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MOTZKIN FACTORIZATION IN ALGEBRAS OF ANALYTIC ELEMENTS

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Abstract. In this work, we show that the Motzkin factorization requires the set of analytic elements to be a K-algebra. Hence, we propose a new presentation of this problem, using a very close relationship, between the Mittag-Leffler series and Motzkin product.

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1.INTRODUCTION.

Let K be an algebraically closed complete ultrametric field whose absolute value is denoted by $|\,.\,|$. Let D be a subset of K, whose closure is denoted by $\overline{D}$. We denote by $R(D)$ the K-algebra of the rational functions $h(x) \in K(x)$ with no pole in D, and we denote by $H(D)$ the completed topological linear space of $R(D)$ for the topology of uniform convergence on D. The elements of $H(D)$ are called the *Analytic Elements* on D. We denote by $H_b(D)$ the set of the bounded elements of $H(D)$ and by $\|.\|_D$ the norm of uniform convergence defined on $H_b(D)$, and then $H_b(D)$ provided with this norm is a K-Banach algebra. Actually $H_b(D)$ is equal to $H(D)$ if and only if D is closed and bounded.

More generally $H(D)$ is provided with a structure of K-subalgebra of K^D if and only if D satisfies the two following conditions :

a) either D or $K\backslash\overline{D}$ is bounded

b) $\overline{D}\backslash D$ is included in the interior of $\overline{D}$.

We denote by $\mathcal{A}$ the set of D subset of K such that $H(D)$ is a K-subalgebra of K^D. It is seen that if D is open and satisfies b) then $\overline{D}$ is open too.

Let us recall that a space $H(D)$ does not contain characteristic functions different from 0 and 1 if and only if D is infraconnected [3],[4]. A set D is said to be *infraconnected* if for every $a \in D$, the mapping from D to $\mathbb{R}_+$ $x \to |x-a|$ has an image whose closure is an interval.

Notation. Let $a \in K$, let D be a set in K. We denote by $\delta(a, D)$ the distance from a to D i.e $inf\{|x-a| \backslash x \in D\}$.

Let $a \in K$ and let $a \in \mathbb{R}_+$. We will denote
by $d(a,r)$ the disk $\{x \in K \backslash |x-a| \leq r\}$
by $d(a,r^-)$ the disk $\{x \in K \backslash |x-a| < r\}$
by $C(a,r)$ the circle $\{x \in K \backslash |x-a| = r\}$
and by $|K|$ the set $\{|x| \backslash x \in K\}$

For every $\alpha \in C(a,r)$, $d(\alpha, r^-)$ is called a class of $C(a,r)$. Let $r' > r$. We will denote by $\Gamma(a,r,r')$ the set $\{x \in K \backslash r < |x-a| < r'\}$, and by $\Delta(a,r,r')$ the set $\{x \in K \backslash r \leq |x-a| \leq r'\}$.

We now recall the definition of the holes.

Definitions. Let D be a closed subset of K. Let R be its diameter and let $a \in D$. We put $\widetilde{D} = d(a,R)$ or $\widetilde{D} = K$ if $R = \infty$ and we now that D is in the form

$$\widetilde{D} \setminus (\cup_{i \in I} d(\alpha_i, r_i^-))$$

with $d(\alpha_i, r_i^-) \cap d(\alpha_j, r_j^-) = \emptyset$ whenever $i \neq j$ and with $r_i = \delta(\alpha_i, D)$ whenever $i \in I$. The disks $d(\alpha_i, r_i^-)$ $(i \in I)$ are named *the holes* of D.

First we have to recall the Mittag-Leffler series for an analytic element on infraconnected closed set [9] [10] [13]. When D is not bounded, we denote by $H_0(D)$ the subspace of the $f \in H(D)$ such that

$$\left\{ \begin{array}{l} \lim\limits_{\substack{|x| \to \infty \\ x \in D}} |f(x)| = 0 \end{array} \right.$$

Let D be an infraconnected closed set and let $f \in H(D)$. We know that f admits a unique series $\sum\limits_{n=0}^{\infty} h_n$ named the Mittag-Leffler series of f on D whose sum is equal to f, where $h_0 \in H(\widetilde{D})$ and for each $n \geq 1$, there exists a hole T_n of D such that $h_n \in H_0(K \backslash T_n)$. (The sequences (T_n), (h_n) are injective [9],[10],[13]).

Moreover for each $n \in \mathbb{N}^*$ we have $\|h_n\|_D = \|h_n\|_{K \backslash T_n}$ and $\lim\limits_{n \to \infty} h_n = 0$.

The holes T_n are called the *f-holes* (assuming $h_n \neq 0$) and h_n is called the Mittag-Leffler term of f associated to the hole T_n.

If $f \in H_b(D)$ then $\|h_0\|_D = \|h_0\|_{\widetilde{D}}$ and we have $\|f\|_D = \sup\limits_{n \in \mathbb{N}} \|h_n\|_D$.

Besides, if D is not bounded and if $f \in H(D)$ then h_0 is a constant.

Now given a *f-hole* T, the Mittag-Leffler series of f associated to T will be denoted by $\overline{f}_T$. Thus in the Mittag-Leffler series of f above, we have $h_n = \overline{f}_{T_n}$ whenever $n \in \mathbb{N}^*$.

In the same way we will always denote by $\overline{f}_0$ the element h_0 of $H(\widetilde{D})$.

We say that f is semi-invertible in $H(D)$ if it factorizes in the form Pg where P is a polynomial whose zeros belong to D and g is an element of $H(D)$ invertible in $H(D)$.

Let us remember the valuation defined on K and the valuation function associated to an analytic element.

Let log be a real logarithm function of base $\theta > 1$. We define a valuation v in K^* by $v(x) = -log|x|$.

Let $a \in K$ and let $r \in \mathbb{R}_+$. We call *the circular filter* of center a of diameter r the filter $\mathcal{F}$ that admits as generating system the annuli $\Gamma(\alpha, r', r'')$ with $\alpha \in d(a, r)$, $r' < r < r''$ [5][8].

Now let D be a set such that $\mathcal{F}$ is secant with D. The intersection $\mathcal{F}_D$ of $\mathcal{F}$ with D is called circular filter of center a, of diameter r, on D [5],[8].

Then we know that for every $f \in H(D)$, $|f(x)|$ has a limit ${}_D\varphi_{a,r}(f)$ along $\mathcal{F}_D$ and the mapping ${}_D\varphi_{a,r}$ is a multiplicative semi-norm in $H(D)$ [5],[6]. Thus we have

$$ {}_D\varphi_{a,r}(f) = \lim_{\begin{cases} |x-a| \to r \\ |x-a| \neq r, x \in D \end{cases}} |f(x)| $$

Besides, given a polynomial or more generally an element f on $H(D \cup d(a,r))$ we have ${}_D\varphi_{a,r}(f) = \|f\|_{d(a,r)} = \|f\|_{d(a,r^-)}$. As a consequence, if $r \in |K|$ so does $\|f\|_{d(a,r^-)}$.

Now, let $\mu \in \mathbb{R}$ be such that the circular filter $\mathcal{F}$ of center a, of diameter $r = \theta^{-\mu}$ is secant with D. We put $v_a(f, \mu) = -log({}_D\varphi_{a,r}(f))$.

In particular when $a = 0$ we just put $v(f, \mu) = v_0(f, \mu)$.

When $D \cap d(a, r^-) \neq \emptyset$ or when $D \cap (K \setminus d(a,r)) \neq \emptyset$ we see that

$$ v_a(f, -log(r)) = \lim_{\begin{cases} |x-a| \to r \\ |x-a| \neq r, x \in D \end{cases}} v(f(x)) $$

Now, let D be an infraconnected set and let $T = d(a, r^-)$ be a hole of D. The circular filter $\mathcal{F}$ of center a, of diameter r is certainly secant with D. In particular, if D has diameter $R > r$ then we have

$$ {}_D\varphi_{a,r}(f) = \lim_{\begin{cases} |x-a| \to r \\ |x-a| > r, x \in D \end{cases}} |f(x)| $$

2. MOTZKIN FACTOR AND PROPERTY.

Henceforth, we suppose that D is infraconnected and closed.

Lemma 2.1 : *Let $T = d(a, r^-)$, with $a \in K$, and $r > 0$, let $E = K \setminus T$ and let $b \in T$. Let $g \in H(E)$ be invertible in $H(E)$. Then there exist $\lambda \in K$, $q \in \mathbb{Z}$, and $h \in H(E)$ invertible in $H(E)$, satisfying $\|h - 1\|_E < 1$, $\lim_{|x| \to +\infty} h(x) = 1$ and $g(x) = \lambda(x-b)^q h(x)$. Besides λ, q, h, are respectively unique, satisfying those relations. Further, both λ, q do not depend on b in T, $\frac{g'}{g}$ belongs to $H_0(E)$.*

Proof : Without loss of generality we may obviously assume $a = 0$. It is easy to show that g is of the form $\widetilde{g} + \widehat{g}$, with $\widetilde{g} \in K[x]$ and $\widehat{g} \in H_b(E)$. Let $q = deg(\widetilde{g})$ and let λ be its coefficient of degree q. Now we put $h(x) = \dfrac{g(x)}{\lambda(x-b)^q}$. By definition it is seen that both λ, q do not depend on b in T. Hence we may also assume $b = 0$. Clearly, as $H(E)$ is a K-algebra, h is invertible in $H(E)$ and satisfies $\lim\limits_{|x|\to+\infty} h(x) = 1$. In particular we notice that h is bounded in E. Now we check that $\|h-1\|_E < 1$. Let $s = \dfrac{1}{r}$, let $A = d(0,s)$ and let $F(u) = h(\frac{1}{u})$ whenever $u \in d(0,s)$, $u \neq 0$. Then F belongs to $H(d(0,s) \setminus \{0\})$. But since h is bounded in E, F is bounded in $d(0,s) \setminus \{0\}$ and therefore F belongs to $H(d(0,s))$ [4]. Besides, the condition $\lim\limits_{|x|\to+\infty} h(x) = 1$ shows that $F(0) = 1$, hence F has no zero in $d(0,s)$. We know that F is invertible in $H(d(0,s))$ and then it satisfies $\|F - F(0)\|_{d(0,s)} < |F(0)|$ [4]. Hence $\|F-1\|_{d(0,s)} < 1$ and therefore we have $\|h-1\|_E < 1$.

Now, h, q, λ are easily seen to be unique. Indeed, let $g(x) = \alpha x^t l(x)$ with l invertible in $H(E)$, satisfying $\lim\limits_{|x|\to+\infty} l(x) = 1$. Then $1 = x^{q-t}\dfrac{\lambda h(x)}{\alpha l(x)}$. Thus, we see that $q = t$, $\lambda = \alpha$ and therefore $h = l$.

Finally, we check that $\dfrac{g'}{g}$ belongs to $H_0(E)$. Indeed $\dfrac{g'}{g} = \dfrac{q}{x-b} + \dfrac{h'}{h}$. Obviously, $\dfrac{q}{x-b}$ belongs to $H_0(E)$. Since $\lim\limits_{|x|\to\infty} |h(x)| = 1$, it is seen that $h(x)$ is of the form $1 + \sum\limits_{n=1}^{\infty} \dfrac{a_n}{x^n}$ with $\lim\limits_{n\to\infty} \dfrac{|a_n|}{s^n} = 0$, and therefore h' is an element of $H(E)$ such that $\lim\limits_{|x|\to\infty} |h'(x)| = 0$. As a consequence $\dfrac{h'}{h}$ belongs to $H_0(E)$. Hence so does $\dfrac{g'}{g}$ and this ends the proof of Lemma 2.1.

We can now give the following definitions.

Definitions. Let $E = K \setminus d(a, r^-)$ with $a \in K$, and $r > 0$. Let $f \in H(E)$ be invertible in $H(E)$ and let $\lambda(x-a)^q h(x)$ be the factorization given in Lemma 2.1. The integer q will be named *the index of* f in the hole $d(a,r^-)$ and will be denoted by $m(f, d(a,r^-))$. The element f will be called *a pure factor associated to* $d(a,r^-)$ if $\lambda = 1$.

Let f belong to $H(D)$. Let T be a hole of D and let h be a pure factor associated to T. Then f will be said *to admit h as a Motzkin factor in the hole T* if $\dfrac{f}{h}$ belongs to $H(D \cup T)$ and has no zero inside T.

Lemma 2.2 : *Let $T = d(a,r^-)$, let $E = K \setminus T$ with $a \in K$, and let f be a pure factor associated to T such that $\|f-1\|_E < 1$. Then $m(f,T) = 0$.*

Proof : Indeed, let $q = m(f, T)$ and let $f = (x - a)^q h$. If $q \neq 0$, this contradicts the unicity of q and h shown in Lemma 2.1.

Denoting by $\mathcal{G}^T$ the group of the invertible elements of $H(K \setminus T)$, by Lemma 2.1 we have Corollary 2.3.

Corollary 2.3 : *Let $T = d(a, r^-)$. The set of the pure factors associated to T is a sub-multiplicative group of the group of $\mathcal{G}^T$. Further, every element of $\mathcal{G}^T$ is of the form λh with h a pure factor associated to T and $\lambda \in K^*$.*

Lemme 2.4 : *Let D satisfy $d(a, r^-) \subset D$ and $d(a, r^-) \neq D$. Let $f \in H(D)$ have no zero inside T. There exists an infraconnected closed bounded set D' satisfying $d(a, r^-) \subset D' \subset D$, $d(a, r^-) \neq D'$, such that f is invertible in $H(D')$.*

Proof : Since f has no zero in T, then $|f(x)|$ is constant $A \neq 0$ [4]. Hence, we have ${}_D\varphi_{a,r}(f) = A$. Let $B \in]0, A[$. We just have to construct a set D' satisfying $d(a, r^-) \subset D' \subset D$, $d(a, r^-) \neq D'$ together with a number B such that $|f(x)| \geq B$ whenever $x \in D'$.

First, we assume $D \cap (K \setminus d(a, r)) \neq \emptyset$. There does exist $s \in]r, diam(D)[$ such that ${}_D\varphi_{a,u}(f) \geq B$ for every $u \in]r, s[$. Then by Proposition [5] there exist $u_1, ..., u_t \in]r, s[$ such that $|f(x)| =_D \varphi_{a,u}(f)$ holds in all $x \in C(a, u) \cap D$ for every $u \in]r, s[\setminus\{u_1, ..., u_t\}$. So we just take $D' = (D \cap d(a, s)) \setminus (\cup_{j=1}^t C(a, u_j))$.

Now we suppose that the condition $D \cap (K \setminus d(a, r)) \neq \emptyset$ is not satisfied. Since $d(a, r^-) \neq D$ and ${}_D\varphi_{a,r}$ don't depend on a in $d(a, r)$, then there exists $b \in D \cap C(a, r)$ such that

$${}_D\varphi_{b,r}(f) =_D \varphi_{a,r}(f) = A.$$

Hence there exists $s \in]0, r[$ such that ${}_D\varphi_{b,u}(f) \geq B$ for every $u \in]s, r[$. Then by Proposition [5] there exists $u_1, ..., u_t \in]s, r[$ such that $|f(x)| =_D \varphi_{b,u}(f)$ holds in all $x \in C(b, u) \cap D$ for every $u \in]s, r[\setminus\{u_1, ..., u_t\}$. Thus we now take

$$D' = \Big((D \cap d(b, r^-)) \setminus (\cup_{j=1}^t C(b, u_j))\Big) \cup d(a, r^-)$$

and then we check that D' is the set we are looking for.

Theorem 2.5 : *Let T be a hole of D and let f have a Motzkin factor h in T. Then h is unique. Further, if T is not a f-hole, h is the polynomial of the zeros of f inside T. Besides, if E is another infraconnected set included in D admitting T as a hole, and if g denotes the restriction of f to E, then g admits a Motzkin factor in the hole T as an element of $H(E)$, and this Motzkin factor is equal to h.*

Proof : Let f have another Motzkin factor l in T, let $F = \frac{f}{h}$ and let $G = \frac{f}{l}$. Since G has no zeros inside T, by lemma 2.4 there exists a closed bounded infraconnected set D' satisfying $T \subset D' \subset (D \cup T)$, $T \neq D'$, such that G is invertible in $H(D')$. Hence in $H(D')$ we have

$$(1)\ \frac{F}{G} = \frac{l}{h}$$

Since $T \neq D'$ it is seen that $D' \cap (K \setminus T)$ is an infraconnected closed bounded set included in D that admits T as a hole. If we see $\frac{l}{h}$ as an element of $H(D' \cap (K \setminus T))$, then by (1) and by uniqueness of Mittag-Leffler term we have $\overline{(\frac{l}{h})_T} = 0$. So $\frac{l}{h}$ belongs to $H(K)$ and therefore is a polynomial P. Since $\frac{F}{G}$ belongs to $H(D')$ and has no zeros inside T, it is seen that $m(h, T) = m(l, T)$, so we have $\lim_{|x| \to \infty} \frac{l(x)}{h(x)} = 1$. Hence $P = 1$ and this proves that h is unique.

Now we assume that T is not a f-hole. Hence f belongs to $H(D \cup T)$. Let Q be the polynomial of the zeros of f inside T. Then $\frac{f}{Q}$ belongs to $H(D \cup T)$ and has no zeros inside T [4]. Since its Motzkin factor h is unique, we have $h = Q$. The last statement about g is obvious, because $\frac{g}{h}$ clearly belongs to $H(E \cup T)$ and has no zero inside T. This ends the proof of Theorem 2.5.

Definitions. We will call *the f-supersequence of D* the sequence of the holes $(T_n)_{n \in I}$ such that either T_n is a f-hole or f belongs to $H(D \cup T)$ and has at least one zero inside T_n. If f admits a Motzkin factor h in a hole T, it will be denoted by f^T. The index of h in T is called *the Motzkin index of f in T* and denoted by $m(f, T)$. For every hole, which does not belong to the f-supersequence, we put $f^T = 1$.

Lemma 2.6 is immediate.

Lemma 2.6 : *Let $D \in \mathcal{A}$, let T be a hole of D, and let $f, g \in H(D)$ admit each one a Motzkin factor in the hole T. Then fg admits a Motzkin factor in T, and we have $(fg)^T = f^T g^T$ and $m(fg, T) = m(f, T) + m(g, T)$.*
Besides, if f is invertible in $H(D)$, then f^{-1} admits a Motzkin factor in T which is $(f^T)^{-1}$, and satisfies $m(f^{-1}, T) = -m(f, T)$.

3.MOTZKIN FACTORIZATION.

Lemma 3.1 : *Let the f-supersequence $(T_n)_{n \in \mathbb{N}}$ be such that for every $n \in \mathbb{N}$, f^{T_n} exists and such that $\lim_{n \to \infty} f^{T_n} - 1 = 0$. Then there exists $N \in \mathbb{N}$ such that $m(f^{T_n}, T_n) = 0$ whenever $n > N$. Besides, if $D \in \mathcal{A}$, the product $(\prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$ does not depend on t whenever $t \geq N$.*

Proof : Indeed, there exists $N \in \mathbb{N}$ such that we have $\|f^{T_n} - 1\|_D < 1$ whenever $n \geq N$, and therefore , by Lemma 2.2, $m(f^{T_n}, T_n) = 0$. Now in $H_b(D)$, we have $(\prod_{n=N+1}^{\infty} f^{T_n}) = (\prod_{n=N+1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$. But then, if D belongs to $\mathcal{A}$, we have

$$(\prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n}) = (\prod_{n=1}^{N} f^{T_n})(\prod_{n=N+1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n}) = (\prod_{n=1}^{N} f^{T_n})(\prod_{n=N+1}^{\infty} f^{T_n}).$$

Definitions. Let $(T_n)_{n \in I}$ be the f-supersequence of D with I a subset of $\mathbb{N}$ which is either finite or equal to $\mathbb{N}$.
If I is finite, f will be said to have *a finite Motzkin factorization* if it factorizes in $H(D)$ in the form $(f^0 \prod_{n \in I} f^{T_n})$ with f^0 an element of $H(\widetilde{D})$ whose zeros belong to D and for each $n \in \mathbb{N}$, f^{T_n} a Motzkin factor of f in T_n.
If I is infinite and equal to $\mathbb{N}$, f will be said to have *an infinite Motzkin factorization* if it admits a sequence of Motzkin factors f^{T_n} satisfying $\lim_{n \to \infty} f^{T_n} - 1 = 0$ such that f factorizes in $H(D)$ in the form $(f^0 \prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$, with f^0 an element of $H(\widetilde{D})$ whose zeros belong to D.
In both cases, f^0 will be called *the principal factor of f.*

Corollary 3.2 : *Let D be bounded and let f have an infinite Motzkin factorization with a f-supersequence $(T_n)_{n \in \mathbb{N}}$. Then we have $f = f^0(\prod_{n=1}^{\infty} f^{T_n})$.*

Corollary 3.3 : *Let f have an infinite Motzkin factorization with a f-supersequence $(T_n)_{n \in \mathbb{N}}$ such that $m(f^{T_n}, T_n) = 0$ for all $n > 0$. Then we have $f = f^0(\prod_{n=1}^{\infty} f^{T_n})$.*

Remark : Let $f \in H(D)$ be unbounded and have Motzkin factorization of the form $(f^0 \prod_{n=1}^{N} f^{T_n})(\prod_{n=N+1}^{\infty} f^{T_n})$. One cannot claim that the product $(f^0 \prod_{n=1}^{\infty} f^{T_n})$ converges in $H(D)$, even if D is closed and belongs to $\mathcal{A}$. Indeed, let $r \in]0, 1[$, let $(a_n)_{n \in \mathbb{N}}$ be a sequence in $d(0, 1)$ such that $|a_n - a_m| = 1$ whenever $n \neq m$, and $a_1 = 0$. For every $n \in \mathbb{N}^*$, we put $T_n = d(a_n, r^-)$ and $E = K \setminus (\bigcup_{n=1}^{\infty} T_n)$. Clearly, the holes of E are the T_n. Let $(\lambda_n)_{n \geq 2}$ be a sequence in $d(0,r^-)$ such that $\lim_{n\to\infty}\lambda_n = 0$. For every $n \geq 2$, we put $g_n = 1 + \frac{\lambda_n}{x - a_n}$. The sequence $(g_n)_{n\geq 2}$ is seen to satisfy $\|g_n - 1\|_E \leq \frac{|\lambda_n|}{r} < 1$, and therefore we have $\lim_{n\to\infty}\|g_n - 1\|_E = 0$. Hence the product $h = \prod_{n=2}^{\infty} g_n$ obviously converges in $H(E)$.

Since E clearly belongs to $\mathcal{A}$, we see that x^2h belongs to $H(E)$ and is invertible in $H(E)$. Besides, f clearly has Motzkin factorization with $f^{T_n} = g_n$ for every $n \geq 2$, $f^{T_1} = x^2$, and $f^0 = 1$. However, we will check that the sequence $(f_n)_{n\in\mathbb{N}^*}$ defined by $f_n = x^2\prod_{j=2}^{n} g_j$ does not converge in $H(E)$. Indeed we have

$$f_{n+1}(x) - f_n(x) = x^2\Big(\prod_{j=2}^{n} g_j(x)\Big)(g_{n+1}(x) - 1).$$

For every $x \in K \setminus d(0,1)$, we have $|x^2\prod_{j=2}^{n} g_j(x)| = |x^2|$, and $|g_{n+1}(x) - 1| = |\frac{\lambda_{n+1}}{x}|$ hence $|f_{n+1}(x) - f_n(x)| = |x||\lambda_{n+1}|$. Thus $f_{n+1} - f_n$ is not bounded in $H(E)$, and therefore the sequence $(f_n)_{n\in\mathbb{N}^*}$ does not converge in $H(E)$. According to Theorem 4 in [11] the product "$\prod_{n=1}^{\infty} f_n$" should converge to x^2h in $H(E)$. Here we see that this is not true in the general case. Actually the proof given in [11] only shows the simple convergence of the sequence (f_n).

By Lemma 2.6, Lemma 3.4 is immediate.

Lemma 3.4 : *Let $D \in \mathcal{A}$, let f, $g \in H(D)$ have Motzkin factorization. Then so does fg. Besides, we have $(fg)^0 = f^0g^0$. Further, if f is invertible, f^{-1} also has Motzkin factorization, and it satisfies $(f^{-1})^0 = (f^0)^{-1}$.*

Corollary 3.5 : *Let $D \in \mathcal{A}$, let f have an infinite Motzkin factorization of the form $(f^0\prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$. Let $N \in \mathbb{N}$ be such that $m(f^{T_n}, T_n) = 0$ for all $n > N$. Then we have*

$$f = f^0\Big(\prod_{n=1}^{N} f^{T_n}\Big)\Big(\prod_{n=N+1}^{\infty} f^{T_n}\Big).$$

Proposition 3.6 : *Let $f \in H(D)$ satisfy $\|f - 1\|_D < 1$ and have Motzkin factorization* *the form* $f^0\left(\prod_{n=1}^{\infty} f^{T_n}\right)$ *with* $(T_n)_{n\in\mathbb{N}^*}$ *the* f*-supersequence of* D*. Then for each* $n \geq 1$ *we have* $m(f^{T_n}, T_n) = 0$.

Proof: For every $n \in \mathbb{N}^*$, we put $q_n = m(f^{T_n}, T_n)$. By Lemma 3.1, we may assume the $(T_n)_{n\in\mathbb{N}^*}$ ranged so that $q_n \neq 0$ for $n \leq N$ while $q_n = 0$ whenever $n > N$. When $n \leq N$, f^{T_n} is of the form $(x-\alpha_n)^{q_n}(1+\omega_n)$ with $\omega_n \in H_0(K \setminus T_n)$, $\| \omega_n \|_{K\setminus T_n} < 1$ and $\alpha_n \in T_n$. When $n > N$, f^{T_n} is just in the form $(1+\omega_n)$ with $\omega_n \in H_0(K \setminus T_n)$ and $\|\omega_n\|_{K\setminus T_n} < 1$. Besides, as f has no zero in D, obviously f^0 has no zero in D and therefore has no zero in $\widetilde{D}$ hence f^0 is of the form $A(1+\omega_0(x))$ with $\omega_0 \in H(\widetilde{D})$, $\|\omega_0\|_D < 1$. Let $h(x) = A\prod_{n=1}^{N}(x-\alpha_n)^{q_n}$. We see that f factorizes in the form $h\prod_{n=0}^{\infty}(1+\omega_n)$. Since $\|\omega_n\|_D < 1$ for every $n \in \mathbb{N}$ and since $\lim_{n\to\infty} \omega_n = 0$ it is seen that h satisfies

(1) $\|h-1\|_D < 1$ as f does.

Let us suppose $q_1 \neq 0$. We may obviously assume $\alpha_1 = 0$. Let $T_1 = d(0, r^-)$. Thus, in T_1, h admits 0 as a zero of order q_1 if $q_1 > 0$ (resp. a pole of order $-q_1$ if $q_1 < 0$) and has neither any zero nor any pole different from 0. Anyway, when $x \in T_1$, we have

(2) $|h(x)| = B|x^{q_1}|$ with $B = |A|\prod_{n=2}^{N}|\alpha_n|^{q_n}$.

First we suppose $r \notin |K|$. Then there exists $r' > r$ such that h has neither any zero nor any pole in $\Delta(0, r, r')$, hence h has neither any zero nor any pole in $d(0, r') \setminus \{0\}$. Therefore $|h(x)|$ is of the form $B|x|^{q_1}$ in all of $d(0, r') \setminus \{0\}$. In particular, this is true in $\Delta(0, r, r') \cap D$ and shows that ${}_D\varphi_{0,r}(h) = Br^{q_1}$ while ${}_D\varphi_{0,r'}(h) = Br'^{q_1}$. Hence we see that $|h(x)|$ is not constant in $\Delta(0, r, r') \cap D$ and therefore this contradicts relation (1).

Now we suppose that r belongs to $|K|$ and then by a classical linear change of variable, we may restrict ourselves to the case where $r = 1$. Hence we have $T_1 = d(0, 1^-)$. By (2) we have ${}_D\varphi_{0,1}(h) = B$. But by definition B belongs to $|K|$ and therefore we may clearly assume $B = 1$ without loss of generality. Hence h is of the form $\frac{P(x)}{Q(x)}$ with

(3) $\|P\|_{d(0,1)} = \|Q\|_{d(0,1)} = 1$ and P prime to Q.

Let us suppose $q_1 > 0$. By definition of h, P has no zero different from 0 in T_1 while Q has no zero in T_1. Hence Q satisfies

(4) $\|Q(x)\| = |Q(0)|$ whenever $x \in T_1$ and then by (3) we have $|Q(0)| = 1$, while obviously $P(0) = 0$. Hence by (3) it is seen that $\|P - Q\|_{d(0,1)} = 1$ and therefore by (4) we have $\|h-1\|_{d(0,1)} = 1$. But we know that for every $g \in R(D \cup d(0,1))$ we have

$$\|g\|_{d(0,1^-)} \leq \|g\|_{D\cup d(0,1^-)} = \|g\|_D$$

hence we see that $\|h-1\|_D \geq 1$ and this contradicts (1).

We now suppose $q_1 < 0$. By definition h is obviously invertible in $R(D)$. Hence we put $F = \frac{1}{h}$ and we see that F satisfies $\|F - 1\|_D < 1$ and admits 0 as a unique zero in T_1 while it has no pole in T_1. Hence the same process lets us get to the same contradiction and finishes showing that $q_n = 0$ for every $n \geq 1$.

Lemma 3.7 : *Let $f \in H(D)$ be invertible in $H(D)$ and have Motzkin factorization, and let $a \in D$. Then f satisfies $\|\frac{f}{f(a)} - 1\|_D < 1$ if and only if for every hole T of the f-supersequence of D we have $m(f, T) = 0$.*

Proof : Without loss of generality, we may obviously assume $f(a) = 1$. By Lemma 3.6, we already know that if f satisfies $\|f - 1\|_D < 1$, then for every hole of the f-supersequence, we have $m(f, T) = 0$. Now we suppose that for every hole T of the f-supersequence we have $m(f, T) = 0$ and will prove that $\|f - 1\|_D < 1$. Indeed, by Lemma 1, for each hole of the f-supersequence, we have $\|f^T - 1\|_D < 1$. Besides since f is invertible, f^0 must also be invertible, hence it is of the form $(1 + \psi(x))$, with $\|\psi\|_D < 1$. Then it is seen that $\|f - 1\|_D < 1$.

4.EXISTENCE OF MOTZKIN FACTOR AND MOTZKIN FACTORIZATION.

We will show all the semi-invertible elements to have Motzkin factorization, step after step, and first we consider rational functions.

Proposition 4.1 : *Let $f \in R(D)$. Then f admits Motzkin factorization.*

Proof: Since the number of zero and pole of f is finite, let $T_1, ..., T_q$ be the f-supersequence of D. For each $n = 1, ..., q$ we denote by f^{T_n} the rational function whose numerator (resp. denominator) is the polynomial of the zeros (resp. of the poles) of f inside T_n. Finally we denote by f^0 the rational function whose numerator (resp. denominator) is the polynomial whose zeros are the zeros (resp. the poles) of f in $D \cup (K \setminus \widetilde{D})$, (resp. in $K \setminus \widetilde{D}$). It is seen that f^0 is the principal factor of f, and that for each $n = 1, ...q$, f^{T_n} is the Motzkin factor of f associated to the hole T_n.

Proposition 4.2 : *Let $\phi \in H(D)$ satisfy $\|\phi - 1\|_D < 1$. Then ϕ admits Motzkin factorization $\phi^0\left(\prod_{n=1}^{\infty} \phi^{T_n}\right)$ with $(T_n)_{n \in \mathbb{N}^*}$ the f-supersequence. For all $n \in \mathbb{N}^*$ we have $\|\phi^{T_n} - 1\|_D = \|\overline{\phi}_{T_n}\|_D$. Besides ϕ^0 satisfies $\|\phi^0 - 1\|_D = \|\overline{\phi}_0 - 1\|_D$.*

Proof : First we suppose $\phi \in R(D)$. Then by Proposition 4.1, ϕ admits Motzkin factorization. Now by Lemma 3.7, for each $n > 0$ we have $m(\phi, T_n) = 0$, and therefore, ϕ^{T_n} is of the form $1 + \omega_n$ with $\|\omega_n\|_D < 1$ whenever $n > 0$ while $\phi^0 = 1 + \omega_0$ with $\|\omega_0\|_D < 1$.

Hence we see that

$$\overline{\phi}_{T_n} = \overline{\Big(\omega_n \prod_{\substack{j\neq n\\ j\in\mathbb{N}}} (1+\omega_j)\Big)}_{T_n}.$$

Clearly, $\prod_{\substack{j\neq n\\ j\in\mathbb{N}}} (1+\omega_j)$ is of the form $1+\psi_n$ with $\|\psi_n\|_D < 1$, hence $\|(\omega_n\psi_n)\|_D < \|\omega_n\|_D$ and we obtain (1) $\|\overline{(\omega_n\psi_n)}_{T_n}\|_D \leq \|\omega_n\psi_n\|_D < \|\omega_n\|_D$. But ω_n is clearly equal to $\overline{(\phi^{T_n})}_{T_n}$ and then we have

$$(2)\quad \|\overline{(\omega_n)}_{T_n}\|_D = \|\omega_n\|_D > \|\omega_n\psi_n\|_D \geq \|\overline{(\omega_n\psi_n)}_{T_n}\|_D.$$

Besides, $\overline{(\omega_n+\omega_n\psi_n)}_{T_n} = \overline{(\omega_n)}_{T_n} + \overline{(\omega_n\psi_n)}_{T_n}$ hence by (1) and (2) we have $\|\overline{(\omega_n(1+\psi_n))}_{T_n}\|_D = \|\omega_n\|_D$ and finally

$$\|\overline{\phi}_{T_n}\|_D = \|\overline{(\omega_n(1+\psi_n))}_{T_n}\|_D = \|\omega_n\|_D = \|\phi^{T_n}-1\|_D.$$

In the same way we put $\prod_{n=1}^{\infty}(1+\omega_n) = 1+\psi$ with (3) $\|\psi\|_D < 1$. It is seen that ψ belongs to $H_0(K\setminus(\bigcup_{n=1}^{\infty} T_n))$. Hence the Mittag-Leffler Theorem [1] applied to ψ shows that (4) $\overline{\psi}_0 = 0$.

Next, we have $\phi = (1+\omega_0)(1+\psi) = 1+\omega_0+\psi+\omega_0\psi$ hence $\overline{\phi}_0 = 1+\overline{(\omega_0)}_0+\overline{\psi}_0+\overline{(\omega_0\psi)}_0$. By defintion $\omega_0 \in H(\widetilde{D})$ hence $\omega_0 = \overline{(\omega_0)}_0$ and then by (4) we have

$$\overline{\phi}_0 = 1+\omega_0+\overline{(\omega_0\psi)}_0.$$

But by (3) it is seen that $\|\overline{(\omega_0\psi)}_0\|_D < \|\omega_0\|_D$ hence finally we obtain $\|\overline{\phi}_0 - 1\|_D = \|\omega_0\|_D = \|\phi^0-1\|_D$. Thus we have proven the inequalities satisfied by the ϕ^{T_n} and by ϕ^0 when ϕ belongs to $R(D)$.

Now we consider the general case when $\phi \in H(D)$. Let $(f_m)_{m\in\mathbb{N}}$ be a sequence in $R(D)$ such that $\lim_{m\to\infty} \|\phi - f_m\|_D = 0$. Let $\varepsilon \in]0,1[$ and let $N \in \mathbb{N}$ be such that $\|f_m - \phi\|_D \leq \varepsilon$ whenever $m \geq N$. Let T be a hole of the ϕ-supersequence. We will show that the sequence $((f_m)^T)_{m\in\mathbb{N}}$ converges in $H(D)$ and that this convergence is uniform with respect to the ϕ-supersequence. We fix $m \geq N$. It is seen that $\|f_m - 1\|_D < 1$ and then by Lemmas 2.1 and 3.6 we have $\|(f_m)^T - 1\|_D < 1$ and in particular $\|(f_m)^T\|_D = 1$. Besides, we remember that in $H(K\setminus T)$, the norm $\|\ .\ \|_D$ is multiplicative and actually equal to ${}_D\varphi_T$. Now let $s \geq N$. We have

$$\|(f_m)^T - (f_s)^T\|_D = \|\frac{(f_m)^T}{(f_s)^T} - 1\|_D \leq \max(\|\frac{(f_m)^T}{(f_s)^T} - (f_m)^T\|_D, \|(f_m)^T - 1\|_D).$$

Therefore we obtain

$$(5)\ \ \|(f_m)^T - (f_s)^T\|_D \le \max(\|(f_m)^T - 1\|_D), \|(f_s)^T - 1\|_D).$$

But, as we just proved about elements of $R(D)$, we have
$\|(f_m)^T - 1\|_D = \|\overline{(f_m)}_T\|_D$ and $\|(f_s)^T - 1\|_D = \|\overline{(f_s)}_T\|_D$.
Hence by (5) and by the Mittag-Leffler Theorem [1], we obtain
(6) $\|(f_m)^T - (f_s)^T\|_D \le \varepsilon$.
Relation (6) does not depend on the hole T and shows that, for each fixed $n \in \mathbb{N}^*$, the sequence $((f_m)^{T_n})_{m\in\mathbb{N}}$ is a Cauchy sequence which converges in $H(K\setminus T_n)$, to an element whose index is equal to 0, and this convergence is uniform with respect to n. For each $n \in \mathbb{N}^*$, we put $\phi_n = \lim\limits_{m\to\infty}(f_m)^{T_n}$. Then it is seen that

$$\prod_{n=1}^{\infty}\phi_n = \lim_{m\to\infty}\prod_{n=1}^{\infty}(f_m)^{T_n}.$$

As a consequence, the sequence $(f_m)^0$ is also convergent in $H(D)$, and actually in $H_b(\widetilde{D})$. Let ϕ_0 be its limit. Then we have this factorization : $\phi = \prod\limits_{n=0}^{\infty}\phi_n$. It is seen that this is the Motzkin factorization for ϕ. Obviously, for each fixed $n > 0$, the equality satisfied by the $(f_m)^{T_n}$ holds for ϕ^{T_n} and shows that

$$\|\overline{(\phi)}_{T_n}\|_D = \|(\phi)^{T_n} - 1\|_D.$$

In the same way, the equality satisfied by the $(f_m)^0$ show that

$$\|\overline{(\phi)_0} - 1\|_D = \|(\phi)^0 - 1\|_D.$$

This ends the proof of Proposition 4.2.

Theorem 4.3 : *Let $a \in D$. Let $\phi \in H_b(D)$ be such that $|\phi(a)| \neq 0$. The following statements* i) , ii) , iii) *are equivalent*

i) $\|\phi - \phi(a)\|_D < |\phi(a)|$.

ii) *For every hole T we have $\|\overline{\phi}_T\|_D < |\phi(a)|$ and $\|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D < |\phi(a)|$.*

iii) *ϕ is invertible, has Motzkin factorization and for every hole T , ϕ^T satisfies $\|\phi^T - 1\|_D < 1$ and ϕ^0 satisfies $\|\phi^0 - \phi^0(a)\|_D < |\phi(a)|$.*

Besides, if statements i) , ii) , iii) *are satisfied then we have*

(u) *$m(\phi, T) = 0$ for every hole T.*

(v) *$\|\overline{\phi}_T\|_D = \|\phi^T - 1\|_D|\phi(a)|$ for every hole T.*

(w) $\|\phi^0 - \phi^0(a)\|_D = \|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D$.

Proof : Without loss of generality we may obviously assume

$$(1) \qquad |\phi(a)| = 1 \text{ and } |\phi(a) - 1| < 1.$$

Let $(T_m)_{m\in I}$ be the f-sequence of D. We notice that when i) is satisfied, ϕ is obviously invertible.

First we suppose i) satisfied and will show that so is ii). By the Mittag-Leffler Theorem we have

$$(2) \qquad \|\overline{(\phi - \phi(a))_{T_m}}\|_D \le \|\phi - \phi(a)\|_D.$$

But it is seen that $\overline{(\phi - \phi(a))_{T_m}} = \overline{\phi}_{T_m}$. Hence by (2) we see that

$$(3) \qquad \|\overline{\phi}_{T_m}\|_D \le \|\phi - \phi(a)\|_D < 1, \text{ whenever } m \in I.$$

In the same way we have $\overline{(\phi - \phi(a))_0} = \overline{\phi}_0 - \phi(a)$ and then by the Mittag-Leffler Theorem[1] we have

$$(4) \qquad \|\overline{\phi}_0 - \phi(a)\|_D < 1.$$

Besides, by (3) we see that $\|\sum_{m=1}^{\infty} \overline{\phi}_{T_m}\|_D < 1$ hence $\|\phi - \overline{\phi}_0\|_D = \|\sum_{m=1}^{\infty} \overline{\phi}_{T_m}\|_D < 1$ and therefore $|\phi(a) - \overline{\phi}_0(a)| < 1$, hence by (4) we see that (5) $\|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D < 1$. Finally by (3) and (5), statement ii) is clearly proven.

Now we will show that each one of the statements ii) and iii) separately implies i). We suppose ii) satisfied. Hence we have

$$(6) \qquad \|\sum_{m\in I} \overline{\phi}_{T_m}\|_D < 1.$$

If D is bounded, by statement ii) and by (6) we obtain i). Now let D be not bounded. Then $\overline{\phi}_0$ is a constant λ. Hence ϕ is in the form $\lambda + \sum_{m=1}^{\infty} \overline{\phi}_{T_m}$ with $\|\overline{\phi}_{T_m}\|_D < 1$ whenever $m \ge 1$ hence (7) $\|\sum_{m=1}^{\infty} \overline{\phi}_{T_m}\|_D < 1$. Now we have

$$\phi - \phi(a) = \sum_{m=0}^{\infty} \overline{\phi}_{T_m} - \overline{\phi}_{T_m}(a) = \sum_{m=1}^{\infty} (\overline{\phi}_{T_m} - \overline{\phi}_{T_m}(a)).$$

By (7) we see that $\|\sum_{m=1}^{\infty} (\overline{\phi}_{T_m} - \overline{\phi}_{T_m}(a))\|_D < 1$ hence finally i) $\|\phi - \phi(a)\|_D < 1$.

We now suppose iii) satisfied. Hence we have (8) $\|\phi^{T_m} - 1\|_D < 1$ for all $m \in I$. If D is bounded we have $\|\phi^0 - \phi^0(a)\|_D < 1$ hence by (8) we directly have i). If D is not bounded then ϕ^0 is a constant ν such that $\phi(a) = \nu \prod_{m \in I} \phi^{T_m}(a)$ hence by (8) and (1) we see that $|\nu - 1| < 1$ hence by (8) we obtain i) again.

Finally, i) is implied as well by ii) as by iii). Obviously by (1), i) implies $\|\phi - 1\|_D < 1$ and therefore we may apply Proposition 4.2. Now we suppose that either ii) or iii) is satisfied. Hence so is i) and so are u) and v) by Proposition 4.2.

Finally, we will show w) and at the same time we will finish proving the equivalence between ii) and iii). Let $\psi = (\overline{\phi_0}(a))^{-1}\phi$. We may apply Proposition 4.2 to ψ and we have

$$(9) \quad \|\overline{\psi}_0 - 1\|_D = \|\psi^0 - 1\|_D.$$

But we have

$$(10) \quad \|\psi^0 - 1\|_D \geq \|\psi^0 - \psi^0(a)\|_D = \|\phi^0 - \phi^0(a)\|_D$$

and

$$(11) \quad \|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D = \|\overline{\psi}_0 - \overline{\psi}_0(a)\|_D = \|\overline{\psi}_0 - 1\|_D.$$

Hence by (9), (10), (11) we obtain

$$(12) \quad \|\phi^0 - \phi^0(a)\|_D \leq \|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D.$$

Now let $\Gamma = \phi^0(a)$ and let $\chi = \Gamma^{-1}\phi$. By (1) and (7) we see that $|\Gamma - 1| < 1$ hence we may apply Proposition 4.2 to χ and we have

$$(13) \quad \|\chi^0 - 1\|_D = \|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D$$

while $\|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D = \|\overline{\chi}_0 - \overline{\chi}_0(a)\|_D \leq \|\overline{\chi}_0 - 1\|_D$ and while

$$\|\chi^0 - 1\|_D = \|\chi^0 - \chi^0(a)\| = \|\phi^0 - \phi^0(a)\|_D.$$

Hence by (13) we see that $\|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D \leq \|\phi^0 - \phi^0(a)\|_D$ and therefore by (12) we obtain w). This finishes proving the equivalence between ii) and iii) and ends the proof of Theorem 4.3.

Remark : When Theorem 4.5 is proven, the Statement iii) is equivalent to :
iii') *ϕ is invertible and for every hole T, ϕ^T satisfies $\|\phi^T - 1\|_D < 1$ and ϕ^0 satisfies $\|\phi^0 - \phi^0(a)\|_D < |\phi(a)|$.*

If D is not bounded, as ϕ is bounded, both ϕ^0, $\overline{\phi}_0$ are constant and therefore, the statements $\|\overline{\phi}_0 - \overline{\phi}_0(a)\|_D < |\phi(a)|$ and $\|\phi^0 - \phi^0(a)\|_D < |\phi(a)|$ are automatically satisfied. Statement ii) is then equivalent to :
ii') *For every hole T we have $\|\overline{\phi}_T\|_D < |\phi(a)|$*

and statement iii) is equivalent to:
iii") *ϕ is invertible and for every hole T , ϕ^T satisfies $\|\phi^T - 1\|_D < 1$.*

Lemma 4.4: *Let D be unbounded and $D \in \mathcal{A}$, let $a \in K \setminus D$ and $f \in H(D)$ be semi-invertible and satisfy $\lim_{|x|\to\infty} f(x) = 0$. Then there exists $q \in \mathbb{N}^*$ and $g \in H(D)$ such that $f = \frac{g}{(x-a)^q}$ and $\lim_{|x|\to\infty} g(x) \neq 0$.*

Proof: We suppose that $a = 0$. Since $D \in \mathcal{A}$ and D is unbounded, we have $K \setminus D$ is bounded [4]. So, since f is semi-invertible, there exists $r > 0$ such that the set $B = \{x \in K \setminus |x| > r\}$ is included in D, and such that $f(x) \neq 0$ whenever $x \in K \setminus d(0,r)$.

Let $F(u) = f(\frac{1}{u})$ whenever $u \neq 0$ and $u \in d(0, \frac{1}{r})$. F is clearly an element of $H(d(0,\frac{1}{r}) \setminus \{0\})$. But as f is bounded we see that F belongs to $H(d(0,\frac{1}{r}))$ and $F(0) = \lim_{|x|\to\infty} f(x)$ Hence there exists $q \in \mathbb{N}^*$ and $G \in H(d(0,\frac{1}{r}))$ such that $F = u^q G$ and $G(0) \neq 0$[4].

In $H(B)$ we have $f = \frac{1}{x^q} G(\frac{1}{x})$, so the lemma is proved if we put $g = x^q f$.

Theorem 4.5: *Let $D \in \mathcal{A}$. Then f has Motzkin factorization if and only if it is semi-invertible .*

Proof: Without loss of generality we may assume the f-supersequence to be infinite. We denote it by $(T_n)_{n\in\mathbb{N}^*}$. Let f have Motzkin factorization $(f^0 \prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$. By definition, f^0 is semi-invertible in $H(\widetilde{D})$ hence in $H(D)$. Besides, $(\prod_{n=1}^{t} f^{T_n})(\prod_{n=t+1}^{\infty} f^{T_n})$ is clearly invertible in $H(D)$. So f is semi-invertible.

Now, we suppose f to be semi-invertible and will show it to have Motzkin factorization. By Lemma 3.4 we may clearly suppose that f is invertible without loss of generality. First, we suppose that there exists M in $\mathbb{R}_+^*$ such that (1) $M \leq |f(x)|$, whenever $x \in D$. Let $h \in R(D)$ satisfy (2) $\|f - h\|_D < \frac{M}{2}$, and let $h^0 \prod_{n=1}^{N} h^{T_n}$ be the Motzkin factorization of h. For every $n = 1, ..., N$, let $q_n = m(h, T_n)$, let $a_n \in T_n$, let $h_n = (x - a_n)^{-q_n} h^{T_n}$ and let $h_0 = h^0$. Let $u(x) = \prod_{n=1}^{N} (x - a_n)^{q_n}$ and let $l(x) = h^0 \prod_{n=1}^{N} h_n$. By (1), (2) it is seen that h has no zero in D and so is h^0 in $\widetilde{D}$. Let $a \in D$. Then h^0 satisfies $\|h^0 - h^0(a)\|_D < |h^0(a)|$, and of course for every $n > 0$, h_n satisfy $\|h_n - 1\|_D < 1$. Hence, we have (3) $\|l - l(a)\|_D < |l(a)|$. Let $b = |l(a)|$. In particular, we have $|l(x)| = b$ whenever $x \in D$. Besides, we notice that we have (4) $\frac{M}{|l(a)|} \leq |u(x)|$. Let $F = \frac{f}{u}$. Then F belongs to $H_b(D)$. By (3) and (4) we check that $|F(x) - l(x)| < \frac{b}{2}$ and therefore by (3) again, we have

$|F(x)| = b$ and $\|F - F(a)\|_D \le \dfrac{|F(a)|}{2}$.

Then we can apply Theorem 4.3 to F and then F has Motzkin factorization $F^0 \prod_{n=1}^{\infty} F^{T_n}$, with $m(F, T_n) = 0$ whenever $n > 0$. As a consequence f also has Motzkin factorization $(f^0 \prod_{n=1}^{N} f^{T_n})(\prod_{n=N+1}^{\infty} f^{T_n})$ with $f^0 = F^0$, and for each $n = 1, ..., N$, $f^{T_n} = (x - a_n)^{q_n} F^{T_n}$, and finally for each $n > N$, $f^{T_n} = F^{T_n}$.

Now we suppose that $\inf\{|f(x)| \backslash x \in D\} = 0$. Since D is closed, and since f is invertible, we see that D is unbounded and that $\lim\limits_{|x| \to \infty} f(x) = 0$. Hence by Lemma 4.4 there exists $q \in \mathbb{N}^*$ such that $x^{-q}\dfrac{1}{f}$ has a non zero limit when $|x|$ tends to $+\infty$, $(x \in D)$. Let $G = \dfrac{1}{f}$, then it is easily seen that there exists $m > 0$ such that $|G(x)| \ge m$ for all $x \in D$. Indeed, on the first hand there exists r such that $|G(x)| \ge 1$ for all $x \in D \setminus d(0, r)$ and on the second hand f is bounded in $D \cap d(0, r)$, hence there does exist $m \in]0, 1[$ such that $|G(x)| \ge m$ whenever $x \in D \setminus d(0, r)$. Thus G admits Motzkin factorization and then by Lemma 3.4 so does f. This ends the proof of the Theorem.

Remark: If a closed set B does not belong to $\mathcal{A}$, there are counter-examples of invertible elements F which don't admit Motzkin factor F^T and by the way dont admit Motzkin factorization. Indeed, don't let B belong to $\mathcal{A}$. Since by hypothesis B is closed we know that $\widetilde{B} \setminus \overline{B}$ is not bounded. By [4] there exists a quasi-minorated element $f \in H_b(B)$ satisfying (1) $\lim\limits_{\substack{|x| \to \infty \\ x \in B}} f(x) = 0$ and such that xf does not belong to $H(B)$. Since $f \in H_b(B)$ we can take it such that $\|f\|_B < 1$. Without loss of generality, we may assume that 0 belongs to a hole of B. Let $T = d(a, r^-)$ be another hole of B, and let $F = \dfrac{x(1+f)}{(x-a)}$. Then it is seen that F belongs to $H_b(B)$ and is invertible in $H_b(B)$ because both $\dfrac{x}{(x-a)}$, $1 + f$ are invertible in $H_b(B)$. Hence F admits Motzkin factorization. In particular, we see that $F^T = \dfrac{1}{(x-a)}$. However we check that $(x-a)F$ does not belong to $H(B)$ because $(x-a)F = x(1+f)$ and by hypothesis, xf does not belong to $H(B)$.

In the same way, let $G = \dfrac{1}{F}$. Since F is invertible in $H_b(B)$, so is G. But then we see that $\dfrac{1}{x-a}G$ does belong to $H_b(B)$ and has no zero in B, but obviously its inverse does not belong to $H(B)$. Therefore $\dfrac{1}{x-a}G$ is not semi-invertible in $H(B)$. Thus, there exist invertible elements h, g in $H(B)$ such that hg is not semi-invertible, although it does belong to $H(B)$. This contradicts Theorem 1 in [11] which apparently states that $\frac{f}{f^T}$ extends to an element of $H(D \cup T)$.

Theorem 4.6 : *Let D belong to $\mathcal{A}$ and let $T = d(a, r^-)$ be a hole of D. Then f admits a Motzkin factor in the hole T* if and only if ${}_D\varphi_{a,r}(f) \neq 0$.

Proof : Let $\mathcal{F}$ be the circular filter of center a, of diameter r, and let $M = {}_D\varphi_{a,r}(f)$. There do exist $a_1, ..., a_q \in d(a, r)$ and s, t satisfying $s < r < t$, such that $|f(x)| \geq M$ whenever $x \in D \bigcap (\bigcap_{j=1}^{q} \Gamma(a_j, s, t))$. Let $A = D \bigcap (\bigcap_{j=1}^{q} \Gamma(a_j, s, t))$. Then T is clearly a hole of A. Besides, the restriction g of f to the infraconnected set $A = D \bigcap (\bigcap_{j=1}^{q} \Gamma(a_j, s, t))$ is invertible in $H(A)$, and therefore, by Theorem 4.5, it admits a Motzkin factor g^T in the hole T. But then, $\frac{f}{g^T}$ belongs to $H(D)$ and to $H(A \cup T)$. So the Mittag-Leffler term is zero and $\frac{f}{g^T}$ belongs to $H(D \cup T)$. Besides, as g^T is the Motzkin factor of g in T, $\frac{g}{g^T}$ has no zero inside T.

Now, if f admits a Motzkin factor f^T in the hole T, then $|(x - a)^{-m(f,T)} f^T(x)| = 1$ for all $x \in K \setminus T$. So ${}_D\varphi_{a,r}(f^T) = r^{m(f,T)}$. As $\frac{f}{f^T}$ has no zero inside T then $|\frac{f}{f^T}(x)|$ is a constant $B \neq 0$ whenever x belongs to T. Hence we have ${}_D\varphi_{a,r}(f) = Br^{m(f,T)} > 0$. This ends the proof.

Corollary 4.7 : *Let $D \in \mathcal{A}$, $f \in H(D)$ and D' be a closed infraconnected subset of D such that D and D' have a same hole T. Then f has Motzkin factor in T if and only if the restiction g of f to D' has a Motzkin factor in T.*

Moreover, if this equivalence is true, we have $f^T = g^T$.

Theorem 4.8 : *Let $D \in \mathcal{A}$ and let $\mathcal{G}$ be the multiplicative group of the invertible elements in $H(D)$. Let $\mathcal{T}$ be the set of the holes of D. Let $\mathcal{G}^0$ be the subgroup of the elements invertible in $H(\widetilde{D})$. Let $\mathcal{H} = \mathcal{G}^0 \prod_{T \in \mathcal{T}} \mathcal{G}^T$. The product $\mathcal{H}$ is a direct product and is dense in $\mathcal{G}$.*

Proof : The product is direct because for each element, Motzkin factorization is unique. Thus $\mathcal{H}$ is the set of the invertible elements whose Motzkin factorization is finite. Since every element of $\mathcal{G}$ has Motzkin factorization, it obviously belongs to the closure of $\mathcal{H}$.

REFERENCES.

[1] **AMICE, Y.** *Les nombres p-adiques* P.U.F. (1975).

[2] **DECOMPS-GUILLOUX, A et MOTZKIN, E.** *Fonctions analytiques univalentes dans un corps ultramétrique algébriquement clos.* C.R.A.S. Paris t.268, p.1531-1533 (1969).

[3] **ESCASSUT, A.** *Algèbres de Krasner.* C.R.A.S. Paris t.272, p.598-601 (1972).

[4] **ESCASSUT, A.** *Algèbres d'éléments analytiques en analyse non archimédienne,* Indagationes Mathematicae, t.36, p.339-335 (1974).

[5] **ESCASSUT, A.** *Eléments analytiques et filtres pércés sur un ensemble infraconnexe,* Annali di Mat. Pura ed Appl. Bologna, t.110 (1976).

[6] **ESCASSUT, A and SARMANT, M.C.** *Sufficient Conditions for injectivity of Analytic Elements.* Bulletin des Sciences Mathématiques, vol. 118, n 1, p.29-46, 1994.

[7] **ESCASSUT, A and SARMANT, M.C.** *Mittag-Leffler series and Motzkin product for Invertible analytic elements.* Rivista Di Mathematica Pura ed Applicata N.12-1992 (61-73)

[8] **GARANDEL, G.** *Les semis normes multiplicatives sur les algèbres d'éléments analytiques au sens de Krasner*, Indagationes Mathematicae, Vol. 37, *n* 4, p.327-341 (1975).

[9] **KRASNER, M.** *Prolongement analytique dans les corps valués complets : préservation de l'analycité par la convergence uniforme et par la dérivation; théorème de Mittag-Leffler généralisé pour les éléments analytiques.* C.R.A.S Paris 224, p.2570-2573 (1957)

[10] **KRASNER, M.** *Prolongement analytique uniforme et multiforme dans les corps valués complets. Les tendances géométriques en algèbre et théorie des nombres.* Clermont-Ferrand , p.94-141 (1964). Centre National de la Recherche Scientifique (1966), (Colloques internationaux de C.N.R.S. Paris,143).

[11] **MOTZKIN, E.** *La décomposition d'un élément analytique en facteurs singuliers.* Annales de l'Institut Fourier 27, 1 (1977) p. 67-82.

[12] **MOTZKIN, E.** *L'arbre d'un quasi-connexe : un invariant conforme p-adique.* Groupe d'Etude d'Analyse Ultramétrique. 9 ième année 1981-82, *n* 3.

[13] **ROBBA, Ph.** *Fonctions analytiques sur les corps valués ultramétriques complet. Prolongement analytique et algèbres de Banach ultramétriques.* Astérisque, t.10, p.109-220 (1973).

[14] **SARMANT, M.C.** *Produits Méromorphes.* Bull.Sc.Math. 109, p.155-178 (1985).

[15] **SARMANT, M.C.** *Factorisation en Produits Méromorphe d'un Elément semi-inversible.* Bull.Sc.Math. t. 115 p.379-394 (1991).

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