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A NEW FAMILY OF FUNCTIONAL SERIES RELATIONS INVOLVING DIGAMMA FUNCTIONS

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Abstract :

The purpose of the present investigation is to obtain certain new families of functional series relations involving the digamma functions. Several types of summation formulae are deducible from the main results. Extension of the results associating the familiar H-functions are also studied.

1 . Introduction and preliminaries.

For any real or complex λ, we denote by $(\lambda)_n$ the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & \text{if } n=0, \\ \lambda(\lambda+1)...(\lambda+n-1), & \forall \ n \in N. \end{cases} \tag{1.1}$$

Using techniques of fractional calculus, Kalla and Ross [7] obtained a summation formula (due to Nörlund [9 a, p. 168]).

$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} = \Psi(\lambda) - \Psi(\lambda-\alpha), \tag{1.2}$$

Re $(\lambda-\alpha)>0, \quad (\lambda \neq 0,-1,-2...)$
involving the digamma (or Psi) function [12]

$$\Psi(z) = \frac{d}{dz}(log\Gamma(z)) = \frac{\Gamma'(z)}{\Gamma(z)}. \tag{1.3}$$

The interesting summation formula (1.2), in fact, was motivated by Ross's formula [11] :

$$\sum_{n=1}^{\infty} \frac{1.3.5...(2n-1)}{n.2^n.n!} = \ell n4, \tag{1.4}$$

which emerges from (1.2) in the special case when $\alpha = \frac{1}{2}, \lambda = 1$.

The result (1.2) and its special cases were revived and studied in recent years as worthwhile illustrations emphasizing the fruitfulness of the approach of fractional calculus in evaluting infinite sums. One may refer to the papers [1] , [4] , [7] , [8] and [12] on the subject.

Alternative proof of (1.2) was furnished by using l'Hospital's theorem on limit by Kalla and Al-Saqabi [6] . Also, direct simple proofs to some of the generalizations of (1.2) have been given by Nishimoto and Srivastava [9] and Srivastava and Nishimoto [13] .

This paper is devoted to obtaining a new family of functional series relations involving the digamma functions without using the techniques of fractional calculus. The approach of derivations is direct and based upon simple series rearrangement methods. The paper is arranged as follows :

Section 2 gives an interesting lemma which is used in the derivation of the main results [Eqns. (2.7) and (2.8) below] . In section 3, a number of examples are deduced, and in the concluding section 4, we study a useful application in associating the well known H-function [3] in functional series relations.

The advantages of our results are that several known and new results on series summations can be deduced from them, and some of these are invariably pointed out in the paper.

In the sequel we use various notations and symbols. The set of positive integers is denoted by N, and $N_0 = NU(0)$. C means the complex number field and $R_+ = (0, \infty)$. As usual (a_p) stands for the array of parameters $a_1, ..., a_p$.

2. Functional series relations.

Before stating our main results giving series relations involving the digamma functions, we require the following result :

LEMMA .

Let $\alpha, \lambda, \mu \in C$ and $\nu \in N_0$ such that $Re(\lambda - \alpha) > 0, Re(\mu) > 0$, then

$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{(\alpha+n)_k(\lambda-\mu)_k(-\nu)_k}{(\alpha)_k(\lambda+n)_k k!}$$

$$= \frac{\Gamma(\lambda)\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\lambda+\nu)} \Big[\Psi(\mu) + \Psi(\lambda+\nu) - \Psi(\lambda-\alpha) - \Psi(\mu+\nu)\Big] , \tag{2.1}$$

provided that both $\lambda, \nu + \lambda \neq 0, -1, -2, ...$; *and $\Psi(z)$ is defined by* (1.3).

Proof.

Let

$$u(n,k)=\frac{(\alpha)_n}{n(\lambda)_n}\frac{(\alpha+n)_k(\lambda-\mu)_k(-\nu)_k}{(\alpha)_k(\lambda+n)_k k!}, \tag{2.2}$$

be the general term of the double series on the L.H.S. of (2.1). Also, put

$$v(n,k)=\frac{(\lambda-\mu)_k}{(\lambda)_k k!}(-\nu)_k\left[\frac{(\alpha+k)_n}{n(\lambda+k)_n}-\frac{(\lambda-\mu+k)_n}{n(\lambda+k)_n}\right], \tag{2.3}$$

and

$$w(n,k)=\frac{(\lambda-\mu)_n}{n(\lambda)_n}\frac{(\lambda-\mu+n)_k(-\nu)_k}{(\lambda+n)_k k!}. \tag{2.4}$$

By using the elementary identity $(a)_n(a+n)_k=(a)_{n+k}=(a)_k(a+k)_n$, then we observe that (2.2) , (2.3) and (2.4) satisfy the following relation :

$$u(n,k)=v(n,k)+w(n,k). \tag{2.5}$$

Now summing separately the v-series and w-series by virtue of Gauss's summation theorem [14, p. 18] and the formula (1.2), we find that

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}v(n,k)=\frac{\Gamma(\lambda)\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\lambda+\nu)}[\Psi(\mu)-\Psi(\lambda-\alpha)]\ , \tag{2.6}$$

and

$$\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}w(n,k)=\frac{\Gamma(\lambda)\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\lambda+\nu)}[\Psi(\lambda+\nu)-\Psi(\mu-\nu)] \tag{2.6a}$$

The result (2.1) follows thus from (2.2) , (2.6) and (2.6 a).

Remark 1.

The formula (2.1) would also follow from a recent known result due to Raina [10, p. 173, Eqn. (1.5)] in which the method of derivation is based on techniques of fractional calculus. The alternative proof given above is a simple and direct one. It may be observed that the summation formula (1.2) is recoverable from (2.1) either when $\nu=0$ or $\lambda=\mu$.

We state now the main results giving a family of functional series relations involving digamma functions which are contained in the following :

Theorem.

Let $A(r)$ be an arbitrary bounded sequence of complex numbers, and the parametric restrictions be same as stated in the hypothesis of (2.1). Then

$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{(\alpha+n)_k(\lambda-\mu)_k(-\nu)_k}{(\alpha)_k(\lambda+n)_k k!} \cdot \sum_{r=0}^{\infty} \frac{A(r)}{(\lambda+n+k)_r} \tag{2.7}$$

$$= \sum_{r=0}^{\infty} A(r) . \frac{\Gamma(\lambda)\Gamma(\mu+\nu+r)}{\Gamma(\mu+r)\Gamma(\lambda+\nu+r)} \cdot [\Psi(\mu+r)+\Psi(\lambda+\nu+r)-\Psi(\mu+\nu+r)- \Psi(\lambda-\alpha+r)]$$

and

$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{(\alpha+n)_k(\lambda-\mu)_k(-\nu)_k}{(\alpha)_k(\lambda+n)_k k!} \cdot \sum_{r=0}^{\infty} \frac{A(r)}{(\alpha+k)_r} . \frac{(\alpha+n+k)_r}{(\lambda+n+k)_r} \tag{2.8}$$

$$= \sum_{r=0}^{\infty} A(r) . \frac{\Gamma(\lambda)\Gamma(\mu+\nu+r)}{\Gamma(\mu+r)\Gamma(\lambda+\nu+r)} .[\Psi(\mu+r)+\Psi(\lambda+\nu+r)-\Psi(\mu+\nu+r)-\Psi(\lambda-\alpha)].$$

Proof.

To prove (2.7), we have by (1.1) and the rearrangement of involved series (which is justified under the stated conditions) :

$$\text{L.H.S. of (2.7)} = \sum_{r=0}^{\infty} \frac{A(r)}{(\lambda)_r} \Big(\sum_{k=0}^{\infty} \frac{(\alpha)_n}{n(\lambda+r)_n} \cdot \sum_{k=0}^{\infty} \frac{(\alpha+n)_k(\lambda-\mu)_k(-\nu)_k}{(\alpha)_k(\lambda+r+n)_k k!} - \Big)$$

Applying appropriately (2.1) (i.e., replacing λ by $\lambda+r$ and μ by $\mu+r$) for getting the closed form expression for the inner double summation, we arrive at the desired R.H.S. of Eqn. (2.7), under of course, the restrictions stated with the theorem.

The result (2.8) can similarly be established.

Remark 2.

When $\lambda = \mu$ or $\nu = 0$, Eqns. (2.7) and (2.8) yield the functional series relations involving digamma functions given by Nishimoto and Srivastava [9, p. 104, Eqns. (4.5) and (4.6)] .

3. Deductions.

Due to the generality of the functional series relations involving the arbitrary coefficients $A(r)$, several known and new series summations involving the digamma functions can be derived from (2.7) and (2.8). To illustrate, we derive the following summation formula :
Let us put

$$A(r) = z^r \Big(\prod_{i=1}^{p}(a_i)_r\Big)\Big(\prod_{i=1}^{q}(b_i)_r \quad r!\Big)^{-1} (p, q \in N), \tag{3.1}$$

in (2.7), then in terms of the triple hypergeometric series $F^3[x, y, z]$ (for details : see [14, p. 44]) , we get the following :

Example 1.

$$\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_r}{\prod_{i=1}^{q}(b_i)_r} \cdot \frac{z^r}{r!} \cdot \frac{\Gamma(\mu+\nu+r)}{\Gamma(\mu+r)\Gamma(\lambda+\nu+r)} \cdot [\Psi(\mu+r)+\Psi(\lambda+\nu+r)-\Psi(\mu+\nu+r)-\Psi(\lambda-\alpha+r)]$$

$$= \frac{\alpha}{\Gamma(\lambda+1)} F^3\left[\begin{matrix} - & :: & 1+\alpha & ;-;-; & 1,1 & : & \lambda-\mu, & -\nu & : & (a_p) & ; \\ 1+\lambda & :: & - & ;-;-; & 2 & : & & \alpha & : & (b_q) & ; \end{matrix}\; 1, 1, z\right], \tag{3.2}$$

where $b_i \neq 0, -1, -2, ...(i = 1, ..., q)$, $Re(\lambda) > Re(\alpha) > 0$, $Re(\mu) > 0$, and $q > p-2, |z| < \infty$

Setting $\lambda = \mu$ in (3.2), we have the simple relation :

Example 2.

$$\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_r}{\prod_{i=1}^{q}(b_i)_r} \cdot \frac{z^r}{r!} \cdot \frac{1}{\Gamma(\lambda+r)} [\Psi(\lambda+r) - \Psi(\lambda-\alpha+r)]$$

$$= \frac{\alpha}{\Gamma(\lambda+1)} F^{0,3,p}_{1,1,q}\left[\begin{matrix} - & : & 1+\alpha, & 1,1 & : & (a_p) & ; \\ 1+\lambda & : & & 2 & : & (b_q) & ; \end{matrix}\; 1, z\right], \tag{3.3}$$

provided that the conditions stated with (3.2) hold true, where the F-function on the right side of (3.3) is the Kampé de Fériet hypergeometric function of two variables, [14, p. 26] .

Again, with the substitution of (3.1) in Eqn (2.8), we have

Example 3.

$$\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_r}{\prod_{i=1}^{q}(b_i)_r} . \frac{z^r}{r!} . \frac{\Gamma(\mu+\nu+r)}{\Gamma(\mu+r)\Gamma(\lambda+\nu+r)}$$

$$.[\Psi(\mu+r)+\Psi(\lambda+\nu+r)-\Psi(\mu+\nu+r)-\Psi(\lambda-\alpha)] \tag{3.4}$$

$$= \frac{\alpha}{\Gamma(\lambda+1)} F^3\left[\begin{matrix} 1+\alpha & :: & -;-;-; & 1,1 & : & \lambda-\mu, & -\nu & : & (a_p) & ; \\ 1+\lambda & :: & -;\alpha;-; & 2 & : & & - & : & (b_q) & ; \end{matrix} 1,1,z\right],$$

under the conditions similar to those given in Example 1. Setting $\lambda = \mu$ in (3.4) , we have

Example 4.

$$\sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_r}{\prod_{i=1}^{q}(b_i)_r} . \frac{z^r}{r!} . \frac{1}{\Gamma(\lambda+r)} [\Psi(\lambda-r)-\Psi(\lambda-\alpha)]$$

$$= \frac{\alpha}{\Gamma(\lambda+1)} F^{1,2,p}_{1,1,q+1}\left[\begin{matrix} \alpha+1 & : & 1,1 & : & (a_p) & ; \\ \lambda+1 & : & 2 & : & \alpha,(b_q) & ; \end{matrix} 1,z\right], \tag{3.5}$$

under the conditions similar to those given with (3.2) .

4. APPLICATIONS TO H-FUNCTION.

It would be interesting to apply our main result (2.7) in deriving a summation formula involving the well known H-function of Fox [3] . In this direction we invoke the following H-function representation due to Braaksma [2] , see also [15, p. 12] .

$$H^{m,n}_{p,q}\left[x\middle| \begin{matrix} (a_1,A_1) & , & \dots & , & (a_p,A_p) \\ (b_1,B_1) & , & \dots & , & (b_q,B_q) \end{matrix}\right] = \sum_{h=1}^{m}\sum_{s=0}^{\infty} \frac{(-1)^s\theta(\sigma)x^\sigma}{s!\ B_h}, \tag{4.1}$$

where

$$\sigma = \sigma(h,s) = \frac{s+b_h}{B_h}\ ;h=1,...,m;\ s\in N \tag{4.2}$$

and

$$\theta(\sigma) = \frac{\prod_{j=1}^{m}\Gamma(b_j-B_j\sigma)\prod_{j=1}^{n}\Gamma(1-a_j+A_j\sigma)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+B_j\sigma)\prod_{j=n+1}^{p}(\Gamma(a_j-A_j\sigma)}. \tag{4.3}$$

Empty product is interpreted as 1, $m \in N$; $n,p,q \in N_0$ such that $0 \leq n \leq p$ and $1 \leq m \leq q$,

with $a_j \in C$, $A_j \in R_+$ $(j = 1, ..., p)$,
$b_j \in C$, $B_j \in R_+$ $(j = 1, ..., q)$.

Also, we have (for convergence) either

$$\Omega = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0, \ 0 < |x| < \infty \tag{4.4}$$

or

$$\Omega = 0, \ 0 < |x| < \Delta = \sum_{j=1}^{p} A_j^{-A_j} \sum_{j=1}^{q} B_j^{-B_j}. \tag{4.5}$$

To make our analysis relatively simple, we prefer a limiting case (2.1) of (2.7) which is attained when $A(r) \to (r \geq 1)$ and $A(0) = 1$ in Eqn. (2.7)). Replacing λ in Eqn. (2.1) by $\lambda + c\sigma$ $(c > 0)$, and then multiplying both sides by

$$\frac{(-1)^s \theta(\sigma) x^\sigma}{s! B_h},$$

where $\sigma = \sigma(h,s)$ and $\theta(\sigma)$ are given by (4.2) and (4.3), respectively. Now summing the resulting equation so obtained from (2.1) first w.r.t. h from $h = 1$ to $h = m$, and then w.r.t. s from $s = 0$ to $s = \infty$. Inverting the order of summation (which is permissible in view of the conditions stated with (2.1) and the conditions (4.4) and (4.5)), then after a little simplification we are lead to the following results :

$$\begin{aligned} &\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n} \sum_{k=0}^{\infty} \frac{(\alpha+n)_k (-\nu)_k}{(\alpha)_k k!} \\ &\cdot H_{p+2,q+2}^{m,n+2} \left[x \middle| \begin{matrix} (1-\lambda, c), (\lambda-\mu+k, c), (a_1, A_1), ..., (a_p, A_p) \\ (b_1, B_1), ..., (b_q, B_q), (\lambda-\mu, c), (1-\lambda-n-k, c) \end{matrix} \right] \\ &= \sum_{h=1}^{m} \sum_{s=0}^{\infty} \frac{(-1)^s \theta(\sigma) s^\sigma}{s! B_h} \cdot \frac{\Gamma(\lambda+cs)\Gamma(\lambda+\nu)}{\Gamma(\mu)\,\Gamma(\lambda+\nu+cs)} \\ &\cdot [\Psi(\mu) + \Psi(\lambda+\nu+c\sigma) - \Psi(\lambda-\alpha+c\sigma) - \Psi(\mu+\nu)] \end{aligned} \tag{4.6}$$

provided that $c > 0, Re(\mu) + \nu > 0, Re(\lambda - \alpha) > 0$, and
$\nu, \lambda + \nu$ (both) $\neq 0, -1, -2, ...$
and the convergence conditions (4.4) and (4.5) are satisfied.

It may be observed that the class of summation formula (4.6) includes the results of Nishimoto and Saxena [8] and Nishimoto and Srivastava [9] . To deduce a simpler summation formula from (4.6) in terms of the Wright's generalised hypergeometric function [14, p. 21]

$$
{}_p\Psi_q\left[\begin{matrix}(a_1,A_1) & , & \cdots & , & (a_p,A_p) & ; \\ (b_1,B_1) & , & \cdots & , & (b_q,B_q) & ;\end{matrix}\; z\right]
= \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}\Gamma(a_j+A_jn)}{\prod_{j=1}^{q}\Gamma(b_j+B_jn)}\cdot\frac{z^n}{n!}, \tag{4.7}
$$

where $A_j \in R_+ \quad (j=1,\ldots,p), \quad B_j \in R_+ \quad (j=1,\ldots,q), \quad 1+\sum_{j=1}^{q}B_j-\sum_{j=1}^{p}A_j \geq 0,$

we obtain the following special case of (4.6) :

$$
\begin{aligned}
&\sum_{n=1}^{\infty}\frac{(\alpha)_n}{n}\sum_{k=0}^{\infty}\frac{(\alpha+n)_k(-\nu)_n}{(\alpha)_k\; k!} \\
&\quad {}_{p+2}\Psi_{q+2}\left[\begin{matrix}(\lambda,c),(1-\lambda+\mu-k,c) & , & (a_1,A_1),\ldots,(a_p,A_p) \\ (1-\lambda+\mu,c),(\lambda+n+k,c) & , & (b_1,B_1),\ldots,(b_q,B_q)\end{matrix}\middle| x\right] \\
&= \sum_{S=0}^{\infty}\frac{\Gamma(\lambda+cs)\Gamma(\mu+\nu)}{\Gamma(\mu)\Gamma(\lambda+\nu+cs)}\cdot\frac{\prod_{j=1}^{p}\Gamma(a_j+A_js)}{\prod_{j=1}^{q}\Gamma(b_j+B_js)} \\
&\quad \cdot[\Psi(\mu)+\Psi(\lambda+\nu+cs)-\Psi(\lambda-\alpha+cs)-\Psi(\mu+\nu)]\,, \\
&\quad (c>0, Re(\lambda-\alpha)>0, \lambda\neq 0,-1,-2,\ldots),
\end{aligned} \tag{4.8}
$$

provided that each member of (4.8) exists.

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