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REPRESENTATIVE SUBALGEBRA OF A COMPLETE ULTRAMETRIC HOPF ALGEBRA

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ABSTRACT. Let (H, m, c, η, σ) be a complete ultrametric Hopf algebra over a complete ultrametric valued field K, e be the unit of H and k the canonical map of K in H. In order words, H is a Banach algebra with multiplication $m : H\widehat{\otimes}H \rightarrow H$, coproduct $c : H \rightarrow H\widehat{\otimes}H$ a continuous algebra homomorphism, inversion or antipode $\eta : H \rightarrow H$ a continuous linear map and counit $\sigma : H \rightarrow K$ a continuous algebra homomorphism. The coassociativity and countinary axioms hold, and

$$m \circ (\eta \otimes 1_H) \circ c = k \circ \sigma = m \circ (1_H \otimes \eta) \circ c.$$

We define the representative subalgebra $\mathcal{R}(H)$ of H, i.e. the subalgebra of H generated by the coefficient "functions" associated with the finite dimensional left H-comodules. Under some conditions on $H, \mathcal{R}(H)$ is a direct sum of finite dimensional subcoalgebras and is dense in H. But in general, $\mathcal{R}(H)$ is not dense in H. The algebra $\mathcal{R}(H)$ is a generalization of the algebra of representative functions on a group. Notice that when the valuation of K and the norm of H are trivial, one obtains the well known fact that H is equal to its representative subalgebra.

INTRODUCTION.

Let (H, m, c, η, σ) be a complete ultrametric Hopf algebra over the complete ultrametric valued field K. An ultrametric Banach space E over K is said to be a *left Banach H-comodule* if there exists a continuous linear map $\Delta_E : E \rightarrow H\widehat{\otimes}E$, called coproduct, such that

(i) $(c \otimes 1_E) \circ \Delta_E = (1_H \otimes \Delta_E) \circ \Delta_E$

(ii) $(\sigma \otimes 1_E) \circ \Delta_E = 1_E$

A closed linear subspace E of E is a (left) Banach *subcomodule* of E if $\Delta_E(M) \subset H\widehat{\otimes}M$.

Let (E, Δ_E) and (F, Δ_F) be two left Banach comodules. A continuous linear map $u : E \to F$ is a *Banach comodule morphism* if $\Delta_F \circ u = (1_H \otimes u) \circ \Delta_E$.

It is associated with any left Banach H-comodule (E, Δ_E) the closed linear subspace $R(\Delta_E)$ of H spaned by the coefficient "functions" $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'$, $x \in E$, where E' if the Banach space dual of E. Furthermore, let $\mathcal{R}(H)$ be the linear subspace of H spaned by all the $R(\Delta_E)$ where (E, Δ_E) is a finite dimensional left H-comodule. Then $\mathcal{R}(H)$ is a (non necessary closed)sub-Hopf-algebra of H ; $\mathcal{R}(H)$ is called the *representative subalgebra* of H. In general, $\mathcal{R}(H)$ is not dense in H (cf. [1] or [5], [6]). However, with additional conditions on H it will be shown that $\mathcal{R}(H)$ is dense in H.

If E and F are ultrametric Banach spaces over K, we denote by $E\widehat{\otimes}F$ the complete tensor product, that is the completion of $E \otimes F$ with respect to the norm $\|z\| = \operatorname*{Inf}_{z=\Sigma x_j \otimes y_j} (\max_j \|x_j\| \, \|y_j\|)$. In the sequel all Banach spaces are ultrametric.

I - LEFT BANACH COMODULES

I - 1 Tensor products of left Banach comodules

Let (E, Δ_E) and (F, Δ_F) be two left Banach comodules. One has the continuous linear map $\Delta_{E\widehat{\otimes}F} : E\widehat{\otimes}F \to H\widehat{\otimes}E\widehat{\otimes}H\widehat{\otimes}F \to H\widehat{\otimes}H\widehat{\otimes}E\widehat{\otimes}F \to H\widehat{\otimes}E\widehat{\otimes}F$ where $\Delta_{E\widehat{\otimes}F} = (m \otimes 1_E \otimes 1_F) \circ (1_H \otimes \tau_{E\widehat{\otimes}F} \otimes 1_F) \circ (\Delta_E \otimes \Delta_F)$ and $\tau_{E\widehat{\otimes}F}(x \otimes a) = a \otimes x$.

Proposition 1 : $\Delta_{E\widehat{\otimes}F} : E\widehat{\otimes}F \to H\widehat{\otimes}E\widehat{\otimes}F$ *is the coproduct of a left Banach H-comodule structure on $E\widehat{\otimes}F$.*

Proof: Put, for $x \in E$ and $y \in F$, $\Delta_E(x) = \sum_{j\geq 1} a_j \otimes x_j \in H\widehat{\otimes}E$ and $\Delta_F(y) = \sum_{\ell \geq 1} b_\ell \otimes y_\ell \in H\widehat{\otimes}F$. Therefore, one has $\Delta_{E\widehat{\otimes}F}(x \otimes y) = \sum_{j\geq 1}\sum_{\ell\geq 1} a_j b_\ell \otimes x_j \otimes y_\ell$.

(i) It follows immediately that $(\sigma \otimes 1_{E\widehat{\otimes}F}) \circ \Delta_{E\widehat{\otimes}F}(x \otimes y) = \sum_{j\geq 1}\sum_{\ell\geq 1} \sigma(a_j b_\ell) x_j \otimes y_\ell =$
$= \sum_{j\geq 1}\sum_{\ell\geq 1} \sigma(a_j)\sigma(b_\ell) x_j \otimes y_\ell = \sum_{j\geq 1} \sigma(a_j) x_j \otimes \sum_{\ell\geq 1} \sigma(b_\ell) y_\ell = x \otimes y = 1_{E\widehat{\otimes}F}(x \otimes y)$. From what, one deduces $(\sigma \otimes 1_{E\widehat{\otimes}F}) \circ \Delta_{E\widehat{\otimes}F} = 1_{E\widehat{\otimes}F}$

(ii) Also, one has for $x \in E$, $y \in F$

α) $(c \otimes 1_E) \circ \Delta_E(x) = \sum_{j\geq 1} c(a_j) \otimes x_j = \sum_{j\geq 1}\sum_{s\geq 1} \alpha^1_{s,j} \otimes \alpha^2_{s,j} \otimes x_j = (1_H \otimes \Delta_E) \circ \Delta_E(x) =$
$= \sum_{j\geq 1} a_j \otimes \Delta_E(x_j) = \sum_{j\geq 1}\sum_{k\geq 1} a_j \otimes \gamma_{k,j} \otimes x_{k,j}$

and

$(c\otimes 1_F)\circ\Delta_F(y)=\sum_{\ell\geq 1}c(b_\ell)\otimes y_\ell=\sum_{\ell\geq 1}\sum_{t\geq 1}\beta^1_{t,\ell}\otimes\beta^2_{t,\ell}\otimes y_\ell=(1_H\otimes\Delta_F)\circ\Delta_F(y)=$

$=\sum_{\ell\geq 1}b_\ell\otimes\Delta_F(y_\ell)=\sum_{\ell\geq 1}\sum_{m\geq 1}b_\ell\otimes\rho_{m,\ell}\otimes y_{m,\ell}$

Let $E_x=E[(x_j,j\geq 1)\cup(x_{k,j},k\geq 1,j\geq 1)]$ be the closed linear subspace of E spaned by $(x_j,j\geq 1)\cup(x_{k,j},k\geq 1,j\geq 1)$, and $F_y=E[(y_\ell,\ell\geq 1)\cup(y_{m,\ell},m\geq 1,\ell\geq 1)]$ be the closed linear subspace of F spaned by $(y_\ell,\ell\geq 1)\cup(y_{m,\ell},m\geq 1,\ell\geq 1)$. It is clear that the Banach spaces E_x and F_y are of countable type. Furthermore, if $x'\in E'_x$ and $y'\in F'_y$ one has

$$\begin{aligned}(1_H\otimes 1_H\otimes x')\circ(c\otimes 1_E)\circ\Delta_E(x) &= \sum_{j\geq 1}\sum_{s\geq 1}<x',x_j>\alpha^1_{s,j}\otimes\alpha^2_{s,j} =\\ = (1_H\otimes 1_H\otimes x')\circ(1_H\otimes\Delta_E)\circ\Delta_E(x) &= \sum_{j\geq 1}\sum_{k\geq 1}<x',x_{k,j}>a_j\otimes\gamma_{k,j}\end{aligned}$$

and

$$\begin{aligned}(1_H\otimes 1_H\otimes y')\circ(c\otimes 1_F)\circ\Delta_F(y) &= \sum_{\ell\geq 1}\sum_{t\geq 1}<y',y_\ell>\beta^1_{t,\ell}\otimes\beta^2_{t,\ell} =\\ = (1_H\otimes 1_H\otimes y')\circ(1_H\otimes\Delta_F)\circ\Delta_F(y) &= \sum_{\ell\geq 1}\sum_{m\geq 1}<y',y_{m,\ell}>b_\ell\otimes\rho_{m,\ell}.\end{aligned}$$

β) On one hand , one has, $(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y) = \sum_{j\geq 1}\sum_{\ell\geq 1}c(a_jb_\ell)\otimes x_j\otimes y_\ell =$

$=\sum_{j\geq 1}\sum_{\ell\geq 1}c(a_j)c(b_\ell)\otimes x_j\otimes y_\ell = \sum_{j\geq 1}\sum_{\ell\geq 1}\left(\sum_{s\geq 1}\alpha^1_{s,j}\otimes\alpha^2_{s,j}\right)\left(\sum_{t\geq 1}\beta^1_{t,\ell}\otimes\beta^2_{t,\ell}\right)\otimes x_j\otimes y_\ell.$

On the other hand, one has

$(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y) = \sum_{j\geq 1}\sum_{\ell\geq 1}a_jb_\ell\otimes\Delta_{E\widehat{\otimes}F}(x_j\otimes y_\ell) = \sum_{j\geq 1}\sum_{\ell\geq 1}\sum_{k\geq 1}\sum_{m\geq 1}a_jb_\ell\otimes$ $\gamma_{k,j}\rho_{m,\ell}\otimes x_{k,\ell}\otimes y_{m,\ell}.$

Hence, if $x'\in E'_x$ and $y'\in F'_y$; first, one has

$(1_H\otimes 1_H\otimes x'\otimes y')\circ(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y) = \sum_{j\geq 1}\sum_{s\geq 1}<x',x_j>\alpha^1_{s,j}\otimes\alpha^2_{s,j}\sum_{\ell\geq 1}\sum_{t\geq 1}<y',y_\ell>$ $\beta^1_{t,\ell}\otimes\beta^2_{t,\ell}=(1_H\otimes 1_H\otimes x')\circ(c\otimes 1_E)\circ\Delta_E(x)\cdot(1_H\otimes 1_H\otimes y')\circ(c\otimes 1_F)\circ\Delta_F(y)=$
$=(1_H\otimes 1_H\otimes x')\circ(1_H\otimes\Delta_E)\circ\Delta_E(x)\cdot(1_H\otimes 1_H\otimes y')\circ(1_H\otimes\Delta_F)\circ\Delta_F(y).$

And, second, one has

$(1_H\otimes 1_H\otimes x'\otimes y')\circ(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y) = \sum_{j\geq 1}\sum_{k\geq 1}<x',x_{k,j}>a_j\otimes\gamma_{k,j}\sum_{\ell\geq 1}\sum_{m\geq 1}<y',y_{m,\ell}>$ $b_\ell\otimes\rho_{m,\ell}=(1_H\otimes 1_H\otimes x')\circ(1_H\otimes\Delta_E)\circ\Delta_E(x)\cdot(1_H\otimes 1_H\otimes y')\circ(1_H\otimes\Delta_F)\circ\Delta_F(y).$

Therefore , for any $x'\in E'_x$ and any $y'\in F'_y$, we have

(a) : $(1_H\otimes 1_H\otimes x'\otimes y')\left[(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)-(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)\right]=0$

γ) Since E_x [*resp.* F_y] is of countable type, there exist $\alpha_0>0,\alpha_1>0$ and

$(e_j)_{j\geq 1}\subset E_x$ [*resp.* $(f_\ell)_{\ell\geq 1}\subset F_y$] such that for $z\in E_x$ [*resp.* $\zeta\in F_y$] one has $z=\sum_{j\geq 1}\lambda_j e_j$ [*resp.* $\zeta=\sum_{\ell\geq 1}\mu\ell f_\ell$] with $\alpha_0 \operatorname{Sup}_{j\geq 1}|\lambda_j|\leq \|z\|\leq \alpha_1 \operatorname{Sup}_{j\geq 1}|\lambda_j|$ [*resp.* $\alpha_0 \operatorname{Sup}_{\ell\geq 1}|\mu_\ell|\leq \|\zeta\|\leq \alpha_1 \operatorname{Sup}_{\ell\geq 1}|\mu_\ell|$] (cf. [4]).

Moreover, one has $E_x\widehat{\otimes}F_y\simeq c_o(\mathbb{N}^*\times\mathbb{N}^*,K)$ and $(H\widehat{\otimes}H)\widehat{\otimes}(E_x\widehat{\otimes}E_y)\simeq$ $\simeq c_o(\mathbb{N}^*\times\mathbb{N}^*,H\widehat{\otimes}H)$ (cf. [7]) ; any Z in $(H\widehat{\otimes}H)\widehat{\otimes}(E_x\widehat{\otimes}E_y)$ can be written in the unique form $Z=\sum_{j\geq\ell}A_{j\ell}\otimes e_j\otimes f_\ell$ with $A_{j\ell}\in H\widehat{\otimes}H$ and $\alpha_0^2 \operatorname{Sup}_{j,\ell}\|A_{j\ell}\|\leq\|Z\|\leq\alpha_1^2\operatorname{Sup}_{j,\ell}|A_{j,\ell}\|$.

Let $e'_j\in E'_x$ [*resp.* $f'_\ell\in F'_y$] be the continuous linear form defined by $<e'_j,e_{j_1}>=\delta_{jj_1}$ [*resp.* $<f'_\ell,f_{\ell_1}>=\delta_{\ell\ell_1}$]. Setting $(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)-(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)=Z_0=\sum_{j,\ell}A^0_{j\ell}\otimes e_j\otimes f_\ell\in H\widehat{\otimes}H\widehat{\otimes}E_x\widehat{\otimes}F_y$, for any $j_1\geq 1$ and any $\ell_1\geq 1$, by (a) , one has $(1_H\otimes 1_H\otimes e'_{j_1}\otimes f'_{\ell_1})(Z_0)=\sum_{j,\ell}A^0_{j\ell}\delta_{j_1 j}\delta_{\ell_1\ell}=A^o_{j_1\ell_1}=0$. It follows that $Z_0=0$, i.e. $(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)=(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}(x\otimes y)$. From what , one deduces that $(c\otimes 1_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}=(1_H\otimes\Delta_{E\widehat{\otimes}F})\circ\Delta_{E\widehat{\otimes}F}$.

Corollary : *Let M [resp. N] be a left Banach H-subcomodule of E [resp. F]. Then $M\widehat{\otimes}N$ is a left Banach subcomodule of $E\widehat{\otimes}F$.*

I - 2 Banach comodule morphisms

I - 2 - 1 Range and kernel

Proposition 2 : *Let $u:E\to F$ be a Banach comodule morphism.*

(i) *If V is a Banach subcomodule of F, then $u^{-1}(V)$ is a Banach subcomodule of E.*

(ii) *The closure $\overline{u(E)}$ of $u(E)$ is a Banach subcomodule of F*

Corollary : *Let V and W be Banach subcomodule of the left Banach H-comodule E ; then $V\cap W$ is a Banach subcomodule of E.*

Proofs : Rather easy, or see [3].

Note : One can also see [3] for the spaces of comodule morphisms.

Remark 1 : *If M is a Banach subcomodule of the left Banach H-comodule E, it is induced on the quotient Banach space E/M a structure of Banach left H-comodule such that the canonical map $E\to E/M$ is a comodule morphism.*

Then, if $u : E \to F$ is a Banach comodule morphism and if u is strict, the Banach comodule $E/\ker u$ and $u(E)$ are isomorphic. Also, one can define the cokernel of u as being $F/\overline{u(E)}$.

I - 2- 2 Comodule morphisms of E into H associated with Δ ; $R(\Delta)$

Put $\Delta = \Delta_E$ the coproduct of the left Banach H-comodule E. Obviously, H is a left Banach H-comodule with respect to its coproduct c.

Proposition 3 : *For any $x' \in E'$, the linear map $A_{x'} = (1_H \otimes x') \circ \Delta : E \to H$ is a Banach comodule morphism.*

Proof : It is easy to see that $c \circ (1_H \otimes x') = c \otimes x' = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E)$. Therefore $c \circ A_{x'} = c \circ (1_H \otimes x') \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (c \otimes 1_E) \circ \Delta = (1_H \otimes 1_H \otimes x') \circ (1_H \otimes \Delta) \circ \Delta =$
$= (1_H \otimes [(1_H \otimes x') \circ \Delta]) \circ \Delta = (1_H \otimes A_{x'}) \circ \Delta$.

Corollary 1 :
(i) *$\ker A_{x'}$ is a closed subcomodule of E.*
(ii) *$\overline{A_{x'}(E)}$ is a left Banach subcomodule(= closed left coideal) of H .*

Corollary 2 : *If E is a space of countable type, one has $\ker A_{x'} \neq E$ for any $x' \in E, x' \neq 0$*

Proof : Indeed, if $x' \in E', x' \neq 0$ and $0 < \alpha < 1$, there exists a α-orthogonal base $(e_j)_{j \geq 1} \subset E$ such that $< x', e_1 >= 1$ and $< x', e_j >= 0, j \geq 2$. Moreover for any $j \geq 1$, $\Delta(e_j) = \sum_{\ell \geq 1} a_{\ell j} \otimes e_\ell$ and $e_j = \sum_{\ell \geq 1} \sigma(a_{\ell j}) e_\ell$; therefore $\sigma(a_{\ell j}) = \delta_{\ell j}$ and $A_{x'}(e_1) = (1_H \otimes x') \circ \Delta(e_1) = a_{11} \neq 0$ since $\sigma(a_{11}) = 1$.

Corollary 3 : *Assume that H is a pseudo-reflexive Banach space ; i.e. $H \to H''$ is isometric.*

Let E be a simple Banach left H-comodule, i.e. E contains no proper closed subcomodule. Then E is a Banach space of countable type and $A_{x'}$ is injective for each $x' \in E'$, $x' \neq 0$.

Proof : If H is pseudo-reflexive, it is shown in [3] that any simple Banach left H-comodule is a space of coutable type. Applying Corollary 2, one sees that $A_{x'}$ is injective for $x' \in E'$, $x' \neq 0$ □

Let $\beta : E \otimes E' \to K$ be the continuous linear form defined upon $\beta(x \otimes x') =< x', x >$. Put $\rho_\Delta = (1_H \otimes \beta) \circ (\Delta \otimes 1_{E'}) \circ \tau : E' \hat{\otimes} E \to H$, where $\tau(x' \otimes x) = x \otimes x'$. Then ρ_Δ is linear and continuous with $\|\rho_\Delta\| \leq \|\Delta\|$. Moreover for $x' \in E'$, $x \in E$, one has $\rho_\Delta(x' \otimes x) = (1_H \otimes x') \circ \Delta(x)$.

Put $R(\Delta) = \overline{\rho_\Delta(E'\widehat{\otimes}E)}$ the closure of $\rho_\Delta(E'\widehat{\otimes}E)$ in H. Obviously, $R(\Delta)$ is the closed linear subspace of H spaned by the elements $(1_H \otimes x') \circ \Delta(x)$, $x' \in E'$, $x \in E$, called the coefficients of the comodule (E, Δ).

Proposition 4 : *$R(\Delta) = \overline{\rho_\Delta(E'\widehat{\otimes}E)}$ is a left Banach subcomodule (= closed left coideal) of H.*

Proof : Since $c : H \to H\widehat{\otimes}H$ is linear and is a homeomorphism of H onto $c(H)$, one has $c(R(\Delta)) = \overline{c(\rho_\Delta(E'\widehat{\otimes}E))}$, a closed linear subspace of H.

It remains to show that if $a = \rho_\Delta(x'\otimes x) = (1_H\otimes x')\circ\Delta(x) = A_{x'}(x)$, $x' \in E'$, $x \in E$; then $c(a) \in H\widehat{\otimes}R(\Delta)$. Writing $\Delta(x) = \sum_{j\geq 1} a_j \otimes x_j$; one has $c(a) = c \circ A_{x'}(x) = (1_H \otimes A_{x'}) \circ \Delta(x) = \sum_{j\geq 1} a_j \otimes A_{x'}(x_j) = \sum_{j\geq 1} a_j \otimes \rho_\Delta(x' \otimes x_j) \in H\widehat{\otimes}R(\Delta)$. □

Proposition 5 : *If the left Banach comodules E and E_1 with coproduct respectively Δ and Δ_1 are isomorphic, then $R(\Delta) = R(\Delta_1)$.*

Proof : Let $u : E \to E_1$ be a comodule isomorphism, in other words, u is linear, continuous and bijective with $\Delta_1 \circ u = (1_H \otimes u) \circ \Delta$. Moreover, the reciprocal map u^{-1} of u satisfies $(1_H \otimes u^{-1}) \circ \Delta_1 = \Delta \circ u^{-1}$ and the transpose of u, ${}^tu : E_1' \to E'$ is linear, continuous and bijective with $({}^tu)^{-1} = {}^tu^{-1}$.

Set $a = \rho_{\Delta_1}(z_1) \in \rho_{\Delta_1}(E_1'\widehat{\otimes}E_1)$ and $z_1 = \sum_{j\geq 1} y_j'\otimes y_j$, $y_j' \in E_1', y_j \in E_1$, $\lim_j \|y_j'\|\,\|y_j\| = 0$. There exist, for $j \geq 1$ unique $x_j' \in E'$ and $x_j \in E$ such that $y_j' = {}^tu^{-1}(x_j') = x_j' \circ u^{-1}$ and $y_j = u(x_j)$; moreover $\lim_j \|x_j'\|\,\|x_j\| = 0$. Therefore $a = \rho_{\Delta_1}(z_1) = \sum_{j\geq 1} \rho_{\Delta_1}(y_j' \otimes y_j) =$

$= \sum_{j\geq 1}(1_H\otimes y_j')\circ\Delta_1(y_j) = \sum_{j\geq 1}(1_H\otimes x_j'\circ u^{-1})\circ\Delta_1\circ u(x_j) = \sum_{j\geq 1}(1_H\otimes x_j')\circ(1_H\otimes u^{-1})\circ\Delta_1\circ u(x_j) = \sum_{j\geq 1}(1_H \otimes x_j') \circ \Delta(x_j) = \sum_{j\geq 1} \rho_\Delta(x_j' \otimes x_j) = \rho_\Delta\Big(\sum_{j\geq 1} x_j' \otimes x_j\Big)$. Hence, $a = \rho_\Delta(z) \in \rho_\Delta(E'\widehat{\otimes}E)$ where $z = \sum_{j\geq 1} x_j' \otimes x_j$; that is $\rho_{\Delta_1}(E_1'\widehat{\otimes}E_1) \subset \rho_\Delta(E'\widehat{\otimes}E)$. Likewise,one has $\rho_\Delta(E'\widehat{\otimes}E) \subset \rho_{\Delta_1}(E_1'\widehat{\otimes}E_1)$.

Therefore $\rho_\Delta(E'\widehat{\otimes}E) = \rho_{\Delta_1}(E_1'\widehat{\otimes}E_1)$ and $R(\Delta) = R(\Delta_1)$. □

Assume that E is a *free Banach space* i.e. $E \simeq c_0(I, K) = \{(\lambda_j)_{j\in I} \subset K \ / \ \lim_j \lambda_j = 0\}$. In other words, there exist $(e_j)_{j\in I} \subset E$, $\alpha_0, \alpha_1 \in \mathbb{R}_+^*$ such that any $x \in E$ can be written in the form $x=\sum_{j\in I}\lambda_j e_j,\ \lambda_j\in K$ and $\alpha_0\sup_{j\in I}|\lambda_j|\le\|x\|\le\alpha_1\sup_{j\in I}|\lambda_j|$.For any continuous linear form $x'\in E'$, one has $\frac{1}{\alpha_1}\sup_{j\in I}|<x',e_j>|\le\|x'\|\le\frac{1}{\alpha_0}\sup_{j\in I}|<x',e_j>|$. Let e'_j be the element of E' defined by $<e'_j,e_\ell>=\delta_{j\ell}$. Put $E'_0=E[(e'_j)_{j\in I}]$, the *closed linear subspace of* E' *spaned by* $(e'_j)_{j\in I}$. Hence each $x'\in E'_0$ can be written in the unique form $x'=\sum_{j\in I}\mu_j e'_j,\ \mu_j\in K,\lim_j|\mu_j|=0$. Moreover, if $v\in E'_0\widehat{\otimes}E\subset E'\widehat{\otimes}E$, one has $v=\sum_{j,\ell}\mu_{\ell j}e'_\ell\otimes e_j,\ \mu_{\ell j}\in K,\ \lim_{(j,\ell)}\mu_{\ell j}=0$ and $\frac{\alpha_0}{\alpha_1}\sup_{j,\ell}|\mu_{\ell,j}|\le\|v\|\le\frac{\alpha_1}{\alpha_0}\sup_{j,\ell}|\mu_{\ell j}|$.

On the other hand, one has $H\widehat{\otimes}E\simeq c_0(I,H)=\{(a_j)_{j\in I}\subset H/\lim_j a_j=0\}$. For any $z\in H\widehat{\otimes}E$ one has $z=\sum_{j\in I}a_j\otimes e_j,a_j\in H$ with $\lim_j\|a_j\|=0$ and $\alpha_0\sup_{j\in I}\|a_j\|\le\|z\|\le\alpha_1\sup_{j\in I}\|a_j\|$. Hence, if (E,Δ) is a left Banach H-comodule, for $x\in E$, one has $\Delta(x)=$ $=\sum_{j\in I}A_j(x)\otimes e_j$. In particular $\Delta(e_\ell)=\sum_{j\in I}A_j(e_\ell)\otimes e_j=\sum_{j\in I}a_{\ell j}\otimes e_j$ and $(c\otimes 1_E)\circ\Delta(e_\ell)=$ $=\sum_{j\in I}c(a_{\ell j})\otimes e_j=(1_H\otimes\Delta)\circ\Delta(e_\ell)=(1_H\otimes\Delta)\Big(\sum_{k\in I}a_{\ell k}\otimes e_k\Big)=\sum_{k\in I}a_{\ell k}\otimes\sum_{j\in I}a_{kj}\otimes e_j=$ $=\sum_{k\in I}\sum_{j\in I}a_{\ell k}\otimes a_{\ell j}\otimes e_j$. Thus one obtains

(1) $c(a_{\ell j})=\sum_{k\in I}a_{\ell k}\otimes a_{kj}\ ;\ \ell,j\in I$

Also, one has

(2) $\sigma(a_{\ell j})=\delta_{\ell j}\ ;\ \ell,j\in I$

(3) $\sum_{k\in I}a_{\ell k}\ \eta(a_{kj})=\delta_{\ell j}\cdot e=\sum_{k\in I}\eta(a_{\ell k})\otimes a_{kj}\ ;\ \ell,j\in I.$

Proposition 6 : $R_0(\Delta)=\overline{\rho_\Delta(E'_0\widehat{\otimes}E)}$ *is a closed subcoalgebra of* H. *In other words* $c(R_0(\Delta))\subset R_0(\Delta)\widehat{\otimes}R_0(\Delta)$

Proof : Since $(e'_j\otimes e_\ell)_{(j,\ell)\in I\times I}$ is a total family of $E'_0\widehat{\otimes}E$ and ρ_Δ is linear and continuous, the family $(\rho_\Delta(e'_j\otimes e_\ell))_{(j,\ell)\in I\times I}$ is total in $\overline{\rho_\Delta(E'_0\widehat{\otimes}E}=R_0(\Delta)=$ the closed linear subspace of H spaned by the $(1_H\otimes x')\circ\Delta(x),\ x'\in E'_0,\ x\in E$.

To see that $c(R_0(\Delta))\subset R_0(\Delta)\widehat{\otimes}R_0(\Delta)$, it suffices to show that for $\ell,j\in I$ one has $c(\rho_\Delta(e'_j\otimes e_\ell))\in R_0(\Delta)\widehat{\otimes}R_0(\Delta)$. However, by definition, $\rho_\Delta(e'_j\otimes e_\ell)=(1_H\otimes e'_j)\circ\Delta(e_\ell)=$ $=a_{\ell j}\in R_0(\Delta)$. Then, one deduces from (1) that $c(\rho_\Delta(e'_j\otimes e_\ell))=c(a_{\ell j})=\sum_{k\in I}a_{\ell j}\otimes a_{kj}\in$

$R_0(\Delta)\widehat{\otimes}R_0(\Delta)$.

Note : *If* $v = \sum_{\ell,j} \mu_{\ell j} e'_j \otimes e_\ell \in E'_0\widehat{\otimes}E$ *, one has* $\rho_\Delta(v) = \sum_{\ell,j} \mu_{\ell j} a_{\ell j}$ *and* $a \in R_0(\Delta)$ *iff there exist* $v_n \in E'\widehat{\otimes}E$ *such that* $a = \lim_{n\to+\infty} \rho_\Delta(v_n)$.

Remark 2 : *Let* (E,Δ) *and* (E_1,Δ_1) *be two isomorphic left Banach comodules that are free Banach spaces. If* $u : E \to E_1$ *is a comodule isomorphism ,* $(e_j)_{j\in I}$ *a base of* E *and* $(\varepsilon_j)_{j\in I}$ *the base of* E_1 *defined by* $\varepsilon_j = u(e_j)$*; then, with the above notations , one has* $R_0(\Delta) = R_0(\Delta_1)$.

Remark 3 : *If* $dim E = n < +\infty$*, one has* $R(\Delta) = R_0(\Delta) = \rho_\Delta(E' \otimes E)$ *and* $dim R(\Delta) \leq n^2$

II - REPRESENTATIVE SUBALGEBRA

II - 1 Conjugate comodule of a finite dimensional comodule

Let (E,Δ) be a (Banach) left H -comodule of finite dimension and $(e_j)_{1\leq j\leq n}$ a K-base of E. As above, for any $x \in E$, one has $\Delta(x) = \sum_{j=1}^{n} A_j(x)\otimes e_j$ and $A_j(x) = (1_H \otimes e'_j)\circ \Delta(x) = \rho_\Delta(e'_j\otimes x); A_j = (1_H\otimes e'_j)\circ\Delta \in \mathcal{L}(E,H)$. In particular $\Delta(e_\ell) = \sum_{j=1}^{n} a_{\ell j}\otimes e_j$ where $a_{\ell j} = A_j(e_\ell) = \rho_\Delta(e'_j \otimes e_\ell)$; and we have the relations (1) , (2) and (3) , with $I = [1,n]$. The relation (3) means here, that the matrix $A = (a_{\ell j})_{1\leq \ell,j\leq n} \in Mat_n(H)$ is invertible with inverse $A^{-1} = (\eta(a_{\ell j})_{1\leq \ell,j\leq n}$.

Fix the base $(e_j)_{1\leq j\leq n}$ of E and define the linear map $\Delta^\vee : E' \to H \otimes E'$ by setting $\Delta^\vee\ (e'_j) = \sum_{\ell=1}^{n} \eta(a_{\ell j}) \otimes e'_\ell,\ 1 \leq j \leq n$. Hence for $x' = \sum_{j=1}^{n} \mu_j e'_j \in E'$, one has $\Delta^\vee(x') = \sum_{\ell=1}^{n}\sum_{j=1}^{n} \mu_j\eta(a_{\ell j}) \otimes e'_j = \sum_{\ell=1}^{n} A^\vee_\ell(x') \otimes e'_\ell$.

Lemma 1 : $(E',\Delta^\vee)$ *is a left* H*-comodule.*

Proof : One verifies that $\sigma\circ\eta = \sigma$; indeed , if $a \in H$, then $c(a) = \sum_{t\geq 1} a^1_t \otimes a^2_t$. Hence, one has $m\circ(\eta\otimes 1_H)\circ c(a) = \sum_{t\geq 1} \eta(a^1_t)a^2_t = \sigma(a)e$ and $a = (1_H\otimes\sigma)\circ c(a) = \sum_{t\geq 1} a^1_t\sigma(a^2_t)$. It follows that $\eta(a) = \sum_{t\geq 1} \eta(a_t^1)\sigma(a_t^2)$ and $\sigma \circ \eta(a) = \sum_{t\geq 1} \sigma(\eta(a_t^1))\sigma(a_t^2) = \sigma\Big(\sum_{t\geq 1} \eta(a_t^1)a_t^2\Big) =$ $\sigma(\sigma(a)e) = \sigma(a)$.

Since $\sigma(a_{\ell j}) = \delta_{\ell j}$, one has $(\sigma \otimes 1_{E'}) \circ \Delta^\vee(e'_j) = \sum_{\ell=1}^{n} \sigma \circ \eta(a_{\ell j})e'_\ell = \sum_{\ell=1}^{n} \sigma(a_{\ell j})e'_j =$ $= e'_j$, $1 \leq j \leq n$. It follows, by linearity, that $(\sigma \otimes 1_{E'}) \circ \Delta^\vee = 1_{E'}$.

Let us remember that $c \circ \eta = \tau \circ (\eta \otimes \eta) \circ c$ where $\tau(a \otimes b) = b \otimes a$. Hence , we have $c \circ \eta(a_{\ell j}) = \sum_{k=1}^{n} \eta(a_{kj}) \otimes \eta(a_{\ell k})$. Therefore $(c \otimes 1_{E'}) \circ \Delta^\vee(e'_j) = (c \otimes 1_{E'})\Big(\sum_{\ell=1}^{n} \eta(a_{\ell j}) \otimes e'_\ell\Big) =$ $= \sum_{\ell=1}^{n}\sum_{k=1}^{n} \eta(a_{kj}) \otimes \eta(a_{\ell j}) \otimes e'_\ell = \sum_{k=1}^{n} \eta(a_{kj}) \otimes \Delta^\vee(e'_k) = (1_H \otimes \Delta^\vee)\Big(\sum_{k=1}^{n} \eta(a_{kj}) \otimes e'_k\Big) =$ $= (1_H \otimes \Delta^\vee) \circ \Delta^\vee(e'_j)$, and $(c \otimes 1_{E'}) \circ \Delta^\vee = (1_H \otimes \Delta^\vee) \circ \Delta^\vee$.

Corollary : $R(\Delta^\vee) = \eta(R(\Delta))$.

Proof : Identifing E'' with E, one has $R(\Delta^\vee) = \rho_{\Delta^\vee}(E \otimes E')$. Set $z = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j} e_\ell \otimes e'_j \in E \otimes E'$; hence $\rho_{\Delta^\vee}(z) = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j}\rho_{\Delta^\vee}(e_\ell \otimes e'_j)$. However $\rho_{\Delta^\vee}(e_\ell \otimes e'_j) = (1_H \otimes e_\ell) \circ \Delta^\vee(e'_j) = \eta(a_{\ell j}) = \eta(\rho_\Delta(e'_j \otimes e_\ell))$; therefore $\rho_{\Delta^\vee}(z) = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j}\eta(\rho_\Delta(e'_j \otimes e_\ell)) =$ $\eta(\rho_\Delta(z_1))$ where $z_1 = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j} e'_j \otimes e_\ell \in E' \otimes E$. It follows that $R(\Delta^\vee) \subset \eta(R(\Delta))$. The same formulae show that if $a = \rho_\Delta(z_1) \in R(\Delta)$, where $z_1 = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j} e'_j \otimes e_\ell \in E' \otimes E$, one has $\eta(a) = \rho_{\Delta^\vee}(z)$ where $z = \sum_{1\leq \ell,j\leq n} \lambda_{\ell j} e_\ell \otimes e'_j \in E \otimes E'$, hence $\eta(R(\Delta)) \subset R(\Delta^\vee)$.

II- 2 Direct sum of Banach comodules

Let $(E_s)_{1\leq s\leq m}$ be a finite family of left Banach H-comodules with Δ_s the coproduct of E_s. The direct sum $E = \bigoplus_{s=1}^{m} E_s$ equiped with any norm equivalent to the norm $\Big\|\sum_{s=1}^{m} x_s\Big\| =$ $= \max_{1\leq s\leq m} \|x_s\|$ is a Banach space. Put $\Delta = \bigoplus_{s=1}^{m} \Delta_s$, i.e. $\Delta\Big(\sum_{s=1}^{m} x_s\Big) = \bigoplus_{s=1}^{m} \Delta_s(x_s)$. It is readily seen that (E, Δ) is a left Banach comodule. Moreover , if $p_s : E \to E$ is the projection of E onto E_s , then $1_H \otimes p_s$ is a projection of $H\widehat{\otimes}E$ onto $H\widehat{\otimes}E_s$ and one has

$H\widehat{\otimes}E = \bigoplus_{s=1}^{m} H\widehat{\otimes}E_s$. On the other hand p_s is a comodule morphism i.e. $(1_H \otimes p_s)\circ\Delta = \Delta\circ p_s$; furthermore $(1_H\otimes p_s)\circ\Delta(x_t) = 0$ for $s \neq t$ and $x_t \in E_t$.

Also, we have $E' = \bigoplus_{s=1}^{m} E'_s$; the projections associated with this direct sum are the ${}^tp_s$, $1 \le s \le m$.

Proposition 7 : *With the above notations, one has* $\rho_\Delta(E'\widehat{\otimes}E) = \sum_{s=1}^{m} \rho_{\Delta_s}(E'_s\widehat{\otimes}E_s)$ *and* $R(\Delta)$ *is the closure of* $\sum_{s=1}^{m} R(\Delta_s)$ *in* H.

Proof: If $x'_s \in E'_s$, $x_t \in E_t$ and $s \neq t$, then $(1_H\otimes x'_s)\circ\Delta(x_t) = (1_H\otimes x'_s)\circ\Delta_t(x_t) = 0$. Set $z = \sum_{j\ge1} x'_j\otimes x_j \in E'\widehat{\otimes}E = \bigoplus_{s=1}^{m}\bigoplus_{t=1}^{m} E'_s\widehat{\otimes}E_t$, one has $z = \sum_{j\ge1}\sum_{s=1}^{m}\sum_{t=1}^{m} x'_{s,j}\otimes x_{t,j}$. It follows that $\rho_\Delta(z) = \sum_{j\ge1}\sum_{s=1}^{m}\sum_{t=1}^{m}\rho_\Delta(x'_{s,j}\otimes x_{t,j}) = \sum_{j\ge1}\sum_{s=1}^{m}\sum_{t=1}^{m}(1_H\otimes x'_{s,j})\circ\Delta(x_{t,j}) =$

$= \sum_{j\ge1}\sum_{s=1}^{m}\sum_{t=1}^{m}\delta_{s,t}(1_H\otimes x'_{s,j})\circ\Delta_t(x_{t,j}) = \sum_{j\ge1}\sum_{s=1}^{m}\rho_{\Delta_s}(x'_{s,j}\otimes x_{s,j}) = \sum_{s=1}^{m}\rho_{\Delta_s}\Big(\sum_{j\ge1} x'_{s,j}\otimes x_{s,j}\Big) =$

$= \sum_{s=1}^{m}\rho_{\Delta_s}(z_s)$. If $z_s \in E'_s\widehat{\otimes}E_s \subset E'\widehat{\otimes}E$, $1\le s\le m$, one has $\rho_\Delta(z_s) = \rho_{\Delta_s}(z_s)$. Therefore, on one hand, $\rho_\Delta(E'\widehat{\otimes}E) \subset \sum_{s=1}^{m}\rho_{\Delta_s}(E'_s\widehat{\otimes}E_s)$, and on the other hand, $\rho_{\Delta_s}(E'_s\widehat{\otimes}E_s) \subset \rho_\Delta(E'\widehat{\otimes}E)$. Hence, one has $\rho_\Delta(E\widehat{\otimes}E) = \sum_{s=1}^{m}\rho_{\Delta_s}(E'\widehat{\otimes}E_s)$. One verifies readily that $R(\Delta)$ is equal to the closure of $\sum_{s=1}^{m} R(\Delta_s)$ in H.

Corollary : *If* $dim E_s < +\infty$, $1\le s\le m$, *then one has* $R(\Delta) = \sum_{s=1}^{m} R(\Delta_s)$ *where* $E = \bigoplus_{s=1}^{m} E_s$, $\Delta = \bigoplus_{s=1}^{m}\Delta_s$.

Remark 3 : *If the comodules* (E_s, Δ_s), $1\le s\le m$, *are pairewise isomorphic, then for the comodule* (E,Δ) *where* $E = \bigoplus_{s=1}^{m} E_s$, $\Delta = \bigoplus_{s=1}^{m}\Delta_s$, *one has* $R(\Delta) = R(\Delta_s), 1\le s\le m$.

II - 3 The representative subalgebra of H

Let $\mathcal{S}(H)$ be the set of all elements of the form $a = (1_H \otimes x') \circ \Delta(x)$ of H where (E, Δ) is a finite dimensional left H-comodule and $x' \in E'$, $x \in E$. Let us put $dimE = dim\Delta$

Lemma 2 : *$\mathcal{S}(H)$ is a multiplicative, unitary submonoid of H.*

Proof : Set $a = (1_H \otimes x') \circ \Delta(x)$ and $b = (1_H \otimes y') \circ \Delta_1(y) \in \mathcal{S}(H)$ where (E, Δ) and (E, Δ_1) are left H-comodules of finite dimension and $x' \in E'$, $x \in E$, $y' \in E_1'$, $y \in E_1$. One has $\Delta(x) = \sum_{j=1}^{p} a_j \otimes x_j$, $\Delta_1(y) = \sum_{\ell=1}^{q} b_\ell \otimes y_\ell$ and $\Delta_{E \otimes E_1}(x \otimes y) = \sum_{j=1}^{p} \sum_{\ell=1}^{q} a_j b_\ell \otimes x_j \otimes y_\ell$.

Hence , $ab = (1_H \otimes x') \circ \Delta(x) \cdot (1_H \otimes y') \circ \Delta(y) = \sum_{j=1}^{p} \sum_{\ell=1}^{q} a_j b_\ell < x', x_j > < y', b_\ell > =$ $(1_H \otimes x' \otimes y') \circ \Delta_{E \otimes E_1}(x \otimes y) \in \mathcal{S}(H)$.

Since $c(e) = e \otimes e$, $E = K.e$ is a left subcomodule of H of dimension 1, one has $e = (1_H \otimes \sigma) \circ c(e) \in \mathcal{S}(H)$. □

Let $\mathcal{R}(H)$ *be the linear subspace of* H *spaned by* $\mathcal{S}(H)$. Then $\mathcal{R}(H)$ is an *unitary subalgebra* of H. Indeed, if $a = \sum_{j=1}^{p} \lambda_j a_j$ and $b = \sum_{\ell=1}^{q} \mu_\ell b_\ell$ are two elements of $\mathcal{R}(H)$, since $a_j b_\ell \in \mathcal{S}(H)$, one has $ab = \sum_{j=1}^{p} \sum_{\ell=1}^{q} \lambda_j \mu_\ell a_j b_\ell \in \mathcal{R}(H)$. One says that $\mathcal{R}(H)$ is the *representative subalgebra* of H.

Note : Put, for the left H-comodule (E, Δ) of finite dimension , $S(\Delta) = \{a = (1_H \otimes x') \circ \Delta(x) \in H;\ x' \in E,\ x \in E\}$. As in Proposition 5, $S(\Delta)$ depends only of the isomorphism class $\widetilde{\Delta}$ of (E, Δ). Furthermore, one has $\mathcal{S}(H) = \bigcup_{dim\Delta < +\infty} S(\Delta)$. □

Also, it is clear that the K-linear vector space $R(\Delta) = \rho_\Delta(E' \otimes E)$ is spaned by $S(\Delta)$. Hence one has $\mathcal{R}(H) = \bigcup_{dim\Delta < +\infty} R(\Delta)$. Moreover, if (E_1, Δ_1) and (E_2, Δ_2) are two comodules, then $R(\Delta_1 \oplus \Delta_2) = R(\Delta_1) + R(\Delta_2)$ contains $R(\Delta_1)$ and $R(\Delta_2)$ i.e. the family $(R(\Delta))_\Delta$ ordered by inclusion is directed upward.

Theorem 1 : *The representative subalgebra $\mathcal{R}(H)$ of H is such that $c(\mathcal{R}(H)) \subset \mathcal{R}(H) \otimes \mathcal{R}(H)$. Moreover $(\mathcal{R}(H), m, c, \eta, \sigma)$ is a Hopf algebra.*

Proof : It follows from Proposition 6 and Remark 3 that if Δ is a coproduct of finite dimension, then $c(R(\Delta)) \subset R(\Delta) \otimes R(\Delta)$: that is $R(\Delta)$ is a coalgebra. Since

$\mathcal{R}(H) = \bigcup_{dim\Delta<+\infty} R(\Delta)$ is the union of coalgebras, it is a coalgebra. On the other hand, one deduces from the Corollary of Lemma 1 that $\eta(\mathcal{R}(H)) \subset \mathcal{R}(H)$. The Theorem 1 is proved.

II - 4 Simple comodules of finite dimension

Let $(e_j)_{1\leq j\leq n}$ be a base of the finite dimensional left H-comodule (E, Δ). Let us remember that $A_j = (1_H \otimes e'_j) \circ \Delta$ is a comodule morphism.One sees that $\bigcap_{1\leq j\leq n} kerA_j = (0)$ and $(A_j)_{1\leq j\leq n}$ is free in $\mathcal{L}(E, H)$. Since $A_j(e_j) = a_{jj}$, one deduces from (2) or from Corollary 2 of Proposition 3 that $e_j \notin kerA_j$ and $kerA_j \neq E$.

Put $H_j = A_j(E)$; then H_j is a left subcomodule of H of dimension $\leq n$. Furthermore, with previous notations, one has $R(\Delta) = \rho_\Delta(E' \otimes E) = \sum_{j=1}^{n} H_j$ and $H_j = \sum_{\ell=1}^{n} K \cdot a_{\ell j}$, also $R(\Delta)$ is a subcoalgebra of dimension $\leq n^2$. One can have $dimR(\Delta) < n^2$; for example, if $E_q = \bigoplus_{t=1}^{q} E$ and $\Delta_q = \bigoplus_{t=1}^{q} \Delta$, $q \geq 2$, one has $R(\Delta_q) = R(\Delta)$ and $dimR(\Delta_q) = dimR(\Delta) \leq n^2 < (qn)^2 = (dim(E_q))^2$.

Definition : *A left Banach H-comodule E is called simple or topologically irreducible if E is not the null space and does not contain any closed subcomodule different from (0) and E.*

Let $Hom.com(E, E_1)$ be the Banach space of the left Banach comodule morphisms of (E, Δ) into (E_1, Δ_1) and $End.com(E) = Hom.com(E, E)$, this later is a Banach algebra.

Remark 4 : Schur's Lemma.

Let (E, Δ) and (E_1, Δ_1) be two simple, finite dimensional left H-comodules.

(i) *If E and E_1 are not isomorphic, one has $Hom.com(E, E_1) = (0)$*

(ii) *In the alternative case, any non null comodule morphism of E into E_1 is an isomorphism. In particular, $End.com(E)$ is a (skew) field of finite dimension $\leq (dimE)^2$. If K is algebraically closed, then $End.com(E) = K.1_E$.*

Proposition 8 : *Let (E, Δ) be a simple Banach left H-comodule of finite dimension n. Let $(e_j)_{1\leq j\leq n}$ be a base of E and $A_j = (1_H \otimes e_j) \circ \Delta$, $1 \leq j \leq n$.*

Then $H_j = A_j(E)$ is a simple left H-comodule of H of finite dimension n. Furthermore, there exists $J \subset [1, n]$ such that $R(\Delta) = \bigoplus_{j\in J} H_j$ (a direct sum of comodules).

Proof: It is the same as in semi-simple module theory. Indeed, since $kerA_j \neq E$ and E is a simple comodule of finite dimension n, the map $A_j : E \to H_j = A_j(E)$ is a comodule isomorphism. Hence H_j is a simple comodule of dimension n with base $(a_{\ell j})_{1\leq \ell \leq n}$. If $1 \leq j, q \leq n$, one has $H_j \cap H_q = (0)$, or $H_j = H_q$. Changing the order if necessary, we my assume that $(H_1, \ldots, H_m)$ is the family of the distinct comodules H_j ; $m \leq n$. Hence $R(\Delta) = \sum_{j=1}^{m} H_j, H_j \neq H_q$ for $j \neq q$.

Since $H_1 \cap H_2 = (0)$, one has the direct sum of comodules $H_1 \oplus H_2$. Let j_0 be the least integer ≥ 3 such that $(H_1 \oplus H_2) \cap H_{j_0} = (0)$. Hence, one has the direct sum $H_1 \oplus H_2 \oplus H_{j_0}$ and for $j < j_0$, one has $(H_1 \oplus H_2) \cap H_j \neq (0)$, therefore $H_j \subset H_1 \oplus H_2$. Hence , by induction, one obtains $J = \{1, 2, j_0, \ldots j_k = m\} \subset [1, m]$ and the direct sum of comodules $\bigoplus_{j\in J} H_j$ such that for $\ell \notin J$, $H_\ell \subset \bigoplus_{j\in J} H_j$. It follows that $R(\Delta) = \bigoplus_{j\in J} H_j$.

Corollary : *Let (E, Δ) and (E_1, Δ_1) be two simple left H-comodules of finite dimension that are not isomorphic; then $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$,a direct sum of comodules.*

Proof : With previous notations, put $R(\Delta) = \bigoplus_{j\in J} H_j$ and $R(\Delta_1) = \bigoplus_{\ell\in L} H^1_\ell$. Let p_j [*resp.* p^1_ℓ] be the projection of $R(\Delta)$ [*resp.* $R(\Delta_1)$] onto H_j [*resp.* H^1_ℓ] . Suppose that $R(\Delta) \cap R(\Delta_1) \neq (0)$; this finite dimensional comodule must contain at least one simple comodule V. There exists $j \in J$ [*resp.* $\ell \in L$] such that $p_j(V) \neq (0)$ [*resp.* $p^1_\ell(V) \neq (0)$] ; therefore $p_j(V) = H_j$ [*resp.* $p^1_\ell(V) = H^1_\ell$]. Since V is simple, $p_{j|V}$ [*resp.* $p^1_{\ell|V}$] is an isomorphism of V onto H_j [*resp.* H^1_ℓ] . It follows that H_j and H^1_ℓ are isomorphic. Hence E and E_1 are isomorphisc ; a contradiction. Therefore $R(\Delta) \cap R(\Delta_1) = (0)$ and $R(\Delta \oplus \Delta_1) = R(\Delta) \oplus R(\Delta_1)$.

Remark 5 : *Notations and hypothesis as above. If $K$is algebraically closed, then the $H_j, 1 \leq j \leq n$, are pairewise distinct.*

Proof: Indeed, if $H_j = H_q$ for $j \neq q$, then $u = A_j \circ A_q^{-1}$ is an automorphism of the finite dimensional simple comodule H_j. By Schur's lemma , one has $u = \lambda \cdot 1_{H_j}, \lambda \in k,\ \lambda \neq 0$. Hence $A_j = \lambda A_q$ and $a_{jj} = A_j(e_j) = \lambda A_q(e_j) = \lambda a_{jq}$. Therefore $\sigma(a_{jj}) = 1 = \lambda\sigma(a_{jq}) = \lambda\delta_{jq} = 0$; a contradiction. □

Let H' be the Banach space dual of H ; if we set for $a', b' \in H', a' * b' = (a' \otimes b') \circ c$, then H' becomes a complete normed algebra with unit σ. If (E, Δ) is a left Banach comodule, setting for $a' \in H'$, and $x \in E, a' \cdot x = (a' \otimes 1_E) \circ \Delta(x)$, one induces on E a complete normed *right H'-module* structure. Moreover, if H is a pseudo-reflexive Banach space, then any closed right H'-submodule of E is a Banach left H-subcomodule of E and reciprocally (cf. [3]).

Let $(E', \Delta^{\vee})$ be the conjugate of the *finite dimensional* left H-comodule (E, Δ). One has for any $a' \in H', x' \in E'$ and $x \in E, < a' \cdot x', x >=< x', {}^t\eta(a') \cdot x >$. Therefore, if M is a H'-submodule of E, then $M^{\perp} = \{x' \in E' / < x', x >= 0, \ x \in M\}$ is a H'-submodule of E'. Reciprocally, if η is *bijective* and if M' is H'-subcomodule of E', then $M'^{\perp}$ is a H'-submodule of E.

Proposition 9 : *Let H be a complete ultrametric Hopf algebra that is a pseudo-reflexive Banach space such that η is bijective.*

Then, a finite dimensional left H-comodule (E, Δ) is simple if and only if $(E', \Delta^{\vee})$ is simple .

Proof : Indeed, suppose that (E, Δ) is simple ; if M' is a left H-subcomodule of $(E', \Delta^{\vee})$ then $M'^{\perp}$ is a left H-subcomodule of E ; therefore $M'^{\perp} = (0)$ or $M'^{\perp}$ and $M' = E'$ or $M' = (0)$. By the same way, one shows the reciprocal.

II - 5 When H admits a left integral

II - 5 - 1 Again some general facts

Lemma 3 : *Let (E, Δ) be a finite dimensional left H-comodule and let Δ_c be the restriction of c to $R(\Delta) = \rho_{\Delta}(E' \otimes E)$; then $R(\Delta_c) = R(\Delta)$.*

Proof : Let $(e_j)_{1 \leq j \leq n}$ be a base of E. One has $\Delta(e_\ell) = \sum_{j=1}^{n} a_{\ell j} \otimes e_j, \ 1 \leq \ell \leq n$ and $(a_{\ell j})_{1 \leq \ell, j \leq n}$ spans $R(\Delta)$. Since $\sigma_{|R(\Delta)} \in R(\Delta)'$, one has , according to (1) , $a_{\ell j} =$
$= (1_H \otimes \sigma) \circ c(a_{\ell j}) = \rho_{\Delta_c}(\sigma \otimes a_{\ell j}) \in R(\Delta_c)$ and $R(\Delta) \subset R(\Delta_c)$. Reciprocally, if $a' \in R(\Delta)'$ and $a = \sum_{1 \leq \ell, j \leq n} \lambda_{\ell j} a_{\ell j} \in R(\Delta)$, one has $(1_H \otimes a') \circ \Delta_c(a) = \sum_{1 \leq \ell, j \leq n} \sum_{k=1}^{n} \lambda_{\ell j} < a', a_{kj} >$ $a_{\ell k} \in R(\Delta)$ and $R(\Delta_c) \subset R(\Delta)$.

Lemma 4 : *Any finite dimensional left H-subcomodule E of H is contained in the representative subalgebra $\mathcal{R}(H)$ of H .*

Proof : If $(e_j)_{1 \leq j \leq n}$ is a base of $E \subset H$, one has $c(e_j) = \sum_{\ell=1}^{n} a_{\ell j} \otimes e_\ell$. Let c_E be the restriction of c to E, then $R(c_E)$ is spaned by $(a_{j\ell})_{1 \leq j, \ell \leq n}$. Since $e_j = (1_H \otimes \sigma) \circ c(e_j) =$
$= \sum_{\ell=1}^{n} \sigma(e_\ell) a_{j\ell} \in R(c_E)$, one has $E \subset R(c_E) \subset R(H)$.

Note : If K and H are discrete, one deduces from the above result and from Theorem 1 - (ii) - of [3] that $\mathcal{R}(H) = H$.

II - 5 - 2 Under the hypothesis : H admits a left integral

Let Ω be the family of the isomorphic classes of the *simple, finite dimensional* left H-comodules ; Ω is not empty : its contains the class of the left subcomodule $K.e$ of H. If $\omega \in \Omega$ is the class of (E, Δ) , we set $R(\omega) = R(\Delta)$ that is independant of (E, Δ). It is readily seen that $\mathcal{R}_s(H) = \sum_{\omega\in\Omega} R(\omega)$ is a subcoalgebra of $\mathcal{R}(H)$. Moreover $\mathcal{R}_s(H) = \bigoplus_{\omega\in\Omega} R(\omega)$, a direct sum of coalgebras. Indeed for any finite subset $(\omega_1, \ldots, \omega_m)$ of Ω, one has $\sum_{t=1}^{n} R(\omega_t) = \bigoplus_{t=1}^{m} R(\omega_t)$: see Corollary of Propositions 8 and its proof. Furthermore, if η is *bijective*, then $\mathcal{R}_s(H)$ is a sub-Hopf-algebra of $\mathcal{R}(H)$. □

By definition, a *left integral* for the complete Hopf algebra H is an element ν of H' such that $\mu * \nu = <\mu, e> \nu$ for all $\mu \in H'$. The complete Hopf algebra H is called *supple* if H is a pseudo-reflexive Banach space and $\eta \circ \eta = 1_H$. For H, a supple complete Hopf algebra that admits a left integral ν such that $<\nu, e>= 1$, we know that *any simple left Banach H-comodule is finite dimensional* (Theorem 3 - [3]) .

Theorem 2 : *Let H a supple complete Hopf algebra that admits a left integral ν such that $<\nu, e>= 1$. Then*

(i) $R(H) = \bigoplus_{\omega\in\Omega} R(\omega)$ *where Ω is the family of the isomorphic classes of simple Banach left H-comodules.*

(ii) *The Hopf algebra $\mathcal{R}(H)$ is dense in H, that is $H = \overline{\mathcal{R}(H)} = \widehat{\bigoplus_{\omega\in\Omega}} R(\omega)$.*

Proof :

(i) One deduces from [2] - Theorem 3 that any finite dimensional H-comodule (E, Δ) is semi-simple i.e. $(E, \Delta) = \bigoplus_{\tau}\bigoplus_{t}(V_{t,\tau}, \Delta_{t,\tau})$ with $V_{t,\tau} \in \omega_\tau$ and $\omega_\tau \in \Omega$. Hence $R(\Delta) = \sum_{\tau}\sum_{t} R(\Delta_{t,\tau}) = \sum_{\tau} R(\omega_\tau) = \bigoplus_{\tau} R(\omega_\tau) \subset \mathcal{R}_s(H)$. It follows that $\mathcal{R}(H) = \mathcal{R}_s(H) = \bigoplus_{\omega\in\Omega} R(\omega)$.

(ii) The Hopf algebra H is naturally a Banach left H-comodule with coproduct c. Let $a \in H$, $a \neq 0$; since H is pseudo-reflexive, the Banach left subcomodule $E(a) = \overline{H' \cdot a}$ of H contains a and is a non null Banach space of countable type (cf. [3]).

With the hypothesis, we know that $E(a)$ contains simple left H-subcomodules (finite dimensional) (cf. [3]).

Let $(V_\tau)_{\tau\in T}$ be the family of all simple subcomodules of $E(a)$. Put $W = \sum_{\tau\in T} V_\tau$, there exists $S \subset T$ such that $W = \bigoplus_{\tau\in S} V_\tau$; one has $c(W) \subset H \otimes W$. Since c is a homeomorphism of H onto $c(H)$, setting $E_0 = \overline{W}$, one has $c(E_0) \subset H\widehat{\otimes}E_0$, i.e. E_0 is a Banach left subcomodule of $E(a)$. In fact $E_0 = E(a)$. Otherwise, one has a direct sum of Banach comodules $E(a) = E_0 \oplus E_1$ with $E_1 \neq (0)$ (cf. [2]). However E_1 must contain at least one simple comodule V and by definition of W, one has $V \subset W$. Hence $E_0 \cap E_1 \neq (0)$; a contradiction.

Let ω_τ be the isomorphic class of the simple comodule $V_\tau, \tau \in T$. By Lemma 4, $V_\tau \subset R(\omega_\tau), \tau \in T$. Hence, we have $W = \sum_{\tau\in T} V_\tau \subset \sum_{\tau\in T} R(\omega_\tau) \subset \bigoplus_{\omega\in\Omega} R(\omega) = \mathcal{R}(H)$. It follows that $a \in E(a) = E_0 = \overline{W} \subset \overline{\mathcal{R}(H)}$. We have proved that $H = \overline{\mathcal{R}(H)} = \widehat{\bigoplus_{\omega\in\Omega}} R(\omega)$.

Note : The above results are abstract version of some results of representation theory of groups. In particular Theorem 2 is Peter-Weyl Theorem (cf. [3]).

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