ANNALES MATHÉMATIQUES BLAISE PASCAL

ANN VERDOODT

Continued fractions for finite sums

Annales mathématiques Blaise Pascal, tome 1, nº 2 (1994), p. 71-84
<http://www.numdam.org/item?id=AMBP_1994__1_2_71_0>

Ann. Math. Blaise Pascal, Vol. 1, N° 2, 1994, pp. 71 - 84

CONTINUED FRACTIONS FOR FINITE SUMS

Ann Verdoodt

Abstract

Our aim in this paper is to construct continued fractions for sums of the type $\sum_{i=0}^{n} b_i\, z^{c(i)}$ or $\sum_{i=0}^{n} b_i/z^{c(i)}$, where (b_n) is a sequence such that b_n is different from zero if n is different from zero, and c(n) is an element of $\mathbb{N}$.

Résumé

Le but est de construire des fractions continues pour des sommes du type $\sum_{i=0}^{n} b_i\, z^{c(i)}$ or $\sum_{i=0}^{n} b_i/z^{c(i)}$, où (b_n) est une suite telle que b_n est différent de zéro pour n différent de zéro, et c(n) est un élément de $\mathbb{N}$.

1. Introduction

$[a_0, a_1, a_2, \ldots]$ denotes the continued fraction $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$,

and $[a_0, a_1, \ldots, a_n]$ denotes $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots\, a_{n-1} + \cfrac{1}{a_n}}}$.

The a_i's are called the partial quotients (or simply the quotients), and $[a_0, a_1, \ldots, a_n]$ is called a finite continued fraction.

Our aim in this paper is to construct continued fractions for sums of the type $\sum_{i=0}^{n} b_i\, z^{c(i)}$ or $\sum_{i=0}^{n} b_i/z^{c(i)}$, where c(i) is an element of $\mathbb{N}$.

In section 2, we find continued fractions for finite sums of the type $\sum_{i=0}^{n} b_i z^i$ ($c(i) = i$)

or $\sum_{i=0}^{n} b_i z^{q^i}$ ($c(i) = q^i$), where (b_n) is a sequence such that b_n is different from zero if n is different from zero, and where q is a natural number different from zero and one.

Therefore, we start by giving a continued fraction for the sum $\sum_{i=0}^{n} b_i T^{3^i}$, where b_i is different from zero for all i different from zero (b_i is a constant in T). This can be found in theorem 1.

If we replace b_i by $b_i z^i$ in theorem 1, and we put T equal to one, we find a continued fraction for $\sum_{i=0}^{n} b_i z^i$ (theorem 2), and if we replace b_i by $b_i z^{q^i}$ in theorem 1, and we put T equal to one, we find a continued fraction for $\sum_{i=0}^{n} b_i z^{q^i}$ (theorem 3) (q is a natural number different from zero and one).

In section 3 we find continued fractions for finite sums of the type $\sum_{i=0}^{n} \frac{b_i}{z^{c(i)}}$, for some sequences (b_n) and ($c(n)$), where $c(n)$ is a natural number.

In theorem 4, we find a result for $c(i)$ equal to 2^i (for all i).

Finally, in theorem 5, we give a continued fraction for $\sum_{i=0}^{v} \frac{b_i}{z^{c(i)}}$, where $c(0)$ equals zero, and $c(n+1) - 2c(n) \geq 0$.

The results in this paper are extensions of results that can be found in [2], [3] and [4].

Acknowledgement : I thank professor Van Hamme for the help and the advice he gave me during the preparation of this paper.

2. Continued fractions for sums of the type $\sum_{i=0}^{n} b_i z^i$

All the proofs in sections 2 and 3 can be given with the aid of the following simple lemma :

Lemma

Let i) $p_0 = a_0$, $q_0 = 1$, $p_1 = a_1 a_0 + 1$, $q_1 = a_1$,

$p_n = a_n p_{n-1} + p_{n-2}$, $q_n = a_n q_{n-1} + q_{n-2}$ $(n \geq 2)$,

then we have

ii) $\frac{p_n}{q_n} = [a_0, a_1, ..., a_n]$

iii) $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ $(n \geq 1)$

iv) $\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, ..., a_1]$ $(n \geq 1)$

These well-known results can e.g. be found in [1] .

First we give a continued fraction for the sum $\sum_{i=0}^{n} b_i T^{3i}$, where b_i is different from zero for all i different from zero (b_i is a constant in T) :

Theorem 1

Let (b_n) be a sequence such that $b_n \neq 0$ for all $n > 0$.

Define a sequence (x_n) by putting $x_0 = [b_0 T]$, $x_1 = [b_0 T, b_1^{-1} T^{-3}]$, and if

$x_n = [a_0, a_1, ..., a_{2n-1}]$ then setting $x_{n+1} = [a_0, a_1, ..., a_{2n-1} , - b_n^2/b_{n+1} T^{-3^n}, -a_{2n-1}, ..., -a_1]$.

Then $x_n = \sum_{i=0}^{n} b_i T^{3i}$ for all $n \in \mathbb{N}$.

Proof

For n = 0 the theorem clearly holds .

If n is at least one , we prove that $x_n = \sum_{i=0}^{n} b_i T^{3i}$ and $q_{2n-1} = b_n^{-1} T^{-3^n}$.

We prove this by induction . For n = 1 the assertion holds .

Suppose it holds for $1 \leq n \leq j$. We then prove the assertion for $n = j+1$.

$x_{j+1} = [\, a_0, a_1, \ldots, a_{2^{j+1}-1}]$

$= \quad [\, a_0, a_1, \ldots, a_{2^j-1}, a_{2^j}, -[a_{2^j-1}, \ldots, a_1\,]] \qquad$ (using the definition of a continued fraction)

$= \quad \dfrac{-q_{2^j-1}\, p_{2^j} + q_{2^j-2}\, p_{2^j-1}}{-q_{2^j-1}\, q_{2^j} + q_{2^j-2}\, q_{2^j-1}} \qquad$ (by i) , ii) and iv) of the lemma)

$= \quad \dfrac{-q_{2^j-1}\,(a_{2^j} p_{2^j-1} + p_{2^j-2}) + q_{2^j-2}\, p_{2^j-1}}{-q_{2^j-1}\,(a_{2^j} q_{2^j-1} + q_{2^j-2}) + q_{2^j-2}\, q_{2^j-1}} \qquad$ (by i) of the lemma)

now we have $p_{2^j-1}\, q_{2^j-2} - p_{2^j-2}\, q_{2^j-1} = (-1)^{2^j-2} = 1 \qquad$ (by iii) of the lemma)

$= \quad \dfrac{p_{2^j-1}}{q_{2^j-1}} - \dfrac{1}{a_{2^j}(q_{2^j-1})^2}$

now $a_{2^j}(q_{2^j-1})^2 = -\,T^{-3^j}\,\dfrac{b_j^2}{b_{j+1}}\,(b_j^{-1}\, T^{-3^j})^2 = -\,T^{-3^{j+1}}\, b_{j+1}^{-1}$

$= \quad [\, a_0, a_1, \ldots, a_{2^j-1}\,] + T^{3^{j+1}}\, b_{j+1} = \displaystyle\sum_{i=0}^{j+1} b_i\, T^{3^i} \qquad$ (by the induction hypothesis)

We still have to prove $q_{2^{j+1}-1} = b_{j+1}^{-1}\, T^{-3^{(j+1)}}$. Let k be at least one .

Then p_k and q_k are polynomials in $U = T^{-1}$. deg $q_k >$ deg q_{k-1} , and the term with the highest degree in q_k is given by $a_k \,.\, a_{k-1} \,.\, \ldots \,.\, a_1$. This follows from i) .

If r is a polynomial in U that divides p_k and q_k , then r must be a constant in U . This immediately follows from iii). If r divides p_k and q_k , then r divides $(-1)^{k-1}$. So r must be a constant.

Since $\displaystyle\sum_{i=0}^{j+1} b_i\, T^{3^i} = [\, a_0, a_1, \ldots, a_{2^{j+1}-1}] = \frac{p_{2^{j+1}-1}}{q_{2^{j+1}-1}}$, we have

$$\frac{p_{2^{j+1}-1}}{q_{2^{j+1}-1}} = \sum_{i=0}^{j+1} \frac{b_i\, T^{3^i}\, T^{-3^{j+1}}}{T^{-3^{j+1}}} = \sum_{i=0}^{j+1} \frac{b_i\, U^{3^{j+1}-3^i}}{U^{3^{j+1}}} = \frac{b_{j+1} + \sum_{i=0}^{j} b_i\, U^{3^{j+1}-3^i}}{U^{3^{j+1}}}$$

and we conclude that $q_{2^{j+1}-1} = C\, U^{3^{j+1}} = C\, T^{-3^{j+1}}$ where C is a constant .

By the previous remark , we have that

$q_{2^{j+1}-1} = C\, T^{-3^{j+1}} = C\, U^{3^{j+1}} = a_1 \,.\, a_2 \,.\, \ldots \,.\, a_{2^{j+1}-1}$

$= \quad (-1)^{2^j-1}(\, a_1 \,.\, a_2 \,.\, \ldots \,.\, a_{2^j-1}\,)^2 \,.\, a_{2^j} = -\,(\, q_{2^j-1}\,)^2 \,.\, a_{2^j}$

(by the induction hypothesis , since $q_{2^j-1} = b_j^{-1}\, T^{-3^j} = a_1 \,.\, a_2 \,.\, \ldots \,.\, a_{2^j-1}$)

$$= \quad -(b_j^{-1}T^{-3j})^2 \cdot \left(-T^{-3j}\frac{b_j^2}{b_{j+1}}\right) = \frac{T^{-3j+1}}{b_{j+1}}$$ which we wanted to prove .

We immediately have the following

Proposition

Let $x_0 = [a_0]$, $x_1 = [a_0, a_1]$ and if $x_n = [a_0, a_1, ..., a_{2^n-1}]$, then

$x_{n+1} = [a_0, a_1, ..., a_{2^n-1}, a_{2^n}, -a_{2^n-1}, ..., -a_1]$.

If n is at least two , then the continued fraction of x_n consists only of the partial quotients

a_{2n-1} , a_{2n-2} , $-a_{2n-2}$, ... , a_1 , $-a_1$ and a_0 .

Then the distribution of the partial quotients for x_n is as follows ($n \geq 2$) :

partial quotient												
a_{2n-1}	a_{2n-2}	$-a_{2n-2}$	a_{2n-3}	$-a_{2n-3}$	...	a_{2i}	$-a_{2i}$	...	a_1	$-a_1$	a_0
number of occurrences											
1	1	1	2	2	...	2^{n-i-2}	2^{n-i-2}	...	2^{n-2}	2^{n-2}	1

Proof

We give a proof by induction on n .

$x_2 = [a_0, a_1, a_2, a_3] = [a_0, a_1, a_2, -a_1]$, so the quotients a_0, a_1 , $-a_1$, a_2 , occur once .

So for n equal to 2 the assertion holds . Suppose it holds for $2 \leq n \leq j$ .Then we prove it holds for $n = j+1$. Since $x_{j+1} = [a_0, a_1, ..., a_{2^{j+1}-1}] = [a_0, a_1, ..., a_{2^j-1}, a_{2^j}, -a_{2^j-1}, ..., -a_1]$, it is clear that the partial quotients a_{2^j} and a_0 occur only once .

In the partial quotients $a_1, ..., a_{2^j-1}$ we have

partial quotient										
a_{2j-1}	a_{2j-2}	$-a_{2j-2}$	a_{2j-3}	$-a_{2j-3}$	...	a_{2i}	$-a_{2i}$	...	a_1	$-a_1$
number of occurrences										
1	1	1	2	2	...	2^{j-i-2}	2^{j-i-2}	...	2^{j-2}	2^{j-2}

so in the partial quotients $-a_1, ..., -a_{2^j-1}$ we have

partial quotient										
$-a_{2j-1}$	a_{2j-2}	$-a_{2j-2}$	a_{2j-3}	$-a_{2j-3}$	...	a_{2i}	$-a_{2i}$	...	a_1	$-a_1$
number of occurrences										
1	1	1	2	2	...	2^{j-i-2}	2^{j-i-2}	...	2^{j-2}	2^{j-2}

This proves the proposition.

Using theorem 1, we immediately have the following :

Theorem 2

Let (b_n) be a sequence such that b_n is different from zero for all n different from zero.

Define a sequence (x_n) by putting $x_0 = [b_0]$, $x_1 = [b_0, b_1^{-1}z^{-1}]$ and if $x_n = [a_0, a_1, ..., a_{2n-1}]$

then setting $x_{n+1} = [a_0, a_1, ..., a_{2n-1}, -b_n^2 / b_{n+1} z^{n-1}, -a_{2n-1}, ..., -a_1]$,

then $x_n = \sum_{i=0}^{n} b_i z^i$ for all $n \in \mathbb{N}$.

Proof

Replace b_i by $b_i z^i$ in theorem 1, and put T equal to one.

Some examples

1) Let $x_n = \sum_{i=0}^{n} x^i$ (i.e. $b_i = 1$ for all i). Then $a_0 = 1$, $a_1 = x^{-1}$ and $a_{2n} = -x^{n-1}$ $(n \geq 1)$

2) Let $x_n = \sum_{i=0}^{n} \frac{x^i}{i!}$ (i.e. $\lim_{n\to\infty} x_n = e^x$).

Then $a_0 = 1$, $a_1 = x^{-1}$ and $a_{2n} = -\frac{n+1}{n!} x^{n-1}$ $(n \geq 1)$

3) Let $x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}$ (i.e. $\lim_{n\to\infty} x_n = \cos x$).

Then $a_0 = 1$, $a_1 = -2x^{-2}$ and $a_{2n} = (-1)^n \frac{(2n+2)(2n+1)}{(2n)!} x^{2n-2}$ $(n \geq 1)$

4) Let $x_n = \sum_{i=0}^{n} \frac{(-1)^i x^{2i+1}}{(2i+1)!}$ (i.e. $\lim_{n\to\infty} x_n = \sin x$).

Then $a_0 = x$, $a_1 = -6x^{-3}$ and $a_{2n} = (-1)^n \frac{(2n+3)(2n+2)}{(2n+1)!} x^{2n-1}$ $(n \geq 1)$

In an analogous way as in the previous theorem, we have

Theorem 3

Let (b_n) be a sequence such that b_n is different from zero for all n different from zero, and let q be a natural number different from zero and one.

Define a sequence (x_n) by putting $x_0 = [b_0 z]$, $x_1 = [b_0 z, b_1^{-1} z^{-q}]$ and if $x_n = [a_0, a_1, \ldots, a_{2^n-1}]$ then setting $x_{n+1} = [a_0, a_1, \ldots, a_{2^n-1}, -b_n^2/b_{n+1} z^{-q^n(q-2)}, -a_{2^n-1}, \ldots, -a_1]$.

Then $x_n = \sum_{i=0}^{n} b_i z^{q^i}$ for all $n \in \mathbb{N}$.

Proof

Replace b_i by $b_i z^{q^i}$ in theorem 1, and put T equal to one.

An Example

In [4] we find the following:

Let $\mathbb{F}_q$ be the finite field of cardinality q. Let $A = \mathbb{F}_q[X]$, $K = \mathbb{F}_q(X)$, $K_\infty = \mathbb{F}_q((1/X))$ and let Ω be the completion of an algebraic closure of K_∞. Then A, K, K_∞, Ω are well-known analogous of $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ respectively.

Let $[i] = X^{q^i} - X$ (the symbol [i] does not have the same meaning as in $x_0 = [a_0]$). This is just the product of monic irreducible elements of A of degree dividing i.

Let $D_0 = 1$, $D_i = [i] D_{i-1}^q$ if $i > 0$. This is the product of monic elements of A of degree i.

Let us introduce the following function: $e(Y) = \sum_{i=0}^{\infty} \frac{Y^{q^i}}{D_i}$ $(Y \in \Omega)$.

Then Thakur gives the following theorem:

Define a sequence x_n by setting $x_1 = [0, Y^{-q}D_1]$ and if $x_n = [a_0, a_1, \ldots, a_{2^n-1}]$ then setting

$x_{n+1} = [a_0, a_1, \ldots, a_{2^n-1}, -Y^{-q^n(q-2)}D_{n+1}/D_n^2, -a_{2^n-1}, \ldots, -a_1]$, then $x_n = \sum_{i=1}^{n} \frac{Y^{q^i}}{D_i}$ for all $n \in \mathbb{N}$.

In particular, $e(Y) = Y + \lim_{n\to\infty} x_n$.

If we put $b_i = D_i^{-1}$ if $i > 0$, and $b_0 = 0$ in theorem 3, then we find the result of Thakur.

3. Continued fractions for sums of the type $\sum_{i=0}^{n} \frac{b_i}{z^{c(i)}}$

In this section , b_i is a constant in z , and c(i) is a natural number . Our first theorem in this section gives the continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2^i}}$ (i.e. $c(i) = 2^i$ for all i) :

Theorem 4

Let (b_n) be a sequence such that b_n is different from zero for all n . A continued fraction for the sum $\sum_{i=0}^{n} \frac{b_i}{z^{2^i}}$ can be given as follows :

Put $x_0 = [\,0, z/b_0]$, $x_1 = [\,0, \frac{z}{b_0} - \frac{b_1}{b_0^2}, \frac{b_0^3 z}{b_1^2} + \frac{b_0^2}{b_1}\,]$ and if $x_k = [\,a_0, a_1, ..., a_{2^k}\,]$ then setting

$x_{k+1} = [\,a_0, a_1, ..., a_{2^k-1}\,, a_{2^k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2^k} - \gamma_{k+1}^{-1}, a_{2^k+2}, ..., a_{2^{k+1}}]$ where $\gamma_{k+1} = b_{k+1} \frac{(b_0)^{2^{k+1}}}{(b_1)^{2^{k+1}}}$,

$a_{2^k+i} = \gamma_{k+1}^{2} a_{2^k-i+1}$ if i is even , and $a_{2^k+i} = \gamma_{k+1}^{-2} a_{2^k-i+1}$ if i is odd ($2 \le i \le 2^k$) ,

then $x_k = \sum_{i=0}^{k} \frac{b_i}{z^{2^i}}$ for all $k \in \mathbb{N}$.

Proof

If we have $x_n = [\,a_0, a_1, ..., a_{2^n}\,] = \frac{p_{2^n}}{q_{2^n}}$, we show by induction that x_n equals $\sum_{i=0}^{n} \frac{b_i}{z^{2^i}}$, and that q_{2^n} equals $z^{2^n} \frac{b_0^{2^n}}{b_1^{2^n}}$. For n = 0 , 1 this follows by an easy calculation .

Suppose the assertion holds for $0 \le n \le k$. Then we show it holds for n = k+1 .

The first part of the proof , i.e. showing that $x_{k+1} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2^i}}$ is analogous to the first part of the proof of [2] , theorem 1 .

$x_{k+1} = [\,a_0, a_1, ..., a_{2^k-1}\,, a_{2^k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2^k} - \gamma_{k+1}^{-1}, a_{2^k+2}, ..., a_{2^{k+1}}]$

$= [\,a_0, a_1, ..., a_{2^k-1}, a_{2^k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2^k} - \gamma_{k+1}^{-1}, \gamma_{k+1}^{2} [a_{2^k-1}, a_{2^k-2}, a_{2^k-3}, ..., a_2, a_1]\,]$

(using the definition of a continued fraction)

Now if $[\,a_0, a_1, ..., a_{2^k}\,] = \frac{p_{2^k}}{q_{2^k}}$, then $[\,a_0, a_1, ..., a_{2^k-1}\,] = \frac{p_{2^k-1}}{q_{2^k-1}}$ and so

$$[\, a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}] = \frac{(a_{2k} + \gamma_{k+1})p_{2k-1} + p_{2k-2}}{(a_{2k} + \gamma_{k+1})q_{2k-1} + q_{2k-2}} = \frac{p_{2k} + \gamma_{k+1}p_{2k-1}}{q_{2k} + \gamma_{k+1}q_{2k-1}}$$

(by i) and ii) of the lemma)

Then $[\, a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1}\,] = \dfrac{(\gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1})(p_{2k} + \gamma_{k+1}p_{2k-1}) + p_{2k-1}}{(\gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1})(q_{2k} + \gamma_{k+1}q_{2k-1}) + q_{2k-1}}$

(by i) and ii) of the lemma)

And so

$$[\, a_0, a_1, \ldots, a_{2k-1}, a_{2k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2k} - \gamma_{k+1}^{-1}, \gamma_{k+1}^{2} [a_{2k-1}, a_{2k-2}, a_{2k-3}, \ldots, a_2, a_1]\,]$$

$$= \frac{a_{2k} q_{2k-1} p_{2k} + \gamma_{k+1} a_{2k} q_{2k-1} p_{2k-1} - \gamma_{k+1} q_{2k-1} p_{2k} + q_{2k-2} p_{2k} + \gamma_{k+1} q_{2k-2} p_{2k-1}}{a_{2k} q_{2k-1} q_{2k} + \gamma_{k+1} a_{2k} q_{2k-1} q_{2k-1} - \gamma_{k+1} q_{2k-1} q_{2k} + q_{2k-2} q_{2k} + \gamma_{k+1} q_{2k-2} q_{2k-1}}$$

(by iv) of the lemma)

If we use the following equalities

$$(p_n - p_{n-2})q_{n-1} = a_n p_{n-1} q_{n-1} \qquad (q_n - q_{n-2})p_n = a_n p_n q_{n-1}$$

$$(q_n - q_{n-2})q_n = a_n q_n q_{n-1} \qquad (q_n - q_{n-2})q_{n-1} = a_n q_{n-1}^2 \qquad \text{(by i) of the lemma)}$$

then we find that the numerator equals $q_{2k} p_{2k} + \gamma_{k+1}$ (by iii) of the lemma) and the denominator equals $(q_{2k})^2$.

So we conclude

$$x_{k+1} = \frac{p_{2k}}{q_{2k}} + \frac{\gamma_{k+1}}{(q_{2k})^2} = \sum_{i=0}^{k} \frac{b_i}{z^{2^i}} + \frac{(b_1)^{2^{k+1}}}{z^{2^{k+1}}(b_0)^{2^{k+1}}}\, b_{k+1} \frac{(b_0)^{2^{k+1}}}{(b_1)^{2^{k+1}}} = \sum_{i=0}^{k+1} \frac{b_i}{z^{2^i}}$$

We still have to show $q_{2^{k+1}} = z^{2^{k+1}} \dfrac{(b_0)^{2^{k+1}}}{(b_1)^{2^{k+1}}}$.

In the same way as in the proof of theorem 1, we find that $q_{2^{k+1}} = C\, z^{2^{k+1}}$ where C is a constant.

Let α_i be the coefficient of z in a_i.

Then for C, the coefficient of $z^{2^{k+1}}$ in $q_{2^{k+1}}$, we have

$$C = \alpha_1\alpha_2 \ldots \alpha_{2k-1}\alpha_{2k}(\gamma_{k+1}^{-2}\alpha_{2k})(\gamma_{k+1}^{2}\alpha_{2k-1})(\gamma_{k+1}^{-2}\alpha_{2k-2})(\gamma_{k+1}^{2}\alpha_{2k-3}) \ldots (\gamma_{k+1}^{2}\alpha_1)$$

$$= (\alpha_1\alpha_2 \ldots \alpha_{2k-1}\alpha_{2k})^2 = (\text{ coefficient of } z^{2^k} \text{ in } q_{2k})^2 = \left(\frac{(b_0)^{2^k}}{(b_1)^{2^k}} \right)^2 = \frac{(b_0)^{2^{k+1}}}{(b_1)^{2^{k+1}}}$$

and we conclude $q_{2^{k+1}} = z^{2^{k+1}} \dfrac{(b_0)^{2^{k+1}}}{(b_1)^{2^{k+1}}}$. This finishes the proof.

Some examples

1) If we put b_i equal to one for all i, and z is an integer at least 3, then we find theorem 1 of [2]:

Let $B(u,v) = \sum_{i=0}^{v} \frac{1}{u^{2^i}} = \frac{1}{u} + \frac{1}{u^2} + \frac{1}{u^4} + \dots + \frac{1}{u^{2^v}}$ $\quad$ ($u \geq 3$, u an integer)

Then $B(u,0) = [0,u]$, $B(u,1) = [0,u-1,u+1]$, and if $B(u,v) = [a_0, a_1, \dots, a_n] = \frac{p_n}{q_n}$

then $B(u,v+1) = [a_0, a_1, \dots, a_{n-1}, a_n+1, a_n-1, a_{n-1}, a_{n-2}, \dots, a_2, a_1]$.

2) Put $b_i = \lambda^i$. Then we have $x_0 = [\,0, u\,]$, $x_1 = [\,0, u-\lambda, \frac{u}{\lambda^2} + \frac{1}{\lambda}\,]$ and if $x_k = [\,a_0, a_1, \dots, a_{2^k}\,]$, then $x_{k+1} = [\,a_0, a_1, \dots, a_{2^k-1}, a_{2^k} + \gamma_{k+1}, \gamma_{k+1}^{-2} a_{2^k} - \gamma_{k+1}^{-1}, a_{2^k+2}, \dots, a_{2^{k+1}}]$, where $\gamma_{k+1} = \lambda^{k+1-2^{k+1}}$,

$a_{2^k+i} = \gamma_{k+1}^{2} a_{2^k-i+1}$ if i is even, and $a_{2^k+i} = \gamma_{k+1}^{-2} a_{2^k-i+1}$ if i is odd ($2 \leq i \leq 2^k$),

then $x_k = \sum_{i=0}^{k} \frac{\lambda^i}{u^{2^i}}$ for all $k \in \mathbb{N}$.

For some some sequences (b_n) and ($c(n)$), we can give a continued fraction for the sum

$\sum_{i=0}^{v} \frac{b_i}{z^{c(i)}}$ as follows:

Theorem 5

Let (b_n) be a sequence such that $b_n \neq 0$ for all n, and $b_0 \neq 0, 1, -1$, and $1/2$, and let ($c(n)$) be a sequence such that $c(0) = 0$, and $c(n+1) - 2c(n) \geq 0$.

Put $x_0 = [\,-b_0^2, \frac{1}{b_0} - 1, \frac{1}{b_0} + 1\,] = [\,a_0, a_1, a_2] = \frac{p_2}{q_2} = \frac{p_{(0)}}{q_{(0)}}$,

and if $x_v = [\,a_0, a_1, \dots, a_n\,] = \frac{p_n}{q_n} = \frac{p_{(v)}}{q_{(v)}}$,

then setting $x_{v+1} = [\,a_0, a_1, \dots, a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, \dots, a_2, a_1]$,

where $d(v) = c(v+1) - 2c(v)$, $\alpha_v = \frac{b_v^2}{b_{v+1}}$ if $v \geq 1$ and $\alpha_0 = \frac{b_0^4}{b_1}$,

then $x_v = \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}}$ for all v in $\mathbb{N}$, and $q_{(v)} = \frac{z^{c(v)}}{b_v}$ if $v \geq 1$, $q_{(0)} = \frac{1}{(b_0)^2}$

Remarks

1) The special form of b_0, $x_0 = b_0 = [\, -b_0^2 \,,\, \frac{1}{b_0} - 1 \,,\, \frac{1}{b_0} + 1 \,] = [\, a_0, a_1, a_2]$ is needed since in the expression $[\, a_0, a_1, \ldots, a_n \,] = \frac{p_n}{q_n}$ the integer n must be even .

2) The value of n is $n = 2^{v+1} + 2^v + 2$ (this can be easily seen by induction)

3) The only partial quotients that appear are $-b_0^2$, $\frac{1}{b_0} - 1$, $\frac{1}{b_0} + 1$, $\frac{1}{b_0}$, $\frac{1}{b_0} - 2$, $\alpha_v z^{d(v)} - 1$, and 1 , so b_0 must be different from 0 , 1 , -1 , and 1/2 .

Proof

For v equal to 0 , 1 or 2 we find this result by an easy computation .

We prove the theorem by induction on v .

Suppose we have $x_v = \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}} = [\, a_0, a_1, \ldots, a_n \,] = \frac{p_n}{q_n} = \frac{p_{(v)}}{q_{(v)}}$ with $q_{(v)} = \frac{z^{c(v)}}{b_v}$

Then we show that $x_{v+1} = [\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, \ldots, a_2, a_1] = \sum_{i=0}^{v+1} \frac{b_i}{z^{c(i)}}$

with $q_{(v+1)} = \frac{z^{c(v+1)}}{b_{v+1}}$.

The first part of the proof , i.e. showing that $x_{v+1} = \sum_{i=0}^{v+1} \frac{b_i}{z^{c(i)}}$, is analogous to the first part of the proof of the theorem in [3] .

Now , by repeated use of i) an ii) of the lemma , we have

$$[\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1] = \frac{(\alpha_v z^{d(v)} - 1)p_n + p_{n-1}}{(\alpha_v z^{d(v)} - 1)q_n + q_{n-1}} \quad ;$$

$$[\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1] = \frac{\alpha_v z^{d(v)} p_n + p_{n-1}}{\alpha_v z^{d(v)} q_n + q_{n-1}} \quad ;$$

$$[\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1] = \frac{a_n \alpha_v z^{d(v)} p_n + a_n p_{n-1} - p_n}{a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n}$$

$$x_{v+1} = [\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, a_{n-1}, \ldots, a_1]$$

$$= [\, a_0, a_1, \ldots, a_n, \alpha_v z^{d(v)} - 1, 1, a_n - 1, [a_{n-1}, \ldots, a_1]]$$

(using the definition of a continued fraction)

$$= \frac{a_n q_{n-1}\alpha_v z^{d(v)} p_n + q_{n-2}\alpha_v z^{d(v)} p_n + a_n q_{n-1} p_{n-1} - q_{n-1} p_n + q_{n-2} p_{n-1}}{a_n q_{n-1}\alpha_v z^{d(v)} q_n + q_{n-2}\alpha_v z^{d(v)} q_n + a_n (q_{n-1})^2 - q_{n-1} q_n + q_{n-2} q_{n-1}}$$

(by i) , ii) and iv) of the lemma)

$$= \frac{p_n}{q_n} + \frac{1}{(q_n)^2 \alpha_v z^{d(v)}}$$ (by i) and iii) of the lemma since n is even)

So $x_{v+1} = \frac{p_n}{q_n} + \frac{1}{(q_n)^2 \alpha_v z^{d(v)}} = \sum_{i=0}^{v} \frac{b_i}{z^{c(i)}} + \frac{(b_v)^2 b_{v+1}}{z^{2c(v)} (b_v)^2 z^{d(v)}}$ since $q_n = q_{(v)} = \frac{z^{c(v)}}{b_v}$, $\alpha_v = \frac{(b_v)^2}{b_{v+1}}$

$$= \sum_{i=0}^{v+1} \frac{b_i}{z^{c(i)}}$$

We still have to prove $q_{(v+1)} = q_{2n+2} = \frac{z^{c(v+1)}}{b_{v+1}}$, and since $\frac{z^{c(v+1)}}{b_{v+1}} = (q_n)^2 \alpha_v z^{d(v)}$, it suffices to prove that $q_{2n+2} = (q_n)^2 \alpha_v z^{d(v)}$.

We can not use the same trick here as in the proofs of theorems 1 and 4 , since we do not necessarily have deg q_{k+1} > deg q_k (q_k as a polynomial in z)

We already know that $q_{n+1} = (\alpha_v z^{d(v)} - 1) q_n + q_{n-1}$, $q_{n+2} = \alpha_v z^{d(v)} q_n + q_{n-1}$

Repeated use of i) of the lemma gives

$q_{n+3} = q_{(n+2)+1} = a_n \alpha_v z^{d(v)} q_n + a_n q_{n-1} - q_n = r_1 \alpha_v z^{d(v)} q_n - q_{n-2}$ (where we put $a_n = r_1$)

$q_{n+4} = q_{(n+2)+2} = (a_{n-1}a_n + 1)\alpha_v z^{d(v)} q_n - a_{n-1} q_{n-2} + q_{n-1} = r_2 \alpha_v z^{d(v)} q_n + q_{n-3}$

(where we put $a_{n-1}a_n + 1 = r_2$)

$q_{n+5} = q_{(n+2)+3} = (a_{n-2}(a_{n-1}a_n + 1) + a_n)\alpha_v z^{d(v)} q_n + a_{n-2} q_{n-3} - q_{n-2}$

$= r_3 \alpha_v z^{d(v)} q_n - q_{n-4}$ (where we put $a_{n-2}(a_{n-1}a_n + 1) + a_n = r_3$)

etc...

Continuing this way , we find

$q_{(n+2)+k} = r_k \alpha_v z^{d(v)} q_n + (-1)^k q_{n-(k+1)}$, $q_{(n+2)+k+1} = r_{k+1} \alpha_v z^{d(v)} q_n + (-1)^{k+1} q_{n-(k+2)}$

Then $q_{(n+2)+k+2} = (a_{n-(k+1)} r_{k+1} + r_k)\alpha_v z^{d(v)} q_n + (-1)^{k+1} a_{n-(k+1)} q_{n-k-2} + (-1)^k q_{n-k-1}$

$= r_{k+2} \alpha_v z^{d(v)} q_n + (-1)^{k+2} q_{n-(k+3)}$

and finally we have $q_{2n} = q_{(n+2)+n-2} = r_{n-2} \alpha_v z^{d(v)} q_n + q_{n-(n-1)}$

$q_{2n+1} = q_{(n+2)+n-1} = r_{n-1}\alpha_v z^{d(v)} q_n - q_{n-n}$ (we remark that n is even)

and so $q_{2n+2} = q_{(n+2)+n} = r_n\alpha_v z^{d(v)} q_n - a_1q_0 + q_1 = r_n\alpha_v z^{d(v)} q_n$

So if we want to show that $q_{2n+2} = (q_n)^2 \alpha_V z^{d(v)}$, we must show that r_n equals q_n.

For the sequence (r_n) we have $r_0 = 1$, $r_1 = a_n$, $r_2 = a_{n-1}a_n + 1 = a_{n-1}r_1 + r_0$,

$r_3 = a_{n-2}(a_{n-1}a_n+1) + a_n = a_{n-2}r_2 + r_1$, and continuing this way we find $r_{k+2} = a_{n-(k+1)}r_{k+1} + r_k$.

From this it follows that $[\,1, a_n, ..., a_1] = [\,1, c_1, ..., c_n] = \frac{t_n}{r_n}$ (we put $a_i = c_{n+1-i}$)

with $t_0 = c_0$, $r_0 = 1$, $t_1 = c_1 c_0 + 1$, $r_1 = c_1$,

$t_n = c_n t_{n-1} + t_{n-2}$, $r_n = c_n r_{n-1} + r_{n-2}$ $(n \geq 2)$,

Now n can be written as n = 2k+2 (see remark 2 following theorem 5) and so

$$[\,a_0, a_1, ..., a_n] = [\,a_0, a_1, ..., a_k, \alpha_{v-1}z^{d(v-1)} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = \frac{p_n}{q_n}$$

and then $[\,1, a_1, ..., a_k, \alpha_{v-1}z^{d(v-1)} - 1, 1, a_k - 1, a_{k-1}, ..., a_1] = [\,1, a_1, ..., a_n] = \frac{p_n'}{q_n}$

where the q_i $(0 \leq i \leq n)$ stay the same since q_i does not depend on a_0.

So $[\,1, a_1, ..., a_{k-1}, a_k - 1, 1, \alpha_{v-1}z^{d(v-1)} - 1, a_k, a_{k-1}, ..., a_1] = [\,1, a_n, ..., a_1] = \frac{t_n}{r_n}$

and we conclude $q_i = r_i$ for $0 \leq i \leq k-1$.

We have to show $q_n = r_n$. Now (by repeated use of i) of the lemma)

$q_k = a_kq_{k-1} + q_{k-2}$, $r_k = q_k - q_{k-1}$;

$q_{k+1} = \alpha_{v-1}z^{d(v-1)} q_k - q_k + q_{k-1}$, $r_{k+1} = q_k$;

$q_{k+2} = \alpha_{v-1}z^{d(v-1)} q_k + q_{k-1}$, $r_{k+2} = \alpha_{v-1}z^{d(v-1)} q_k - q_{k-1}$;

$q_{k+3} = q_{(k+2)+1} = \alpha_{v-1}z^{d(v-1)} a_kq_k + a_kq_{k-1} - q_k = a_k\alpha_{v-1}z^{d(v-1)} q_k - q_{k-2}$

$= R_1\alpha_{v-1}z^{d(v-1)} q_k - q_{k-2}$, where we put $a_k = R_1$,

$r_{k+3} = r_{(k+2)+1} = a_k\alpha_{v-1}z^{d(v-1)} q_k + q_{k-2} = R_1\alpha_{v-1}z^{d(v-1)} q_k + q_{k-2}$;

$q_{k+4} = q_{(k+2)+2} = (a_{k-1}a_k+1)\alpha_{v-1}z^{d(v-1)} q_k - a_{k-1}q_{k-2} + q_{k-1}$

$= (a_{k-1}a_k+1)\alpha_{v-1}z^{d(v-1)} q_k + q_{k-3}$

$= R_2\alpha_{v-1}z^{d(v-1)} q_k + q_{k-3}$ where we put $(a_{k-1}a_k+1) = R_2$,

$$r_{k+4} = r_{(k+2)+2} = (a_{k-1}a_k+1)\alpha_{v-1}z^{d(v-1)}q_k + a_{k-1}q_{k-2} - q_{k-1}$$

$$= (a_{k-1}a_k+1)\alpha_{v-1}z^{d(v-1)}q_k - q_{k-3} = R_2\alpha_{v-1}z^{d(v-1)}q_k - q_{k-3}$$

....

If we continue this way , we find $q_{(k+2)+i} = R_i\alpha_{v-1}z^{d(v-1)}q_k + (-1)^i q_{k-(i+1)}$, and $r_{(k+2)+i} = R_i\alpha_{v-1}z^{d(v-1)}q_k - (-1)^i q_{k-(i+1)}$ ($0 \le i \le k$, $R_0 = 1$) , and so we have $q_{2k} = q_{(k+2)+k-2} = R_{k-2}\alpha_{v-1}z^{d(v-1)}q_k + q_{k-(k-1)}$, $q_{2k+1} = q_{(k+2)+k-1} = R_{k-1}\alpha_{v-1}z^{d(v-1)}q_k - q_{k-k}$ (we remark that k is even) and thus $q_{2k+2} = q_{(k+2)+k} = R_k\alpha_{v-1}z^{d(v-1)}q_k - a_1q_0 + q_1 = R_k\alpha_{v-1}z^{d(v-1)}q_k$, and $r_{2k} = r_{(k+2)+k-2} = R_{k-2}\alpha_{v-1}z^{d(v-1)}q_k - q_{k-(k-1)}$, $r_{2k+1} = r_{(k+2)+k-1} = R_{k-1}\alpha_{v-1}z^{d(v-1)}q_k + q_{k-k}$ and thus $r_{2k+2} = r_{(k+2)+k} = R_k\alpha_{v-1}z^{d(v-1)}q_k + a_1q_0 - q_1 = R_k\alpha_{v-1}z^{d(v-1)}q_k$,

So we conclude that $q_{2k+2} = q_n$ equals $r_{2k+2} = r_n$. This finishes the proof .

The case b_i equal to one , where z is an integer at least two , is studied by Shallit ([3]) :
Let (c(k)) be a sequence of positive integers such that $c(v+1) \ge 2c(v)$ for all $v \ge v'$, where v' is a non-negative integer . Let $d(v) = c(v+1) - 2c(v)$. Define S(u,v) as follows :

$S(u,v) = \sum_{i=0}^{v} u^{-c(i)}$, where u is an integer , $u \ge 2$. Then Shallit proved the following theorem :

Suppose $v \ge v'$. If $S(u,v) = [\, a_0, a_1, \ldots, a_n]$ and n is even , then
$S(u,v+1) = [\, a_0, a_1, \ldots, a_n, u^{d(v)}-1, 1, a_n-1, a_{n-1}, a_{n-2}, \ldots , a_2, a_1]$.

References

[1] G. H. Hardy and E. M. Wright , An Introduction to the Theory of Numbers, Oxford University Press , 1979 .

[2] J. O. Shallit , Simple Continued Fractions for Some Irrational Numbers , Journal of Number Theory , vol. 11 (1979) , p. 209-217 .

[3] J. O. Shallit , Simple Continued Fractions for Some Irrational Numbers II , Journal of Number Theory , vol. 14 (1982) , p. 228-231.

[4] D. S. Thakur , Continued Fraction for the Exponential for $\mathbf{F}_q[t]$, Journal of Number Theory , vol. 41 (1992), p. 150-155 .

Ann VERDOODT
Vrije Universiteit Brussel,
Faculty of Applied Sciences
Pleinlaan 2, B - 1050 Brussels
Belguim

Manuscrit reçu le 20 Juin 1994