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Radonification of Cylindrical Semimartingales on Hilbert Spaces

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Abstract

In this work we prove that a cylindrical semimartingale on a Hilbert space becomes a semimartingale with values in the aforementioned Hilbert space if we compose it with three Hilbert-Schmidt operators.

Résumé

Dans ce travail nous prouvons qu'une semimartingale cylindrique sur un espace de Hilbert devient une semimartingale à valeurs dans cet espace hilbertien si on la compose avec trois opérateurs Hilbert-Schmidt.

1 Introduction

This paper represents a portion of a joint research project which is uncompleted due to the untimely death of the first author. The result (cf. Theorem 3.2) derived here was first shown in the thesis of the second author in the framework of nuclear spaces and was extended to Banach spaces by Laurent Schwartz (cf. [4]). In our project we had chosen, for the pedagogical reasons, to present it in the simple setting of Hilbert spaces, where everything can be done by straightforward calculations.

2 Notations and Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a probability space equipped with a right continuous, increasing filtration of sub-sigma algebras of $\mathcal{F}$, denoted by $\{\mathcal{F}_t;\ t \in [0,1]\}$. We suppose that $\mathcal{F}_0$ contains all the P-negligeable sets and smallest sigma algebra containing all the elements of this filtration is $\mathcal{F}$.

We shall denote by S^0 the space of real-valued semimartingales defined relative to the filtration mentioned above, we suppose that they are equipped with the metric topology defined by

$$q(X,0) = \sup_{H\in\Phi} E\left[1 \wedge \left|\int_0^1 H_s dX_s\right|\right]$$

where Φ denotes the set of predictable, simple processes. Recall that under this topology S^0 is a (non-locally convex) Fréchet space (cf. [2], [3]). Let H a separable Hilbert space any $\tilde{X}$ continuous, linear mapping from H into S^0 will be called a cylindrical semimartingale. We will say that $\tilde{X}$ is radonified to an H-valued semimartingale with a linear operator u on H if there exists an H-valued semimartingale, say K such that $\tilde{X}(u(h)) = (K, h)$ for any $h \in H$, where the equality is to be understood with respect to the equivalence classes of S^0.

3 Radonification results

We begin with some technical lemmas:

Lemma 3.1 *Suppose that $\tilde{X} : H \to R^0$ is a linear, continuous mapping, where R^0 denotes the space of cadlag processes under the topology of uniform convergence in probability. Then, for any $\epsilon > 0$, there exists some $\delta > 0$ such that*

$$E\left[\sup_{t\in[0,1]} |1 - e^{i\tilde{X}(\phi)_t}|\right] \leq \epsilon + \frac{2}{\delta^2}\|\phi\|^2,$$

for any $\phi \in H$.

Proof: Let $\delta_1 > 0$ be chosen such that $|1-\exp i\alpha| \leq \epsilon/2$ if $|\alpha| \leq \delta_1$, choose δ_2 as $\inf(\delta_1, \frac{1}{2}(-1+\sqrt{1+\epsilon})$. Let

$$d(\tilde{X}(\phi), 0) = E\left[\frac{\sup_t |\tilde{X}(\phi)_t|}{1 + \sup_t |\tilde{X}(\phi)_t|}\right].$$

Since $\phi \mapsto d(0, \tilde{X}(\phi))$ is continuous, there exists some $\delta > 0$ such that $\|\phi\| < \delta$ implies $d(\tilde{X}(\phi), 0) < \delta_2^2$. From the Chebytchev inequality, we have

$$\begin{aligned} P\{\sup_t |\tilde{X}(\phi)_t| \geq \delta_2\} &\leq \frac{1+\delta_2}{\delta_2} d(\tilde{X}(\phi), 0) \\ &\leq \delta_2(1+\delta_2) \\ &\leq \frac{\epsilon}{4}, \end{aligned}$$

for any $\|\phi\| \leq \delta$. Consequently, for any $\|\phi\| \leq \delta$ we have

$$\begin{aligned} E\left[\sup_t |1 - e^{i\tilde{X}(\phi)_t}|\right] &= E\left[\sup_t |1 - e^{i\tilde{X}(\phi)_t}|; \sup_t |\tilde{X}(\phi)_t < \delta_2\right] \\ &\quad + E\left[\sup_t |1 - e^{i\tilde{X}(\phi)_t}|; \sup_t |\tilde{X}(\phi)_t \geq \delta_2\right] \\ &\leq \frac{\epsilon}{2} + 2\cdot\frac{\epsilon}{4} = \epsilon. \end{aligned}$$

For $\|\phi\| \geq \delta$, since we have $\sup_t |1 - \exp i\tilde{X}(\phi)_t| \leq 2$, it follows that

$$d(\tilde{X}(\phi), 0) \leq 2\frac{\|\phi\|^2}{\delta^2}.$$

□

Lemma 3.2 *Let $u : H \to H$ be a Hilbert-Schmidt operator. Then we have*

$$P\left\{\sup_t \sum_{j=1}^{\infty} (\tilde{X} \circ u)^2(e_j)_t > c^2\right\} \leq \frac{\sqrt{e}}{\sqrt{e}-1}(\epsilon + \frac{2\|u\|_2^2}{c^2\delta^2}),$$

where (e_j) is a complete, orthonormal basis in H, $\epsilon > 0$ is arbitrary and δ is chosen as in the Lemma 3.1 and $\|u\|_2$ denotes the Hilbert-Schmidt norm of u.

Proof: We have, using the last lemma

$$\begin{aligned}
P\left\{\sup_t \sum_{j=1}^{\infty} (\tilde{X} \circ u)^2(e_j)_t > c^2\right\} &= \lim_n P\left\{\sup_t \sum_{j=1}^{n} (\tilde{X} \circ u)^2(e_j)_t > c^2\right\} \\
&\leq \lim_n \frac{\sqrt{e}}{\sqrt{e}-1} E\left[1 - \exp - \sup_t \sum_1^n \frac{(\tilde{X} \circ u)(e_j)_t^2}{2c^2}\right] \\
&= \lim_n \frac{\sqrt{e}}{\sqrt{e}-1} E\left[\sup_t \left(1 - \exp - \sum_1^n \frac{(\tilde{X} \circ u)(e_j)_t^2}{2c^2}\right)\right] \\
&= \lim_n \frac{\sqrt{e}}{\sqrt{e}-1} E\left[\sup_t \left\{\int_{\mathbb{R}^n} (1 - \exp i \sum_1^n (\tilde{X} \circ u)(e_j)_t y_j) \mu_{n,c}(dy)\right\}\right] \\
&\leq \lim_n \frac{\sqrt{e}}{\sqrt{e}-1} \int_{\mathbb{R}^n} E\left[\sup_t |1 - \exp i \sum_1^n y_j (\tilde{X} \circ u)(e_j)_t|\right] \mu_{n,c}(dy) \\
&\leq \lim_n \frac{\sqrt{e}}{\sqrt{e}-1} \int_{\mathbb{R}^n} \left(\epsilon + \frac{2}{\delta^2}\|\sum_1^n y_j u(e_j)\|^2\right) \mu_{n,c}(dy) \\
&= \frac{\sqrt{e}}{\sqrt{e}-1}\left(\epsilon + \frac{2}{\delta^2}\frac{1}{c^2}\sum_{j=1}^{\infty} \|u(e_j)\|^2\right) \\
&= \frac{\sqrt{e}}{\sqrt{e}-1}\left(\epsilon + \frac{2}{\delta^2}\frac{1}{c^2}\|u\|_2^2\right),
\end{aligned}$$

where $\mu_{n,c}$ denotes the Gauss measure on $\mathbb{R}^n$ whose density is given by $(2\pi)^{-n/2}c^n \exp -\frac{1}{2}c^2|y|^2$ and for the first inequality cf. [1].

□

Corollary 3.1 *With the notations of lemma 3.2, $\tilde{X} \circ u$ defines an H-valued cadlag process Y.*

Proof: Since $\epsilon > 0$ is arbitrary, we see that

$$P\left\{\sup_t \sum_{j=1}^{\infty}(\tilde{X}(u(e_j))_t^2 < +\infty\right\} = 1,$$

hence the sum

$$\sum_{j=1}^{\infty}(\tilde{X}(u(e_j))_t e_j$$

converges in H almost surely uniformly with respect to $t \in [0,1]$. □

Let us denote by Y^2 the H-valued process defined by

$$Y_t^2 = \sum_{s\le t} \Delta Y_s 1_{\{\|\Delta Y_s\|>1\}},$$

with the convention $Y_{0-} = 0$. Since Y is a cadlag process, Y^2 is of finite variation (it has finite number of jumps on each bounded interval). Let us denote by Y^1 the process

$$Y_t^1 = Y_t - Y_t^2.$$

Then Y^1 has uniformly bounded jumps.

Theorem 3.1 *Suppose that $\tilde{X} : H \to S^0$ is a linear, continuous mapping. Then the process Y^1 constructed above defines a continuous, linear mapping from H into the space of special semimartingales. Furthermore, if we denote by $M(\phi)$ and $A(\phi)$ respectively, the local martingale and predictable, finite variation parts of (Y^1, ϕ), $\phi \in H$, then $\phi \mapsto M(\phi)$ and $\phi \mapsto A(\phi)$ are linear, continuous maps from H into S^0.*

Proof: We have, for any ϕ, $\psi \in H$,

$$\begin{aligned}
(Y^1, \phi+\psi) &= (Y^1,\phi) + (Y^1,\psi) \\
&= M(\phi) + A(\phi) + M(\psi) + A(\psi) \\
&= M(\phi+\psi) + A(\phi+\psi),
\end{aligned}$$

hence

$$A(\phi+\psi) - A(\phi) - A(\psi) = M(\phi) + M(\psi) - M(\phi+\psi).$$

Since $A(\xi)$ is predictable for any $\xi \in H$, the left hand side of the last equality is a predictable process of finite variation and it is also a local martingale, hence both sides are constant, since all the processes are zero at $t = 0$ (this follows from the construction of Y^1 and the definition of special semimartingales), this constant is equal to zero, and this proves the linearity of the maps $\phi \mapsto A(\phi)$ and $\phi \mapsto M(\phi)$.

Let us now show the continuity of these mappings: let $\phi \in H$, by definition of a local martingale, there exists a sequence of stopping times $(S_k : k \in \mathbb{N})$ increasing to one such that, for any $k \in \mathbb{N}$, $\{M(\phi)_{t\wedge S_k}; t \in [0,1]\}$ is a uniformly integrable martingale and $\{A(\phi)_{t\wedge S_k}; t \in [0,1]\}$ is of integrable, total variation. Since $A(\phi)$ is predictable we have

$$\begin{aligned} E\left[(\Delta Y^1_{t\wedge S_k}, \phi)|\mathcal{F}_{t-}\right] &= E\left[\Delta A(\phi)_{t\wedge S_k} + \Delta M(\phi)_{t\wedge S_k}|\mathcal{F}_{t-}\right] \\ &= \Delta A(\phi)_{t\wedge S_k} . \end{aligned}$$

By construction $\|\Delta Y^1_t\| \le 1$ for any $t \ge 0$, hence

$$|\Delta A(\phi)_t| \le \|\phi\|$$

almost surely and

$$|\Delta M(\phi)_t| \le 2\|\phi\|$$

almost surely. Suppose now that $\phi_k \to 0$ in H and $(M(\phi_k); k \in \mathbb{N})$ converges to some m in S^0. $(A(\phi_k); k \in \mathbb{N})$ converges then to some $a \in S^0$. From the above majorations, we know that the processes $(A(\phi_k); k \in \mathbb{N})$ and $(M(\phi_k); k \in \mathbb{N})$ have uniformly bounded jumps. It is well-known that (cf. [2] and [3]) the set of local martingales and the predictable processes of finite variation having uniformly bounded jumps are closed in S^0. Therefore m is a local martingale and a is a process of finite variation, moreover, we have

$$m + a = 0\,,$$

consequently $m = a = 0$, and the closed graph theorem implies the continuity of M and A.
□

From the Corollary 3.1, if $u : H \to H$ is a Hilbert-Schmidt operator, then $M \circ u_1$ and $A \circ u_1$ are two cadlag processes, in fact since $M \circ u_1$ has uniformly bounded jumps, it is an H-valued local martingale. In fact we have

Corollary 3.2 *Suppose that $\tilde{M}$ is a cylindrical local martingale with uniformly bounded jumps. Then, for any Hilbert-Schmidt operator u on H, $\tilde{M} \circ u$ is an H-valued local martingale with bounded jumps.*

We proceed with the following result which is of independent interest:

Lemma 3.3 *Let us denote by V^0 the space of real-valued cadlag processes of almost surely finite variation endowed with the metric topology defined by*

$$d'(A,0) = E\left[\frac{\int_0^1 |dA_s|}{1 + \int_0^1 |dA_s|}\right] .$$

Suppose that $B : H \to V^0$ is a linear, continuous map and v is a Hilbert-Schmidt operator on H. Then the set

$$\left\{\omega \in \Omega : \int_0^1 |dB(v(\phi))_s| = +\infty \text{ for some } \phi \in H\right\}$$

is a negligeable set.

Proof: Let Δ be a finite partition of $[0,1]$ of order $n \in \mathbb{N}$. Let us denote by B^Δ the linear, continuous map from H into $L^0(l^1_n)$, where $l^1_n = \mathbb{R}^n$ with l^1-topology and $L^0(l^1_n)$ denotes the space of the equivalence classes of l^1_n-valued random variables under the topology of convergence in probability.

We have a similar result to that of Lemma 3.1: for any $\epsilon > 0$, there exists a $\delta > 0$ such that

$$\sup_\Delta E\left[\sup_{|\xi|_{l^1_n}\le 1} |1-\exp i(B^\Delta(\phi)|\xi)|\right] \le \epsilon + \frac{2}{\delta^2}\|\phi\|^2\,, \tag{3.1}$$

for any $\phi \in H$, where the sup with respect Δ is taken on a sequence of increasing partitions of $[0,1]$ in such a way that it can be obtained as a monotone limit and $(\cdot|\cdot)$ denotes the scalar product in $\mathbb{R}^n$. To see 3.1, we proceed as in the proof of Lemma 3.1, namely, suppose that $\epsilon > 0$ is given. Choose $\delta_1 > 0$ such that $|1-\exp i\alpha| < \epsilon/2$ if $|\alpha| < \delta_1$. Let also $\delta_2 = \inf(\delta_1, \frac{1}{2}(\sqrt{1+\epsilon}-1))$. Since the map

$$\phi \mapsto d'(B(\phi),0)$$

is continuous, there is some $\delta > 0$ such that for any $\|\phi\| < \delta$, we have $d'(B(\phi),0) < \delta_2^2$. From the Chebytchev inequality,

$$P\left\{\int_0^1 |dB(\phi)_s| > \delta_2\right\} \le \frac{1+\delta_2}{\delta_2} d'(B(\phi),0) \le \frac{\epsilon}{4}\,.$$

Now, denoting by $|B(\phi)|_1$ the total variation of $B(\phi)$, we have

$$\begin{aligned}
E\left[\sup_{|\xi|\le 1}|1-e^{i(B^\Delta(\phi)|\xi)}|\right] &= E\left[\sup_{|\xi|\le 1}|1-e^{i(B^\Delta(\phi)|\xi)}|; |B(\phi)|_1 < \delta_2\right] &(3.2)\\
&\quad +E\left[\sup_{|\xi|\le 1}|1-e^{i(B^\Delta(\phi)|\xi)}|; |B(\phi)|_1 \ge \delta_2\right] &(3.3)\\
&\le \frac{\epsilon}{2}+\frac{\epsilon}{2} = \epsilon\,, &(3.4)
\end{aligned}$$

for any ϕ with $\|\phi\| < \delta$, since $|(B^\Delta(\phi)|\xi)| \le |B(\phi)|_1$ for any $|\xi|_{l^1_n} \le 1$. If $\|\phi\| > \delta$, we have trivially

$$E\left[\sup_{|\xi|\le 1}|1-e^{i(B^\Delta(\phi)|\xi)}|\right] \le 2\frac{\|\phi\|^2}{\delta^2}\,, \tag{3.5}$$

hence, combining the inequalities 3.2 and 3.5 and noting that all of them are controlled with $|B(\phi)|_1$, we can pass to the limit on the partitions and we obtain the inequality 3.1.

As in the Lemma 3.2, we have

$$\begin{aligned}
P\left\{\|B^\Delta \circ v\|_{H\otimes l^1_n} > c\right\} &\le \frac{\sqrt{e}}{\sqrt{e}-1} E\left[\sup_{\|\phi\|\le 1}\sup_{|\xi|_{l^1_n}\le 1}\left(1-\exp -\frac{1}{2c^2}|(B^\Delta\circ v(\phi)|\xi)|^2\right)\right]\\
&\le 6E\left[\sup_{|\xi|_{l^1_n}\le 1}\left(1-\exp -\frac{1}{2c^2}\sum_{j=1}^\infty (B^\Delta\circ v(e_j)|\xi)^2\right)\right],
\end{aligned}$$

where $(e_j; j \in \mathbb{N})$ is a complete, orthonormal basis of H. Using the Gaussian trick as in the proof of Lemma 3.2, the last term above can be majorated by

$$I_\Delta = \lim_{m\to\infty} \frac{6c^m}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} E\left[\sup_{|\xi|_{l_n^1}\le 1} \left|1 - \exp i\sum_{j=1}^m (B^\Delta \circ v(e_j)|\xi) y_j\right|\right] \exp -\frac{c^2|y|^2}{2} dy.$$

From the inequality 3.1, we have

$$\sup_\Delta E\left[\sup_{|\xi|_{l_n^1}\le 1} \left|1 - \exp i(B^\Delta(\sum_{j=1}^m v(e_j y_j))|\xi)\right|\right] \le \epsilon + \frac{2}{\delta^2}\|v(\sum_1^m e_j y_j)\|^2,$$

hence

$$I_\Delta \le 6\epsilon + \frac{12}{c^2\delta^2}\|v\|_2^2,$$

uniformly with respect to Δ. Therefore, we have

$$\begin{aligned}
\sup_\Delta P\left\{\|B^\Delta \circ v\|_{H\otimes l_n^1} > c\right\} &= P\left\{\sup_\Delta \sup_{\|\phi\|\le 1} |B^\Delta \circ v(\phi)|_{l_n^1} > c\right\} \\
&= P\left\{\sup_{\|\phi\|\le 1} \sup_\Delta |B^\Delta \circ v(\phi)|_{l_n^1} > c\right\} \\
&= P\left\{\sup_{\|\phi\|\le 1} |B(\phi)|_1 > c\right\} \\
&\le 6\epsilon + \frac{12}{c^2\delta^2}\|v\|_2^2,
\end{aligned}$$

where $\sup_\Delta$ means, as explained before, that we take the supremum with respect to an increasing sequence of partitions. Since ϵ is arbitrary, it follows that

$$P\left\{\sup_{\|\phi\|\le 1} \int_0^1 |dB(v(\phi))_s| < +\infty\right\} = 1.$$

□

We can now announce the main result:

Theorem 3.2 *Suppose that $\tilde{X} : H \to S^0$ is a linear, continuous mapping and that u_i, $i = 1,2,3$ are three Hilbert-Schmidt operators on H. Then there exists an H-valued semimartingale Z such that*

$$(Z, \phi) = \tilde{X} \circ u_1 \circ u_2 \circ u_3(\phi),$$

almost surely, for any $\phi \in H$.

Proof: From the Corollary 3.1, $\tilde{X} \circ u_1$ defines an H-valued, cadlag process Y that we write as $Y = Y^1 + Y^2$, where Y^2 is defined by

$$Y_t^2 = \sum_{s\le t} \Delta Y_s 1_{\{\|\Delta Y_s\|>1\}}.$$

Using the Theorem 3.1, we can write Y^1 as

$$(Y^1, \phi) = \tilde{M}(\phi) + \tilde{A}(\phi),$$

where $\tilde{M}$ is a cylindrical local martingale and $\tilde{A}$ is a cylindrical predictable process of finite variation and both have uniformly bounded jumps. As we have seen in the Corollary 3.2, $\tilde{M} \circ u_2$ gives an H-valued local martingale. Lemma 3.3 implies that almost surely the process $\tilde{A} \circ u_2$ is of finite weak variation. Let us denote by $V[0,1]$ the space of finite measures on $[0,1]$ endowed with the total variation norm. Then from the Lemma 3.3, we see that there exists a negligeable subset N of Ω such that for any $\omega \in N^c$ the map $\phi \mapsto \tilde{A} \circ u_2(\omega, \phi)$ is a linear, continuous mapping from H into $V[0,1]$. Therefore, there exists some $\rho(\omega) \geq 0$ such that

$$\left|\tilde{A} \circ u_2(\omega, \phi)\right|_1 \leq \rho(\omega)\|\phi\|,$$

for any $\phi \in H$ and $\omega \in N^c$. Let us denote the process $\tilde{A} \circ u_2$ by L. Using an auxiliary Gaussian measure on H with covariance operator $u_3 \circ u_3^\star$, it is easy to see that

$$\begin{aligned}\sum_\Delta \|L_{t_{i+1}}(\omega) \circ u_3 - L_{t_i}(\omega) \circ u_3\| &\leq \sup_{\|\phi\| \leq 1} \sum_\Delta |L_{t_{i+1}}(\omega)(\phi) - L_{t_i}(\omega)(\phi)| \\ &\leq \|u_3\|_2 \rho(\omega),\end{aligned}$$

since the above majoration is uniform in Δ, we have obviously

$$\sup_\Delta \sum_\Delta \|L_{t_{i+1}} \circ u_3 - L_{t_i} \circ u_3\| < +\infty,$$

almost surely and this completes the proof of the theorem.

□

Remark Let us note that, if $\tilde{X}$ is a cylindrical local martingale, its radonification $\tilde{X} \circ u_1 \circ u_2 \circ u_3$ is not necessarily an H-valued local martingale but it is a semimartingale, this phenomena is due to the "unbounded jumps" of $\tilde{X}$.

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