equality which already follows from the other relations. Thus the diagrams must be equal on any subexpression with a zero. Meanwhile, both sides send $[11 \dots 1] \mapsto [11 \dots 1]$ because each $2m$ -valent vertex $G_{s,t}$ does the same.

Note that there is considerable subtlety in the Zamolodchikov relations, and sweeping it under the rug entirely would be a disservice. The fact that precomposition with a dot gives an equality should be viewed as rather surprising! For example, the Zamolodchikov

relation in type A_3 or B_3 can be stated as follows. There is a particular reduced expression for the longest element, and there are two different sequences of braid relations one can apply to get an “antipodal” reduced expression for the longest element. Turning these sequences of braid relations into sequences of $2m$ -valent vertices, the two resulting diagrams are equal. However, this statement of equality is false if we chose the wrong reduced expression to begin with, and one can check this by precomposing with a dot and confirming that the diagrams are *not* equal.

3.6. The status of type H_3

This section is an informal discussion of the current status of the Hecke category in types containing H_3 . For further discussion see [17, §1.4.3].

Given a reduced expression, and a sequence of braid relations to another reduced expression, one obtains a diagram by transforming each braid relation into a $2m$ -valent vertex. Such a diagram will be called a *rex move*. By construction, rex moves induce the identity map $[11 \dots 1] \mapsto [11 \dots 1]$. By applying the functor to algebraic Soergel bimodules, we obtain morphisms between Bott–Samelson bimodules which we also call rex moves. We assume for this discussion that Soergel bimodules categorify the Hecke algebra and obey the Soergel hom formula, since this is the behavior that we wish the diagrammatic Hecke category to mimic.

Let X and Y be two rex moves with the same source and target, both reduced expressions for an element $w \in W$. Let $P(X, Y)$ be the property that $X - Y$ is a linear combination of double leaves which factor through lower terms. Using the Soergel Hom formula, one can deduce a priori that the $P(X, Y)$ holds for morphisms between Soergel bimodules. Given $P(X, Y)$ for all comparable X and Y , [17] gives a diagrammatic proof that double leaves span the diagrammatic Hecke category. Moreover, all (non-trivial) cycles in reduced expression graphs come from the longest element in a rank 3 parabolic subgroup (this has been noticed by several authors, and is the subject of [18]). Thanks to this one can prove as in [17] that $P(X_0, Y_0)$ for a single pair X_0 and Y_0 (certain rex moves between reduced expressions for the longest element, which are not homotopic as paths in the reduced expression graph) is sufficient to deduce $P(X, Y)$ for all pairs X and Y . The goal of the Zamolodchikov relation is to impose $P(X_0, Y_0)$ for one such pair.

In types $I_2(m) \times A_1$ and A_3 and B_3 , one is lucky: there exists a pair X_0 and Y_0 such that $X_0 - Y_0 = 0$. No lower terms are needed for this difference of rex moves. As noted in Remark 3.15, this requires a special choice of X_0 and Y_0 , and there is no a priori reason it should be true.

In type H_3 , it was checked by computer that there is no pair X_0 and Y_0 such that $X_0 - Y_0 = 0$. (It somewhat non-trivial to check that this is the case. Proceeding naively

involves the multiplication of many sparse matrices with $2^{15} = 32768$ rows and columns, which is far beyond our hardware’s capabilities. However, for each pair X_0 and Y_0 we were able to find a block of this sparse matrix for which $X_0 - Y_0$ was nonzero.)

To define the Hecke category, we would need to find a linear combination L of double leaves through lower terms, such that $X_0 - Y_0 = L$, and impose this as the H_3 Zamolodchikov relation. Any relation of this form is sufficient to imply that double leaves span the diagrammatic category. However, to prove that double leaves are linearly independent one requires a functor (either the functor to bimodules, or the functor Λ defined herein), and for this functor to be well-defined, the relation $X_0 - Y_0 = L$ must hold. Picking some X_0 and Y_0 , we were unable to discover the correct linear combination L so that $X_0 - Y_0 = L$ is preserved by Λ . This remains an interesting computational challenge.

Such a relation is all that is missing for a definition of the diagrammatic Hecke category in types containing H_3 .

4. Heuristics for the definition of the localization functor

Let us give some details on where the definition of Λ came from, by reminding the reader of some technology developed in [17, §5.4 and following].

We continue to use B_s to refer both to a Bott–Samelson bimodule, and to a generating object in the diagrammatic Hecke category. Similarly, we use R to refer both to the regular R -bimodule, and the monoidal unit of the Hecke category.

4.1. Mixed calculus and the splitting of B_s

Firstly, the category $\Omega_Q W$ (or rather, its monoidal strictification) was given a diagrammatic presentation by generators and relations in [18]. Instead of having objects r_w for each $w \in W$, the objects are sequences $\underline{w} = (s_1, \dots, s_d)$ of simple reflections, meant to embody the object $r_{s_1} \otimes \dots \otimes r_{s_d}$. Below we draw the object r_s (or rather, its identity map) as a squiggly line, of the same color as the solid line which represents B_s :

$$\text{id}_{B_s} := \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}, \quad \text{id}_{r_s} := \begin{array}{c} \text{---} \\ \color{red}{\text{~\textasciitilde\textasciitilde\textasciitilde}} \\ \text{---} \end{array}.$$

Morphisms in the category are linear combinations of diagrams built out of squiggly cups and caps, and squiggly $2m$ -valent vertices. These diagrams are subject to the relations spelled out in [17, §4.1] (see also [18, §6]). In particular, squiggly cups and caps define isomorphisms between the unit and $r_s^{\otimes 2}$.

Then in [17, §5.4] we developed a diagrammatic calculus which mixed the objects B_s and r_s . For sake of discussion we call this the *mixed calculus*. This is a model for the

monoidal subcategory of R -bimodules containing both the standard bimodules R_s and the Bott–Samelson bimodules B_s . In addition to the diagrams appearing in \mathcal{H} and in $\Omega_Q W$ individually, we also have new generating morphisms

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}, \quad (4.1)$$

both of degree +1.

Remark 4.1. In the category of R -bimodules there are important non-split short exact sequences

$$0 \longrightarrow R_s(-1) \longrightarrow B_s \longrightarrow R_{\text{id}}(+1) \longrightarrow 0$$

and

$$0 \longrightarrow R_{\text{id}}(-1) \longrightarrow B_s \longrightarrow R_s(+1) \longrightarrow 0$$

which describe how B_s is filtered by standard/costandard bimodules. The maps between B_s and R_{id} are maps between Bott–Samelson bimodules, so they appear already in \mathcal{H} as the two dot maps. The maps between B_s and R_s are new in this mixed calculus, and they are the new maps of (4.1). After localization to Q , these two short exact sequences “split each other” up to the invertible element α_s (see [17, §3.6]).

Inside the mixed calculus we have the following relations (these are not all of them, just enough to illustrate a point):

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \alpha \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}, \quad \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \alpha \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}, \quad (4.2)$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array} = 0 = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \text{---} \end{array}, \quad (4.3)$$

$$\alpha \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array}. \quad (4.4)$$

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} f = \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} s f \quad (4.5)$$

Now pass to the localization Q where $\frac{1}{\alpha}$ is defined. Combining the relations above we see that

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} = \frac{1}{\alpha} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \\ | \\ \text{---} \end{array} + \frac{1}{\alpha} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \quad (4.6)$$

is actually a decomposition of id_{B_s} into the sum of two orthogonal idempotents. More precisely, set

$$i_s = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array}, \quad p_s = \frac{1}{\alpha} \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array}, \quad i_{\text{id}} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array}, \quad p_{\text{id}} = \frac{1}{\alpha} \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \\ \text{---} \end{array}. \quad (4.7)$$

Then these are the inclusion and projection maps of complementary direct summands:

$$p_s i_s = \text{id}_{r_s}, \quad p_{\text{id}} i_{\text{id}} = \text{id}_{r_{\text{id}}},$$

$$p_s i_{\text{id}} = 0, \quad p_{\text{id}} i_s = 0,$$

$$\text{id}_{B_s} = i_s p_s + i_{\text{id}} p_{\text{id}}.$$

This makes explicit the direct sum decomposition $B_s \otimes_R Q \cong r_s \oplus r_{\text{id}}$, and we will use these projection and inclusion maps below.

Remark 4.2. While we will not recall all the relations in the mixed calculus, we wish to warn the reader of an important technicality which appears in the mixed calculus but not in \mathcal{H} or in $\Omega_Q W$ itself. Namely, the new morphisms of (4.1) are *not* mates under adjunction. Instead, this is only true up to a sign. We have relations without signs

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}. \quad (4.8)$$

and *with* signs

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} = - \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = - \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}. \quad (4.9)$$

(Note that both pairs of relations have a vertical symmetry.)

As a consequence the new morphisms (4.1) are mates under adjunction only up to a sign:

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} = - \begin{array}{c} \text{---} \\ | \\ \text{---} \\ \text{---} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array}. \quad (4.10)$$

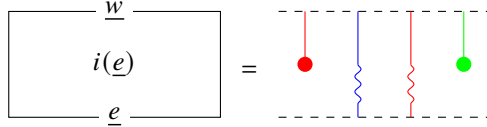
(Another pair of relations is obtained by performing a vertical flip.)

4.2. Localizing morphisms

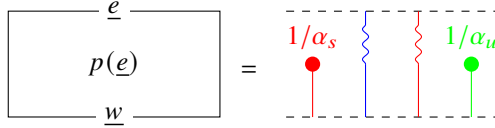
Here we elaborate on the content of [17, §5.5].

For a word $\underline{w} = (s_1, \dots, s_d)$ in S and a subexpression $\underline{e} \subset \underline{w}$, we take the tensor product of the inclusion and projection maps of (4.7) to produce the inclusion and projection maps for the summand $r_{\underline{w}^e} \subset B_{\underline{w}}$ (see (2.6)). These were denoted $i(\underline{e})$ and $p(\underline{e})$ in Section 2.1.

Example 4.3. For example, if $\underline{w} = (s, t, s, u)$ and $\underline{e} = (0, 1, 1, 0)$ then

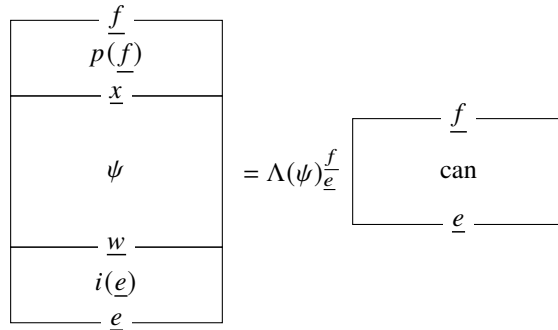


and



In [17] and [18] we have proven that every morphism in the mixed calculus whose boundary consists only of squiggly lines is itself a Q -linear combination of squiggly diagrams (i.e. one does not need the new morphisms from (4.1) or any diagrams from \mathcal{H} , but only squiggly cups, caps, and $2m$ -valent vertices.) Moreover, any two squiggly diagrams with the same boundary are equal. Squiggly diagrams are isomorphisms, and any squiggly diagram represents the same isomorphism; we refer to any morphism represented by a squiggly diagram as the canonical isomorphism from its source to its target.

If $\psi: B_{\underline{w}} \rightarrow B_{\underline{x}}$ is a morphism between Bott–Samelson objects, and $\underline{e} \subset \underline{w}$ and $\underline{f} \subset \underline{x}$, then the entry in column \underline{e} and row \underline{f} of $\Lambda(\psi)$ should be the coefficient of the canonical isomorphism in the composition $p(\underline{f}) \circ \psi \circ i(\underline{e})$:



Here can denotes the canonical isomorphism.

Remark 4.4. If the reader desires a mnemonic, they should remember that all the complicated business happens on top of morphisms! This is because we have included the factors $\frac{1}{\alpha}$ into the projection maps, not the inclusion maps.

We can compute these matrix entries explicitly using the mixed calculus, as we do in several examples below.

Example 4.5. The dot map $B_s \rightarrow R = r_{\text{id}}$ from (2.7c) should be precomposed with the two inclusion maps i_{id} and i_s to determine its matrix entries after applying Λ . We have

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \alpha \quad \text{and} \quad \begin{array}{c} \bullet \\ | \\ \text{---} \\ \text{---} \end{array} = 0.$$

This gives us the matrix $\begin{pmatrix} \alpha & 0 \end{pmatrix}$ from (2.7c).

Example 4.6. The dot map $R \rightarrow B_s$ from (2.7b) should be postcomposed with the two projection maps p_{id} and p_s to determine its matrix entries after applying Λ . The calculation is almost identical to the previous example (upside-down) except that p_{id} and p_s have an additional factor of $\frac{1}{\alpha}$, whence the matrix in (2.7b).

For the trivalent vertices, the reader is encouraged to use relation (4.4) to prove that

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \bullet \\ | \\ \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \quad (4.11)$$

From this it becomes easy to compute the matrices associated to trivalent vertices, see e.g. the proof of [17, (5.29)].

When a $2m$ -valent vertex is precomposed with $i(\underline{e})$ and \underline{e} contains a zero somewhere, then the $2m$ -valent vertex meets a dot, and we can apply the Jones–Wenzl relation (3.3). Thus we can continue our computation using only the one-color relations. If instead it is precomposed with $i(\underline{11 \dots 1})$ then we can use [17, (5.28)], which gives the relationship between solid and squiggly $2m$ -valent vertices. In formulas [17, (5.28)] says that

$$G_{s,t} \circ i(\underline{11 \dots 1}) = i(\underline{11 \dots 1}) \circ \text{can},$$

where in this case can is the squiggly version of $G_{s,t}$. Consequently

$$p(\underline{11 \dots 1}) \circ G_{s,t} \circ i(\underline{11 \dots 1}) = \text{can}$$

and $(G_{s,t})_{(\underline{11 \dots 1})}^{(\underline{11 \dots 1})} = 1$, agreeing with (2.31).

5. On the error(s) in our previous work

Our intention in [17] was to define the diagrammatic category \mathcal{H} and prove that it behaved nicely without needing to use the functor \mathcal{F} to Soergel bimodules or any theorems about Soergel bimodules.

Suppose however that there is a functor \mathcal{F} from \mathcal{H} to Soergel bimodules. Then one should have the following commutative diagram:

$$\begin{array}{ccc}
 \mathcal{H} & \xrightarrow{\mathcal{F}} & \mathbb{S}\text{Bim} \\
 \downarrow \Lambda & & \downarrow (-) \otimes_R Q \\
 \Omega_Q W & \xrightarrow{\mathcal{F}_{\text{std}}} & \text{StdBim}_Q
 \end{array} \tag{5.1}$$

The functor \mathcal{F}_{std} was discussed in the introduction; it is always a well-defined faithful functor, and it is an equivalence when the realization is faithful. The functor $(-) \otimes_R Q$ is also always well-defined, and it is faithful when hom spaces between Soergel bimodules are free³ as right R -modules. One could use this commutative diagram to attempt to define either \mathcal{F} or Λ in the presence of the other. For example, if Λ is well-defined, then $\mathcal{F}_{\text{std}} \circ \Lambda$ is a functor $\mathcal{H} \rightarrow \text{StdBim}$. If hom spaces between Soergel bimodules are free then $(-) \otimes_R Q$ is faithful, and one can lift $\mathcal{F}_{\text{std}} \circ \Lambda$ to a functor \mathcal{F} if and only if the image of $\mathcal{F}_{\text{std}} \circ \Lambda$ lies inside the image of $(-) \otimes_R Q$, i.e. if the map between Q -bimodules can be defined without the use of fractions. This lift is unique if it exists.

Remark 5.1. In [32] Soergel proves that hom spaces between Soergel bimodules are free when the realization is reflection faithful and the base ring \mathbb{k} is an infinite field for which $2 \neq 0$. The assumption that the field is infinite can easily be removed (it is only used to identify functions with elements in the symmetric algebra). The other assumptions, particularly reflection faithfulness, appear more serious, and we do not know in what generality to expect them to be true. This is one of the motivations for studying the diagrammatic incarnation of the Hecke category, rather than Soergel bimodules. For an exciting recent approach which is more bimodule theoretic, see [3].

In [17] we did not define the functor Λ explicitly as we have done in this paper. Instead we defined the mixed calculus, and used it to argue that Λ could be defined explicitly (as

³In fact, for faithfulness of $(-) \otimes_R Q$, we only need that hom spaces are torsion free. Torsion freeness is often much easier to establish. However for applications an essential role is played by Soergel's hom formula, which only makes sense when hom spaces are free.

in Section 4) using the mixed calculus. In our own unpublished computations we had computed the matrices associated to the univalent, trivalent, and $2m$ -valent vertices for $m \leq 5$, and we checked all the relations involving these generators; this much we asserted in [17]. At that time, we did not know the formula (2.30) for the $2m$ -valent vertex in general, and as such, we could not have checked the two color relations explicitly for all m .

Instead, we argued in [17] that the two color relations were satisfied by referencing the earlier work [8], which defined and studied \mathcal{H} for dihedral groups. In doing so, we implicitly (and unwittingly) defined Λ as the unique lift of $(-) \otimes_R Q \circ \mathcal{F}$, where the functor \mathcal{F} was also defined on the $2m$ -valent vertex in [8]. The reason this argument is faulty is that \mathcal{F} was only defined under certain assumptions on the realization (which we recall below), while we claimed that Λ was defined in greater generality. We also made an error in [17, §5.3] in that we only assumed Demazure Surjectivity when we claimed that \mathcal{F} was well-defined, forgetting that additional assumptions were used in [8].

To set the record straight, here is the list of assumptions that should be made about realizations in order to use various theorems, now that this paper has constructed Λ in generality.

5.1. Definition of \mathcal{H}

Which assumptions do we need for \mathcal{H} to be well defined? All the generators and relations in the category make sense without any assumptions, with the exception of the Jones–Wenzl relation. For this relation to make sense we need the existence of a particular rotatable Jones–Wenzl projector in the two-colored Temperley–Lieb category, one for each finite dihedral parabolic subgroup. If these exist, then \mathcal{H} is well defined.

The existence and rotatability of Jones–Wenzl projectors is a relatively mild constraint on a realization, and is primarily a condition on the base ring \mathbb{k} . For example, suppose that the restriction of the realization to a finite dihedral parabolic subgroup generated by $\{s, t\}$ (with $m = m_{st} < \infty$) is a faithful representation of that dihedral subgroup. Then $JW_{(m-1)_s}$ and $JW_{(m-1)_t}$ are defined and rotatable so long as \mathbb{k} is a field, as we prove in Theorem 6.21. Thus if a realization over a field is *dihedrally faithful* (meaning that it is faithful after restriction to any dihedral parabolic subgroup), then \mathcal{H} is well defined.

However, giving precise algebraic conditions for the existence of Jones–Wenzl projectors, and for their rotatability, is extremely subtle! Conditions have been put forth in previous works [8, 13], with some handwaving arguments that we no longer believe are justified (though many of these arguments suffice in all but the most degenerate realizations). In Section 6 we do the best we can at the moment, providing necessary conditions on \mathbb{k} . In particular, in Section 6.8 we prove (generalizing the argument of [8])

that only even-balanced realizations can admit rotatable Jones–Wenzl projectors. Thus even-balancedness is a prerequisite to the well-definedness of \mathcal{H} .

Remark 5.2. The condition that a Jones–Wenzl projector is negligible is much more common in the literature. If a projector is rotatable it is also negligible, but not vice versa, see Remark 6.24. The paper [8] often works under the assumption that the realization is dihedrally faithful (as defined above). Under this assumption, when the base ring is a domain, the Jones–Wenzl projector in question is both rotatable and negligible, and the realization is even-balanced (we show this in Section 6). In papers that followed, which did not make the dihedrally faithful assumption, sometimes the rotatability assumption was erroneously replaced with weaker assumptions like negligibility or even-balancedness.

Remark 5.3. In this paper, we give examples of realizations, all of which are not dihedrally faithful, and all of which have negligible Jones–Wenzl projectors JW_{m-1} , where:

- The realization is not even-balanced, and the Jones–Wenzl projector is not rotatable (Example 6.22).
- The realization is even-balanced, and the Jones–Wenzl projector is not rotatable (Examples 6.32 and 6.29).
- The realization is even-balanced, and the Jones–Wenzl projector is rotatable (Example 6.31).

Because non-dihedrally faithful realizations with rotatable Jones–Wenzl projectors do exist, we make no assumptions of dihedral faithfulness in this paper.

Remark 5.4. Although one might attempt to define \mathcal{H} even in the absence of a rotatable Jones–Wenzl projector, it fails to satisfy enough crucial properties that we do not believe it deserves the name “Hecke category.” For example, its Grothendieck group is not isomorphic to the Hecke algebra.

Remark 5.5. Note that Demazure surjectivity is not a necessary requirement for the well-definedness of \mathcal{H} , or for the construction of the functor Λ in this paper.

5.2. The Soergel categorification theorem

Which assumptions on the realization do we need to assert that double leaves form a basis for \mathcal{H} , and hence that the Soergel Categorification Theorem holds for \mathcal{H} ? The argument that double leaves are linearly independent (because they remain so after applying Λ) requires no additional assumptions. For our diagrammatic arguments that double leaves

span, we use polynomial forcing extensively, and this requires that R is a Frobenius extension over R^s (with Frobenius trace ∂_s) for each simple reflection s . This is equivalent to the assumption of Demazure Surjectivity.

5.3. Existence of \mathcal{F}

Which assumptions on the realization do we need to define \mathcal{F} ? We certainly need Demazure surjectivity, because the image of one of the dots under \mathcal{F} requires the existence of dual bases for R over R^s . Beyond this, the essential difficulty is to provide an explicit formula for the image of the $2m$ -valent vertex directly.

Recent progress on this issue has been made by Abe [1]. Rather than try to define an explicit formula for the morphism between Bott–Samelson bimodules (which seems very hard), he instead checks that a morphism given by the explicit formulas of Lemma 2.11 preserves the Bott–Samelson bimodules. (In other words, he checks that a morphism between localizations arises as the localization of a morphism.) His proof requires [1, Assumption 1.1], which in the language of two coloured quantum numbers (see Section 6.1) is the requirement that

$$\begin{bmatrix} m_{st} \\ k \end{bmatrix}_s = \begin{bmatrix} m_{st} \\ k \end{bmatrix}_t = 0$$

$k = 1, 2, \dots, m_{st} - 1$ (and all $s, t \in S$). This assumption is natural (and holds in all Kac–Moody realizations, as is easily checked).

The original approach to defining the image of the $2m$ -valent was found in [8]. There, the $2m$ -valent vertex is described as an explicit composition of more elementary morphisms between singular Soergel bimodules. This description has many uses, but minimizing assumptions on the realization is not one of them. These elementary morphisms are built using various Frobenius extension structures, and require that the ring extensions $R^{s,t} \subset R^s, R^t \subset R$ are Frobenius extensions. The underlying assumption⁴ is *dihedral Demazure surjectivity*, which states that R is a Frobenius extension over $R^{s,t}$ with Frobenius trace $\partial_{s,t}$. Dihedral Demazure surjectivity implies that the action of the finite dihedral group generated by $\{s, t\}$ is faithful, because if it factors through a smaller dihedral group then $\partial_{s,t} = 0$.

⁴In [8] one also makes an assumption called (\star) from [16, Definition 1.7]. This assumption is used to ensure that the diagrammatics which describe singular Soergel bimodules is well-behaved, but is not strictly necessary for defining the image of the $2m$ -valent vertex.

6. On two-colored Jones–Wenzl projectors

In this section we develop some of the facts we need about Jones–Wenzl projectors; in particular, the two-colored Jones–Wenzl projectors which appear in the two-colored Temperley–Lieb category $2\mathcal{TL}$. The literature on Jones–Wenzl projectors is quite rich, but less so in the two-colored case. Many of the facts in this section were stated in [8] but without proof, by analogy with the uncolored case. We will prove the facts we need, and clearly state some additional properties as conjectures⁵. Starting in this section, we develop the needed technology to address the unbalanced case. We hope that this section fills some gaps in the literature.

6.1. Two-colored quantum numbers

Let $A = \mathbb{Z}[x_s, x_t]$ be the integral polynomial ring in two variables. We define the *two-colored quantum numbers*, which are elements of A , as follows. First set $[2]_s = x_s$ and $[2]_t = x_t$ and $[1]_s = [1]_t = 1$. Now define the remaining quantum numbers inductively by the formulas

$$[n+1]_t = [n]_s [2]_t - [n-1]_t, \quad [n+1]_s = [n]_t [2]_s - [n-1]_s. \quad (6.1)$$

For example, $[3]_s = [3]_t = [2]_s [2]_t - 1$, and $[4]_s = [2]_s^2 [2]_t - 2[2]_s$. One can also use these same formulas to define $[n]_s$ and $[n]_t$ for $n \leq 0$, and one can easily confirm that

$$[0]_s = [0]_t = 0, \quad (6.2)$$

$$[-n]_s = -[n]_s, \quad [-n]_t = -[n]_t. \quad (6.3)$$

Indeed, these formulas hold for $n = 0$ and $n = 1$, and follow by induction since both $[n]$ and $-[-n]$ satisfy the recursive formulas (6.1).

A straightforward induction proves that

$$[k]_s = [k]_t \quad \text{whenever } k \text{ is odd}, \quad (6.4)$$

so we may merely write it as $[k]$ rather than $[k]_s$ or $[k]_t$.

Many facts about ordinary quantum numbers will generalize easily to the two-colored case. For example, if k and n are odd one can prove that

$$[k][n] = [k+n-1] + [k+n-3] + \cdots + [k-n+3] + [k-n+1]. \quad (6.5a)$$

Meanwhile, if k is even and n is odd then

$$[k]_s [n] = [k+n-1]_s + [k+n-3]_s + \cdots + [k-n+3]_s + [k-n+1]_s. \quad (6.5b)$$

⁵Most of these conjectures are within reach, but we did not feel it was appropriate for this paper.

Finally, if both k and n are even then

$$[k]_s[n]_t = [k+n-1] + [k+n-3] + \cdots + [k-n+3] + [k-n+1]. \quad (6.5c)$$

All these results can be proven by induction en masse. An additional consequence is that $[k]_s$ divides $[n]_s$ whenever k divides n . For example, if k is even then $[2k]_s = [k]_s([k+1] - [k-1])$, and if k is odd then $[2k]_s = [k]_s([k+1]_s - [k-1]_s)$. We leave these elementary properties as exercises for the reader.

Because of the divisibility of quantum numbers, fractions like $\frac{[8]_s}{[2]_s}$ can be viewed as polynomials in A as well. In fact, if k is even then

$$\frac{[k]_s}{[2]_s} = \frac{[k]_t}{[2]_t} = [k-1] - [k-3] + [k-5] - \cdots \pm 1. \quad (6.6)$$

Thus the dependence of an even quantum number on the coloring is no more than a factor of either $[2]_s$ or $[2]_t$.

Definition 6.1. Define the *two-colored quantum binomial coefficients* as

$$\begin{bmatrix} n \\ k \end{bmatrix}_s := \frac{[n]_s[n-1]_s \cdots [n-k+1]_s}{[1]_s[2]_s \cdots [k]_s} \in A. \quad (6.7)$$

Because A is a UFD, one can use standard arguments to prove that binomial coefficients are indeed elements of A .

6.2. Two-colored quantum numbers modulo $[m]$

Let \mathbb{k} be an A -algebra. The *two-colored quantum numbers* in \mathbb{k} are the specializations of the two-colored quantum numbers in A . For any realization of a Coxeter group with base ring \mathbb{k} , and any pair of simple reflections $s \neq t \in S$, we obtain an A -algebra structure on \mathbb{k} by specializing

$$x_s \mapsto -a_{st} = -\alpha_s^\vee(\alpha_t), \quad x_t \mapsto -a_{ts} = -\alpha_t^\vee(\alpha_s). \quad (6.8)$$

One of the requirements of a realization is that both $[m]_s$ and $[m]_t$ specialize to 0 when $m = m_{st} < \infty$.

Let us discuss the consequences of imposing the relation $[m]_s = [m]_t = 0$ on the generic ring A . Let $A_m = A/([m]_s, [m]_t)$. Note that A_m is not a domain and not local when m is not prime. For example, $[10]_s$ has three irreducible factors: $[2]_s$, $[5]$, and $\frac{[10]_s}{[2]_s[5]} = [5] - 2[3] + 2$. In particular, we can specialize further from A_m to A_k whenever k divides m .

In the same way one proves (6.3) one can deduce that

$$[m+k]_s = -[m-k]_s, \quad [m+k]_t = -[m-k]_t. \quad (6.9)$$

Using the inductive definition of quantum numbers, but working down from m rather than up from 0, one can prove that

$$[m-1]_s[k] = [m-k]_s \text{ for } k \text{ odd, } [m-1]_s[k]_t = [m-k]_t \text{ for } k \text{ even.} \quad (6.10)$$

One can also prove this by applying (6.5) and cancelling terms with (6.9). In particular (taking $k = m-1$) we have

$$[m] = 0, m \text{ odd} \implies [m-1]_s[m-1]_t = 1, \quad (6.11)$$

$$[m]_s = [m]_t = 0, m \text{ even} \implies [m-1]^2 = 1. \quad (6.12)$$

Either way, $[m-1]_s$ and $[m-1]_t$ are invertible.

Remark 6.2. If m is odd then $[2]_s$ and $[2]_t$ are invertible, since $[2]_s$ divides $[m-1]_s$. Whenever $[2]_s$ and $[2]_t$ are invertible, $\frac{[2]_s}{[2]_t}$ is an invertible scalar factor which gives the ‘‘ratio’’ between $[2k]_s$ and $[2k]_t$ for any $k \geq 0$, in the sense that $[2k]_t \frac{[2]_s}{[2]_t} = [2k]_s$, see (6.6).

6.3. Faithful and unbalanced realizations

Definition 6.3. A realization is called *dihedrally faithful* if, for every $s \neq t \in S$ with $m_{st} \neq \infty$, the action on \mathfrak{h}^* of the finite dihedral group generated by $\{s, t\}$ is faithful.

Definition 6.4. A realization is called *balanced for the pair $\{s, t\}$* if $[m-1]_s = [m-1]_t = 1$. The realization is called *balanced* if it is balanced for every pair which generates a finite dihedral subgroup. It is *even-balanced* if it is balanced for every pair with m_{st} even, and *odd-balanced* if it is balanced for every pair with m_{st} odd.

Remark 6.5. Note that a balanced realization is not necessarily dihedrally faithful. For example, if $a_{st} = a_{ts} = -1$ then $[2] = 1$ and $[3] = 0$, so one has a balanced faithful realization if $m = 3$. Continuing, $[4] = [5] = -1$ and $[6] = 0$ so one has an unbalanced non-faithful realization if $m = 6$. Continuing, $[7] = [8] = 1$ and $[9] = 0$ so one has a balanced non-faithful realization if $m = 9$.

It was proven in [8, Claim 3.5, and p. 63] that $[k]_s = [k]_t = 0$ implies that $(st)^k$ acts by the identity on the span of α_s and α_t . In particular, if a realization is dihedrally faithful then either $[k]_s \neq 0$ or $[k]_t \neq 0$ for any $k < m_{st}$. (Of course $[m]_s = [m]_t = 0$ for $m = m_{st}$, by the definition of a realization). It was also claimed in [8] that the converse holds when \mathbb{k} is a domain. This result has a slight error, because it neglects to carefully study the case when $[2]_s = 0$ but $[2]_t \neq 0$. For completeness, we prove the correct result here, after a lemma.

Lemma 6.6. *Suppose that $[2]_s = 0$. Then for all integers k we have*

$$[2k + 1] = (-1)^k, \quad [2k]_s = 0, \quad [2k]_t = (-1)^{k-1} k [2]_t. \quad (6.13)$$

Note the appearance of the ordinary integer k as a factor of $[2k]_t$.

Proof. These equations are easily shown to satisfy the recursion (6.1). □

Remark 6.7. One is often interested in the case when $[2]_s = 0$ but $[2]_t \neq 0$, as happens for example in type B_2 in characteristic 2, or in type G_2 in characteristic 3. Thus Lemma 6.6 is relevant in small characteristic Kac–Moody realizations.

Theorem 6.8. *Consider a realization of a dihedral group over the base ring \mathbb{k} . If $[k]_s = [k]_t = 0$ then $(st)^k$ acts as the identity on the span of α_s and α_t . If \mathbb{k} is a domain, and $(st)^k$ acts as the identity on the span of α_s and α_t , then exactly one of the following options holds*

- $[k]_s = [k]_t = 0$, or
- the characteristic of \mathbb{k} is 2, $[2]_s = [2]_t = 0$, and k is odd,
- the characteristic of \mathbb{k} is 2, exactly one of $[2]_s$ and $[2]_t$ is zero, and k is equivalent to 2 modulo 4.

Proof. We recall a computation from [8, Claim 3.5 and p. 63]. The action of $(st)^k$ on the span of α_s and α_t is given by the matrix

$$(st)^k = \begin{pmatrix} [2k + 1] & -[2k]_s \\ [2k]_t & -[2k - 1] \end{pmatrix}. \quad (6.14)$$

If $[k]_s = [k]_t = 0$ then one can use (6.9) to deduce that $[2k]_s = [2k]_t = 0$, and $[2k + 1] = 1 = -[2k - 1]$. Hence the matrix above is the identity matrix.

Conversely, suppose that the matrix above is the identity, and that \mathbb{k} is a domain. Then $[2k]_s = [2k]_t = 0$, and $[2k - 1] = -1$. Using (6.10) with $2k$ replacing m , we deduce (regardless of whether k is even or odd, c.f. (7.1)) that

$$[k]_s = [2k - 1][k]_s = -[k]_s, \quad [k]_t = [2k - 1][k]_t = -[k]_t, \quad (6.15)$$

and

$$[k + 1]_t = [2k - 1][k - 1]_t = -[k - 1]_t, \quad (6.16)$$

$$[k + 1]_s = [2k - 1][k - 1]_s = -[k - 1]_s. \quad (6.17)$$

In particular, if $2 \neq 0$ then $[k]_s = [k]_t = 0$.

For the rest of the proof we assume $2 = 0$. We also have

$$[2]_s[k]_t = [k+1]_s + [k-1]_s = 0, \quad [2]_t[k]_s = [k+1]_t + [k-1]_t = 0. \quad (6.18)$$

If $[2]_s \neq 0$ and $[2]_t \neq 0$ then $[k]_s = [k]_t = 0$.

Suppose that $[2]_s = [2]_t = 0$. If k is even then $[k]_s = [k]_t = 0$. If k is odd, then $[k]_s = [k]_t = 1 \neq 0$ by (6.13) in characteristic 2, but nonetheless the matrix from (6.14) is the identity.

The only remaining case is that exactly one of $[2]_s$ and $[2]_t$ is zero. Without loss of generality, $[2]_s = 0$ and $[2]_t \neq 0$. By (6.13) in characteristic 2, $[2k]_t = k[2]_t = 0$, so $k = 2\ell$ is even. Then $[k]_t = (-1)^{\ell-1}\ell[2]_t$, which (in characteristic 2) is zero if and only if ℓ is even. Thus $[k]_t$ is nonzero if and only if k is equivalent to 2 modulo 4. \square

6.4. Two-colored Temperley–Lieb

Let $2\mathcal{T}\mathcal{L} = 2\mathcal{T}\mathcal{L}_A$ denote the two-colored Temperley–Lieb category with base ring A . We remind the reader of the definition.

Definition 6.9. The objects of $2\mathcal{T}\mathcal{L}$ are associated with finite alternating sequences $(\dots, s, t, s, t, \dots)$. For an integer $n \geq 0$ we let ${}_s n$ denote the sequence of length $n+1$ which begins with s , and n_s the sequence of length $n+1$ which ends in s , and similarly for t . Thus

$${}_s 2 = (s, t, s) = 2_s \quad \text{and} \quad {}_s 3 = (s, t, s, t) = 3_t.$$

We visualize this object as a sequence of n dots on a line; removing these points from the line we have $n+1$ connected components or regions, which are colored with alternating colors.

There are no morphisms between n_s and n'_t in either direction, or between ${}_s n$ and ${}_t n'$, for any n, n' . If n and m have the same parity then $\text{Hom}_{2\mathcal{T}\mathcal{L}}(n_s, m_s)$ has an A -basis given by two-colored (n, m) -crossingless matchings which have the color s on the far right. Recall that a two-colored crossingless matching is a crossingless matching where the regions are colored either s or t , and adjacent regions have different colors. By convention, an (n, m) -crossingless matching has n boundary points on bottom and m boundary points on top.

Composition of morphisms is as in the Temperley–Lieb category. A blue circle inside a red region evaluates to $-x_s$ while a red circle inside a blue region evaluates to $-x_t$.

For any commutative A -algebra A' we define $2\mathcal{T}\mathcal{L}_{A'}$ to be the base change of $2\mathcal{T}\mathcal{L}_A$ from A to A' . Morphism spaces still have a basis given by crossingless matchings.

Definition 6.10. We let $2\mathcal{T}\mathcal{L}_{A'}^{\text{Kar}}$ denote the Karoubi envelope of the additive envelope of $2\mathcal{T}\mathcal{L}_{A'}$.

For any A' , the category $2\mathcal{T}\mathcal{L}_{A'}$ is an *object adapted cellular category*, as originally defined in [12]. See [15, Chapter 11] for background reading on object adapted cellular categories (the definition is found in [15, 11.3.1]). The object adapted structure on the Temperley–Lieb category is discussed in [15, Example 11.31]. We assume the reader is familiar with this background material below. We use the following general result about object adapted cellular categories, whose proof morally derives from [17, Theorem 6.25], and was generalized in [12, Proposition 2.24] and [15, Chapter 11.3.4].

Theorem 6.11. *Suppose that A' is an A -algebra, a commutative domain, and a Henselian local ring.*

- (1) $2\mathcal{T}\mathcal{L}_{A'}$ is an object-adapted cellular category with its cellular basis of crossing-less matchings.
- (2) In any decomposition of the identity of n_s (resp. n_t) into indecomposable orthogonal idempotents

$$1_{n_s} = e_1 + \cdots + e_d,$$

there is a unique idempotent in the sum for which the coefficient of the identity (in the basis of crossingless matchings) is nonzero. This coefficient is 1.

For each $n \geq 0$ we pick such an indecomposable idempotent and denote it $e_{n_s}^{A'}$ (resp. $e_{n_t}^{A'}$).

- (3) Let $V_{n_s}^{A'}$ denote the image of $e_{n_s}^{A'}$, an object in $2\mathcal{T}\mathcal{L}_{A'}^{\text{Kar}}$, and similarly for $V_{n_t}^{A'}$. Up to isomorphism, the indecomposable objects in $2\mathcal{T}\mathcal{L}_{A'}^{\text{Kar}}$ are precisely enumerated by $\{V_{n_s}^{A'}, V_{n_t}^{A'}\}_{n \geq 0}$.
- (4) For any idempotent $e \in \text{End}(n_s)$, its image V in $2\mathcal{T}\mathcal{L}_{A'}^{\text{Kar}}$ decomposes as

$$V \cong (V_{n_s}^{A'})^{\oplus \delta} \oplus \bigoplus_{0 \leq k = n - 2l} (V_k^{A'})^{\oplus \mu_k} \tag{6.19}$$

for some multiplicities μ_k . The multiplicity δ of $V_{n_s}^{A'}$ is equal to zero if e has zero coefficient of the identity, and is equal to 1 if e has nonzero coefficient of the identity.

The summand $V_{n_s}^{A'}$ inside n_s is called the *top summand*, and $e_{n_s}^{A'}$ the *top idempotent*, because all other summands in n_s are appear as summands of k_s for $k < n$. While the top idempotent itself may depend on the choice of decomposition of 1_{n_s} , its image $V_{n_s}^{A'}$ is independent of this choice up to isomorphism.

Proof. The theorem is a straightforward adaption of [12, Proposition 2.24] and [15, Example 11.31]. \square

Note that the behavior of $e_{n_s}^{A'}$ and $V_{n_s}^{A'}$ is quite dependent on the choice of the base ring A' . For example, 2_s has a nontrivial top summand (the complement of a summand isomorphic to 0_s) when $[2]_s$ is invertible, and is indecomposable otherwise.

6.5. Two-colored Jones–Wenzl projectors

Until otherwise stated we fix an A -algebra A' as in Theorem 6.11, and omit A' from the notation for idempotents and summands.

Definition 6.12. Fix $n \geq 0$. If e_{n_s} is a top idempotent which satisfies the following property, then we call it a *Jones–Wenzl projector* and denote it JW_{n_s} :

$$e_{n_s} \circ \text{Hom}(k_s, n_s) = 0, \quad \text{for all } 0 \leq k < n. \quad (6.20)$$

We now state some properties of Jones–Wenzl projectors. To be walked through the proofs of these facts, see [15, Exercise 9.25].

Whether JW_{n_s} exists depends on the choice of A' . If it exists, then it is unique. Note that (6.20), which states that the image of JW_{n_s} is orthogonal to all the objects k_s for $k < n$, is equivalent to the condition that JW_{n_s} is orthogonal to the objects V_{k_s} for $k < n$, using (6.19).

Every (n, n) -crossingless matching except for the identity has a cap on bottom and a cup on top. Similarly, every (k, n) -crossingless matching for $k < n$ has a cup on top. For an endomorphism $f \in \text{End}(n_s)$, being orthogonal to k_s for $k < n$ as in (6.20) is equivalent to being orthogonal to all cups which could be placed below, i.e.

$$f \circ \text{cup} = 0,$$

which we call *death by cups*. So e_{n_s} is a Jones–Wenzl projector if and only if it satisfies death by cups. Conversely, any endomorphism which satisfies death by cups and whose coefficient of the identity equals 1 is an idempotent, and hence is a Jones–Wenzl projector.

Jones–Wenzl projectors are also killed by all caps on top, and death by caps can replace death by cups as an equivalent definition of a Jones–Wenzl projector. In other words,

$$\text{Hom}(n_s, k_s) \circ e_{n_s} = 0, \quad \text{for all } 0 \leq k < n. \quad (6.21)$$

When JW_{n_s} exists, then any endomorphism of n_s which satisfies death by caps or death by cups is a scalar multiple of JW_{n_s} , and this scalar is determined by the coefficient of the identity.

Using death by caps and/or cups, it is easy to prove that the endomorphism ring of the image of JW_{n_s} (namely V_{n_s}) is precisely A' . All non-identity crossingless matchings in $\text{End}(n_s)$ are killed when composed with JW_{n_s} .

By swapping the roles of s and t , the same things can be said about the morphism JW_{n_t} . Of course, $JW_{s,n}$ is equal to JW_{n_s} or JW_{n_t} , depending on the parity of n .

The horizontal reflection of a Jones–Wenzl projector still satisfies the same conditions. Thus the horizontal reflection of JW_{n_s} is $JW_{s,n}$. Similarly, the vertical reflection still satisfies death by cups instead of death by caps, so it is still a Jones–Wenzl projector. Thus the vertical reflection of JW_{n_s} is JW_{n_s} . The rotation of a Jones–Wenzl projector by 180 degrees is thus also a Jones–Wenzl projector (composing these two reflections).

We say that JW_n exists when both JW_{n_s} and JW_{n_t} exist. It is possible for JW_n to exist even when JW_k does not exist for various $k < n$. Clearly JW_0 and JW_1 exist. It is easy to confirm by hand that JW_2 exists if and only if $[2]_s$ and $[2]_t$ are invertible, and a reasonable exercise to confirm that JW_3 exists if and only if $[3]$ is invertible. It might be that $[2]_s = [2]_t = 0$ and thus $[3] = -1$, in which case JW_3 exists but JW_2 does not.

6.6. Two-colored Jones–Wenzl projectors: the generic case

When all quantum numbers are invertible then all Jones–Wenzl projectors exist. This is a familiar story, which we elaborate on here mostly to pin down the colorings on quantum numbers precisely.

Definition 6.13. Let $n \geq 1$. The *partial trace* $\text{pTr}(f)$ of an endomorphism f of n_s is an endomorphism of $(n-1)_t$ defined as follows.

$$\text{pTr}(f) = \text{Diagram} \tag{6.22}$$

(The purplish color on the left is meant to represent either red or blue, depending on parity.) Let $\text{pTr}_1(f)$ denote the coefficient of the identity when $\text{pTr}(f)$ is expanded in the basis of crossingless matchings.

Now suppose that JW_n exists, and consider $\text{pTr}(JW_{n_s})$. This morphism is clearly orthogonal to all cups and caps. If JW_{n-1} exists then

$$\text{pTr}(JW_{n_s}) = \text{pTr}_1(JW_{n_s}) \cdot JW_{(n-1)_t}. \tag{6.23}$$

Conversely, if $\text{pTr}_1(JW_{n_s})$ is invertible then $JW_{(n-1)_t}$ exists, by this same formula. On the other hand, if JW_{n-1} exists and $\text{pTr}_1(JW_{n_s}) = 0$ then $\text{pTr}(JW_{n_s}) = 0$ by (6.23). If JW_{n-1} does not exist, it is possible that $\text{pTr}_1(JW_{n_s}) = 0$ but $\text{pTr}(JW_{n_s}) \neq 0$, see Remark 6.23.

Assuming that JW_n exists and $\text{pTr}_1(JW_{n_s})$ is invertible, $JW_{(n+1)_t}$ also exists, and can be defined by the recursive formula:

$$\begin{array}{|c|} \hline \cdots \\ \hline JW_{(n+1)_t} \\ \hline \cdots \\ \hline \end{array} = \begin{array}{|c|} \hline \cdots \\ \hline JW_{n_s} \\ \hline \cdots \\ \hline \end{array} - \text{pTr}_1(JW_{n_s})^{-1} \begin{array}{|c|} \hline \cdots \\ \hline JW_{n_s} \\ \hline \cdots \\ \hline \end{array} \quad (6.24)$$

Meanwhile, if JW_n exists but $\text{pTr}_1(JW_{n_s})$ is not invertible, one can deduce that $JW_{(n+1)_t}$ can not exist. It would have to be defined by a formula just like (6.24), but there is no choice of coefficients which would work.

The standard way which one computes $\text{pTr}(JW_n)$ is to assume that JW_{n-1} exists, and that JW_n is constructed by (6.24) for $n - 1$. This allows one to deduce that

$$\text{pTr}(JW_{n_s}) = (-[2]_t + \text{pTr}_1(JW_{(n-1)_t})^{-1}) \cdot JW_{(n-1)_t} \quad (6.25)$$

giving the recursive formula

$$\text{pTr}_1(JW_{n_s}) = (-[2]_t + \text{pTr}_1(JW_{(n-1)_t})^{-1}). \quad (6.26)$$

Thus assuming that all JW_k are defined for $0 \leq k < n$, one can solve the recursion and compute that

$$\text{pTr}_1(JW_{n_s}) = \frac{-[n+1]_t}{[n]_s}. \quad (6.27)$$

Let us summarize.

Theorem 6.14. *Suppose that A' is an commutative A -algebra for which $[k]_s$ and $[k]_t$ are invertible for all $k \geq 1$. Then JW_n exists for all n , and*

$$\begin{array}{|c|} \hline \cdots \\ \hline JW_{(n+1)_t} \\ \hline \cdots \\ \hline \end{array} = \begin{array}{|c|} \hline \cdots \\ \hline JW_{n_s} \\ \hline \cdots \\ \hline \end{array} + \frac{[n+1]_t}{[n]_s} \begin{array}{|c|} \hline \cdots \\ \hline JW_{n_s} \\ \hline \cdots \\ \hline \end{array} \quad (6.28)$$

and (6.27) holds.

Remark 6.15. Note that there are two-colored analogs of other formulas for the Jones–Wenzl formula as well. For example, there is a two-colored single clasp expansion:

$$JW_{(n+1)_t} = JW_{n_s} + \sum_{a=1}^n \frac{[a]_u}{[n+1]_t} JW_{n_s}$$

(6.29)

In the a -th diagram in the sum there is a cup connecting strands a and $a + 1$ on top, and the coefficient $[a]$ in the numerator has the same color as the interior of this cup, which alternates between blue and red. We expect there is a two-colored analog of Morrison’s closed formula [28] as well.

6.7. Two-colored Jones–Wenzl projectors: from generic to special

There are a number of special properties of Jones–Wenzl projectors that one can deduce from (6.28) and (6.27). Unfortunately, for a base ring A' where some quantum numbers are not invertible, one can no longer crank the induction to define JW_n , and these properties become difficult to prove. What if $[113] = 0$ so that JW_{113} is not defined; then $[226]_s = 0$ and $[225]$ is invertible and we might hope that JW_{225} exists; do our computations using (6.28) in the generic case still tell us anything about JW_{225} ? The goal of this section is to deduce properties of any existant Jones–Wenzl projector from the generic case.

Let \mathbb{k} be an A -algebra which is a commutative domain and a local Henselian ring. In particular, it contains elements $[2]_s$ and $[2]_t$ which are the images of x_s and x_t . Let $A_{\mathbb{k}} = \mathbb{k}[x_s, x_t]$, made an A -algebra via $x_s \mapsto x_s$ and $x_t \mapsto x_t$. Now let \widehat{A} denote the completion of $A_{\mathbb{k}}$ at the ideal $(x_s - [2]_s, x_t - [2]_t)$. If we let $z_u = x_u - [2]_u$ for $u = s, t$ then we have an isomorphism

$$\widehat{A} = \mathbb{k}[[z_s, z_t]].$$

(6.30)

The ring \widehat{A} is also a commutative domain, and is also local and Henselian [29, 31]. Now let Q denote the field of fractions of \widehat{A} .

It might be the case that some quantum numbers vanish in \mathbb{k} . Regardless, we claim that all two-colored quantum numbers are invertible in Q . After all, the natural maps $A_{\mathbb{k}}$ to \widehat{A} and finally to Q are all injective. The map $A \rightarrow A_{\mathbb{k}}$ is not necessarily injective, as \mathbb{k} might have finite characteristic. But the two-colored quantum numbers are all monic

polynomials (with distinct leading monomials), and they are sent to (distinct) nonzero elements of $A_{\mathbb{k}}$. Since Q is a field, all nonzero elements are invertible.

Thus we can apply Theorem 6.11 for $A' \in \{\mathbb{k}, \widehat{A}, Q\}$, and we can apply Theorem 6.14 for $A' = Q$.

We have functors of extension of scalars

$$2\mathcal{T}\mathcal{L}_{\mathbb{k}} \xleftarrow{(-) \otimes_{\widehat{A}} \mathbb{k}} 2\mathcal{T}\mathcal{L}_{\widehat{A}} \xrightarrow{(-) \otimes_{\widehat{A}} Q} 2\mathcal{T}\mathcal{L}_Q . \quad (6.31)$$

These extend (by the universal property) to functors between the corresponding Karoubi envelopes.

Proposition 6.16. *We have isomorphisms:*

- (1) $V_{n_s}^{\widehat{A}} \otimes \mathbb{k} \cong V_{n_s}^{\mathbb{k}}$;
- (2) $V_{n_s}^{\widehat{A}} \otimes Q \cong V_{n_s}^Q \oplus \bigoplus_{0 \leq k = n - 2l} (V_{k_s}^Q)^{\oplus \mu_k}$.

Similar statements hold for n_t . In other words, the idempotent $e_{n_s}^{\widehat{A}}$ specializes over \mathbb{k} to be $e_{n_s}^{\mathbb{k}}$, and specializes over Q to be $e_{n_s}^Q$ plus idempotents corresponding to lower cells.

Proof. Property (1) is a consequence of idempotent lifting: every idempotent in $\text{End}(n_s)$ over \mathbb{k} lifts to an idempotent over \widehat{A} . Hence indecomposable idempotents lift to indecomposable idempotents. The idempotent $e_{n_s}^{\mathbb{k}}$ thus lifts to an idempotent over \widehat{A} which satisfies the conditions to be $e_{n_s}^{\widehat{A}}$. Property (2) follows directly from the last part of Theorem 6.11. \square

We will not use Property (2), we merely wish to emphasize that the specialization of $e_{n_s}^{\widehat{A}}$ to Q is not necessarily $e_{n_s}^Q$, because it need not be indecomposable.

Theorem 6.17. *Suppose that $JW_{n_s}^{\mathbb{k}}$ exists. Then it lifts to $JW_{n_s}^{\widehat{A}}$, which specializes to $JW_{n_s}^Q$.*

Proof. Suppose that $JW_{n_s}^{\mathbb{k}}$ exists. By Proposition 6.16, it lifts to an idempotent $e_{n_s}^{\widehat{A}}$, and

$$V_{n_s}^{\mathbb{k}} = V_{n_s}^{\widehat{A}} \otimes_{\widehat{A}} \mathbb{k}. \quad (6.32)$$

For any $0 \leq k < n$ we have

$$\text{Hom}_{2\mathcal{T}\mathcal{L}_{\widehat{A}}}(V_{k_s}^{\widehat{A}}, V_{n_s}^{\widehat{A}}) \otimes_{\widehat{A}} \mathbb{k} = \text{Hom}_{2\mathcal{T}\mathcal{L}_{\mathbb{k}}}(V_{k_s}^{\mathbb{k}}, V_{n_s}^{\mathbb{k}}) = 0$$

where the first equality uses (1) in Proposition 6.16 and second equality again uses (6.20). As $\text{Hom}_{2\mathcal{T}\mathcal{L}_{\widehat{A}}}(V_{k_s}^{\widehat{A}}, V_{n_s}^{\widehat{A}})$ is certainly finitely-generated over \widehat{A} , we conclude that

$$\text{Hom}_{2\mathcal{T}\mathcal{L}_{\widehat{A}}}(V_{k_s}^{\widehat{A}}, V_{n_s}^{\widehat{A}}) = 0 \quad \text{for any } 0 \leq k < n \quad (6.33)$$

by Nakayama’s lemma. Hence $e_{n_s}^{\widehat{A}}$ satisfies (6.20) and is a Jones–Wenzl projector.

Now let f be the specialization of $e_{n_s}^{\widehat{A}}$ to \mathcal{Q} . Property (6.20) (i.e. death by cups) for $e_{n_s}^{\widehat{A}}$ specializes to (6.20) for f . This in turn implies that the endomorphism ring of the image of f is one-dimensional over \mathcal{Q} (spanned by the identity map), so f is an indecomposable idempotent. Hence $f = e_{n_s}^{\mathcal{Q}}$. \square

The upshot of Theorem 6.17 is that it implies that, whenever $JW_n^{\mathbb{k}}$ exists, it has the same “coefficients” as $JW_n^{\mathcal{Q}}$, in the sense that both come from the same idempotent $JW_n^{\widehat{A}}$.

Example 6.18. Using (6.28), we can compute that $JW_{3_t}^{\mathcal{Q}}$ is a linear combination of five crossingless matchings: the identity with coefficient 1, two others with coefficient $\frac{1}{[3]}$, one with coefficient $\frac{[2]_s}{[3]}$, and one with coefficient $\frac{[2]_t}{[3]}$. More precisely, all these coefficients should be interpreted as rational functions in x_s and x_t . (See [8, (A.3)], matching notation via $x = x_s$ and $y = x_t$.) Now suppose that $JW_{3_s}^{\mathbb{k}}$ exists. Since the idempotent $JW_{3_s}^{\mathcal{Q}}$ lifts to $JW_{3_s}^{\widehat{A}}$, the ring \widehat{A} must have some element which specializes to $\frac{1}{[3]}$, so that $[3] = x_s x_t - 1$ must be invertible in \widehat{A} (by the injectivity of $\widehat{A} \rightarrow \mathcal{Q}$). In turn, this implies that its specialization $[3]$ is invertible in \mathbb{k} . The exact same formula for the coefficients of the Jones–Wenzl projector must hold over \mathbb{k} as well. This is true even in the unusual situations where $[2]_s = 0$ and/or $[2]_t = 0$ in \mathbb{k} , where JW_2 need not exist, so we can not deduce the coefficients immediately from (6.28). Consequently, (6.27) still holds over \mathbb{k} , telling us that $\text{pTr}_1(JW_{3_t}) = -\frac{[4]_s}{[3]}$. In particular, $\text{pTr}_1(JW_{3_t}) = 0$ when $[2]_s = 0$.

6.8. Rotatable Jones–Wenzl projectors

In this section, we fix an A -algebra \mathbb{k} which is a commutative domain and a local Henselian ring. When omitted, the base ring of any idempotents is \mathbb{k} .

Let us assume that JW_n exists. We now ask whether it is possible that the (counter-clockwise) rotation of JW_{n_s} by one strand is equal to a scalar multiple of JW_{n_t} :

Lemma 6.19. *Suppose that JW_n exists. The rotation of JW_{n_s} by one strand, as pictured in (6.34), is a scalar multiple of JW_{n_t} if and only if $\text{pTr}(JW_{n_s}) = 0$.*

Proof. Being equal to a scalar multiple of a Jones–Wenzl projector is equivalent to death by caps. The rotation of a Jones–Wenzl projector automatically killed by all but the

rightmost cap. It is killed by the rightmost cap if and only if the original diagram has zero partial trace. \square

Definition 6.20. When $\text{pTr}(JW_{n_s}) = 0$ and $\text{pTr}(JW_{n_t}) = 0$ we say that JW_n is *rotatable*.

The following result implies that the appropriate Jones–Wenzl projector is rotatable for most dihedrally-faithful realizations. In most cases this is proven in [8], but the edge cases are treated carefully here.

Theorem 6.21. *Suppose that $m = m_{st} < \infty$, that (st) has order exactly m on the span of α_s and α_t , and that the base ring is a field. Then JW_{m-1} exists and is rotatable.*

Proof. By Theorem 6.8, either $[k]_s \neq 0$ or $[k]_t \neq 0$ for all $k < m$. In particular, m is the first positive integer such that $[m]_s = [m]_t = 0$.

Suppose that $[k]_s, [k]_t \neq 0$ for all $k < m$. Since the base ring is a field, $[k]_s$ and $[k]_t$ are invertible for all $k < m$. Hence we can apply the inductive formulas to construct all JW_k for $k < m$, and we can use (6.27). By definition of a realization, $[m]_s = [m]_t = 0$, so $\text{pTr}_1(JW_{m-1}) = 0$ regardless of coloring. Because JW_{m-2} exists, (6.23) implies that $\text{pTr}(JW_{m-1}) = 0$, thus JW_{m-1} is rotatable by definition. This handles the case when $m = 2$, so we assume $m > 2$.

In the remaining case, $[k]_s = 0$ but $[k]_t \neq 0$, or vice versa, for some $k < m$. Choose d minimal such that this situation arises for $k = d$. Clearly d is even, because if d is odd then $[d]_s = [d]_t$. If $d > 2$ then $[2]_s$ and $[2]_t$ are nonzero, hence invertible. But then (6.6) implies that $[d]_s$ and $[d]_t$ are related by a unit, a contradiction since one is zero and the other isn't. The only remaining possibility is that $d = 2$.

Now we treat the case when $[2]_s = 0$ and $[2]_t \neq 0$. All quantum numbers were computed in (6.13). If the base field has characteristic zero, then $[k]_t \neq 0$ for all k , a contradiction since $[m]_t = 0$. If the base ring has characteristic p , then $[2p]_s = [2p]_t = 0$, and $2p$ is the first positive integer for which both quantum numbers are zero. Consequently, $m = 2p$. It remains to construct JW_{m-1} in this case, and prove it is rotatable, which is the content of Theorem 6.36. \square

We are still interested in realizations which are not dihedrally faithful, so that the above theorem does not apply.

Example 6.22. Suppose that $[2]_s = [2]_t = 0$. This arises for a non-faithful realization when $m = 4$. The Jones–Wenzl projector JW_{3_s} exists and is given by

$$\begin{array}{c} \boxed{JW_{3_s}} \end{array} = \begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} - \begin{array}{c} \text{Diagram 3} \end{array} . \quad (6.35)$$

(The reader can either verify death by caps directly, or follow the discussion of Example 6.18.) One can define JW_{3_t} by reversing the roles of the two colors. In particular, the rotation of JW_{3_s} is not a scalar multiple of JW_{3_t} . The partial trace is

$$\begin{array}{|c|} \hline JW_{3_s} \\ \hline \end{array} = -2 \begin{array}{|c|} \hline \\ \hline \end{array}, \tag{6.36}$$

which is nonzero when the characteristic of the base ring is not 2. Note that $\text{pTr}_1(JW_{3_s}) = 0$ as predicted by (6.27).

Remark 6.23. When JW_{n-1} is defined then (6.23) holds, so that $\text{pTr}(JW_n) = 0$ if and only if $\text{pTr}_1(JW_n) = 0$. The previous example indicates that $\text{pTr}(JW_n) = 0$ is not equivalent to $\text{pTr}_1(JW_n) = 0$ when JW_{n-1} does not exist.

Remark 6.24. A common concept is that of the *negligible* Jones–Wenzl projector. Negligibility is the property that, no matter how the morphism is glued into a larger closed diagram (i.e. an endomorphism of 0_s or 0_t), the result is zero. Rotatable Jones–Wenzl projectors are negligible, but the converse is false. In Example 6.22, JW_{3_s} is negligible but not rotatable.

Here is the main result of this section, which was stated without proof in [8].

Lemma 6.25. *Suppose that JW_n exists and is rotatable. Then the rotation of JW_{n_s} by one strand (clockwise or counterclockwise) is equal to $[n]_t JW_{n_t}$, and vice versa. Moreover, $[n]_s [n]_t = 1$, implying that the rotation of JW_{n_s} by two strands is again JW_{n_s} . If n is odd then $[n]_s = [n]_t = 1$.*

Proof. Suppose that (counterclockwise) rotation of JW_{n_s} by one strand is equal to a times JW_{n_t} , and that rotation of JW_{n_t} by one strand is equal to b times JW_{n_s} . That is:

$$\begin{array}{|c|} \hline JW_{n_s} \\ \hline \end{array} = a \begin{array}{|c|} \hline JW_{n_t} \\ \hline \end{array}, \tag{6.37}$$

$$\begin{array}{|c|} \hline JW_{n_t} \\ \hline \end{array} = b \begin{array}{|c|} \hline JW_{n_s} \\ \hline \end{array}. \tag{6.38}$$

Using invariance of Jones–Wenzl projectors under vertical flips, we see that clockwise rotation of JW_{n_s} is also equal to a times JW_{n_t} . Now using adjunction, one can argue

that $ab = 1$. This also implies that rotation of a rotatable Jones–Wenzl projector by two strands will recover the Jones–Wenzl projector on the nose. This isn't surprising, owing to the existence of crossingless matchings which are invariant under counterclockwise rotation by two strands⁶.

Rotation by n strands will send JW_{n_s} to JW_{s_n} , implying that $abab \cdots = 1$ where this word has length n . In particular, if n is odd then $a = b = 1$.

In fact, the rotational scalars a and b can be determined explicitly by the following argument. Let c_i denote the coefficient inside JW_{n_s} of the following diagram, which we denote y_i :

(6.39)

(There are i strands to the right of the cup on top. For example, y_1 is invariant under a vertical flip.) After counterclockwise rotation by one strand, y_{n-1} becomes the identity, so $a = c_{n-1}$. We claim that $a = c_{n-1} = \frac{1}{[n]_s}$, which exists in \mathbb{k} . Similarly, $b = \frac{1}{[n]_t}$. Since $ab = 1$ we deduce that $[n]_s [n]_t = 1$, and that $a = [n]_t$ and $b = [n]_s$. Thus this claim finishes the proof of the lemma.

One way to verify that $c_{n-1} = \frac{1}{[n]_s}$ is to verify this fact in the generic Jones–Wenzl projector $JW_{n_s}^Q$. This can be done most easily using (6.29). Using Theorem 6.17 we deduce that $[n]_s$ is invertible in \widehat{A} and hence in \mathbb{k} , and that $\frac{1}{[n]_s}$ is also the coefficient inside $JW_{n_s}^{\mathbb{k}}$.

There is also an elementary proof that $c_{n-1} = \frac{1}{[n]_s}$, which we briefly outline. Set y_0 to be the identity diagram, and $c_0 = 1$ (the coefficient of the identity diagram), and $c_n = 0$. For $1 \leq i \leq n - 1$, adding a i -th cap from the right to y_i (right on top of the cup in y_i) yields $-[2]_i$ times a particular diagram Y . Here $[2]_i$ is $[2]_s$ for i odd and $[2]_t$ for i even. Meanwhile, adding this i -th cap to $y_{i\pm 1}$ yields Y with coefficient 1. No other diagram in JW_n will yield Y after the placement of any cap. By isolating the coefficient of Y in a death by caps relation, we deduce that $[2]_i c_i = c_{i+1} + c_{i-1}$. Now induction gives $c_{n-i} = [n - i]_i c_{n-1}$ where $[n - i]_i$ is either $[n - i]_s$ or $[n - i]_t$. In conclusion, $c_0 = [n]_s c_{n-1}$. □

⁶Generically, all crossingless matchings have nonzero coefficients in the Jones–Wenzl projector.

6.9. Evaluating Jones–Wenzl projectors

Consider a realization of (W, S) with base ring \mathbb{k} , pick a pair $s \neq t \in S$, and make \mathbb{k} an A -algebra via (6.8). Given a two-colored crossingless matching we can associate a monomial in $R = \mathbb{k}[\alpha_s, \alpha_t]$, namely $\alpha_s^c \alpha_t^d$, where c is the number of s -colored regions and d is the number of t -colored regions. To a morphism f in $2\mathcal{TL}$, thought of as a linear combination of crossingless matchings, we can associate the corresponding linear combination of monomials, which we denote $\text{poly}(f)$.

Lemma 6.26. *Suppose that JW_{m-1} exists. We have $\text{poly}(JW_{(m-1)_s}) = \pi_{s,t}$ and $\text{poly}(JW_{(m-1)_t}) = \pi_{t,s}$.*

The polynomials $\pi_{s,t}$ and $\pi_{t,s}$ were defined in (2.22) and (2.23) respectively. We make no balanced-ness assumption on the realization, so it is possible that $\pi_{s,t} \neq \pi_{t,s}$; further discussion of these polynomials is found in Section 7.1.

Proof. Note that we can also define realizations of (W, S) with base rings \widehat{A} and Q respectively, where $\alpha_s^\vee(\alpha_t) := -x_s$ and $\alpha_t^\vee(\alpha_s) := -x_t$. We need only prove the result when the base ring is Q ; then the formula will lift to \widehat{A} and specialize to \mathbb{k} by Theorem 6.17.

Over Q we can use (6.28). In [8], it is inductively proven that $\text{poly}(JW_{k_s})$ is the product of the first $k + 1$ roots in the formula for $\pi_{s,t}$. More precisely, this is proven when $[2]_s = [2]_t$ and $[m - 1]_s = [m - 1]_t = 1$ in [8, Proposition 4.14], and adapted to the general case in [8, Chapter A.6]. The proof in [8] relies upon the recursive formula (6.28), although that paper fails to mention this assumption at times. \square

Lemma 6.26 is critical to our proof of the well-definedness of Λ . Note the compatibility of Lemma 6.26 with Lemma 6.25 and (7.11).

6.10. The existence of Jones–Wenzl projectors

So when does $JW_n^{\mathbb{k}}$ exist? Is there a precise criterion? When precisely is JW_n rotatable? We have given necessary conditions above, but no sufficient conditions.

Let us first ask the analogous question for the Jones–Wenzl projector in the ordinary (uncolored) Temperley–Lieb algebra, where the circle has value $-[2]_q = -(q + q^{-1})$. In [13, Theorem A.2], Ben Webster proved that the Jones–Wenzl projector JW_n exists so long as the quantum binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_q$ are invertible for all $1 \leq k \leq n$. The proof uses representation theory, so it does not immediately adapt to the two-colored case. However, it suggests that the appropriate expectation is that the two-colored Jones–Wenzl projectors JW_n exist so long as the two-colored quantum binomial coefficients $\begin{bmatrix} n \\ k \end{bmatrix}_s$ and $\begin{bmatrix} n \\ k \end{bmatrix}_t$ are invertible. This was stated as a theorem in [13] without proof, supposedly as a consequence of [13, Theorem A.2]. Let us revise its status to a conjecture now.

Conjecture 6.27. *The Jones–Wenzl projectors JW_{n_s} and JW_{n_t} exist over \mathbb{k} if and only if $\begin{bmatrix} n \\ k \end{bmatrix}_s$ and $\begin{bmatrix} n \\ k \end{bmatrix}_t$ are invertible in \mathbb{k} .*

Remark 6.28. As we remarked in the introduction, it appears this conjecture has been resolved completely in recent work of Hazi [21].

Just because $[m-1]_s$ is invertible does not imply that $\begin{bmatrix} m-1 \\ k \end{bmatrix}_s$ is invertible. We do not know a precise criterion for when $\begin{bmatrix} m-1 \\ k \end{bmatrix}_s$ and $\begin{bmatrix} m-1 \\ k \end{bmatrix}_t$ are invertible for all $0 \leq k \leq m-1$.

Example 6.29. Suppose that $[2]_s = [2]_t = 0$ (let's ignore the colors then), so that $[3] = -1$, $[4] = 0$, $[5] = 1$, and $[6] = 0$. Note that $\frac{[4]}{[2]} = [3] - 1 = -2$, and hence $\begin{bmatrix} 5 \\ 2 \end{bmatrix} = -2$. Let $m = 6$. Note that $[m-1]$ is invertible but $\begin{bmatrix} m-1 \\ 2 \end{bmatrix}$ is not unless 2 is invertible. One can verify that JW_5 is defined in this case, so long as 2 is invertible. It fails to be rotatable.

Remark 6.30. One possible way to prove this conjecture would be to reproduce Morrison's closed formula and its proof from [28] in the two-colored case over \mathcal{Q} . Then one could use Theorem 6.17 to deduce that the same formula holds over \mathbb{k} for any \mathbb{k} . In particular, the well-definedness of the coefficients in that formula would be equivalent to the existence of the Jones–Wenzl projector. It seems likely that Morrison's formula could be rephrased so that the only denominators required are quantum binomial coefficients, but this too has not been done.

Above we argued that the rotatability of JW_n implies that $[n]_s[n]_t = 1$, and that $[n]_s = 1$ when n is odd. We did this by examining very particular coefficients in the Jones–Wenzl projector. We noted that generically the partial trace of $JW_{n_s}^{\mathcal{Q}}$ is $\frac{-[n+1]_t}{[n]_s} JW_{(n-1)_t}^{\mathcal{Q}}$, so that one can prove that JW_n is rotatable when $[n+1]_s = [n+1]_t = 0$ and JW_{n-1} exists. However, when JW_{n-1} does not exist, the problem is quite subtle, Example 6.29 demonstrates that the conditions $[n+1] = 0$ and $[n] = 1$ are not sufficient to imply the rotatability of JW_n .

Example 6.31. A realization where $[2]_s = [2]_t = 0$ can be thought of as a faithful realization of the dihedral group $A_1 \times A_1$ with $m = 2$, or a non-faithful realization for $m = 4$ (or any higher even value of m). When $[2]_s = [2]_t = 0$, the explicit formula from Example 6.22 shows that JW_3 is rotatable if and only if the base ring has characteristic 2. In characteristic 2, this gives an example of a rotatable Jones–Wenzl projector even when the realization (for $m = 4$) is not dihedrally faithful.

Example 6.32. Similarly, suppose that $[2]_s = [2]_t = 1$ and hence $[3] = 0$, and suppose that 2 is invertible. One can compute that JW_8 exists, that $\text{pTr}_1(JW_8) = 0$, but that $\text{pTr}(JW_8)$ is $\frac{3}{2}$ times some particular nonzero morphism. We have omitted the lengthy

computation. In characteristic 3 this gives a rotatable Jones–Wenzl projector, but outside of characteristic 3 this gives a non-rotatable Jones–Wenzl projector. Note that $[8] = 1$, so either way this is a balanced non-faithful dihedral realization when $m = 9$.

6.11. Jones–Wenzl projectors in special cases

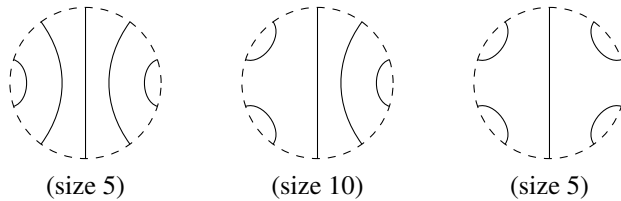
In this section we describe Jones–Wenzl projectors JW_{m-1} , and prove that they are rotatable, in two special cases of interest:

- When $[2]_s = 0$ but $[2]_t \neq 0$ (or vice versa), and $m = 2p$ in characteristic p . This handles the Kac–Moody realizations of type B_2 in characteristic 2, and G_2 in characteristic 3.
- When $[2]_s = [2]_t = 1$ in characteristic 2, and $m = 6$. This handles the Kac–Moody realization of type G_2 in characteristic 2.

We begin with type G_2 in characteristic 2.

Theorem 6.33. *Suppose that $[2]_s = [2]_t = 1$ and \mathbb{k} has characteristic 2. Then JW_5 exists over \mathbb{k} , and is rotatable.*

Proof. Because $[2]_s = [2]_t$ we will ignore the colors (as JW_{n_s} becomes JW_{n_t} after applying color swap). Note that if JW_5 is rotatable, it must be invariant under rotation by Lemma 6.25. It must also be invariant under the vertical and horizontal flip. The 42 crossingless $(5, 5)$ matchings split into 6 orbits under these flips and rotations. Here are three representatives of orbits, together with the sizes of the orbits under flips and rotations:

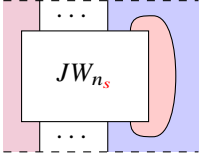


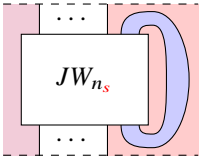
The reader can check that there are three remaining orbits (of sizes 10, 10 and 2 respectively). We claim that JW_5 is the sum over the three orbits displayed above (with coefficient 1); the remaining orbits (not displayed) have coefficient zero. It is an exercise to show that this sum satisfies death by cap. The coefficient of the identity is 1, so it is the Jones–Wenzl projector, and it is rotation invariant by construction. \square

Now we wish to treat the case when $[2]_s = 0$ and $[2]_t \neq 0$. In such circumstances one expects that JW_n does not exist when n is even. For example, JW_{2_s} does not exist

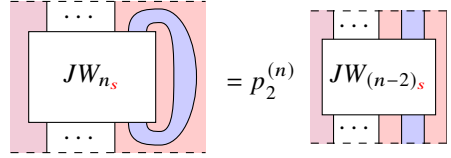
since $[2]_s$ is not invertible. However, generically one expects that JW_n exists when n is odd. In order to study odd Jones–Wenzl projectors in the absence of even Jones–Wenzl projectors, we prove a two-step recursion formula.

In this section, when JW_{n_s} exists, we write

$$p_1^{(n)} = \text{coefficient of the identity in } \text{tr}(JW_{n_s}) , \quad (6.40)$$


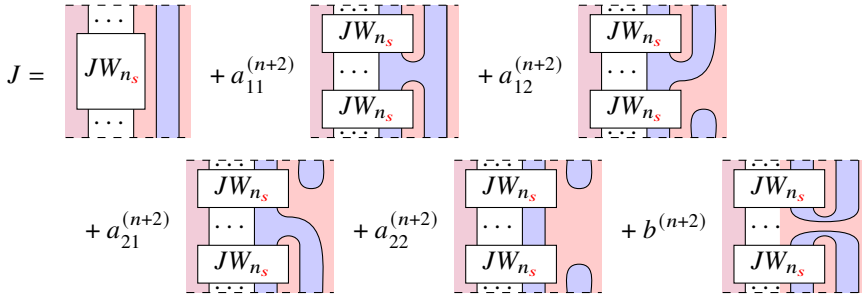
$$p_2^{(n)} = \text{coefficient of the identity in } \text{tr}(JW_{n_s}) . \quad (6.41)$$


Just as the diagram in (6.40) is called the *partial trace* of JW_{n_s} , the diagram in (6.41) will be called the *double (partial) trace* of JW_{n_s} . Note that this double trace satisfies death by caps. Thus if $JW_{(n-2)_s}$ exists, then

$$\text{tr}(JW_{n_s}) = p_2^{(n)} JW_{(n-2)_s} . \quad (6.42)$$


Conversely, if $p_2^{(n)}$ is invertible, then (6.42) can be used to define $JW_{(n-2)_s}$ from JW_{n_s} .

Theorem 6.34. *Suppose that JW_{n_s} exists, and consider the morphism J defined below.*

$$J = \text{tr}(JW_{n_s}) + a_{11}^{(n+2)} \text{tr}(JW_{n_s}) + a_{12}^{(n+2)} \text{tr}(JW_{n_s}) + a_{21}^{(n+2)} \text{tr}(JW_{n_s}) + a_{22}^{(n+2)} \text{tr}(JW_{n_s}) + b^{(n+2)} \text{tr}(JW_{n_s})$$


Then J satisfies death by cap (and is therefore equal to $JW_{(n+2)_s}$) if and only if the following equations hold.

$$1 + a_{12}^{(n+2)} - [2]_s a_{22}^{(n+2)} = 0, \quad 1 + a_{21}^{(n+2)} + p_1^{(n)} a_{11}^{(n+2)} = 0, \quad (6.43a)$$

$$a_{22}^{(n+2)} + p_1^{(n)} a_{12}^{(n+2)} = 0, \quad a_{11}^{(n+2)} - [2]_s a_{21}^{(n+2)} = 0, \quad (6.43b)$$

$$1 + p_1^{(n)} a_{11}^{(n+2)} + p_2^{(n)} b^{(n+2)} = 0. \quad (6.43c)$$

These equations have a solution if and only if $1 + [2]_s p_1^{(n)}$ is invertible and $p_2^{(n)}$ is invertible. In this case, letting

$$\kappa^{(n+2)} = \kappa := \frac{-1}{1 + [2]_s p_1^{(n)}}, \quad (6.44)$$

the unique solution is

Moreover, we have

$$p_1^{(n+2)} = -[2]_t - p_1^{(n)} \kappa, \quad p_2^{(n+2)} = [3] + \kappa(-[3] + \frac{1}{p_2^{(n)}}). \quad (6.46)$$

Proof. Let us ignore the superscripts $(n + 2)$ in the proof. If J satisfies death by cap, then it is killed by composition with any of the three diagrams with coefficients a_{21} , a_{22} , and b . Relatively straightforward computations (exercises for the reader) show that these three compositions are zero if and only if (6.43) holds.

Let us compute the solution to (6.43). Using two equations we get $a_{11} = [2]_s a_{21}$ and $a_{22} = -p_1^{(n)} a_{12}$. Plugging these back in to the first equation we get

$$a_{12}(1 + [2]_s p_1^{(n)}) = -1,$$

from which we deduce that $a_{12} = \kappa$. Similarly, $a_{21} = \kappa$. The last equation becomes

$$p_2^{(n)} b = -1 - [2]_s p_1^{(n)} a_{21} = \frac{-1 - [2]_s p_1^{(n)} + [2]_s p_1^{(n)}}{1 + [2]_s p_1^{(n)}} = \kappa.$$

Thus $b = \frac{\kappa}{p_2^{(n)}}$.

Computing $p_1^{(n+2)}$ from (6.45) is easy. The only diagrams which could possibly yield a multiple of the identity after taking the partial trace are the first diagram (which produces $-[2]_t$), and the diagram with coefficient a_{22} .

Computing $p_2^{(n+2)}$ from (6.45) is relatively straightforward. Once is computing the coefficient of JW_{n_s} in the double trace of $JW_{(n+2)_s}$. The first diagram yields $[2]_s[2]_t$, the second diagram yields $-[2]_t a_{11}$, the third diagram yields a_{12} and the fourth diagram yields a_{21} , the fifth diagram yields $-[2]_s a_{22}$, and the final diagram yields b . Adding these together and performing easy simplifications, one obtains the desired formula (6.46). \square

Now we solve the recursion set up in Theorem 6.34, to deduce the recursive formula for odd Jones–Wenzl projectors.

Theorem 6.35. *Suppose that n is odd. Suppose that either $[k]_t$ or $[k]_s$ is invertible for all $1 \leq k \leq n$. For $\ell \geq 3$ odd, set $\frac{[\ell-3]}{[\ell-1]}$ to be either $\frac{[\ell-3]_s}{[\ell-1]_s}$ or $\frac{[\ell-3]_t}{[\ell-1]_t}$, whichever formula has an invertible denominator. Note that if both denominators are invertible then the two formulas agree by (6.6). Then JW_{n_s} exists, and*

$$\kappa^{(n)} = \frac{[n-2]}{[n]}, \quad b^{(n)} = \frac{[n-2][n-3]}{[n][n-1]}, \quad p_1^{(n)} = \frac{-[n+1]_t}{[n]}, \quad p_2^{(n)} = \frac{[n+1]}{[n-1]}. \quad (6.47)$$

By swapping the roles of the two colors, JW_{n_t} also exists.

If $[n+1]_s = [n+1]_t = 0$, then JW_{n_s} and JW_{n_t} are rotatable.

Proof. We prove that JW_{n_s} exists by induction on n odd, using Theorem 6.34. Note that JW_1 exists, and we set $a_{ij}^{(1)} = 0$ and $b^{(1)} = 0$. It is easy to verify that (6.47) solves the recursive formulas (6.44), (6.45) and (6.46).

To prove the rotatability statement, we need to prove that the rotation of JW_n satisfies death by caps. Without loss of generality, we assume that $[n-1]_t$ is invertible. We know that JW_{n-2} exists, and $p_1^{(n-2)} = \frac{-[n-1]_t}{[n-2]}$, which is invertible. Thus we can use (6.24) to prove that $JW_{(n-1)_t}$ exists (although $JW_{(n-1)_s}$ need not)! Now $p_1^{(n)} = 0$, which combined with (6.23) implies that the partial trace of JW_{n_s} is the zero morphism. Thus the rotation of JW_{n_s} by one strand satisfies death by caps, and is a multiple of JW_{n_t} . Note that the rotation of JW_{n_s} by two strands also must satisfy death by caps, and thus is a multiple of JW_{n_s} . This implies that the rotation of JW_{n_t} by one strand is a multiple of JW_{n_s} (though we could not use the same argument with colors swapped to deduce this). In conclusion, the Jones–Wenzl projectors JW_n are rotatable. \square

Now we apply Theorem 6.35 in the special case where $[2]_s = 0$.

Theorem 6.36. *Suppose that $[2]_s = 0$ and $[2]_t \neq 0$, and \mathbb{k} is a field. If the characteristic of \mathbb{k} is zero, then all odd Jones–Wenzl projectors exist. If the characteristic of \mathbb{k} is p , then JW_{2k+1} exists for all $0 \leq k < p$, and JW_{2p-1} is rotatable.*

Proof. By (6.13), all odd quantum numbers are invertible, and $[2k]_t$ is invertible for k less than the characteristic. In characteristic p , $[2p]_t = [2p]_s = 0$. Now the result is an immediate consequence of Theorem 6.35. \square

6.12. Jones–Wenzl projectors in crystallographic type

Because they appear most frequently, we wish to summarize the conditions needed for JW_{m-1} to exist and be rotatable, when given a realization of $I_2(m)$ over a domain, with $m \in \{2, 3, 4, 6\}$. Thanks to Simon Riche for inspiring and assisting with this section.

Lemma 6.37. *In type $A_1 \times A_1$, JW_1 always exists and is always rotatable and balanced.*

Proof. This is obvious. Note that $\text{pTr}(JW_{1_s}) = -[2]_s$ and $\text{pTr}(JW_{1_t}) = -[2]_t$, and these are both zero by the definition of a realization. \square

Lemma 6.38. *In type A_2 , JW_2 always exists and is rotatable.*

Proof. Note in general that JW_2 exists when $[2]_s$ and $[2]_t$ are invertible. Since JW_1 always exists, JW_2 is rotatable if $\text{pTr}_1(JW_2) = 0$, which holds when $[3] = 0$. By the definition of a realization, $[3] = 0$, and consequently $[2]_s[2]_t = 1$, so both are invertible. \square

Note however when $m = 3$ that the realization need not be balanced! There is no requirement that $[2]_s = [2]_t = 1$.

Lemma 6.39. *In type B_2 , JW_3 always exists, and it is rotatable if and only if it is balanced if and only if $[3] = 1$. Moreover, any realization over a domain \mathbb{k} where JW_3 is not rotatable satisfies:*

- $[2]_s = [2]_t = 0$, and $2 \neq 0$.

Proof. In general, JW_3 exists when $[3]$ is invertible. One can directly compute that $\text{pTr}(JW_{3_s})$ is equal to the identity diagram with coefficient $\frac{[4]_t}{[3]}$, and the cup-cap with coefficient $\frac{1-[3]}{[3]}$. For a realization with $m = 4$, we have $[4]_s = [4]_t = 0$ and $[3]^2 = 1$, so JW_3 exists, and it is rotatable if $[3] = 1$, which is the definition of balanced.

Since $[3]^2 = 1$ and \mathbb{k} is a domain, $[3] = \pm 1$. In characteristic 2, this implies $[3] = 1$ and the realization is balanced. Otherwise, suppose that $[3] = -1 \neq 1$. Then $[2]_s[2]_t = 0$, so at least one of them is zero. If $[2]_s = 0$ and $[2]_t \neq 0$ then by (6.13), $[4]_t = -2[2]_t \neq 0$, a contradiction. Thus $[2]_s = [2]_t = 0$. \square

Lemma 6.40. *In type G_2 , JW_5 exists if and only if $[3] - 1$ is invertible. Moreover, for any realization over a domain \mathbb{k} where JW_5 exists, it is rotatable if and only if $[3] = 2$ (which implies that $[5] = 1$, so the realization is balanced). Moreover, any realization over a domain for which JW_5 does not exist or is not rotatable satisfies one of the following:*

- $[2]_s = [2]_t = 0$ and 2 is not invertible (JW_5 does not exist).
- $[2]_s = [2]_t = 0$ and 2 is invertible but $3 \neq 0$ (JW_5 exists but is not rotatable).

- $[3] = 0$ and $2 \neq 0$ (JW_5 exists but is not rotatable).

Proof. It is straightforward if time-consuming to compute JW_5 generically. An explicit formula for JW_{5_t} can be found in [4, Figure 2.1]. One deduces that JW_5 exists if and only if $[5]$ and $\frac{[4]_t}{[2]_t}$ are invertible. They must be invertible for existence, since two of the coefficients appearing are $\mu = \frac{1}{[5]}$ and $\nu = \frac{[2]_t}{[4]_t}$. The remaining coefficients live in $\mathbb{Z}[[2]_s, [2]_t] \cdot \mu\nu$. Note that $\frac{[4]_t}{[2]_t} = [3] - 1 = \frac{[4]_s}{[2]_s}$, so by symmetry JW_{5_s} exists if and only if JW_{5_t} exists.

When $[6]_s = [6]_t = 0$ (as in any realization of type G_2), $[5]$ is invertible since $[5]^2 = 1$. So JW_5 exists if and only if $[3] - 1$ is invertible.

Note that $[6]_t = [2]_t \cdot [3] \cdot ([3] - 2)$. Over a domain, at least one of these factors must be zero. If $[3] = 0$ or $[3] = 2$ then $[3] - 1$ is invertible and JW_5 exists. Otherwise, we must have $[2]_s = [2]_t = 0$, in which case $[3] = -1$. Then $[3] - 1 = -2$ is invertible if and only if 2 is invertible. Note that $[2]_s = [2]_t = 0$ in characteristic 2 yields a well-defined, balanced realization in type G_2 , but one where JW_5 does not exist.

Assume now that $[3] - 1$ is invertible. One can explicitly compute $\text{pTr}(JW_{5_s})$. Every coefficient appearing is either a multiple of $[6]_t$, a multiple of $\frac{[6]_t}{[2]_t}$, or a multiple of $\frac{[6]_t}{[3]}$. Therefore, JW_{5_s} is rotatable if and only if $\frac{[6]_t}{[2]_t}$ and $\frac{[6]_t}{[3]}$ are both zero. (Note that this should also be equivalent to the condition that JW_{5_t} is rotatable, which requires that $\frac{[6]_s}{[3]} = 0$ as well. We will see below that this is a consequence.)

Both $\frac{[6]_t}{[2]_t}$ and $\frac{[6]_t}{[3]}$ (and $\frac{[6]_s}{[3]}$) have a common factor $[3] - 2$, so if $[3] = 2$ then JW_5 is rotatable. Also, if $[3] = 2$ then $[3]^2 = 1 + [3] + [5]$ implies that $[5] = 1$.

Over a domain, if $[3] \neq 2$, then $\frac{[6]_t}{[3]} = 0$ implies $[2]_t = 0$. In this case, using (6.13), $\frac{[6]_t}{[2]_t} = [5] - [3] + 1 = 3$, so $3 = 0$. But then $[3] = -1 = 2$. So, however you slice it, $[3] = 2$ is required for rotatability. Note that we get a rotatable Jones–Wenzl projector in characteristic 3 if $[2]_t = 0$ or $[2]_s = 0$ or both, a result slightly more general than Theorem 6.36.

Finally, if $[6]_t = [6]_s = 0$ over a domain, but $[3] \neq 2$, then either $[3] = 0 \neq 2$, or $[2]_s = [2]_t = 0$. In the latter case, $[3] = -1 \neq 2$, so $3 \neq 0$. \square

7. The unbalanced case

The definition of a balanced and unbalanced realization was given above in Section 6.3. Let us describe the modifications to the construction and proofs above required for the unbalanced case. For more details on everything in this section, see the appendix of [8].

7.1. Positive roots

It is more convenient to write (6.10) in a fashion which depends on the parity of m rather than that of k . One has

$$[m-1][k]_s = [m-k]_s, \quad [m-1][k]_t = [m-k]_t, \quad \text{if } m \text{ is even,} \quad (7.1)$$

$$[m-1]_s[k]_t = [m-k]_s, \quad [m-1]_t[k]_s = [m-k]_t, \quad \text{if } m \text{ is odd.} \quad (7.2)$$

Recall from Section 2.4 the definition of X_s and X_t . Let us compare the sets

$$\{\beta_{\underline{w}} \mid \underline{w} \in X_s\} \quad \text{and} \quad \{\beta_{\underline{w}} \mid \underline{w} \in X_t\},$$

which (morally speaking) represent two different ways one might attempt to enumerate the “positive roots.”

Let $\underline{x} = (s, t, s, \dots)$ be an alternating expression which starts in s , and has length $k > 0$. One can verify that (see Section 2.4 for the definition of $\beta_{\underline{x}}$)

$$\beta_{\underline{x}} = [k]_s \alpha_s + [k-1]_t \alpha_t. \quad (7.3)$$

In particular, if $k = m$ so that \underline{x} is a reduced expression for the longest element, then

$$\beta_{\underline{x}} = [m-1]_t \alpha_t. \quad (7.4)$$

This is an indication of the importance of the quantum number $[m-1]$.

Similarly, let $\underline{y} = (t, s, t, \dots)$ be an alternating expression which starts in t , and has length $m+1-k > 0$. Then by reversing the roles of s and t we have

$$\beta_{\underline{y}} = [m-k]_s \alpha_s + [m+1-k]_t \alpha_t. \quad (7.5)$$

Now let $1 \leq k \leq m$ so that both (7.3) and (7.5) hold. The comparison between $\beta_{\underline{x}}$ and $\beta_{\underline{y}}$ is different based on whether m is even or odd. If m is even then

$$\beta_{\underline{y}} = [m-1] \cdot ([k]_s \alpha_s + [k-1]_t \alpha_t) = [m-1] \beta_{\underline{x}}. \quad (7.6)$$

Meanwhile, if m is odd then the comparison depends on whether k is even or odd. If m is odd and k is odd then

$$\beta_{\underline{y}} = [m-k]_s \alpha_s + [m+1-k]_t \alpha_t = [m-1]_s [k]_s \alpha_s + [m-1]_s [k-1]_t \alpha_t = [m-1]_s \cdot \beta_{\underline{x}}. \quad (7.7)$$

If m is odd and k is even then

$$\beta_{\underline{y}} = [m-k]_s \alpha_s + [m+1-k]_t \alpha_t = [m-1]_t [k]_s \alpha_s + [m-1]_t [k-1]_t \alpha_t = [m-1]_t \cdot \beta_{\underline{x}}. \quad (7.8)$$

In a balanced realization, $\beta_{\underline{y}} = \beta_{\underline{x}}$.

Define $\pi_{s,t}$ and $\pi_{t,s}$ as in (2.22) and (2.23). Thus $\pi_{s,t}$ is the product of terms like $\beta_{\underline{x}}$ above, and $\pi_{t,s}$ is the product of terms like $\beta_{\underline{y}}$ above. Consequently,

$$\pi_{t,s} = [m-1]^m \pi_{s,t} = \pi_{s,t}, \quad \text{if } m \text{ is even,} \quad (7.9)$$

where we used that $[m-1]^2 = 1$. Similarly,

$$\pi_{t,s} = [m-1]_s [m-1]_t \dots [m-1]_s \pi_{s,t} = [m-1]_s \pi_{s,t}, \quad \text{if } m \text{ is odd,} \quad (7.10)$$

where we used that $[m-1]_s [m-1]_t = 1$. Equivalently,

$$\pi_{s,t} = [m-1]_t \pi_{t,s} \quad \text{if } m \text{ is odd.} \quad (7.11)$$

Example 7.1. Suppose that $m_{st} = 3$ and $[2]_t = q^{-1}$ and $[2]_s = q$. Then $s(\alpha_t) = \alpha_t + q\alpha_s$ and $t(\alpha_s) = \alpha_s + q^{-1}\alpha_t$, which differ by a factor of q . We have $\pi_{s,t} = q^{-1}\alpha_s\alpha_t s(\alpha_t)$ and $\pi_{t,s} = \alpha_s\alpha_t s(\alpha_t)$.

Example 7.2. Suppose that $m_{st} = 4$ and $[2]_s = [2]_t = 0$, so that $[3] = -1$. Then $s(\alpha_t) = \alpha_t$ and $t(\alpha_s) = \alpha_s$. Both $\pi_{s,t}$ and $\pi_{t,s}$ are equal to $\alpha_s\alpha_t(-\alpha_s)(-\alpha_t) = \alpha_s^2\alpha_t^2$.

Lemma 7.3. *The polynomial $\pi_{s,t}$ (resp. $\pi_{t,s}$) is anti-invariant, meaning that*

$$s(\pi_{s,t}) = t(\pi_{s,t}) = -\pi_{s,t}, \quad (7.12)$$

if and only if the realization is even-balanced.

Proof. Since $\pi_{s,t}$ and $\pi_{t,s}$ are colinear, we need only investigate $\pi_{s,t}$. Apply s to

$$\pi_{s,t} = \prod_{\underline{x} \in X_s} \beta_{\underline{x}}.$$

The first term in the product (when X_s is sorted by length) is α_s , which goes to $-\alpha_s$. The remaining $m-1$ terms in the product go to the first $m-1$ terms in $\pi_{t,s}$, by the definition of these terms. The final missing term in $\pi_{t,s}$ is $[m-1]_s \alpha_s$. Consequently,

$$s(\pi_{s,t}) = \frac{-\alpha_s}{[m-1]_s \alpha_s} \pi_{t,s} = \frac{-1}{[m-1]_s} \pi_{t,s}. \quad (7.13)$$

This leads to a major difference depending on the parity of m . If m is odd then $\pi_{t,s} = [m-1]_s \pi_{s,t}$ so

$$s(\pi_{s,t}) = -\pi_{s,t}, \quad \text{if } m \text{ is odd.} \quad (7.14)$$

Meanwhile, if m is even then $\pi_{s,t} = \pi_{t,s}$ so

$$s(\pi_{s,t}) = -[m-1]_s \pi_{s,t}, \quad \text{if } m \text{ is even.} \quad (7.15)$$

So when m is even, $\pi_{s,t}$ is s -antiinvariant if and only if the realization is balanced for $\{s, t\}$. This can be witnessed in the examples for $m = 3$ and $m = 4$ above. \square

7.2. The definition of the category

Now we remind the reader about some features of $2m$ -valent vertices, and how the definition of the category \mathcal{H} changes in the unbalanced case. To motivate this story, we hold as self-evident that the Soergel Hom formula should describe the dimension of the morphism spaces in \mathcal{H} .

Pick a pair $s \neq t \in S$ with $m = m_{st} < \infty$. Let w_0 denote the longest element of this dihedral parabolic subgroup. By the Soergel Hom formula, the space of degree zero morphisms between any two of these three objects should be one-dimensional: $B_s B_t \dots$, $B_t B_s \dots$, and B_{w_0} (both expressions have length m). If the composition map $B_s B_t \dots \rightarrow B_{w_0} \rightarrow B_t B_s \dots$ were zero then the Soergel categorification theorem would fail, as these two expressions would not have a common top summand. By construction, the $2m$ -valent vertex will span the one-dimensional space of maps $B_s B_t \dots \rightarrow B_t B_s \dots$, so it agrees with the projection to the common summand B_{w_0} up to scalar. Similarly, there is another $2m$ -valent vertex spanning the degree zero morphisms from $B_t B_s \dots$ to $B_s B_t \dots$.

Here are four features of the $2m$ -valent vertices which must be true if the Soergel Categorification Theorem is to hold.

- Composing the $2m$ -valent vertices should yield, up to invertible scalar, the idempotent projecting to the common summand B_{w_0} .
- Rotating the $2m$ -valent vertex by two strands will produce a scalar multiple of itself. Rotating by one strand will produce a scalar multiple of the other $2m$ -valent vertex.
- The $2m$ -valent vertex satisfies death by pitchfork.
- Placing 2 adjacent dots on the $2m$ -valent vertex will yield a degree +2 morphism built from univalent and trivalent vertices, which must be a deformation retract of a linear combination of crossingless matchings (see [8, §5.3.3]). So there is some relation like (3.3).

What is this linear combination of crossingless matchings appearing in (3.3)? It must satisfy death by cap, in order for the $2m$ -valent vertex to satisfy death by pitchfork. It must have a rotational eigenvalue, in order for the $2m$ -valent vertex to have one as well. It must have an invertible coefficient of the identity map, in order for the composition of the $2m$ -valent vertices to be an idempotent up to invertible scalar. With these constraints, it must be a scalar multiple of a rotatable Jones–Wenzl projector JW_{m-1} . If there is no

rotatable Jones–Wenzl projector, then there is no way to define \mathcal{H} such that the properties listed above hold.

Henceforth we assume that JW_{m-1} exists and is rotatable. By Lemma 6.25, this implies that our realization is even-balanced. There is now nothing to discuss for dihedral groups with m even, so we focus on the case where m is odd.

The source of the difficulty in the unbalanced case is that rotating $JW_{(m-1)_s}$ by one strand yields $JW_{(m-1)_t}$ (and vice versa) only up to the scalar factor $[m-1]_s^{\pm 1}$. The appendix of [8] suggests two diagrammatic options for keeping track of this discrepancy. The first option involves fixing once and for all a scalar multiple of the Jones–Wenzl projector, and adding scalars to various relations in \mathcal{H} to make them match this choice. This makes the presentation much less intuitive and easy to remember (although this option is to be preferred for singular Soergel calculus). The second option involves keeping track of two different versions of the Jones–Wenzl projector, and two different versions of the $2m$ -valent vertex, which agree with each other up to a scalar. This second option is our preference here. The details of how \mathcal{H} should be constructed in this second option can be found in [17, §5.6], having extrapolated the answer from [8, Appendix]. Because some statements (e.g. the analog of (3.19)) are not stated explicitly anywhere and would require extrapolation from multiple sources to piece together, we will recall everything in one place here. We use a different convention than [17] for shading the vertices.

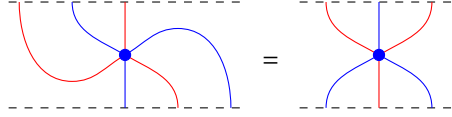
In the definition of \mathcal{H} by generators and relations, there are now two different versions of the $2m$ -valent vertex as morphism generators, one shaded s and one shaded t . Both are rotation-invariant. We have

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} = [m-1]_s \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array} \begin{array}{c} \text{---} \\ \text{---} \end{array}, \quad (7.16)$$

so these differently-shaded vertices are equal up to an invertible scalar. Because of this relation any diagram with s -shaded vertices can be replaced with a diagram with only t -shaded vertices by rescaling. It helps to have both shadings in practice, and the relations of the category will be preserved by the symmetry that swaps s with t (in diagrams and in coefficients, so $[m-1]_s$ is swapped with $[m-1]_t$).

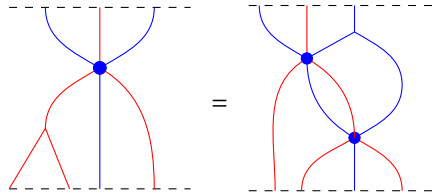
Remark 7.4. The normalization on these vertices designed as follows. When the t -shaded one is rotated so that it gives a morphism from $B_t B_s \dots$ to $B_s B_t \dots$, then after applying \mathcal{F} it should send the 1-tensor to the 1-tensor. The same is true switching s and t . This is opposite to the convention used in [17].

Now we explain how to modify the relations of \mathcal{H} accordingly. The cyclicity relation now states that the t -shaded $2m$ -valent vertex rotates to itself:


(7.17)

After rescaling this implies that the s -shaded $2m$ -valent vertex also rotates to itself.

The two-colored associativity relation now states the following:


(7.18)

To state the analog of (3.3), we set

$$\mathcal{JW}_{s,t} := \Sigma(JW_{(m-1)_s}), \quad \mathcal{JW}_{t,s} := \Sigma(JW_{(m-1)_t}). \quad (7.19)$$

Note that $(m-1)_s = {}_s(m-1)$, because m is odd. So $\mathcal{JW}_{s,t}$ is an endomorphism of the alternating Bott–Samelson object $B_s B_t \dots$ of length m , and the coefficient of the identity is 1. Just as in Section 3.3, we can define morphisms ${}^2\mathcal{JW}_{s,t}$, $JW_{s,t}$, and $JW'_{s,t}$.

For example, $JW_{s,t}$ is a degree +2 morphism from $B_s B_t \dots$ (alternating of length $2(m-1)$) to the monoidal identity. The fact that $JW_{(m-1)_s}$ has coefficient +1 of the identity implies that $JW_{s,t}$ has coefficient +1 of the diagram


(7.20)

Meanwhile, the coefficient in $JW_{s,t}$ of the diagram


(7.21)

is exactly the coefficient c_{n-1} computed in the proof of Lemma 6.25, namely $[m-1]_t$.

Lemma 6.25 implies that

$$\text{---} \quad \text{---}$$

$$\begin{array}{c} \text{---} \\ \circlearrowleft \\ \boxed{JW_{s,t}} \\ \text{---} \end{array} = [m-1]_t \begin{array}{c} \boxed{JW_{t,s}} \\ \text{---} \end{array} \quad (7.22)$$

Lemma 6.26 implies that

$$\begin{array}{c} \text{---} \\ \bullet \dots \bullet \\ \boxed{JW_{s,t}} \\ \bullet \dots \bullet \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \bullet \dots \bullet \\ \boxed{JW_{s,t}} \\ \bullet \dots \bullet \\ \text{---} \end{array} = \pi_{s,t}, \quad (7.23)$$

exactly as for (3.16).

Now we can finally state the relation which resolves what happens when a dot is placed on a $2m$ -valent vertex. Analogous to (3.21) we have

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowleft \\ \boxed{JW_{s,t}} \\ \text{---} \end{array} \quad (7.24)$$

We also have

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \circlearrowright \\ \boxed{JW_{s,t}} \\ \text{---} \end{array} \quad (7.25)$$

Note that the color swap of (7.24) gives the relation for a red dot on a blue vertex, which can be rescaled to give the relation for a red dot on a red vertex, yielding (7.25). We need only check (7.24) below, and the other relation will follow by symmetry.

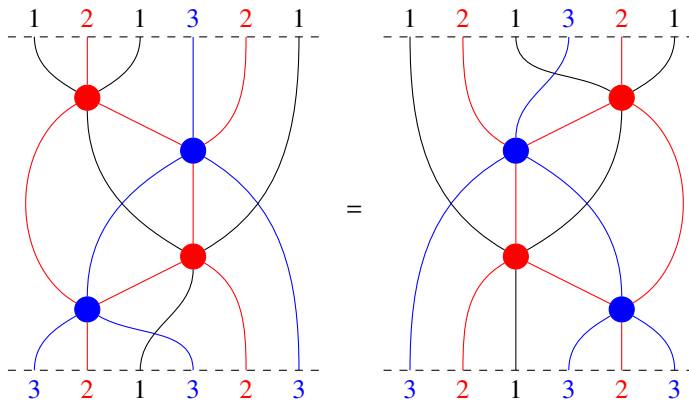
The analog of (3.24) is

$$\begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} = \begin{array}{c} \text{---} \\ \bullet \\ \text{---} \end{array} + \begin{array}{c} \text{---} \\ \boxed{P} \\ \text{---} \end{array} \quad (7.26)$$

Meanwhile,

$$\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ (crossing with red dot)} = [m-1]_t \left[\begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ (strand with blue dot)} + \begin{array}{c} \text{---} \\ \text{---} \end{array} \text{ (strand with box P)} \right]. \tag{7.27}$$

Finally, one also has the 3-color relations, which are shaded so that the 1-tensor is sent to the 1-tensor. For example, here is the A_3 Zamolodchikov relation (again, note that we use the opposite convention from [17] for the shadings).



7.3. The definition of the functor Λ

We have two versions of $E_{s,t}$: the version $E_{s,t}^s$ which uses the s -shaded $2m$ -valent vertex, and the version $E_{s,t}^t$ which uses the t -shaded $2m$ -valent vertex (the source of the morphism $E_{s,t}$ has not changed, only the shading of the vertex). Define $\Lambda(E_{s,t}^s)$ exactly as in (2.27), using the polynomial $\pi_{s,t}$. Define $\Lambda(E_{s,t}^t)$ in the analogous way using the polynomial $\pi_{t,s}$.

Note that the definition is symmetric under swapping s and t , so we only need check one version of each relation and the other follows by symmetry.

7.4. Checking the relations

The proof of Lemma 3.2 found in Section 3.2 goes through with only one major change. The morphisms $E_{s,t}^s$ and $E_{t,s}^s$ being compared have the same shading, and lead to the same polynomial $\pi_{s,t}$, so every instance of $\pi_{t,s}$ in that proof should be replaced with $\pi_{s,t}$. The proof of Lemma 3.2 splits the problem into two cases. The second case uses the fact

that $s(\pi_{s,t}) = -\pi_{s,t}$, which we proved for the unbalanced case in Lemma 7.3. The first case uses the equality⁷ $\pi_{s,t} = \pi_{s,t}$.

The proof of Lemma 3.9 goes through verbatim.

The proof of Lemma 3.11 also goes through verbatim, thanks to (7.26).

The analysis of the 3-color relations goes through verbatim.

References

- [1] Noriyuki Abe. A homomorphism between Bott–Samelson bimodules. <https://arxiv.org/abs/2012.09414>, 2020.
- [2] Noriyuki Abe. A Hecke action on G_1T -modules. *J. Inst. Math. Jussieu*, 2023. first view.
- [3] Noriyuki Abe. On one-sided singular Soergel bimodules. *J. Algebra*, 633:722–753, 2023.
- [4] Pramod N. Achar, Shotaro Makisumi, Simon Riche, and Geordie Williamson. Free-monodromic mixed tilting sheaves on flag varieties. <https://arxiv.org/abs/1703.05843>, 2017.
- [5] Pramod N. Achar, Shotaro Makisumi, Simon Riche, and Geordie Williamson. Koszul duality for Kac–Moody groups and characters of tilting modules. *J. Am. Math. Soc.*, 32(1):261–310, 2019.
- [6] Roman Bezrukavnikov and Simon Riche. Modular affine Hecke category and regular centralizer. <https://arxiv.org/abs/2206.03738>, 2022.
- [7] Chris Bowman, Anton Cox, and Amit Hazi. Path isomorphisms between quiver Hecke and diagrammatic Bott–Samelson endomorphism algebras. <https://arxiv.org/abs/2005.02825>, 2020.
- [8] Ben Elias. The two-color Soergel calculus. *Compos. Math.*, 152(2):327–398, 2016.
- [9] Ben Elias. Quantum Satake in type A. Part I. *J. Comb. Algebra.*, 1(1):63–125, 2017.
- [10] Ben Elias and Matthew Hogancamp. Categorical diagonalization of full twists. <https://arxiv.org/abs/1801.00191>, 2017.
- [11] Ben Elias and Mikhail Khovanov. Diagrammatics for Soergel categories. *Int. J. Math. Math. Sci.*, 2010:58, 2010.

⁷The proof found in Section 3.2 appears to use the equality $\pi_{s,t} = \pi_{t,s}$, an equality which holds in the balanced case and fails in the unbalanced case. However, this is a red herring. When using shadings appropriately, both sides should be the same polynomial $\pi_{s,t}$.

- [12] Ben Elias and Aaron D. Lauda. Trace decategorification of the Hecke category. *J. Algebra*, 449:615–634, 2016.
- [13] Ben Elias and Nicolas Libedinsky. Indecomposable Soergel bimodules for universal Coxeter groups. *Trans. Am. Math. Soc.*, 369(6):3883–3910, 2017. with an appendix by Ben Webster.
- [14] Ben Elias and Ivan Losev. Modular representation theory in type A via Soergel bimodules. Preprint. <https://arxiv.org/abs/1701.00560>.
- [15] Ben Elias, Shotaro Makisumi, Ulrich Thiel, and Geordie Williamson. *Introduction to Soergel bimodules*, volume 5 of *RSME Springer Series*. Springer, 2020.
- [16] Ben Elias, Noah Snyder, and Geordie Williamson. On cubes of Frobenius extensions. In *Representation theory—current trends and perspectives*, EMS Series of Congress Reports, pages 171–186. European Mathematical Society, 2017.
- [17] Ben Elias and Geordie Williamson. Soergel calculus. *Represent. Theory*, 20:295–374, 2016.
- [18] Ben Elias and Geordie Williamson. Diagrammatics for Coxeter groups and their braid groups. *Quantum Topol.*, 8(3):413–457, 2017.
- [19] Joel Gibson, Lars Thorge Jensen, and Geordie Williamson. Calculating the p -canonical basis of Hecke algebras. *Transform. Groups*, 28:1121–1148, 2023.
- [20] Eugene Gorsky and Matthew Hogancamp. Hilbert schemes and y -ification of Khovanov–Rozansky homology. <https://arxiv.org/abs/1712.03938>, 2017.
- [21] Amit Hazi. Existence and rotatability of the two-colored Jones–Wenzl projector. <https://arxiv.org/abs/2302.14476>, 2023.
- [22] Xuhua He and Geordie Williamson. Soergel calculus and Schubert calculus. *Bull. Inst. Math., Acad. Sin. (N.S.)*, 13(3):317–350, 2018.
- [23] Lars Thorge Jensen and Geordie Williamson. The p -canonical basis for Hecke algebras. In *Categorification and higher representation theory*, volume 683 of *Contemporary Mathematics*, pages 333–361. American Mathematical Society, 2017.
- [24] Nicolas Libedinsky. Sur la catégorie des bimodules de Soergel. *J. Algebra*, 320(7):2675–2694, 2008.
- [25] Nicolas Libedinsky. Presentation of right-angled Soergel categories by generators and relations. *J. Pure Appl. Algebra*, 214(12):2265–2278, 2010.

- [26] Nicolas Libedinsky and Geordie Williamson. The anti-spherical category. *Adv. Math.*, 405: article no. 108509 (34 pages), 2022.
- [27] George Lusztig and Geordie Williamson. Billiards and tilting characters for SL_3 . *SIGMA, Symmetry Integrability Geom. Methods Appl.*, 14: article no. 015 (22 pages), 2018.
- [28] Scott Morrison. A Formula for the Jones–Wenzl Projections. <https://arxiv.org/abs/1503.00384>, 2015.
- [29] Francesco Polizzi. Formal power series over a Henselian ring. MathOverflow, <https://mathoverflow.net/q/122170> (version: 2014-10-23).
- [30] Simon Riche and Geordie Williamson. *Tilting modules and the p-canonical basis*, volume 397 of *Astérisque*. Société Mathématique de France, 2018.
- [31] N. Sankaran. A theorem on Henselian rings. *Can. Math. Bull.*, 11:275–277, 1968.
- [32] Wolfgang Soergel. Kazhdan-Lusztig–Polynome und unzerlegbare Bimoduln über Polynomringen. *J. Inst. Math. Jussieu*, 6(3):501–525, 2007.
- [33] Geordie Williamson. Schubert calculus and torsion explosion. *J. Am. Math. Soc.*, 30(4):1023–1046, 2017. with a joint appendix with Alex Kontorovich and Peter J. McNamara.
- [34] Geordie Williamson. Parity sheaves and the Hecke category. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 979–1015. World Scientific, 2018.

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