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ON THE TOPOLOGY OF COMPACTOID CONVERGENCE IN NON-ARCHIMEDEAN SPACES

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Abstract

Some of the properties, of the topology of uniform convergence on the compactoid subsets of a non-Archimedean locally convex space E, are studied. In case E is metrizable, the compactoid convergence topology coincides with the finest locally convex topology which agrees with $\sigma(E', E)$ on equicontinuous sets.

[1]

1 Introduction

In [7] some of the properties of the topology of uniform convergence on the compactoid subsets, of a non-Archimedean locally convex space, are investigated. In the same paper, the authors defined the ϵ-product $E\epsilon F$ of two non-Archimedean locally convex spaces E and F. $E\epsilon F$ is the space of all continuous linear operators of E'_{co} to F equipped with the topology of uniform convergence on the equicontinuous subsets of E', where E'_{co} is the dual space E' of E endowed with the topology of uniform convergence on the compactoid subsets of E. In this paper, we continue with the investigation of the compactoid convergence topology τ_{co}. Among other things, we show that, for metrizable E, τ_{co} coincides with the topology τ_σ, where τ_σ is the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on equicontinuous

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sets. We also prove that τ_{co} has a base at zero all sets $\overline{W}^{\sigma(E',E)}$, where W is a τ_σ-neighborhood of zero and $\overline{W}^{\sigma(E',E)}$ denotes the $\sigma(E',E)$-closure of W. If $T : E \mapsto F$ is a nuclear (resp. compactoid) operator, then $T' : F'_{co} \mapsto E'_{co}$ is nuclear (resp. compactoid). Also, if $T_i : E_i \mapsto F_i$, $i = 1, 2$, are nuclear, then

$$T = T_1 \epsilon T_2 : E_1 \epsilon E_2 \mapsto F_1 \epsilon F_2, \quad Tu = T_2 u T_1',$$

is nuclear. Finally we show that τ_{co} is compatible with the dual pair $< E', E >$ iff every closed compactoid subset of E is complete.

2 Preliminaries

Throughout this paper, $\mathbf{K}$ will stand for a complete non-Archimedean valued field, whose valuation is non-trivial, and $\mathbf{N}$ for the set of natural numbers. By a seminorm, on a vector space E over $\mathbf{K}$, we will mean a non-Archimedean seminorm.

Let now E be a locally convex space over $\mathbf{K}$. The collection of all continuous seminorms on E will be denoted by $cs(E)$. The algebraic dual, the topological dual, and the completion of E will be denoted by E^*, E' and $\widehat{E}$ respectively. A seminorm p on E is called polar if

$$p = \sup\{|f| : f \in E^*, |f| \leq p\},$$

where $|f|$ is defined by $|f|(x) = |f(x)|$. The space E is called polar if its topology is generated by a collection of polar seminorms. The edged hull A^e, of an absolutely convex subset A of E, is defined by:
$A^e = A$ if the valuation of $\mathbf{K}$ is discrete and $A^e = \cap\{\lambda A : |\lambda| > 1\}$ if the valuation is dense (see [10]). For a subset S of E, we denote by $co(S)$ the absolutely convex hull of S. A subset B of E is called compactoid if, for each neighborhood V of zero in E, there exists a finite subset S of E such that

$$B \subseteq co(S) + V.$$

The space E is said to be of countable type if, for each $p \in cs(E)$, there exists a countable subset S of E, such that the subspace $[S]$ spanned by S is p-dense in E.

A linear map $T : E \mapsto F$ is called:
1) compactoid if there exists a neighborhood V of zero in E such that $T(V)$ is a compactoid subset of F.
2) compactifying if $T(B)$ is compactoid in F for each bounded subset of E.

3) nuclear if there exist a null sequence (λ_n) in $\mathbf{K}$, a bounded sequence (y_n) in F and an equicontinuous sequence (f_n) in E' such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$$

for all $x \in E$.

We will denote by E'_{co} the dual space E' of E equipped with the topology of uniform convergence on the compactoid subsets of E. The ϵ-product $E\epsilon F$, of two locally convex spaces E, F is the space of all continuous linear maps from E'_{co} to F endowed with the topology of uniform convergence on the equicontinuous subsets of E'. For other notions, concerning non-Archimedean locally convex spaces and for related results, we will refer to [10].

We will need the following

Lemma 2.1 *([7, Lemma 2.6]). Let E, F be Hausdorff polar quasi-complete spaces and let $T : E' \mapsto F$ be a linear map. If T is continuous with respect to the weak topologies $\sigma(E', E)$ and $\sigma(F, F')$, then $T \in E\epsilon F$ iff T maps equicontinuous subsets of E' into compactoid subsets of F.*

3 The topology τ_σ

Let E be a Hausdorff polar space. We will denote by τ_σ the finest locally convex topology on E' which agrees with $\sigma(E', E)$ on equicontinuous sets. It is easy to see that τ_σ is the locally convex topology which has as a base at zero all absolutely convex subsets W of E' with the folowing property: For every equicontinous subset H of E' there exists a finite subset S of E such that $S^0 \cap H \subseteq W$, where S^0 is the polar of S in E'. In case E is a normed space, τ_σ coincides with the bounded weak star topology bw' (see [12] or [13]).

Since a linear functional f on E' is τ_σ-continuous iff its restriction to every equicontinuous subset of E' is $\sigma(E', E)$-continuous we have the following

Proposition 3.1 *If E is a Hausdorff polar space, then $(E', \tau_\sigma)' = \widehat{E}$.*

Proof. See the proof of Theorem 2 in [5].

The following Proposition for normed spaces was proved by Schikhof in [12, Proposition 3.2].

Proposition 3.2 *If E is a metrizable polar space, then (E', τ_σ) is of countable type.*

Proof. Let (V_n) be a decreasing sequence of convex neighborhoods of zero in E which is a base for the neighborhoods of zero. Then

$$E' = \bigcup_{n=1}^{\infty} V_n^0.$$

Let now q be a τ_σ-continuous seminorm on E' and set

$$W_m = \{x' \in E' : q(x') \leq 1/m\}.$$

Each V_n^0 is a $\sigma(E', E)$-compactoid and hence a τ_σ- compactoid since V_n^0 is absolutely convex and $\tau_\sigma = \sigma(E', E)$ on V_n^0. Thus, for each $m \in \mathbf{N}$, there exists a finite S_{nm} of E' such that

$$V_n^0 \subseteq co(S_{nm}) + W_m.$$

Now, the set $S = \bigcup_{m,n} S_{nm}$ is countable and the space $[S]$ is q-dense in E'. This completes the proof.

Let now E be a Hausdorff polar space and let $j_E : E \mapsto E''$ the canonical map. In the following Theorem, we will consider E as a vector subspace of E'' identifying E with its image under the canonical map. For a subset A of E'' we will denote by A^0 and A^{00}, respectively, the polar and the bipolar of A with respect to the pair $< E'', E' >$. If we consider on E'' the topology of uniform convergence on the equicontinuous subsets of E', then E will be a topological subspace of E''. In this case E'' will have as a base at zero all sets V^{00} where V is a convex neighborhood of zero in E.

The proof of the next Proposition is an adaptation of the corresponding proof for normed spaces given by Schikhof in [12, Proposition 3.3].

Proposition 3.3 *Let E be a Hausdorff polar space and consider on E'' the topology of uniform convergence on the equicontinuous subsets of E'. If F is the dual space of (E', τ_σ) then $F \cap E''$ coincides with the closure of E in E''. Thus, if $F \subseteq E''$ (e.g if τ_σ is coarser than the topology of the strong dual of E), then $F = \overline{E}$.*

Proof. Let $x'' \in \overline{E}$ and consider the set

$$W = \{x' \in E' : | < x', x'' > | \leq 1\}.$$

For each convex neighborhood V of zero in E, there exists $x_\nu \in E$ such that $x'' - x_\nu \in V^{00}$. Indexing the convex neighborhoods of zero in E by inverse inclusion, we get a net (x_ν) in E. Let now V_0 be a convex neighborhood of zero in E and let $\mu \in \mathbf{K}$, $\mu \neq 0$. If $V \subseteq \mu V_0$, then $x'' - x_\nu \in \mu V^{00}$ and so $| < x'' - x_\nu, x' > | \leq |\mu|$ for all $x' \in V^0$.

$$< x_\nu, x' > \rightarrow < x'', x' >$$

uniformly on V_0^0. Since each of the functions $x' \mapsto < x_\nu, x' >$ is $\sigma(E', E)$ -continuous on V_0^0, it follows that the restriction of x'' to V_0^0 is $\sigma(E', E)$-continuous. This clearly proves that x'' is τ_σ- continuous.
On the other hand, let $x'' \in F \cap E''$ and let V be a convex neighborhood of zero in E. Let $|\lambda| > 1$ and set

$$D = \{x' \in E' : | < x', x'' > | \leq 1\}.$$

There exists a finite subset S of E such that

$$S^0 \cap V^0 \subseteq \lambda^{-1} D.$$

The set $A = co(S)$ is a complete metrizable compactoid in $(E'', \sigma(E'', E'))$. Since V^{00} is absolutely convex and $\sigma(E'', E')$-closed, it follows that $(A + V^{00})^e$ is $\sigma(E'', E')$-closed by [11, Theorem 1.4]. Since

$$S^0 \cap V^0 = (A + V)^0,$$

we get that

$$\lambda D^0 \subseteq (A + V)^{00} = \left(A + V^{00}\right)^{00} = \left(\overline{A + V^{00}}^{\sigma(E'', E')}\right)^e = \left(A + V^{00}\right)^e$$

and so $D^0 \subseteq A + V^{00} \subseteq E + V^{00}$. Since $x'' \in D^0$, it follows that $x'' \in \overline{E}$, which completes the proof.

As we will see in the next section, if E is metrizable, then τ_σ is coarser than the strong topology on E' and so in this case $(E', \tau_\sigma)' = \overline{E}$, a result proved by Schikhof in [12] for normed spaces.

4 The Topology of Compactoid Convergence

For a locally convex space (E, τ), we will denote by τ_{co} the topology of compactoid convergence, i.e the topology on E' of uniform convergence on the compactoid subsets of E. We will denote (E', τ_{co}) by E'_{co}. By [7, 3.3], every equicontinuous subset of E' is τ_{co}-compactoid.

Proposition 4.1 *([10, Lemma 10.6]) If E is a Hausdorff polar space, then $\tau_{co} = \sigma(E', E)$ on equicontinuous subsets of E'.*

Proposition 4.2 *If every compactoid subset of E is metrizable, then τ_{co} is the topology of uniform convergence on the null sequences in E.*

Proof. It follows from [10, Proposition 8.2], since for a metrizable compactoid A, there exists a null sequence (x_n) such that $A \subseteq \overline{co}(X)$ where $X = \{x_n : n \in \mathbf{N}\}$.

Corollary 4.3 $\sigma(E', E) \leq \tau_{co} \leq \tau_\sigma$.

Example If $E = c_0$ with the usual norm topology, then $E' = l_\infty$ and τ_{co} is the topology generated by the seminorms p_z, $z = (z_n) \in c_0$ where $p_z(x) = \max_\kappa |z_\kappa x_\kappa|$ for $x = (x_n) \in l_\infty$. This follows from the fact that a subset A of c_0 is compactoid iff

$$A \subseteq \hat{z} = \{x \in c_0 : |x_n| \leq |z_n| \ \forall n\}$$

for some $z \in c_0$.

Notation For a locally convex topology γ on E', we will denote by $\overline{\gamma}^\sigma$ the locally convex topology on E' which has as a base at zero all sets of the form $\overline{W}^{\sigma(E',E)}$, where W is a γ-neighborhood of zero.

Theorem 4.4 *If (E, τ) is a Hausdorff polar space, then $\tau_{co} = \overline{\tau_\sigma}^\sigma$.*

Proof. Since $\tau_{co} \leq \tau_\sigma$, we have that

$$\tau_{co} = \overline{\tau_{co}}^\sigma \leq \overline{\tau_\sigma}^\sigma.$$

On the other hand, let W be a convex τ_σ-neighborhood of zero. If V is a polar neighborhood of zero in E and $|\lambda| > 1$, then there exists a finite subset S of E such that $S^0 \cap V^0 \subseteq \lambda^{-1} W$. Since $S^0 \cap V^0 = (co(S) + V)^0$, it follows that

$$\lambda W^0 \subseteq (co(S) + V)^{00} = (co(S) + V)^e \subseteq \lambda\,(co(S) + V)$$

(by [10, Corollary 5.8]). Thus

$$W^0 \subseteq co(S) + V,$$

which shows that W^0 is a compactoid subset of E. Thus W^{00} is a τ_{co}-neighborhood of zero. Since

$$W^{00} = \left(\overline{W}^{\sigma(E',E)}\right)^e \subseteq \lambda \overline{W}^{\sigma(E',E)},$$

and so $\overline{W}^{\sigma(E',E)}$ is a τ_{co}-neighborhood of zero. This completes the proof.

The following is a Banach-Dieudonné type Theorem for non-Archimedean spaces (see [3, Theorem 10.1]).

Theorem 4.5 *If (E,τ) is metrizable polar space, then $\tau_{co} = \tau_\sigma$.*

Proof. Let (V_n) be a decreasing sequence of convex neighborhoods of zero in E which is a base at zero and let D be a convex τ_σ-neighborhood of zero in E'. Since τ_σ is the finest locally convex topology on E' which agrees with $\sigma(E',E)$ on the sets V_n^0, $n \in \mathbf{N}$, we may assume that there exists (by [4, Theorem 5.2]) a sequence $(S_n)_{n=0}^{\infty}$ of finite subsets of E such that for $W_n = S_n^0$ we have

$$D = W_0 \cap \left(\bigcap_{n=1}^{\infty} \left(W_n + V_n^0 \right) \right).$$

Since each $W_n + V_n^0$ is $\sigma(E',E)$-closed and since W_0 is also $\sigma(E',E)$-closed, it follows that $D = \overline{D}^{\sigma(E',E)}$. Now since $\tau_{co} = \overline{\tau_\sigma}^{\sigma}$ it follows that $\tau_\sigma \leq \tau_{co}$. This clearly completes the proof.

Corollary 4.6 *Let E be a Hausdorff polar space and consider on E'' the topology of uniform convergence on the equicontinuous subsets of E'. Then:*
a) τ_σ is polar and coarser than the strong topology on E'.
b) $(E',\tau_\sigma)' = \hat{E} = \overline{E}$, where $\overline{E}$ is the closure of E in E''.

Open Problems .
1) Is τ_σ always a polar topology ?
2) Is it always true that $\tau_\sigma = \tau_{co}$?
3) Is it always true that $(E',\tau_\sigma)' \subseteq E''$?

The following Theorem gives a necessary and sufficient condition for the topology τ_{co} to be compatible with the pair $< E', E >$.

Theorem 4.7 *For a Hausdorff polar space E, the following are equivalent:*
(1) τ_{co} is compatible with the pair $< E', E >$, i.e. $(E',\tau_{co})' = E$.
(2) Every closed (or equivalently weakly closed) compactoid subset of E is complete.
(3) Every closed (or equivalently weakly closed) absolutely convex subset of E is weakly complete.

Proof. First of all we observe that a compactoid subset of E is closed iff it is weakly closed and that an absolutely convex compactoid is complete iff it is weakly complete (by [10, Theorem 5.13]).
$(1) \Rightarrow (2)$. Let A be a closed compactoid subset of E. Since τ_{co} is compatible with the pair $< E', E >$, it is the topology of uniform convergence on some

special covering (by [12, Proposition 7.4]). Thus, there exists a weakly bounded, weakly complete edged subset B of E such that $B^0 \subseteq A^0$. Thus

$$A \subseteq A^{00} \subseteq B^{00} = B.$$

Since A^{00} is an absolutely convex weakly complete subset of E, it is complete and hence A is complete.
(2) $\Rightarrow$ (3). It is trivial.
(3) $\Rightarrow$ (1). The proof is included in the proof of [6, Proposition 4.2].

Proposition 4.8 *Let E be a Hausdorff polar space and let G be the dual space of E'_{co}. Then*
(1) $$G = \bigcup_A \overline{A}^{\sigma(E'',E')}$$
where A ranges over the family of all absolutely convex compactoid subsets of E.
(2) If we consider on G the topology of uniform convergence on the equicontinuous subsets of E', then E is a dense topological subspace of G.

Proof. (1) Since the topology of E'_{co} is coarser than the strong topology on E', G is a vector subspace of E''. For a subset B of G we denote by B^0 and B^{00}, respectively, the polar and the bipolar of B with respect to the pair $< G, E' >$. Let now $x'' \in G$. There exists an absolutely convex compactoid subset A of E such that

$$A^0 \subseteq \{x' \in E : | < x', x'' > | \leq 1\}.$$

If $|\lambda| > 1$, then
$$x'' \in A^{00} \subseteq \lambda \overline{A}^{\sigma(E'',E')}.$$
On the other hand, if $x'' \in \overline{A}^{\sigma(E'',E')}$, for some absolutely convex compactoid subset A of E, then $x'' \in A^{00}$ and so $| < x', x'' > | \leq 1$ for $x' \in A^0$, which implies that $x'' \in G$.
(2) Since the topology of E'_{co} is finer then the topology $\sigma(E', E)$ and since E is Haudorff and polar, it follows that E is a topological subspace of G. It only remains to show that E is dense in G. So let $x'' \in G$. By (1), $x'' \in \overline{A}^{\sigma(G,E')}$ for some absolutely convex compactoid subset A of E. Given a convex neighborhood V of zero in E and $|\lambda| > 1$, there exists a finite subset S of E such that

$$A \subseteq co(S) + \lambda^{-1}V \subseteq co(S) + \lambda^{-1}V^{00}.$$

Now
$$x'' \in A^{00} \subseteq \left(co(S) + \lambda^{-1}V^{00}\right)^{00} = \left(co(S) + \lambda^{-1}V^{00}\right)^{e}$$

and so

$$x'' \in \lambda co(S) + V^{00}.$$

This clearly completes the proof.

By [7, 3.1], every equicontinuous subset of E' is a compactoid set in E'_{co}. Also, by Proposition 4.1, the topology of E'_{co} coincides with the topology $\sigma(E', E)$ on equicontinuous sets. We have the following

Proposition 4.9 *Let (E, τ) be a Hausdorff polar space and let γ be a polar locally convex topology on E' for which every equicontinuous subset of E' is a compactoid set. If γ is compatible with the pair $< E', E >$, then γ is coarser than τ_{co}.*

Proof. Since $(E', \gamma)' = E$ and every equicontinuous subset H of E' is γ-compactoid, we have that $\gamma = \sigma(E', E)$ on H and so $\gamma \leq \tau_\sigma$. Thus

$$\gamma = \overline{\gamma}^{\sigma(E',E)} \leq \overline{\tau_\sigma}^{\sigma(E',E)} = \tau_{co}.$$

Proposition 4.10 *Let E, F be polar Hausdorff spaces and let $T : E \mapsto F$ be a continuous linear map. Then:*
a) T is compactifying iff the map

$$T' : F'_{co} \mapsto E'_b$$

is continuous, where E'_b is the strong dual of E.
b) If T is compactifying and each closed compactoid subset of F is complete, then $T''(E'') \subseteq F$.

Proof. a) If T is compactifying and B is a bounded subset of E, then $D = T(B)$ is a compactoid subset of F and $T'(D^0) \subseteq B^0$, which proves that $T' : F'_{co} \mapsto E'_b$ is continuous. Conversely, let $T' : F'_{co} \mapsto E'_b$ be continuous and let B be a bounded subset of E. There exists a compactoid subset D of F such that $T'(D^0) \subseteq B^0$. Now $T(B) \subseteq D^{00}$ and so $T(B)$ is compactoid since D^{00} is compactoid by [10, Theorem 5.3].
b) By [1], we have

$$E'' = \bigcup_B \overline{B}^{\sigma(E'',E')}$$

where B ranges over the family of all bounded subsets of E. Let now B be a bounded absolutely convex subset of E. Since T'' is continuous with respect to the topologies $\sigma(E'', E')$ and $\sigma(F'', F')$, we have

$$T''\left(\overline{B}^{\sigma(E'',E')}\right) \subseteq \overline{T''(B)}^{\sigma(F'',F')} = \overline{T(B)}^{\sigma(F'',F')}.$$

Let $A = \overline{T(B)}$ be the closure of $T(B)$ in F. Since T is compactifying, the set A is compactoid in F and hence A is complete by our hypothesis. Since A is absolutely convex, it is $\sigma(F, F')$-complete and hence it is $\sigma(F'', F')$-complete. Thus A is $\sigma(F'', F')$-closed and so

$$\overline{T(B)}^{\sigma(F'',F')} \subseteq \overline{T(B)} \subseteq F.$$

This clearly completes the proof.

Proposition 4.11 *Let $T : E \mapsto F$ be a linear operator, where E and F are Hausdorff polar spaces. Then: (1) If T is continuous, then the adjoint map*

$$T' : F'_{co} \mapsto E'_{co}$$

is continuous.
(2) If T is compactoid, then

$$T' : F'_{co} \mapsto E'_{co}$$

is compactoid.

Proof. (1) If A is a compactoid subset of E, then $B = T(A)$ is compactoid in F and $T'(B^0) \subseteq A^0$.
(2) Assume that T is compactoid and let $p \in cs(E)$ be such that the set $A = T(V_p)$ is compactoid in F where

$$V_p = \{x \in E : p(x) \leq 1\}.$$

We will finish the proof by showing that $T'(A^0)$ is a compactoid subset of E'_{co}. So, let B be a compactoid subset of E. Since E is polar, it has the approximation property (by [9, Theorem 5.4]). Thus there are $g_1, \dots, g_n$ in E' and $e_1, \dots, e_n$ in E such that

$$p\left(x - \sum_{\kappa=1}^{n} g_\kappa(x)e_\kappa\right) \leq 1$$

for all $x \in B$. Let $\phi_\kappa \in (E'_{co})'$, $\phi_\kappa(x') = x'(e_\kappa)$.
Claim: For all $y' \in A^0$ we have

$$T'y' - \sum_{\kappa=1}^{n} \phi_\kappa(T'y')g_\kappa \in B^0.$$

Indeed, let $y' \in A^0$ and $x \in B$. Then

$$x - \sum_{\kappa=1}^{n} g_\kappa(x)e_\kappa \in V_p$$

and so

$$Tx - \sum_{\kappa=1}^{n} g_\kappa(x)T(e_\kappa) \in A.$$

Thus,

$$< T'y' - \sum_{\kappa=1}^{n} \phi_\kappa(T'y')g_\kappa\,, x > = < y', Tx > - \sum_{\kappa=1}^{n}(T'y')(e_\kappa)g_\kappa(x)$$

$$= < y', Tx > - \sum_{\kappa=1}^{n} g_\kappa(x) < y', Te_\kappa > = < y'\,,\, Tx - \sum_{\kappa=1}^{n} g_\kappa(x)Te_\kappa >$$

which clearly proves our claim.

Now, there exists $\mu \in \mathbf{K}$ such that $e_\kappa \in \mu V_p$ for $\kappa = 1, 2, \dots, n$. If $y' \in A^0$, then

$$|\phi_\kappa(T'y')| = |< y', Te_\kappa >| \leq |\mu|.$$

Replacing ϕ_κ by $\mu^{-1}\phi_\kappa$ and g_κ by μg_κ, we may assume that $|\phi_\kappa(T'y')| \leq 1$ for all $y' \in A^0$ and that

$$\sum_{\kappa=1}^{n} \phi_\kappa(T'y')g_\kappa \in co(g_1, \dots, g_n).$$

It follows that

$$T'(A^0) \subseteq co(g_1, \dots, g_n) + B^0$$

which completes the proof.

Proposition 4.12 *If E, F are Hausdorff polar spaces and $T : E \mapsto F$ a nuclear linear operator, then $T' : F'_{co} \mapsto E'_{co}$ is nuclear.*

Proof. There exist a bounded sequence (y_n) in F, an equicontinuous sequence (f_n) in E' and a null sequence (λ_n) in $\mathbf{K}$ such that

$$Tx = \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n$$

for all x in E. For $y' \in F'$ and $x \in E$, we have

$$< T'y', x > = < y', Tx > = < y', \sum_{n=1}^{\infty} \lambda_n f_n(x) y_n > = \sum_{n=1}^{\infty} \lambda_n f_n(x) y'(y_n).$$

Let $|\lambda| > 1$ and choose $\mu_n \in \mathbf{K}$ with

$$|\mu_n| \leq \sqrt{|\lambda_n|} \leq |\lambda \mu_n|.$$

Let $\gamma_n \in \mathbf{K}$, where $\gamma_n = 0$ if $\lambda_n = 0$ and $\gamma_n = \lambda_n \mu_n^{-1}$ otherwise. Let

$$\phi_n : F'_{co} \mapsto \mathbf{K}, \quad \phi_n(y') = \mu_n y'(y_n).$$

Since $A = \{\mu_n y_n : n \in \mathbf{N}\}$ is a compactoid subset of F, it follows that the sequence (ϕ_n) is equicontinuous in $(F'_{co})'$. Also, (f_n) is a bounded sequence in E'_{co}. Indeed, the set

$$V = \{x \in E : |f_n(x)| \leq 1 \ \forall n\}$$

is a neighborhood of zero in E. If A is a compactoid (and hence bounded) subset of E, then $A \subseteq \mu V$ for some μ in $\mathbf{K}$, and so $f_n \in \mu A^0$. Finally,

$$T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y') f_n$$

in E'_{co}. In fact, let $p \in cs(E)$ be such that $|f_n| \leq p$ for all n. Let $|\mu| > \sup\{|\lambda_n y'(y_n)| : n \in \mathbf{N}\}$. Set

$$s_n = \sum_{\kappa=1}^{n} \gamma_\kappa \phi_\kappa(y') f_\kappa.$$

If $V = \{y \in E : p(y) \leq 1\}$, then $s_n \in \mu V^0$. Moreover $s_n(x) \to < T'y', x >$ for all $x \in E$. Thus $s_n \to T'y'$ in E'_{co} since the topology of E'_{co} coincides with $\sigma(E', E)$ on μV^0 by proposition 4.1. Thus

$$T'y' = \sum_{n=1}^{\infty} \gamma_n \phi_n(y') f_n$$

in E'_{co}. Since (γ_n) is a null sequence, the result follows.

5 On the ϵ-product

Proposition 5.1 *([10, 5.1]) If E, F are Hausdorff polar spaces, then $F\epsilon E$ is a Hausdorff polar space.*

As it is shown in [7], the ϵ-product of two polar complete spaces is complete. The following proposition shows that the same is true for quasi-complete spaces.

Proposition 5.2 *Let E, F be Hausdorf polar spaces. If E and F are quasicomplete, then $E\epsilon F$ is quasicomplete.*

Proof. Let (u_α) be a bounded Cauchy net in $E\epsilon F$. For each $f \in E'$, the net $((u_\alpha(f))$ is bounded and Cauchy in F and thus the limit $\lim u_\alpha(f)$ exists. Define

$$u_0 : E'_{co} \mapsto F\,,\; u_0(f) = \lim u_\alpha(f).$$

Since the map $u \mapsto u'$ is a topological isomorphism between $E\epsilon F$ and $F\epsilon E$ (by [7, Theorem 3.3]), the net (u'_α) is bounded in $F\epsilon E$. Define

$$v_0 : F'_{co} \mapsto E\,,\; v_0(g) = \lim u'_\alpha(g).$$

Claim 1: u_0 is continuous with respect to the weak topologies $\sigma(E', E)$ and $\sigma(F, F')$. Indeed, let S be a finite subset of F' and $T = v_0(S)$. For $f \in E'$ and $g \in F'$, we have

$$\lim < u_\alpha(f), g > = \lim < f, u'_\alpha(g) >$$

and so

$$< u_0(f), g > = < f, v_0(g) > .$$

It follows from this that $u_0(T^0) \subseteq S^0$.
Claim 2: For each equicontinuous subset H of E', $u_0(H)$ is a compactoid subset of F. In fact, let W be a convex neighborhood of zero in F. The set

$$D = \{u \in E\epsilon F : u(H) \subseteq W\}$$

is a zero neighborhood in $E\epsilon F$. Thus, there exists α_0 such that $u_\alpha - u_\beta \in D$ for $\alpha, \beta \succeq \alpha_0$. Since W is closed in F, it follows that $u_\alpha(f) - u_0(f) \in W$ for all $f \in H$ and all $\alpha \succeq \alpha_0$. Since $u_{\alpha_0}(H)$ is a compactoid subset of F, there exsits a finite subset S of F such that

$$u_{\alpha_0}(H) \subseteq co(S) + W.$$

Thus

$$u_0(H) \subseteq co(S) + W.$$

Now by claims 1, 2 and Lemma 2.1, we have that $u_0 \in E\epsilon F$. Finally it is easy to see that $u_\alpha \to u_0$ in $E\epsilon F$.

For a Hausdorff polar space F, we denote by $\widetilde{F}$ the dual space of F'_{co} equipped with the topology of uniform convergence on the equicontinuous subsets of F'. It is easy to see that if $u \in F\epsilon E$, then the adjoint u' belongs to $E\epsilon\widetilde{F}$. We will consider F as a topological subspace of $\widetilde{F}$.

Proposition 5.3 *Let E, F be Hausdorff polar spaces. Then, the map $u \mapsto u'$, from $F\epsilon E$ to $E\epsilon\widetilde{F}$, is linear, continuous and one-to-one.*

Proof. For a convex neighborhood V of F, we will let V^{00} denote the bipolar of V with respect to the dual pair $< \widetilde{F}, F >$. Sets of the form V^{00} form a base at zero in $\widetilde{F}$. Let now W and V be convex neighborhoods of zero in E and F respectively and let

$$D = \{v \in E\epsilon\widetilde{F} : v(W^0) \subseteq V^{00}\}.$$

If $u \in F\epsilon E$ is such that $u(V^0) \subseteq W$, then $u' \in D$. This proves that the map $u \mapsto u'$ is continuous. The rest of the proof is trivial.

Proposition 5.4 *Let E, F be Hausdorff polar spaces and let D be a compactoid subset of $F\epsilon E$. Then:*
(1) For every equicontinuous subset H of F', the set

$$D(H) = \bigcup_{u \in D} u(H)$$

is a compactoid subset of E.
(2) If every closed compactoid subset of F is complete, then D is an equicontinuous subset of $L(F'_{co}, E)$.
(3) If in both E and F the closed compactoid subsets are complete, then the closure $\overline{D}$ of D in $F\epsilon E$ is complete.

Proof. (1) Let H be an equicontinuous subset of F'. For each $u \in F\epsilon E$ the set $u(H)$ is compactoid. Let now W be a convex neighborhood of zero in E. The set

$$U = \{u \in F\epsilon E : u(H) \subseteq W\}$$

is a neighborhood of zero in $F\epsilon E$ and thus

$$D \subseteq co(S) + U$$

for some finite set S. If $T = co(S)$, then $T(H)$ is a compactoid subset E and hence
$$T(H) \subseteq co(B) + W$$
for some finite subset B of E. Now
$$D(H) \subseteq co(B) + W.$$
(2) If every closed compactoid subset of F is complete, then $\widetilde{F} = F$ (by Theorem 4.7) and so the set $D' = \{u' : u \in D\}$ is a compactoid subset of $E \epsilon F$ by the preceding Proposition. Given a polar neighborhood W in E, the set W^0 is an equicontinuous subset of E' and so $A = D'(W^0)$ is a compactoid subset of F by the first part of the proof. Moreover, for $u \in D$, we have
$$u(A^0) \subseteq W^{00} = W$$
which completes the proof of (2).
(3) The set $\overline{D}$ is a compactoid subset of $F \in E$. Let (u_α) be a Cauchy net in $\overline{D}$. For each $x' \in F'$, the set $\overline{D}(x')$ is compactoid in E and $(u_\alpha(x'))$ is a Cauchy net. By our hypothesis, the limit $\lim u_\alpha(x')$ exists in E. Define
$$u : F' \mapsto E,\ u(x') = \lim u_\alpha(x').$$
Claim: $u \in F \epsilon E$. Indeed, u is linear. Also, given a polar neighborhood W of zero in E, the set $B = \overline{D}'(W^0)$ is compactoid in F and $\overline{D}(B^0) \subseteq W$. If $x' \in B^0$, there exists α_0 such that $u(x') - u_\alpha(x') \in W$, for $\alpha \succeq \alpha_0$, and so $u(x') \in u_\alpha(x') + W \subseteq W$, which proves that $u \in F \epsilon E$. If H is an equicontinuous subset of F', then there exists β_0 such that $(u_\alpha - u_\beta)(H) \subset W$ for $\alpha \succeq \beta \succeq \beta_0$, and so $(u_\alpha - u)(H) \subset W$ for $\alpha \succeq \beta_0$. This proves that $u_\alpha \to u$ in $F \epsilon E$ and the result follows.

Theorem 5.5 *Let E_1, E_2, F_1, F_2 be Hausdorff polar spaces and let $T_i : E_i \mapsto F_i$, $i = 1, 2$, be continuous linear operators. Then: 1) The map*
$$T = T_1 \epsilon T_2 : E_1 \epsilon E_2 \mapsto F_1 \epsilon F_2,\ \ Tu = T_2 u T_1'.$$
is continuous.
2) If both T_1 and T_2 are nuclear, then T is nuclear.

Proof. First of all we notice that, since
$$T_1' : (F_1')_{co} \mapsto (E_1')_{co}$$

is continuous, we have that $Tu \in F_1 \epsilon F_2$ for $u \in E_1 \epsilon E_2$. To show that T is continuous, let W_i be a convex neighborhood in F_i, $i = 1, 2$, and let

$$U = \{w \in F_1 \epsilon F_2 : w(W_1^0) \subseteq W_2\}.$$

Let $V_i = T_i^{-1}(W_i)$, $i = 1, 2$, and set

$$D = \{u \in E_1 \epsilon E_2 : u(V_1^0) \subseteq V_2\}.$$

Then D is a neighborhood of zero in $E_1 \epsilon E_2$ and $T(D) \subseteq U$. This proves that T is continuous.

2) Assume that both T_1 and T_2 are nuclear. There are null sequences (λ_i), (μ_i) in $\mathbf{K}$, bounded sequences (y_i) and (w_i) in F_1, F_2, respectively, and equicontinuous sequences (f_i), (g_i) in E_1' and E_2' such that

$$T_1 x = \sum_i \lambda_i f_i(x) y_i \,, \; T_2 z = \sum_j \mu_j g_j(z) w_j.$$

As it is shown in the proof of proposition 4.12, we have

$$T_1' y' = \sum_i \lambda_i y'(y_i) f_i \,, \;\; y' \in F_1',$$

where the series converges in $(E_1')_{co}$. Thus, for $u \in E_1 \in E_2$ and $y' \in F_1'$, we have

$$< Tu, y' > = < T_2 u, \sum_i \lambda_i y'(y_i) f_i > = \sum_i \lambda_i y'(y_i) T_2(u(f_i))$$

$$= \sum_i \lambda_i y'(y_i) \left(\sum_j \mu_j g_j(u(f_i)) w_j \right).$$

Let $v_{ij} \in F_1 \epsilon F_2$, $v_{ij}(y') = y'(y_i) w_j$. The double sequence (v_{ij}) is bounded in $F_1 \epsilon F_2$. Indeed, let W and V be convex neighborhoods of zero in F_2 and F_1 respectively. Set

$$D = \{v \in F_1 \epsilon F_2 : v(V^0) \subset W\}.$$

Let $\mu \in \mathbf{K}$ be such that $y_i \in \mu V$ and $w_j \in \mu W$ for all i, j. Now, for $y' \in V^0$, we have

$$v_{ij}(y') = y'(y_i) w_j \in \mu^2 W$$

which proves that $v_{ij} \in \mu^2 D$. Also, let

$$h_{ij} : E_1 \epsilon E_2 \mapsto \mathbf{K} \,, \; h_{ij}(u) = g_j(u(f_i)).$$

The double sequence (h_{ij}) is equicontinuous in $(E_1 \epsilon E_2)'$. Indeed, let V_1, W_1 be convex neighborhoods of zero in E_1, E_2, respectively, such that $f_i \in V_1^0$ and $g_j \in W_1^0$ for all i, j. If

$$D_1 = \{u \in E_1 \epsilon E_2 : u(V_1^0) \subseteq W_1\},$$

then $h_{ij} \in D_1^0$.

Let now $\sigma = \sigma_1 \times \sigma_2 : \mathbf{N} \mapsto \mathbf{N} \times \mathbf{N}$ be any bijection. Set

$$\gamma_n = \lambda_{\sigma_1(n)}\mu_{\sigma_2(n)}\,,\ g_n(u) = h_{\sigma_1(n)\sigma_2(n)}\,,\ \phi_n = v_{\sigma_1(n)\sigma_2(n)}.$$

We will show that

$$Tu = \sum_{n=1}^{\infty} \gamma_n g_n(u)\phi_n,$$

where the series converges in $F_1 \epsilon F_2$. To this end, we may assume that $|\lambda_i|, |\mu_j| \leq 1$ for all i, j. Let V, W, V_1, W_1, D and μ be as above. For $y' \in V^0$, we have $|y'(y_i)| \leq |\mu|$ for all i. By Proposition 5.4, the set $A = u(V_1^0)$ is compactoid and hence bounded in E_2. Since $g_j \in W_1^0$ and $f_i \in V_1^0$, there exists $\gamma \in K$ such that $|g_j(u(f_i))| \leq |\gamma|$ for all i, j. It is now clear that there exists n_0 such that if either $i \geq n_0$ or $j \geq n_0$, then

$$\lambda_i y'(y_i)\mu_j g_j(u(f_i))w_j \in W$$

for all $y' \in V^0$. Since W is closed, we get that $\lambda_i y'(y_i)T_2(u(f_i)) \in W$, for $i > n_0$, and so, for $y' \in V^0$, we have

$$< Tu, y' >= \sum_{i=1}^{n_0} \lambda_i y'(y_i)T_2(u(f_i)) + v\,,\ v \in W.$$

For an analogous reason, we get that

$$< Tu, y' >= \sum_{i=1}^{n_0}\sum_{j=1}^{n_0} \lambda_i y'(y_i)\mu_j g_j(u(f_i))w_j + v_1$$

with $v_1 \in W$. Let now m_0 be such that $\sigma_1(n) > n_0$ or $\sigma_2(n) > n_0$ if $n \geq m_0$. It is easy to see that for $n \geq m_0$ we have

$$\sum_{\kappa=1}^{n} \gamma_\kappa g_\kappa(u)\phi_\kappa(y') - \sum_{i=1}^{n_0}\sum_{j=1}^{n_0} \lambda_i\mu_j h_{ij}(u)v_{ij}(y') \in W$$

and so

$$< Tu, y' > - \sum_{\kappa=1}^{n} \gamma_\kappa g_\kappa(u)\phi_\kappa(y') \in W$$

for all $y' \in V^0$, i.e.

$$Tu - \sum_{\kappa=1}^{n} \gamma_\kappa g_\kappa(u)\phi_\kappa \in D$$

for $n \geq m_0$. This clearly completes the proof.

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