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ION ARMEANU

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ABOUT AMBIVALENT GROUPS

par ION ARMEANU

Abstract :

In this note we shall prove some properties of the ambivalent groups and will completely determine the ambivalent solvable groups with one conjugacy class of involutions. Also we shall study the structure of the ambivalent groups having abelian Sylow 2-subgroups.

All groups will be finite. The notations and definitions will be those of [3]and [4].

Definition.

i) *An ambivalent group is a group all whose characters are real valued.*

ii) *A rational group is a group all of whose irreducible characters are rational valued.*

Proposition 1. (see [3]pp. 31

i) *A group G is ambivalent iff for every $x \in G$ there is a $t \in G$ such that $x^t = x^{-1}$.*

ii) *A group G is rational iff for every x in G the generators of $< x >$ are conjugate in G.*

Proposition 2.

Let G be an ambivalent group. Then :

a) If N is a normal subgroup of G, then G/N is ambivalent

b) Every nonidentity 2-central element of G is an involution.

c) $Z(G)$ is an elementary abelian 2-group.

d) If G is abelian, then G is an elementary abelian 2-group.

e) G/G' is an elementary abelian 2-group.

f) $0^2(G) = 0^2(G')$.

g) If p is an odd prime and P a Sylow p-subgroup of G, then $P \leq [P, G]$.

h) G is generated by its 2-elements.

Proof.

a) The irreducible characters of G/N are in fact irreductible characters of G.

b) Let $x \in G$ be a 2-central element and set $|x| = 2^k m$, where $(2, m) = 1$. Then $S \leq C(x) \leq N(< x >)$ for some $S \in Syl_2(G)$, hence $(2, |N(< x >) : C(x)|) = 1$. Therefore $m = k = 1$.

c) The nontrivial elements of $Z(G)$ are 2-central.
d) Follows from c).
e) Follows from d).
f) Observe that $|G : G'| = 2^n$.
g) Let $Foc_G(P)$ be the focal subgroup of P in G. $Foc_G(P)$ is generated by the commutators $[x, y], y \in G$ which lie in P. Let $\tau : G \to P/Foc_G(P)$ stand for the transfer map. Then (see [4]chap. 10) τ is onto and therefore $P/Foc_G(P)$ is an abelian ambivalent group. Since $p \neq 2$, it follows that $P = Foc_G(P)$. Hence $P = Foc_G(P) \leq P \cap [P, G] \leq P$ and therefore $P \leq [P : G]$.
h) Let H be the subgroup of G generated by the 2-elements of G. Then H is normal in G and G/H is an ambivalent group of odd order.

Corollary 3.
Let G be an ambivalent group and $S \in Syl_2(G)$. Then $C_G(S) = Z(S)$.

Proof. Every element of $C_G(S)$ is 2-central and 2-central elements must be involutions by Prop. 2.b). Since $Z(S)$ is the set of 2-elements of $C_G(S)$, then $Z(S)$ is the Sylow 2-subgroup of $C_G(S)$. By Burnside Transfer Theorem (see [4]pp. 244) $Z(S)$ has a normal complement N in $N_G(S)$ and therefore $C_G(S) = Z(S)N$. Furthermore, since $Z(S)$ is the Sylow 2-subgroup of $C_G(S), (2, |N|) = 1$. Hence N must be trivial.

Lemma 4.
Let G be a solvable group and $S \in Syl_2(G)$. Let $x \in N_G(S)$ be an element off odd order. Then x is non-real in G.

Proof. We prove by induction on the order of G. Let N be a minimal subgroup of G. Since G is solvable, then N is an elementary abelian p-group, for a prime p. If $x \notin N$ the image of x in G/N is non-real by induction. So x is non-real in this case. If $x \in N$, we have that $[x, S] \subseteq S \cup N = 1$. Since $C_G(x)$ contains a sylow 2-subgroup of G, we have that the order of $N_G(< x >)/C_G(x)$ is odd. Hence x is a non-real element.

By Lemma 4 it follows immediately the following.

Proposition 5.
Let G be an ambivalent solvable group and $S \in Syl_2(G)$. Then $N_G(S) = S$.

Corollary 6.
Be G be an ambivalent solvable group and $S \in Syl_2(G)$. Then $N_G(H) = H$ for every subgroup H of G such that $S \leq H$.

Proof. Let $x \in N_G(H)$. Then S and xSx^{-1} are Sylow 2-subgroups of H and hence $xSx^{-1} = ySy^{-1}$ for some $y \in H$. Therefore, $y^{-1}x \in N_G(S) = s \leq H$ hence $x \in H$.

Theorem 7.
Suppose G is a solvable ambivalent group with one conjugacy class of involutions. Then, the Sylow 2-subgroups of G are isomorphic either th $\mathbb{Z}_2$ or to a generalised quaternion group. Furthermore if the Sylow 2-subgroups of G are isomorphic to $\mathbb{Z}_2$, then $G = 0_{2'}(G)\mathbb{Z}_2$ inverts all the elements of $0_{2'}(G)$ and $0_{2'},(G) = G'$.

Proof. Clearly we can suppose that $0_{2'}(G)$ is trivial. Let $I(G) = \{x \in G | x^2 = 1\}$. Let A be a minimal subgroup of G. Since G is solvable and $0_2(G)$ is trivial, it follows that A is an elementary abelian 2-group and hence $A = I(G)$.

By Thompson Theorem (see [1]pp. 511) if G contains more than one involution, then, Sylow 2-subgroups of G are either homocyclic or Suzuki 2-groups. Let $S \in Syl_2(G)$. If S is homocyclic, then $A \subseteq Z(S)$. If S is a Suzuki 2-group then (see [1]pp. 313) $S' = Z(S) = A = I(S)$. In both cases, by Burnside Transfer Theorem, for $x, y \in Z(S)$ such that $x^t = y$, with t, in G then there is a $z \in N_G(S) = S$ such that $x^Z = y$. Since $A \subseteq Z(S)$ this is a contradiction. Therefore G contains only one involution, hence S is either cyclic or a generalised quaternion group.

IF S is a cyclic group, by Burnside Transfer Theorem and since $N_G(S) = S$, it follows that every element of S must be real in S, hence S is isomorphic to $\mathbb{Z}_2$.

By Glauberman Z^* Theorem (see [5]), if $0_{2'}(G)$ is trivial, then $Z(G) \neq 1$ and by Prop. 2 it follows that $S = Z(G) = G$.

If $0_{2'}(G)$ is not trivial, then $G = 0_2,(G)\mathbb{Z}_2$ and $\mathbb{Z}_2$ inverts elements of $0_{2'}(G) = G'$.

Remark.
In [2]Feit and Seitz proved that the only groups all whose elements of same order are conjugate are the symmetric groups S_1, S_2 and S_3. The proof of this theorem depends deeply on the classification theory of simple groups. The next corollary offer a proof of this theorem for the solvable case without using classification theory of simple groups.

Corollary 8.
Let G be a solvable groups all of whose elements of the same order are conjugate. Then G is isomorphic to S_1, S_2 and S_3.

Proof. Clearly such a group is a rational group and all involutions are conjugate in G. By theorem 7 the Sylow 2-subgroups of G are either $\mathbb{Z}_2$ or generalized quaternion groups. Denote $0_{2'}(G) = 0(G)$.

Case $S = \mathbb{Z}_2$. By Theorem 7, $G = 0(G)\mathbb{Z}_2$ and $0(G) = G'$. If $0(G)$ is trivial we have the statement. Suppose $0(G) \neq 1$. We shall prove now that $0(G)$ is an elementary abelian 3-group. Let $t \in S$ be the element of order 2. Since t inverts all elements of G' for every $u, v \in G'$ we have $uvu^{-1}v^{-1} = uvtut^{-1}tvt^{-1} = uvtuvt^{-1} = uv(uv)^{-1} = 1$ thus G' is abelian. Let x in G' of order $|x| = p^k$ with p an odd prime. Since $|Aut(< x >)|p_{k-1}(p-1)$ and G' is abelian it follows that $k = 1$ and $p = 3$. Thus G' is an elementary abelian 3-group and since all elements of order 3 must be conjugate in G we have $G' \simeq \mathbb{Z}_3$ and $G \simeq S_3$.

Case S quaternionic. Let x, y generators for S with relations $x^{2^{n-1}} = 1$, $y^2 = x^{2^{n-2}}$ and $yxy^{-1} = x^{-1}$. Then $|N_G(< x >) : C_G(x)|2^{n-2}$. Let $N(x)_2$ resp. $C(x)_2$ a Sylow 2-subgroup of $N_G(< x >)$ resp. $C_G(x)$. Then $|N(x)_2| = 2^{n-2}|C(x)_2| \geq 2^{n-2} \times |x| = 2^{2n-3}$. Since S is a Sylow 2-subgroup of G, $2n - 3 \leq n$ and hence $n = 3$. Thus S must be the quaternionic group Q_8 of order 8.

Following [2]we shall prove now that the factor group of a group all whose elements of the same order are conjugate have the same property. Let N be a normal subgroup of G. Let $\underline{x}, \underline{y} \in G/N$ of same order and chosse $x, y \in G$ minimal order coset representatives for $\underline{x}, \underline{y}$. If $|x| = |y|$ then x and y are conjugate in G and hence $\underline{x}$ and $\underline{y}$ are conjugate in G/N. Suppose $|x| \neq |y|$ and let p a prime such that $|x| = p^i k$, $|y| = p^j m$ with $i < j$, $(p, km) = 1$. set $u = k^{km}$, $v = y^{km}$. Then $\underline{u}$ and $\underline{v}$ have the same order. Clearly u and $w = v^{p^{j-i}}$ hazve both order p^i hence they are conjugate in G. So $\underline{u}$ and $\underline{w}$ are conjugate in G/N hence $|\underline{u}| = |\underline{w}| = |\underline{v}|$ and $|\underline{v}| = \underline{v}^{p^{j-i}}|$. It follows that p do not divide the orders of $\underline{v}$ and $\underline{u}$. Set $t = x^{p^i}$ and $r = y^{p^j}$. Then $\underline{t} = \underline{x}$ and $\underline{r} = \underline{y}$ with contradicts the minimality of x and y.

Set $Z(G) =< z >, z = y^2$. By the previous $G/ < z >= H$ is a group with the same property as G and with abelian Sylow 2-subgroup $Q_8/ < z >\simeq \mathbb{Z}_2 \times \mathbb{Z}_2$ contradiction.

Theorem 9.

Let G be an ambivalent group having abelian Sylow 2-subgroups. Let $S \in Syl_2(G)$. Then G is 2-nilpotent and splits over G' with S as complement.

Proof. By Walter [6], G has a normal subgroup $N \geq 0_{2'}(G)$ such that G/N has off order and $N/0_{2'}(G) \simeq M \times P$ where M is a 2-group and P is a direct product of simple groups of the form $L_2(q), q > 3, q = 3, 5(mod 8)$ or $q = 2^n$, or the Janko simple group $J(11)$, or is of Ree type. By Prop. 2, G/N is an ambivalent group, hence $G = N$ and $G/0_{2'}(G) \simeq M \times P$. Since no of the previous simple groups is ambivalent it follows that $G/0_{2'}(G) \simeq M$. Since M is a 2-group and by Prop. 1 $0_{2'}(G) < (G')$ the statement follows.

Corollary 10.

Let G be an ambivalent group and let $S \in Syl_2(G)$ abelian. Then S is elementary abelian and all elements of G are strong real.

Proof. Suppose $G' \cap S = 1$. By Prop. 1 $G = G'S$ and hence $G/G' \simeq S/(G' \cap S) \simeq S$ is abelian.

Conversaly, suppose that S is abelian. By theorem 2 $G' = 0_{2'}(G)$, hence $G' \cap S = 1$.

Corollary 11.
Let G be an ambivalent group with $S \in Syl_2(G)$ abelian. Then S is elementary abelian and all element of G are strong real.

Proof. By theorem 9, $S \simeq G/G'$, and the statement follows by Prop. 2.

Theorem 12.
Let G be an ambivalent group having abelian Sylow 2-subgroups. Then all irreducible characters of G have Schur index 1 over $\mathbb{R}$.

Proof. By Brauer-Speiser theorem (see [3]) $m_{\mathbb{R}}(\xi) \leq 2$. Let $\xi \in Irr(G)$ with $m_{\mathbb{R}}(\xi) = 2$. From Brauer-Witt theorem (see [3]pp. 162) there is a subgroup W of G, $W = AH$ (semidirect product) and a real valued irreducible character φ of W, such that (φ, ξ_W) is odd, where $A =< a >$ is a cyclic group of odd order and H is a 2-group. By the general properties of the Schur index $m_{\mathbb{R}}(\varphi) = 2$. If W is abelian, then φ is linear and then $m_{\mathbb{R}}(\varphi) = 1$. Thus there is some involution $h \in H$ such that $a^h = a^{-1}$. since $W < C^*(a)$, there is a Sylow 2-subgroup T of $C^*(a)$ such that $H < T$. Let $V < T$, $V \in Syl_2(C(a))$ and $U = AT$. It is easy to see that the irreducible characters of U have Schur index 1 and degree at most 2. Let

$$\xi_U = \sum a_i(\lambda_i + \overline{\lambda}_i) + \sum b_j \nu_j,$$

where λ_i are not real-valued and ν_j are real valued irreducible characters of U. All b_j must be even, otherwise (ξ_U, ν_j) is odd and $m_{\mathbb{R}}(\xi) = 1$. Any irreducible character of U whose restriction to W is φ is an extension of φ. Therefore , if $\lambda_{i|U} = \varphi$ then $\overline{\lambda}_{i|U} = \varphi$ hence $\xi_W = (\xi_U)_W$ must contain φ an even number of times which is not the case. Hence $m_{\mathbb{R}}(\xi) = 1$.

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University of Bucharest
Physics Faculty
Mathematics Dept.
Bucharest-Magurele
P.O. Box MG - 11
ROMANIA