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COMPACTIFICATION AND COMPACTOIDIFICATION

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Abstract. After discussing some of the many ways to get the Banaschewski compactification $\beta_0 T$ of an arbitrary ultraregular space T, we develop another construction of $\beta_0 T$ in Th. 2.1. Using those ideas, we develop an analog of $\beta_0 T$—what we call a *compactoidification* κT of an ultraregular space T in Sec. 3; κT is, in essence, a complete absolutely convex compactoid 'superset' of T to which continuous maps of T with precompact range into any complete absolutely convex compactoid subset may be 'continuously extended.'

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1 The Many Faces

For any topological spaces X and Y, $C(X,Y)$ and $C^*(X,Y)$ denote the spaces of continuous maps of X into Y and the continuous maps of X into Y with relatively compact range, respectively. To say that a topological space X is *ultraregular* or *ultranormal* means, respectively, that the clopen sets are a basis or disjoint closed subsets of X may be separated by clopen sets. A synonym for *ultraregular* is *0-dimensional.* We have a slight preference for the former in order to avoid confusion with other notions of dimension. Throughout the discussion, T denotes at least a Hausdorff space. For an ultraregular space E containing at least two points and ultraregular T, B. Banaschewski [2] discovered a compactification $\beta_0 T$ of T in which every $x \in C^*(T,E)$ may be continuously extended to $\beta_0 x \in C(\beta_0 T, E)$. $\beta_0 T$ is nowadays usually called the *Banaschewski compactification* of T. It functions as the natural analog of the Stone-Čech compactification ($\beta_0 T$ *is* βT for ultranormal T) in non-Archimedean analysis. Like the Stone-Čech compactification, the Banaschewski compactification is a protean entity, assuming many different guises. We discuss some of them in this section and then develop a new one in Sec. 2.

1.1 As a completion

Let E be an ultraregular space containing at least two points and let T be ultraregular. Let $C^*(T,E)$ denote the weakest uniform structure on T making each $x \in C^*(T,E)$ uniformly continuous into the compact space cl $x(T)$ equipped with its unique compatible uniform structure. By [1], pp. 92-93, since T is ultraregular, $\mathsf{C}^*(T,E)$ is compatible with the topology on T and $\mathsf{C}^*(T,E)$ is a precompact uniform structure on T. Since $\mathsf{C}^*(T,E)$ is precompact, its completion $\beta_0 T$ is compact and is called the *Banaschewski compactification* of T. $\beta_0 T$ is ultranormal ([2], p. 131, Satz 2 or [1], p. 93, Theorem 1)—hence ultraregular—and, by the usual process of extension by continuity function from a dense subspace to the whole space, each $x \in C^*(T,E)$ may be continuously extended to a unique continuous function $\beta_0 x \in C^*(\beta_0 T, E)$. $\beta_0 T$ is unique in a sense we discuss in the context of *E-compactifications* (Th. 1.6). At this point the reader may find the notation $\beta_0 T$ curious. Why $\beta_0 T$ and not $\beta_E T$? As long as E is ultraregular and contains at least two points ([1], p. 93, [8], pp. 240-243), the uniformity $\mathsf{C}^*(T,E)$ does not depend on E! A fundamental system of entourages for $\mathsf{C}^*(T,E)$, no matter what E is, is defined by the sets

$$\mathsf{V}_{\mathcal{P}} = \bigcup \{V \times V : V \in \mathcal{P}\}$$

where $\mathcal{P}$ is any finite open (therefore clopen) cover of T by pairwise disjoint sets. The completion of T with respect to this uniformity is the way Banaschewski obtained $\beta_0 T$. The definition of $\beta_0 T$ as the completion of $\mathsf{C}^*(T,E)$ where E is the discrete space of integers was first given in [7], though the idea of treating compactifications as completions is due to Nachbin. The connection with the Stone-Čech compactification is the following.

Definition 1.1 *Let $\mathcal{P}$ be a finite clopen cover of a topological space S by pairwise disjoint sets and let V denote the uniformity generated by $\mathsf{V}_{\mathcal{P}}$. We say that S is* strongly ultraregular *if* $\mathsf{V} = \mathsf{C}^*(T,\mathbf{R})$.

Theorem 1.2 ([8], pp. 251-2) *(a) Every ultranormal T_1-space S is strongly ultraregular.*
(b) If a topological space S is strongly ultraregular then $\beta_0 S = \beta S$.

1.2 As an E-Compactification

Tihonov proved that a completely regular space T may be characterized as one that is homeomorphic to a subspace of a product $[0,1]^m$ of unit intervals. Even though his name is not associated with it, he created the first version of the Stone-Čech compactification βT of T by then taking the closure of T in $[0,1]^m$. Engelking and Mrówka [5] developed analogous notions of *E-completely regular* space T and *E-compactification* $\beta_E T$. Let S and E be two topological spaces. S is called *E-completely regular* if it is homeomorphic to a subspace of the m-fold topological product E^m for some cardinal m. If $E = \mathbf{R}$ or $[0,1]$, this is the familiar notion of complete regularity. With $\mathbf{2}$ denoting the discrete space $\{0,1\}$, it happens that

Theorem 1.3 ([16], p. 17) *A topological space S is $\mathbf{2}$-completely regular if and only if it is an ultraregular T_0-space.*

An *E-compact* space is one which is homeomorphic to a closed subspace of a topological product E^m for some cardinal m. The $\mathbf{2}$-compact spaces are characterized as follows:

Theorem 1.4 ([5]. p.430, Example (iii)) *A topological space S is $\mathbf{2}$-compact if and only if it is compact and ultraregular.*

An *E-compactification* $\beta_E T$ of an E-completely regular space T is

(1) an E-compact space which contains T as a dense subset and

(2) ('the E-extension property') each $x \in C(T, E)$ may be extended to $\beta_E x \in C(\beta_E T, E)$.

The following analogs of properties of the Stone-Čech compactification obtain for E-compactifications.

Theorem 1.5 ([5], p. 433, Theorem 4, [16], pp. 25-27, 4.3 and 4.4). *An E-completely regular (Hausdorff) space T has a Hausdorff E-compactification $\beta_E T$ with the following properties:.*

(a) If S is an E-compact space then every continuous function $x : T \to S$ has a continuous extension $\bar{x} : \beta_E T \to S$.

(b) The space $\beta_E T$ is unique in the sense that if S is an E-compact space containing T as a dense subset and such that every continuous $x : T \to E$ has a continuous extension to S, then S is homeomorphic to $\beta_E T$ under a homeomorphism that is the identity on T.

(c) T is E-compact if and only if $T = \beta_E T$.

How does this apply to $\beta_0 T$? Ultraregular spaces T are **2**-completely regular by Th. 1.3. Since $\beta_0 T$ is compact and ultranormal, it follows that $\beta_0 T$ is **2**-compact by Th. 1.4. Therefore, by Th. 1.5(b) it follows that

Theorem 1.6 UNIQUENESS OF $\beta_0 T$. *$\beta_0 T$ is homeomorphic to $\beta_2 T$ under a homeomorphism that is the identity on T, as would be any ultraregular compactification of an ultraregular T with the E-extension property.*

1.3 As a Space of Characters

Let F be an ultraregular Hausdorff topological field so that $X = C^*(T, F)$ may be considered as an F-algebra. A *character* of X is a nonzero algebra homomorphism from X into F. Let the set H of characters of X be equipped with the weakest topology for which the maps $H \to F$, $h \longmapsto h(x)$, are continuous for each $x \in C^*(T, F)$. For each $p \in \beta_0 T$, let $p\hat{}$ denote the *evaluation map* at p, the map $C^*(T, F) \to F$, $x \longmapsto \beta_0 x(p)$. It is trivial to verify that each $p\hat{}$ is a character of $C^*(T, F)$. But more is true: You get all the characters of $C^*(T, F)$ this way. In fact, the map

$$\begin{array}{rccc} A: & \beta_0 T & \longrightarrow & H \\ & p & \longmapsto & p\hat{} \end{array}$$

establishes a homeomorphism between $\beta_0 T$ and H. The details may be found in [1], Theorem 3 and [8], Theorem 8.15.

1.4 Characters Again

Once again $\beta_0 T$ is realized as a space of nonzero homomorphisms—ring homomorphisms this time—into the very simple (discrete) field $\mathbf{2}$ with 2 elements.

A commutative ring X with identity in which each element is idempotent is called a *Boolean ring.* A subcollection X of the set of subsets of a given set T which is closed under union, intersection and set difference of any two of its members is called a *ring* of sets. Such a collection forms a ring in the usual algebraic sense if addition and multiplication are taken to be symmetric difference and intersection, respectively. If the sets in X cover T then X is called a *covering ring.* Since X must have a multiplicative identity (i.e., with respect to intersection) any covering ring must contain T as an element. Any covering ring X generates (in the sense that it is a subbase for) a ultraregular topology on T; the topology is ultraregular since the complement $T - A$ of any open set (member of X) must belong to X. In the converse direction, the class $\mathsf{Cl}(T)$ of clopen subsets obviously constitutes a covering ring of any topological space T.

Let X be a Boolean ring and endow $\mathbf{2}^X$ with the product topology. The *Stone space* $S(X)$ of the Boolean ring X is the subspace of $\mathbf{2}^X$ of all nonzero ring homomorphisms of X into $\mathbf{2}$. $S(X)$ is called the Stone space because of Stone's use of it in his remarkable characterization of compact ultraregular spaces.

> THE STONE REPRESENTATION THEOREM ([12], Theorem 4, [12], [4] p.227 or [6], pp. 77-80) If T is a compact ultraregular space, then T is homeomorphic to the Stone space of the Boolean ring $\mathsf{Cl}(T)$ of clopen subsets of T. Conversely, the Stone space $S(X)$ of any Boolean ring X is a compact ultraregular Hausdorff space and X is ring-isomorphic to the Boolean ring $\mathsf{Cl}(T)$ of clopen subsets of $S(X)$.

If T is ultraregular then $\beta_0 T$ is the Stone space of $\mathsf{Cl}(T)$. Indeed, the map $\beta : T \to S(\mathsf{Cl}(T))$, $t \longmapsto \beta t$, defined for $t \in T$ and $K \in \mathsf{Cl}(T)$ by

$$(\beta t)(K) = \begin{cases} 1 \in \mathbf{2} & t \in K \\ 0 \in \mathbf{2} & t \notin K \end{cases}$$

is a homeomorphism of T onto a dense subset of the compact ultraregular Hausdorff space $S(\mathsf{Cl}(T))$.

1.5 As a Space of Measures

Let T be ultraregular and let $\mathsf{Cl}(T)$ be the ring (algebra, actually, since $T \in \mathsf{Cl}(T)$) of clopen subsets of T, and let F be an ultraregular Hausdorff topological field. A *0-1 measure on T* is a finitely additive set function $m : \mathsf{Cl}(T) \to \{0,1\} \subset F$ satisfying the condition:

$$m(U) = 0 \quad \textit{and} \quad U \supset V \in \mathsf{Cl}(T) \Longrightarrow m(V) = 0$$

in other words, that clopen subsets of sets of measure 0 also have measure 0. Measures m_t 'concentrated at points $t \in T$' (also called 'purely atomic' or 'the point mass at t')) which are 1 on a clopen set U if $t \in U$ and 0 otherwise are 0-1 measures on T. The *weak clopen topology* for the collection M of all 0-1 measures on T has as a neighborhood base $m_0 \in \mathsf{M}$ sets of the form

$$V(m_0; S_1, \ldots, S_n) = \{m \in \mathsf{M} : m(S_j) = m_0(S_j), j = 1, \ldots n\}$$

where the S_j are clopen sets and $n \in \mathbb{N}$. It is trivial to verify that the map $t \to m_t$ is a homeomorphism of T into M. Using the techniques of [1] one can demonstrate that M is a compact ultranormal Hausdorff space to which any $x \in C^*(T, F)$ may be continuously extended. It follows that $\beta_0 T = \mathsf{M}$ in the sense of Th. 1.6.

Last, let us mention that $\beta_0 T$ may also be realized as a Wallman compactification utilizing the lattice of clopen subsets of T..

2 A New Approach

A construction of $\beta_0 T$ using the methods of non-Archimedean functional analysis is presented in Theorem 2.1. The proof hinges on the fact that, for a local field F, if U is a neighborhood of 0 in a locally F-convex space X then its polar U^o is $\sigma(X', X)$-compact ([15], Th. 4.11). Note that $\sigma(X', X)$ is ultraregular since the seminorms $p_x(f) = |f(x)|$, $x \in X$, $f \in X'$, are non-Archimedean.

Theorem 2.1 *Let F be a local field, let T be ultraregular and let $C^*(T, F)$ denote the* sup-*normed space of all continuous F-valued functions on T with relatively compact range. There is an ultranormal compactification $\beta_0 T$ of T such that any $x \in C^*(T, F)$ may be continuously extended to a function $\beta_0 x \in C(\beta_0 T, F)$.*

Proof. For $t \in T$, let $t\hat{}$ denote the evaluation map $x \longmapsto x(t)$ for any $x \in C^*(T, F)$. We note that each such $t\hat{}$ is a continuous linear form (algebra homomorphism, actually) and is of norm one. Thus $T\hat{} = \{t\hat{} : t \in T\} \subset U$ where U denotes the unit ball of the norm-dual $C^*(T, F)'$ of $C^*(T, F)$. Furthermore, the map $i : T \to C^*(T, F)'$, $t \longmapsto t\hat{}$, embeds T homeomorphically in $C^*(T, F)'$ endowed with its weak-* topology by the following argument. The map i is obviously injective. If a net $t_s \to t \in T$ then $x(t_s) \to x(t)$ for any $x \in C^*(T, F)$; hence $t_s\hat{} \to t\hat{}$ and therefore i is continuous. To see that i is a homeomorphism onto $i(K)$, let K be a closed subset of T. Since T is ultraregular, if $t \notin K$ then there exists $x \in C^*(T, F)$ such that $x(t) = 0$ and $|x(K)| = r > 1$. Hence the polar $\{x\}^o$ of $\{x\}$ is a neighborhood of $t\hat{}$ disjoint from $K\hat{}$ and $K\hat{}$ is a closed subset of $i(K)$. As U is the polar of the unit ball of $C^*(T, F)$, it follows that U is weak-*-compact ([15], Th. 4.11). Therefore the closure cT in U of (the homeomorphic image of) $T\hat{}$ is compact in $C^*(T, F)'$ endowed with the weak-* topology. As to the continuous extendiblity of $x \in C^*(T, F)$, consider the canonical image Jx of x in the second algebraic dual of $C^*(T, F)$, i.e., for any $f \in C^*(T, F)'$, $Jx(f) = f(x)$. Clearly Jx is weak-*-continuous on $C^*(T, F)'$; so, therefore, is its restriction $\beta_0 x = Jx\,|_{cT}$. Should this be called $c_F T$ rather than cT? No topologically significant changes occur for different F's: the compactness of the ultraregular space cT and the fact that T is C^*-embedded in cT imply that $cT = \beta_0 T$ by Th. 1.6.

3 Compactoidification

In this section we construct a *compactoidification* κT of an ultraregular space T. $(F, |\cdot|)$ denotes a complete nontrivially ultravalued field throughout. As usual, we abbreviate 'F-convex' to 'convex.' A map f defined on an absolutely convex subset A of a vector space over F with values in some absolutely convex set in a vector space over F is called *affine* if $f(ax+by) = af(x) + bf(y)$ for all $x, y \in A$ and all $a, b \in F$ with $|a| \le 1$ and $|b| \le 1$.

Definition 3.1 *A* compactoidification *of an ultraregular space T is a pair $(i, \kappa T)$ where κT is a complete absolutely convex compactoid subset of some Hausdorff locally convex space E over F and $i : T \to \kappa T$ is a continuous map with precompact range for which following extendibility property holds: For any complete absolutely convex compactoid subset A of some Hausdorff locally convex space E over F and any continuous map $j : T \to A$ with precompact range, there exists a unique continuous affine map $J : \kappa T \to A$ such that $J \circ i = j$.*

$$\begin{tikzcd} \kappa T \arrow[dr, "J"] & \\ T \arrow[u, "i"] \arrow[r, "j"] & A \end{tikzcd}$$

Theorem 3.2 *A compactoidification is unique in the following natural sense: if $(i_1, \kappa_1 T)$ and $(i_2, \kappa_2 T)$ are compactoidifications of* T *then there exists a unique affine homeomorphism $J_1 : \kappa_1 T \to \kappa_2 T$ such that $J_1 \circ i_1 = i_2$. Moreover, the map* i *must be injective.*

Proof. By definition, there exist unique continuous affine maps J_1 and J_2 such that $J_2 \circ i_1 = i_2$ and $J_1 \circ i_2 = i_1$. Thus, $J_1 \circ (J_2 \circ i_1) = J_1 \circ i_2 = i_1$.

$$\begin{tikzcd} \kappa_1 T \arrow[dr, "J_2"] & \\ T \arrow[u, "i_1"] \arrow[r, "i_2"] & \kappa_2 T \end{tikzcd}$$

Since the identity map $I_1 : t \longmapsto t$ of $\kappa_1 T$ onto $\kappa_1 T$ also satisfies $I_1 \circ i_1 = i_1$, it follows from the uniqueness that $I_1 = J_1 \circ J_2$. Similarly, $I_2 = J_2 \circ J_1$ where I_2 is the identity map of $\kappa_2 T$ onto $\kappa_2 T$. It follows that J_1 is a homeomorphism of $\kappa_1 T$ onto $\kappa_2 T$ and J_2 is its inverse. If $i_1(t_1) = i_1(t_2)$ then $i_2(t_1) = J_1 \circ i_1(t_1) = J_1 \circ i_1(t_2) = i_2(t_2)$ so if one of the maps i is 1-1, all such i must be. As shown in Theorem 3.3, there is an i that is 1-1.

In the notation of Sec. 2:

Theorem 3.3 *Let* T *be ultraregular and let the continuous dual $C^*(T,F)'$ of $C^*(T,F)$ carry the weak-* topology. Then*

(a) the closed absolutely convex hull κT of $T^{\wedge}$ is the unit ball U of $C^*(T,F)'$ and

(b) the pair $(i, \kappa T)$ is a compactoidification of T.

Proof. Clearly the absolute convex hull B of $T^{\wedge}$ is contained in the unit ball U of $C^*(T,F)'$. Since U is a complete compactoid by the p-adic Alaoglu theorem ([9], Prop.

3.1), so, therefore, is the closed absolutely convex hull κT of the compact set cl $T^\wedge$. It follows from [10], Prop. 1.3 that B is edged (i.e., if the valuation of F is dense then cl $B = \cap \{a(\ clB) : a \in F, |a| > 1\}$) and therefore ([9], Th. 4.7) a polar set in $C^*(T, F)'$. If cl $B \neq U$ there must exist $g \in C^*(T, F)''$ such that $|g| \leq 1$ on B and $|g(f)| > 1$ for some $f \in U-$cl B. Since g must be an evaluation map determined by some point $x \in C^*(T, F)$ by [9], Lemma 7.1, we have found an x such that $|x(t)| = |t^\wedge(x)| \leq 1$ for all $t \in T$ but $|f(x)| > 1$. As this contradicts $\|f\| \leq 1$, the proof of (a) is complete.

(b) As in the proof of Th. 2.1, i is a homeomorphism onto the precompact set $T^\wedge$. To verify the extendibility requirement, let A be a complete absolutely convex compactoid and let $j : T \to A$ be continuous with precompact range. We define the affine extension J of j on the absolutely convex hull B of $T^\wedge$by taking $J(\sum_{i=1}^{n} a_i t_i^\wedge) = \sum_{i=1}^{n} a_i j(t_i)$ for $a_i \in F, |a_i| \leq 1, i = 1, \ldots, n$. The definition makes sense because the $t_i^\wedge$ are linearly independent for distinct t_i. Evidently $j = J \circ i$. To prove the continuity of J, let $s \to \mu_s = \sum_{i=1}^{n} a_i^s t_i^{s\wedge}$ be a net in B convergent to 0 in the weak-* topology. Let $[A]$ denote the linear span of A and note that for any $f \in [A]'$, the map $f \circ j \in C^*(T, F)$,since $j(T)$ is precompact. Thus,

$$f(J(\mu_s)) = f\left(\sum_{i=1}^{n_s} a_i^s j(t_i^s)\right) = \sum_{i=1}^{n_s} a_i^s f(j(t_i^s)) = \mu_s(f \circ j) \to 0$$

and we conclude that $J(\mu_s) \to 0$ in the weak topology of $[A]$. As A is of countable type, hence a polar space, the weak topology coincides with the initial one on the compactoid A ([9], Th. 5.12) so $J(\mu_s) \to 0$ in A. By continuity and 'affinity,' J extends uniquely to a continuous affine map of cl $B = \kappa T$ into A, since A is complete.

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