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Measurable linear mappings from a Wiener space

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CHAPTER ONE

Gaussian probabilities

We want to describe a situation involving the Wiener measure and the white noise measure and we shall work with a structure more general than the "abstract Wiener space". More precisely instead of handling only separable Banach space, we shall work with Lusin spaces. It is not the maximum of generality, but we give only the preliminaries needed for our purpose.

1 - General definitions and results

Let for a moment E be a topological vector space (on $\mathbb{R}$ if it is not precised), locally convex and Hausdorff and let us denote E' its dual.

As it is well known a Borelian probability P on E is said ***"Gaussian centered"*** if for every $x' \in E', x'(P)$ is a Gaussian centered probability on $\mathbb{R}$ (eventually equal to δ_0). This is equivalent to say that for every $x' \in E'$, the random variable : $< x', \bullet >_{E',E}$ on (E, P) is Gaussian centered.
In the following we shall omit the mention "centered".

If P is a Gaussian probability on E, we associate to it a linear map j from E' into $L^2(E, P)$ defined as :

$$j(x') =< x', \bullet >_{E',E}$$

(we have used the same notation for a map and its P-equivalence class, as we shall do often in the sequel).
We have not equipped E' with a topology, so we cannot speak about the continuity of j. The (algebraic) transpose of j defines a linear map from $L^2(E, P)$ into the algebraic dual $(E')^*$ of E'. We shall denote by S this transpose.

We have therefore :

$$\text{for } f \in L^2(E, P) : \quad \big(j(x'), f\big)_{L^2(E,P)} = \int_E < x', x > f(x)P(dx)$$
$$=< Sf, x' >_{(E')^*,E'}.$$

Now we can give another definition for S :
for $f \in L^2(E,P)$, let us consider the vector function : $x \rightsquigarrow f(x)x$. It belongs scalarly in $L^1(E,P)$, that is to say,

$$\text{for every x}' \in E' : \ x \rightsquigarrow f(x) < x', x > \quad \text{is integrable with respect to P.}$$

Then we define its weak integral as the unique element from $(E')^*$ such that,

$$\text{for every x}' \in E' : < \int f(x)xP(dx), x' >_{E'^*,E'} := \int_E < x', x > f(x)P(dx).$$

Now Sf is the barycenter of the ***measure*** $f(x)P(dx)$.

We shall denote by $H(P)$ (or simply H if there is no ambiguity) the subspace $S(L^2(E,P))$ of $(E')^*$.
$H(P)$ is called the ***"Cameron-Martin space"*** of P.
The closure of $j(E')$ in $L^2(E,P)$ is called the ***"Gaussian space"*** of P (or the ***"first Wiener chaos"***) and will be denoted by $E_2'(P)$. It is an Hilbert space for the topology induced by the topology of $L^2(E,P)$.

Let us notice that :

- $H(P) = S(E_2'(P))$ and that :
- S, when restricted to $E_2'(P)$, is injective, whence bijective, with values in $H(P)$.

Actually :

Suppose first that $f \in L^2(E,P)$ is orthogonal to $E_2'(P)$; that means that :

$$\text{for every } x' \in E' : \quad \int_E j(x') \ (x) \ f(x)P(dx) = 0.$$

Then we have

$$\text{for every } x' \in E' : \int_E < x', x > f(x)P(dx)$$
$$= < \int_E f(x)xP(dx), x' > = 0.$$

But this means that :

$$Sf = 0.$$

The first assertion is therefore proven.

For proving the second assertion it suffices to notice that if

$$f \in E'^2(P)$$

is such that

$$S(f) = 0,$$

then :

$$f \perp E'_2(P) \quad \text{and} \quad f \perp f, \quad \text{whence } f = 0.$$

Now we can put on $H(P)$ an Hilbertian structure : the image of the Hilbertian structure of $E'_2(P)$ by the linear isomorphism S.
Then we have :

a) $(Sf, Sg)_{H(P)} = \int_E f(x)g(x)P(dx), \quad \forall (f,g) \in \left(E'_2(P)\right)^2,$

b) if $f \in E'_2(P)$ and $y' \in E'$:

$$< Sf, y' >_{(E')^*, E'} = \int_E f(x) < y', x > P(dx) = \left(f, j(y')\right)_{E'_2(P)},$$

c) if $(x', y') \in E' \times E'$:

$$\left(S \circ j(x'), S \circ j(y')\right)_{H(P)} = \int_E < x', x >< y', x > P(dx).$$

Remark : If $\operatorname{Supp} P = E$, the map $j : E' \to E'_2(P)$ is injective. Actually, if $j(x') = 0$, then :

$$\int < x', x >^2 P(dx) = 0\,;$$

therefore :

$$< x', \bullet > \quad \text{is null, almost everywhere.}$$

But $< x', \bullet >$ being continuous, we have :

$$< x', \bullet > = 0 \quad \text{everywhere.}$$

For the questions about which we shall be concerned, the case where $H(P) \subset E$ is of paramount importance . We shall give a sufficient condition under which $H(P) \subset E$.

If $h \in H(P)$ we shall denote by $\widetilde{h}$ the element $S^{-1}(h)$ of $E'_2(P)$.

Lemma 1 : *Let us suppose $H(P) \subset E$. If $h \in H(P)$, let us denote by P_h the translate of P by h, ($P_h(A) = P(A+h)$). Then we have :*

- $P_h \equiv P$
- $\frac{dP_h}{dP} = \exp\{\widetilde{h} - \frac{1}{2}\|h\|^2_{H(P)}\}$.

Proof :

(We shortly write H instead of $H(P)$) . $\widetilde{h}$ being a (P-class of) Gaussian random variable with variance $\|h\|^2_H$, we have:

$$\int_E \exp(\widetilde{h})dP = \exp\{\frac{1}{2}\|h\|^2_H\}.$$

Therefore :

$$\exp\{\widetilde{h} - \frac{1}{2}\|h\|^2_H\}.P \quad \text{is a probability.}$$

Now, let us recall some results about the Fourier transform (or characteristic function) of a Borelian probability on E, ***not necessarily Gaussian centered*** :

If Q is a Borelian probability on E, we define its characteristic function as the map : $E' \to \mathbb{C}$ defined by :

$$x' \rightsquigarrow \int e^{i<x',x>_{E',E}} Q(dx) := \widehat{Q}(x').$$

Then two probabilities coincide if and only if their characteristic functions coincide.

In our case, where P is Gaussian, we have :

$$\widehat{P}(x') = \exp\{-\frac{1}{2}\|j(x')\|^2_{L^2(E,P)}\} = \exp\{-\frac{1}{2}\|Sj(x')\|^2_{H(P)}\}.$$

Now :

$$\widehat{P}_h(x') = \exp\{i < x', h >\}\widehat{P}(x').$$

Moreover we have :

$$\int_E \exp\{i < x', x >\} \ \exp\{\widetilde{h}(x) - \frac{1}{2}\|h\|^2_H\} \ P(dx)$$

$$= \exp\{-\frac{1}{2}\|h\|^2_H\} \int \exp\{i(< x', x > -i\widetilde{h}(x))\}P(dx)$$

$$= \exp\{-\frac{1}{2}\|h\|^2_H\} \ \exp\{\frac{1}{2}\|h\|^2_H - \frac{1}{2}\|jx'\|^2_{E'_2(P)} + 2i\big(j(x')\,,\,\widetilde{h}\big)_{L^2(E,P)}\}.$$

Actually, the 2-dimension random vector $(\widetilde{h}, j(x'))$ is Gaussian (centered) and its covariance matrix is equal to :

$$\begin{pmatrix} \|j(x')\|^2_{E'_2} & (j(x'),\widetilde{h})_{E'_2} \\ (j(x'),\widetilde{h})_{E'_2} & \|h\|^2_H \end{pmatrix}$$

From what precedes we deduce immediately that P_h and $\exp\{\widetilde{h}-\frac{1}{2}\|h\|^2_H\}P$ have the same characteristic functional.

— *Lemma 1 is proven.*—

Corollary : *If F is a P-measurable vector subspace, carrying P, then : $H(P) \subset F$. (We suppose that $H(P) \subset E$).*

Proof :

Let us suppose the contrary : there exists $h \in H(P)$ such that $h \notin F$. Then :

$$(h+F)\cap F = \emptyset.$$

But

$$P(h+F) > 0, \qquad \text{by Lemma 1}$$

and since : $P(F) = 1$, we have :

$$P\big((h+F)\cap F\big) > 0.$$

There is a contradiction.

— *The Corollary is proven.*—

Remark : we shall see later that for every P-measurable vector space we have : $P(F) = 0$ or 1 (we shall give the proof if E is Lusin).

Lemma 2 : *Let E and F be two Hausdorff l.c.v.s., and P_1 be a Gaussian probability on E. Let $u : E \to F$ continuous linear and $P_2 = u(P_1)$, then P_2 is Gaussian. Let $H_i := H(P_i), (i = 1, 2)$ then :*

- *if $H_1 \subset E$ we have : $H_2 \subset F$.*

Moreover :

- $u(H_1) = H_2$,
- *$u_{|H_i}$ is continuous from H_1 into H_2 (for the Hilbertian topologies).*
- *the image of the unit ball of H_1 by u'^* is the unit ball of H_2.*

Proof :

The map u induces a linear map U from $L^2(F, P_2)$ into $L^2(E, P_1)$ defined by :

$$Uf = f \circ u.$$

It is easy to see that U is an isometry from $F_2'(P_2)$ ***into*** $E_2'(P_1)$; let $G = U\big(F_2'(P_2)\big)$.

Let S_1 (resp. S_2) the barycenter map considered as a map from $E_2'(P_1)$ $\big($ resp. $F_2'(P_2)\big)$ onto H_1 (resp. H_2). S_i is an isomorphism between Hilbert spaces.
I assert that :

$$u'^* \circ S_1 \circ U = S_2 \,.$$

In fact, let $g \in F_2'(P_2)$ then :

$$\begin{aligned} u'^* \circ S_1 \circ U(g) &= u'^*\Big(\int_E g\big(u(x)\big)xP_1(dx)\Big) \\ &= \int_E g\big(u(x)\big)u(x)P_1(dx) \\ &= \int_F g(y)\; yP_2(dy) \\ &= S_2(g)\,. \end{aligned}$$

We have proven that :

$$u'^*(H_1) \supset u'^*(S_1(G)) = H_2\,.$$

Moreover, if $h \in H_1$ with $\widetilde{h}$ orthogonal to G, we have

$$u'^*(h) = 0 \text{ , so } u'^*(H_1) = H_2\,.$$

In fact,

$$\begin{aligned} \text{for each } y' \in F' : \quad < u'^*(S_1 h), y' > &= u'(y')\int_E \widetilde{h}(x)xP_1(dx) \\ &= \int_E \widetilde{h}(x)y'\big(u(x)\big)P_1,(dx) \\ &= 0\,. \end{aligned}$$

Now it is clear that

$$u'^*_{|H_1} \text{ is continuous from } H_1 \text{ to } H_2, \text{ with norm one.}$$

— *Lemma 2 is proven.* —

In the sequel we shall do the following hypothesis about P :

(H) for every $\varepsilon > 0$, there exists a compact $K_\varepsilon \subset E$ such that :

$$P(K_\varepsilon) > 1 - \varepsilon.$$

A probability (not necessarily Gaussian) satisfying the hypothesis (H) is said a ***"Radon probability"***. A Borelian probability will be, ipso facto, a Radon probability in the following cases (that we shall consider in the sequel) :

- E is a separable Banach space
- more generally : E is a Lusin space.

Lemma 3 : *If P is a Radon probability (not necessarily Gaussian), $\widehat{P}$ is continuous from E'_c (E' with the topology of compact convergence) into $\mathbb{C}$.*

Proof :

Let $\varepsilon \in]0,1[$ be given and let K_ε be a compact subset of E such that :

$$P(K_\varepsilon) > 1 - \varepsilon.$$

Let at last $p_{K_\varepsilon^\circ}$ the gauge of the absolute polar of K_ε, that is to say :

$$K_\varepsilon^\circ = \{x' \in E', \quad |< x', x >| \leq 1, \quad \forall x \in K_\varepsilon\},$$

$$p_{K_\varepsilon^\circ}(x') = inf\{\lambda, \quad \lambda > 0, \quad x' \in \lambda K_\varepsilon^\circ\}.$$

I assert that :

$$| 1 - \widehat{P}(x') | \leq 2\varepsilon + p_{K_\varepsilon^\circ}(x'), \quad \forall x'.$$

Actually, on one hand we have :

$$x'(K_\varepsilon) \subset [-p_{K_\varepsilon^\circ}(x'), \ p_{K_\varepsilon^\circ}(x')], \quad \text{by definition;}$$

and on the other hand, if we denote by ν the probability $x'(P)$:

$$\begin{aligned} | 1 - \widehat{P}(x') | &= | \int_{\mathbb{R}} (1 - e^{it}) \nu(dt) | \\ &\leq \int_{\mathbb{R}} | 1 - e^{it} | \nu(dt). \end{aligned}$$

But :

$$\int_{\mathbb{R}} \quad = \int_{|t| \leq p_{K_\varepsilon^\circ}(x')} + \int_{|t| > p_{K_\varepsilon^\circ}(x')}$$

and

$$| 1 - e^{it} | \leq inf(2, | t |), \quad \forall t \in R.$$

Therefore :

$$\int_{|t| \leq p_{K_\varepsilon^\circ}(x')} | 1 - e^{it} | \; d\nu(t) \; \leq \; p_{K_\varepsilon^\circ}(x')$$

and :

$$\int_{|t| > p_{K_\varepsilon^\circ}(x')} | 1 - e^{it} | \; d\nu(t) \leq \; 2\varepsilon$$

and the assertion is proven.

We deduce then immediately that :

$$x' \in \varepsilon K_\varepsilon^\circ \Rightarrow | 1 - \widehat{P}(x') | \leq 3\,\varepsilon,$$

— *Lemma 3 is proven.*—

As a corollary we deduce that, if P is Gaussian, the map $j : E' \to L^2(E, P)$ is continuous for E'_c.

Lemma 4 : *Let P be a Gaussian (Radon) probability on E. If E is quasi-complete (this means that the closed disked hull of every bounded subset is complete), then we have: $H(P) \subset E$.*

Proof :

We have noticed that :

$$j : E'_c \to L^2(E, P) \quad \text{is continuous.}$$

Therefore, for every $f \in L^2(E, P)$, the linear form :

$$x' \rightsquigarrow \int_E f(x) < x', x > P(dx) \quad \text{is continuous on } E'_c.$$

By Mackey's theorem, we deduce that :

$$\int f(x) x P(dx) \in E.$$

— *Lemma 4 is proven.*—

Remark : a complete space being trivially quasi-complete, Lemma 4 can be applied when E is complete.

Now we have the main following result :

THEOREM 1 : *If P is Radon, Gaussian, then $H(P) \subset E$.*

Proof :

Let $\widetilde{E}$ be the completion of E and i be the canonical injection $E \to \widetilde{E}$. Let $\widetilde{P} = i(P)$. By Lemma 2 and Lemma 4, we have :

$$i(H(P)) = H(\widetilde{P}) \subset \widetilde{E}.$$

But $E = i(E)$ is a subspace carrying $\widetilde{P}$. Therefore, by corollary of Lemma 1 :

$$H(\widetilde{P}) \subset E$$

— *Q.E.D.*—

It is easy to see that $H(P)$ and $H(\widetilde{P})$ are isometric.

Remark : under the hypothesis of the theorem, the canonical injection $H(P) \to E$ is continuous.

For proving this fact it suffices to prove that, for every $f \in L^2(E,P)$ and every continuous semi-norm q, we have :

$$q\Big(\int f(x)xP(dx)\Big) \leq c\|f\|_{L^2(E,P)}, \quad \text{where } c \text{ is a constant.}$$

By a result of Fernique and Skorokhod, we know that, for every continuous semi-norm q on E, we have :

$$\int_E |\, q(x)\,|^2 \, P(dx) < \infty.$$

(A non-trivial fact if E is infinite-dimensional).
Therefore :

$$q\Big(\int_E \big(xf(x)P(dx)\big) \leq \int |f(x)|\, q(x)P(dx)$$
$$\leq \|f\|_{L^2_{(E,P)}} \, \|q\|_{L^2_{(E,P)}}.$$

The assertion is proven.

We shall call a ***"Gauss space"*** (or ***"abstract Wiener space"***) a couple (E,P) where E is a Hausdorff l.c.s. and P is a centered Gaussian Radon measure on E.

2 - Cameron-Martin space of a Lusin Gauss space

From now on, E will denote a **Lusin** locally convex vector space and P a Gaussian Borelian probability on E (P is automatically Radon).

We shall suppose that $\mathbf{Supp}\, P = E$. (This is not a loss of generality). We shall suppose moreover (unless the contrary is specified) that $\dim E = +\infty$. (The case where $\dim E < \infty$ being well known).

As a first example of this situation we have the ***"discrete white noise"*** or the canonical Gaussian measure on $\mathbb{R}^{\mathbb{N}}$ which is described as follows :

Let $E = \mathbb{R}^{\mathbb{N}}$ with the product topology (it is a polish space. therefore Lusinian). Its dual E' is the space $\mathbb{R}_0^{\mathbb{N}}$ of sequences of real numbers "almost null". Let $P = \gamma_1^{\otimes \mathbb{N}}$ the canonical Gaussian measure on E (γ_1 is the $\mathcal{N}(0,1)$ probability on R).

Here we have $E' \subset E$. If $x' = (x'_n) \in E', j(x')$ is the (P-class) function $x \rightsquigarrow \sum x_n x'_n$ where $x = (x_n)_{n \in \mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$).
Obviously :

$$\|j(x')\|^2_{L^2(\mathbb{R}^{\mathbb{N}},P)} = \sum |x'_n|^2 = \|x'\|^2_{\ell^2}.$$

Therefore

$$E'_2(P) \simeq \ell^2.$$

Let us now determine the Cameron-Martin space of P.
Let $x' \in E'$, then

$$S\big(j(x')\big) = \int_{\mathbb{R}^{\mathbb{N}}} \big(\sum_n x'_n x_n\big) x \; P(dx).$$

Therefore :

$$S\big(j(x')\big)_k = \int_{\mathbb{R}^{\mathbb{N}}} \big(\sum_n x'_n x_n\big) \; x_k \; P(dx) = x'_k. \quad \forall k.$$

We have then

$$S\big(j(E')\big) = E' \quad \text{and} \quad H(P) = \ell^2.$$

The above example is actually the model for a Gaussian probability on a Lusin space, as we shall now see.

THEOREM 2 : *Let E be a Lusin l.c.s. (Hausdorff) with infinite dimension and let P be a Gaussian probability on E,* ***whose support is equal to E.*** *Then there exists a continuous linear injection from E into $\mathbb{R}^{\mathbb{N}}$, denoted π, such that $\pi(P)$ is the canonical Gaussian measure on $\mathbb{R}^{\mathbb{N}}$.*

Proof :

The family of continuous functions on E :

$$\{\langle x', \bullet\rangle_{E',E}\,,\ x' \in E'\} \quad \text{is separating.}$$

E being Lusin and $\dim E = +\infty$ there exists an ***infinite*** sequence $(x'_n)_{n\in\mathbb{N}}$ in E' separating points of E. Then $\sigma\{x'_n, n \in \mathbb{N}\}$ is identical to the Borelian σ-field of E, and the $j(x'_n)$ generate $E'_2(P)$.

Let (y'_n) be the orthonormal sequence obtained from the (x'_n) by application of the Schmidt orthogonalization procedure to $\big(j(x'_n)\big)_n$. The (y'_n) separate points of E, and

$$\text{card}\ \{y'_n,\ n \in \mathbb{N}\} = +\infty.$$

Let π be the map $x \rightsquigarrow (\langle y'_n, x\rangle)_n$ from E into $\mathbb{R}^{\mathbb{N}}$. It is trivial to see that π is linear, continuous, injective and that :

$$\pi(P) = \gamma_1^{\otimes\mathbb{N}}.$$

— *Q.E.D.* —

Remark 1 : π being continuous, it is clearly Borelian. Moreover π being injective, E and $\mathbb{R}^{\mathbb{N}}$ being Lusinian, π is ***bimeasurable*** (not necessarily onto).

Remark 2 : It is easy to see that if E is finite dimensional (or more generally $H(P)$ is finite dimensional) there exists n and a linear continuous (not necessarily injective) application $\pi : E \to \mathbb{R}^n$, such that $\pi(P) = \gamma_n$.

Coming back to the situation $\dim E = +\infty$, $\text{Supp}\, P = E$, we see that $H(P)$ is separable, therefore a Polish space. Therefore, $H(P)$ is a Borel subset of E, and we can speak of $P\big(H(P)\big)$.

Proposition 1 : *If H denotes the Cameron-Martin space of E, then $P(H) = 0$.*

Proof :

We can suppose that (E, P) is the canonical Gauss space $\mathbb{R}^{\mathbb{N}}$. Then :

$$H = \ell^2.$$

Now the random variables $x \rightsquigarrow x_n$ on $\mathbb{R}^{\mathbb{N}}$ are independent and $\mathcal{N}(0,1)$. Therefore :

$$\sum x_n^2 = +\infty, \quad \text{almost surely},$$

or equivalently :

$$P(\ell^2) = 0.$$

— *Q.E.D.* —

Remark : The result is clearly false if dim $H < \infty$. For instance, $\gamma_n(\mathbb{R}^n) = 1$ (in this case $H = \mathbb{R}^n$).

We can actually prove that $P(H) = 0$ if and only if dim $H = \infty$. Let us notice that in any case $P(H) = 0$ or 1.

Proposition 2 : *If $(\widetilde{x}_n)$ is an orthonormal basis of $E_2'(P)$ and $x_n = S(\widetilde{x}_n)$, then for every $x' \in E'$: $\left(\sum_{n \leq k} \widetilde{x}_n(\bullet)\ \langle x', x_n\rangle_{E',E}\right)_k \longrightarrow \langle x', \bullet\rangle$, almost surely.*

Proof :

The random variables $\widetilde{x}_n(\bullet)\langle x', x_n\rangle_{E',E}$, $(n \in \mathbb{N})$ are independent Gaussian with variances

$$\langle x', x_n\rangle_{E',E}^2 .$$

But

$$\langle x', x_n\rangle_{E',E} = (Sj(x'), x_n).$$

Therefore :

$$\sum_n |\langle x', x_n\rangle|^2 = \|Sj(x')\|_H^2 < \infty$$

and

$$\sum_n \widetilde{x}_n(\bullet)\ \langle x', x_n\rangle \text{ converges almost surely.}$$

We have to prove that the limit is $j(x')$.
Actually we have

$$\mathbb{E}\Big\{j(x') \mid \widetilde{x}_i\,, \quad {\scriptstyle 0\le i\le n}\Big\} = \sum_{i=0}^{n} \mathbb{E}\Big\{j(x')\ \widetilde{x}_i\Big\}\ \widetilde{x}_i$$

$$= \sum_{i=0}^{n} (Sj(x')\,,\, x_i)\ \widetilde{x}_i = \sum_{i=0}^{n} \langle x'\,, x_i\rangle_{E',E}\ \widetilde{x}_i.$$

Now it suffices to apply the martingale convergence theorem (for real martingales).

— *Q.E.D.* —

Remark : The preceding result is very poor. We did not prove that

$$\sum_{n\le k} \widetilde{x}_n.x_n \longrightarrow Id \text{ for the weak topology,}$$

the exceptional set of non convergence being a priori dependent of x'.

In chapter two we shall prove that $\sum_{n\le k} \widetilde{x}_n.x_n \longrightarrow Id$, for the topology of E.

Proposition 3 : *The canonical injection $i : H(P) \to E$ is compact. We have even more if K denotes the unit ball of $H(P), i(K)$ is a compact subset of E.*

Proof :

Let us suppose first that E is a Banach space (separable since E is Lusin).

First we see that $i(K)$ is closed in E. Actually, i being continuous, it is weakly continuous. Therefore $i(K)$ is weakly compact in E, then closed for the topology of E, since it is convex.

We shall now prove that $i(K)$ is relatively compact.

Identifying the dual of $H(P)$ with $E_2'(P)$, the transpose of i is equal to j, and it suffices to prove that $j : E' \to E_2'(P)$ is a compact map. This last fact may be verified as follows :

Let (x_n') a sequence of elements of $B_{E'}$, (the unit ball of E') , converging weakly to $x' \in B_{E'}$, therefore converging uniformly on every compact of E ($B_{E'}$ being an equicontinuous set).

By Fernique's theorem :

$$\int_E \|x\|_E^2 \; P(dx) < \infty\,;$$

applying Lebegue's dominated convergence theorem, we obtain :

$$\left(\int_E |\langle x'_n - x', x\rangle_{E',E}|^2 \; P(dx)\right)_n \longrightarrow 0.$$

Therefore

$$j(x'_n) \longrightarrow j(x') \text{ in } E'_2(P).$$

— *In this case the proposition is proven.* —

In the general case, let us give a sketch of the proof :

E is a projective limit of a family (E_i) of Banach spaces, (indexed by a filtering set). Let $\pi_i : E \longrightarrow E_i$ (we can suppose that every π_i is dense) and let $P_i = \pi_i(P)$. We know that $\pi_i(K)$ is the unit ball of $H(P_i) \subset E_i$.

It suffices to notice that the unit ball K of $H(P)$ is compact if and only if the $\pi_i(K)$ are. But it is the case by the first part of the proof.

— *Q.E.D.* —

Proposition 4 : *Let K be a Hilbert space and $u : E \to K$ continuous linear. Then $u_{|H(P)}$ is an Hilbert-Schmidt mapping from $H(P)$ into K.*

Proof :

There exists a Banach F, such that u has the following factorization :

$$E \xrightarrow{\pi} F \xrightarrow{\beta} K$$

$$u = \beta \circ \pi$$

with β and π linear continuous, π a dense map.
Let $Q = \pi(P)$; Q is Gaussian and r sends $H(P)$ onto $H(Q)$. Then $u_{|H(P)}$ can be factorized as follows :

$$H(P) \xrightarrow{\pi'} H(Q) \xrightarrow{\beta'} K$$

$$u_{|H(P)} = \beta' \circ \pi'$$

(π' is the application π when it is considered as a mapping from $H(P)$ into $H(Q)$ and $\beta' = \beta_{|H(Q)}$). It is sufficient to prove that β' is Hilbert-Schmidt and we can therefore suppose that E is a separable Banach space.

We already know that $u_{|H}$ is compact. There exists then an orthonormal basis (e_n) of $H(P)$, an orthonormal sequence (f_n) in K and a sequence (λ_n) of real numbers, converging to zero such that :

$$u_{|H}(h) = \sum_n \lambda_n (h\,,\, e_n)_H \; f_n, \quad \forall h \in H$$

(we shortly write H instead of $H(P)$).

It remains to prove that $\Sigma\, \lambda_k^2 < \infty$.
Let for $n \in \mathbb{N}$:

$$X_n(\bullet) = \sum_{k=0}^{n} \lambda_k \; \widetilde{e}_k(\bullet) \; f_k$$

(with $\widetilde{e}_k = S^{-1}(e_k)$).
X_n is a K-valued random variable defined on (E, P). Clearly :

$$\|X_n(\bullet)\|_K^2 = \sum_{k=0}^{n} \lambda_k^2 \; |\widetilde{e}_k(\bullet)|^2.$$

The random variables $\widetilde{e}_k(\bullet)$ are independent and $\mathcal{N}(0,1)$.

I assert that there exists a constant K_1 (independent from n and c) such that for every $c > 0$:

$$P\Big\{x; \Big|\|X_n(x)\|_K^2 - \sum_{k=0}^{n} \lambda_k^2\Big| \geq c\Big\} \leq \frac{K_1}{c^2} \sum_{k=0}^{n} \lambda_k^4\,. \tag{A}$$

In fact if we apply the Tchebychev's inequality to the centered random variable :

$$\sum_{k=0}^{n} (\widetilde{e}_k^{\,2}(\bullet)\; \lambda_k^2 - \lambda_k^2) = \sum_{k=0}^{n} \lambda_k^2 (\widetilde{e}_k^{\,2}(\bullet) - 1),$$

we obtain (A) (the constant K_1 equals $\mathbb{E}\{(\widetilde{e}_k^{\,2}(\bullet) - 1)^2\}$).
We deduce from (A) that for every $c > 0$:

$$P\Big\{x; \Big|\|X_n(x)\|_K^2 - \sum_{k=0}^{n} \lambda_k^2\Big| < c\Big(\sum_{k=0}^{n} \lambda_k^2\Big)^{\frac{1}{2}}\Big\} \geq 1 - \frac{K_1}{c^2} \frac{\sum_{k=0}^{n} \lambda_k^4}{\sum_{k=0}^{n} \lambda_k^2}$$

$$\geq 1 - \frac{K_1}{c^2} \sup_{0 \leq k \leq n} \lambda_k^2. \tag{B}$$

Now let

$$\rho_{n,c} := \sum_{k=0}^{n} \lambda_k^2 - c\Big(\sum_{k=0}^{n} \lambda_k^2\Big)^{\frac{1}{2}};$$

if $\rho_{n,c} > 0$ then let :

$B_{n,c}$ the open ball in K with center 0 and radius $\sqrt{\rho_{n,c}}$

and if it is not, then :

$$B_{n,c} = \emptyset.$$

It is clear that :

$$\Big\{y \in K, \quad \Big|\|y\|_K^2 - \sum_{k=0}^{n} \lambda_k^2\Big| < c\Big(\sum_{k=0}^{n} \lambda_k^2\Big)^{\frac{1}{2}}\Big\} \cap B_{n,c} = \emptyset.$$

From (B) we deduce :

$$P\Big\{x; \Big|\|X_n(x)\|_K^2 - \sum_{k=0}^{n} \lambda_k^2\Big| < c\Big(\sum_{k=0}^{n} \lambda_k^2\Big)^{\frac{1}{2}}\Big\} \geq 1 - \frac{K_1}{c^2} \sup_{k\in\mathbb{N}} \lambda_k^2.$$

Let us choose now c such that :

$$1 - \frac{K_1}{c^2} \sup_{k\in\mathbb{N}} \lambda_k^2 = \frac{1}{2};$$

we have then :

$$P\Big\{x \in E; \quad X_n(x) \in B_{n,c}\Big\} \leq \frac{1}{2}, \quad \text{for every } n \text{ such that } B_{n,c} \neq \emptyset. \tag{C}$$

Now let us suppose that $u_{|H}$ is ***not*** Hilbert-Schmidt.
Then

$$\sum_{k=0}^{n} \lambda_k^2 - c\Big(\sum_{k=0}^{n} \lambda_k^2\Big)^{\frac{1}{2}} \xrightarrow[n\to\infty]{} \infty.$$

Therefore

$B_{n,c} \neq \emptyset$ if n is sufficiently large.

Moreover

$$B_{n,c} \uparrow K, \ (n \longrightarrow \infty).$$

But this last result contradicts (C).

Actually we shall see (Chapter two) that :

$$\Big(\sum_{n\leq k}\widetilde{e}_n(\bullet)\ e_n\Big)_k \longrightarrow Id_E\,,\quad \text{almost surely.}$$

Therefore

$$P\big\{x; X_n(x) \in B_{n,c}\big\} \xrightarrow[n\to\infty]{} 1.$$

There is a contradiction with (C) , therefore :

$$u_{|H} \text{ is Hilbert-Schmidt.}$$

— *Q.E.D.*—

Corollary 1 : *Let K be an Hilbert space and $v : K \to E'$ linear and continuous when E' is equipped with the strong topology. Then $j \circ v : K \to E'_2(P)$ is Hilbert-Schmidt.*

Proof :

Let us identify K with its dual and let v' be the transpose of v ; v' maps the bidual E'' of E into K. Let Q be the image of P by the canonical injection $E \to E''$. Identifying E with a subset of E'', we know that :

$$H(P) = H(Q) \subset E''.$$

But

$$v'_{|H(Q)} = v'_{|H(P)}$$

and

$$v'_{|H(Q)} \text{ is Hilbert-Schmidt.}$$

Therefore

$$v'_{|H(P)} \text{ is Hilbert-Schmidt, and also } j \circ v.$$

— *Corollary 1 is proven.*—

Remark : Clearly $S \circ j \circ v$ is Hilbert-Schmidt from K into $H(P)$.

Corollary 2 : *Let K_1 and K_2 be two Hilbert spaces and $u_1 : K_1 \to E'$, $u_2 : E \to K_2$ linear continuous (E' being equipped with the strong topology). Then :*

$$u_2 \circ i \circ S \circ u_1 : K_1 \to K_2 \text{ is nuclear}$$

$$(K_1 \xrightarrow{u_1} E' \xrightarrow{S} H \xrightarrow{i} E \xrightarrow{u_2} K_2).$$

Therefore we can speak of the trace of the above application.

We have already seen that if F is a P-measurable vector subspace of E such that $P(F) = 1$, then $H \subset F$. We have seen too that if dim $H = +\infty$: $P(H) = 0$.

— *Corollary 2 is proven.*—

Now we shall give some "zero or one laws".

Proposition 5 : *Let G be an additive subgroup of E, P-measurable. Then for every $a \in E$ we have*

$$P\{a + G\} = \begin{cases} 0 \\ 1 \end{cases}$$

(A trivial fact if dim $E < \infty$).

Proof :

We have supposed E Lusin, $\operatorname{Supp} P = E$ and $\dim E = \infty$! Now we may suppose that $E = \mathbb{R}^{\mathbb{N}}$, **with the canonical Gaussian measure on it**. We need several lemmas :

Lemma 1 : *If G is a P-measurable additive subgroup of $\mathbb{R}^{\mathbb{N}}$, containing $H(P)$ (in our case $H(P) = \ell^2$), then for every $a \in \mathbb{R}^{\mathbb{N}}$:*

$$P\{a + G\} = \begin{cases} 0 \\ 1 \end{cases}$$

Proof :

Let (X_n) be the projection of index n from $\mathbb{R}^{\mathbb{N}}$ into $\mathbb{R}$. Then the (X_n) are independent and $\mathcal{N}(0,1)$ and moreover $\mathcal{B}_{\mathbb{R}^{\mathbb{N}}}$ is equal to $\sigma(X_n, n \in \mathbb{N})$.

Now I assert that

$$(a + G) \text{ is P-independent from } \mathcal{B}_{\mathbb{R}^{\mathbb{N}}}.$$

Actually let:

$$Y_n = \sum_{i \leq n} X_i \, e_i \quad \text{and} \quad Z_n(\bullet) = Id_{\mathbb{R}^{\mathbb{N}}} - Y_n.$$

($(e_i)_{i \in \mathbb{N}}$ is the canonical "basis" of $\mathbb{R}^{\mathbb{N}}$).
Since $e_i = S(X_i)$: Y_n takes its values in $H(P)$. Then, since $H(P) \subset G$ we have :

$$a + G = \{Y_n + Z_n \in a + G\} = \{Z_n \in a + G\}$$

Now it is clear that for each n, $\{Z_n \in a + G\}$ is independent from $\sigma(X_i, i \in n\}$, therefore $(a + G)$ is independent of $\sigma(X_n, n \in \mathbb{N})$.

Our assertion is then proven. In particular $a + G$ is independent from itself.

— *Lemma 1 is proven.* —

Now we shall prove that the conclusion of the preceding lemma does not need the hypothesis $H \subset G$. For this we shall need the following lemma :

Lemma 2 : *Let $A \subset \mathbb{R}^{\mathbb{N}}$ be a P-measurable set such that $P(A) > 0$. For every $h \in H$, there exists a real number $r_h > 0$ with property :*

$$|r| < r_h \Rightarrow rh \in A - A.$$

Proof :

The result is true if $h = 0$. So we suppose $\|h\|_{\ell^2} = 1$. To h there corresponds a random variable $\widetilde{h}$ in $E_2'(P)$ as seen before.
Let

$$Y(\bullet) = \widetilde{h}(\bullet)h \quad \text{and} \quad Z = Id - Y.$$

They are two random variables with values in $\mathbb{R}^{\mathbb{N}}$, independent. Therefore :

$$P(A) = P\{Y + Z \in A\} = \int_{\mathbb{R}^{\mathbb{N}}} P\{Y \in -z + A\} \; P_Z(dz) > 0$$

(P_Z is the law of Z).
Therefore there exists some $z_0 \in \mathbb{R}^{\mathbb{N}}$ such that :

$$P\{Y \in -z_0 + A\} > 0.$$

Now let

$$C := \{r \in \mathbb{R}, \quad rh \in -z_0 + A\}.$$

Clearly

$$\gamma_1(C) = P\{Y \in -z_0 + A\} > 0.$$

Therefore $C - C$ is a neighbourhood of zero : there exists $r_h > 0$ such that :

$$]-r_h, r_h[\subset C - C \subset \{r \in \mathbb{R}, \quad rh \in A - A\}.$$

— *Lemma 2 is proven.* —

Now we are able to prove the proposition :

Let us suppose that $P(a+G) > 0$.
Then for every $h \in H$ there exists an ***integer*** n_h such that

$$h \in n_h \ ((a+G) - (a+G)) = n_h(G), \quad \text{(since } G - G = G).$$

Therefore $H \subset G$, and by Lemma 1 we conclude that

$$P(a+G) = 1.$$

— *Q.E.D.* —

Remark : In particular if F is a P-measurable vector subspace, $P(F) = 0$ or 1.

We can now see from another way that if $P(F) = 1$, then $H \subset F$:
In fact, if $h \in H$, there exists $r_h > 0$ such that

$$rh \in F - F = F, \quad \text{if } |r| \le r_h.$$

But F being a vector space, $rh \in F$ for every r.

In the next chapter we shall see that the reproducing kernel space is the intersection of the subspaces carrying P.

Definition : *We shall call **"abstract Wiener space"** (or shortly **"Wiener space"**) a triple (E, H, P) where E is a Lusin l.c.s., P is a centered Gaussian measure on E and $H = H(P)$ is the Cameron-Martin space of P.*

3 - Some examples

First let us give a trivial example :

Let $E = \mathbb{R}^n$ and $p \leq n$, a natural number ; let γ_p the normal measure on $\mathbb{R}^p$ having the density $(2\pi)^{\frac{-p}{2}} \exp\{-\frac{1}{2}\|x\|^2\}$. Let P the image of γ_p by the canonical injection $\mathbb{R}^p \to \mathbb{R}^n$. P is a Gaussian measure on $\mathbb{R}^n$. Moreover :

$$E_2'(P) = \mathbb{R}^p \text{ and } H(P) = \operatorname{Supp} P.$$

(Therefore $P\big(H(P)\big) = 1$, $H(P)$ is finite dimensional)

$H(P)$ is dense in $\mathbb{R}^n$ if and only if $p = n$.

Now let us give some less trivial examples.

Before doing this, we shall recall the notion of reproducing kernel Hilbert space of a centered real Gaussian process $(X_t)_{t\in T}$, ***indexed by an arbitrary set*** T **:**
Let us denote K the covariance :

$$K(s,t) = \mathbb{E}\{X_s X_t\}.$$

Then $\mathcal{H}(T,K)$ is the subspace of $\mathbb{R}^T$ generated by the functions : $K(s,\bullet), s\ (s \in T)$.
If $u = \sum_{i=1}^{m} \alpha_i K(s_i,\bullet)$ and $v = \sum_{j=1}^{n} \beta_j K(t_j,\bullet)$ are elements of $\mathcal{H}(T,K)$ we set :

$$(u,v)_{\mathcal{H}(T,K)} := \sum_{ij} \alpha_i \beta_j K(s_i,t_j).$$

It is easily proven that the right member is independent from the representations chosen of u and v, and that $(\bullet,\bullet)_{\mathcal{H}}$ is symmetric non-negative definite (not necessarily a scalar product). $\mathcal{H}(T,K)$ is therefore a prehilbertian vector space, whose the associated Hilbert space is called the ***"reproducing kernel Hilbert space of the process*** (X_t)***"*** and is denoted by $H(T,K)$. Moreover, the elements in $H(T,K)$ are functions on T (a non trivial fact, a priori) :

$$H(T,K) \subset \mathbb{R}^T.$$

Note that "reproducing property" is satisfied, that is to say :

$$\big(f(\bullet), K(s,\bullet)\big)_{H(T,K)} = f(s), \quad \forall s \in T, \quad \forall f \in H(T,K).$$

This relation generalizes this one : $\big(K(s\,,\,\bullet)\,,\,K(t\,,\,\bullet)\big) = K(s\,,\,t)$.
For all what precedes about reproducing kernel spaces of a Gaussian process, we can see Neveu [Séminaire, Université de Montréal, Eté 1968], for instance.

The situation "abstract Wiener space" is an important example of what precedes. If (E, H, P) is an abstract Wiener space then the process indexed by E' :

$$x' \rightsquigarrow \langle x', \bullet\rangle_{E',E} \text{ is a Gaussian process}$$

with covariance

$$(x', y') \rightsquigarrow \mathbb{E}\{\langle x', \bullet\rangle\langle y', \bullet\rangle\}.$$

Then it is easy to see that the Cameron-Martin space of P is identical to the reproducing kernel space of the process.

Now we give some more concrete examples of this situation.

Let us begin by a general consideration :
Let T be a compact metrisable space and let $E = \mathcal{C}(T)$ (E is a separable Banach space). Let P be a Gaussian probability on $\mathcal{C}(T)$. Then the process $(X_t)_{t\in T}$ defined by :

$$X_t(\omega) = \omega_t \quad \text{if } \omega \in \mathcal{C}(T)$$

is a Gaussian process under P. In this case it is easily proven that :

- for $t \in T \ : \ S(\delta_t) = K(t, \bullet)$, (if $K(s,t) = \mathbb{E}\{X_s X_t\}$)
- if μ is a ***real*** measure on T : $S(\mu)(\bullet) = \int_T K(s, \bullet)\, d\mu(s)$
- if $x = S(\mu),\ y = S(\nu)$ are elements of $H(P)$then

$$(x\,,\,y)_{H(P)} = \int_{T\times T} K(s,t)\, d\mu(s)\, d\nu(t).$$

We shall apply what precedes to the following cases.

Example 1 : Brownian motion on $[0,1]$

P is the Wiener measure on $\mathcal{C}([0,1])$, carried by the space $\mathcal{C}_0([0,1])$ of functions null to zero, and continuous. The covariance of the process is

$$K(s,t) = s \wedge t.$$

The Cameron-Martin space is the completion of the space of functions on $[0,1]$ from the form :

$$\int_{[0,1]} (s \wedge \bullet)\; d\mu(s)$$

where μ is a real measure on $[0,1]$.
Now if F_μ is the distribution function of μ, by the generalised integration by parts formula, we have, for every t :

$$\begin{aligned}\int_{[0,1]} (s \wedge t)\; dF_\mu(s) &= [(s \wedge t)\; F_\mu(s)]_0^1 - \int_0^1 (s \wedge t)'\; F_\mu(s)\; ds \\ &= tF_\mu(1) - \int_0^t F_\mu(s)\; ds \\ &= \int_0^t \mu(]s,1])\; ds\end{aligned}$$

Therefore

$$S(\mu)(\bullet) = \int_0^\bullet \mu(]s,1])\; ds.$$

Now the $H(P)$-norm of $S(\mu)$ is equal to :

$$\begin{aligned}\int_{[0,1]^2} (s \wedge t)\; d\mu(s)\; d\mu(t) &= \int_{[0,1]} d\mu(t) \Big(\int_{[0,1]} (s \wedge t)\; dF_\mu(s)\Big) \\ &= \int_{[0,1]} \Big(\int_0^t \mu(]s,1])\; ds\Big)\; dF_\mu(t) \quad \text{(by what precedes).}\end{aligned}$$

Integrating by parts once more, the above expression is seen equal to :

$$\begin{aligned}&\Big[F_\mu(t) \int_0^t \mu(]s,1]\; ds\Big]_0^1 - \int_0^1 F_\mu(t)\; \mu(]t,1])\; dt \\ &= F_\mu(1) \int_0^1 \mu(]s,1])\; ds - \int_0^1 F_\mu(t)\; \mu(]t,1])\; dt \\ &= \int_0^1 \mu(]t,1])^2\; dt.\end{aligned}$$

The function $\int_0^\bullet \mu(]s,1])\, ds$ is null at zero, absolutely continuous ; its derivative is Lebesgue-almost everywhere equal to $\mu(]\bullet,1])$.

Now let us notice that the functions $s \rightsquigarrow \mu(]s,1])$ are dense in $L^2([0,1],dt)$, when μ describes the set of real measures on $[0,1]$ We can conclude that the Cameron-Martin space of the Wiener measure is the vector space of functions from $[0,1]$ into $\mathbb{R}$, absolutely continuous, vanishing at zero and whose the distributional derivative belongs to $L^2([0,1],dt)$, equipped with the scalar product :

$$(f,g) \rightsquigarrow \int \dot{f}(t)\ \dot{g}(t)\ dt.$$

Example 2 : The Brownian bridge

Now let P the measure on $\mathcal{C}([0,1])$ corresponding to the Brownian bridge on $[0,1]$. The covariance of the Brownian bridge process is :

$$(s,t) \rightsquigarrow K(s,t) = s \wedge t - st.$$

For μ, a real measure on $[0,1]$ we have for each t :

$$\int_{[0,1]} (s \wedge t - st)\ d\mu(s) = \int_{[0,t]} \mu(]s,1]\ ds - t \int_{[0,1]} \mu(]s,1])\ ds.$$

Moreover :

$$\int_{[0,1]^2} (s \wedge t - st)\ d\mu(s)\ d\mu(t) = \int_0^1 \Big(\mu(]s,1])\Big)^2\ ds - \Big(\int_0^1 \mu(]s,1])\ ds\Big)^2.$$

Now let us notice that the function

$$t \rightsquigarrow \int_0^t \mu(]s,1])\ ds - t \int_0^1 \mu(]s,1])\, ds \text{ is null for } t = 0 \text{ and } t = 1\,.$$

It is absolutely continuous and with derivative equal almost everywhere to :

$$F(t) = \mu(]t,1]) - \int_0^1 \mu(]s,1])\, ds.$$

The $L^2(dt)$ norm of F is equal to :

$$\int_{[0,1]} \Big(\mu(]t,1])\Big)^2\ dt - \Big(\int_0^1 \mu(]t,1])\ dt\Big)^2.$$

From this we can easily see that the Cameron-Martin space of P is the space of absolutely continuous functions on $[0,1]$ vanishing at zero and one, and whose derivative belongs to $L^2([0,1])$ equipped with scalar product :

$$(f,g) \rightsquigarrow \int \dot{f}(t)\ \dot{g}(t)\ dt.$$

Example 3 : The Brownian motion on $\mathbb{R}_+$

Let P be the Gaussian measure on $E = \mathcal{C}(\mathbb{R}_+)$ associated to it. $\mathcal{C}(\mathbb{R}_+)$ will be equipped with the topology of uniform convergence on compacts sets : it is a separable Fréchet space (not Banach), whence a Lusin space.

E' is the space of signed Radon measures on $\mathbb{R}_+$ with compact support.
Then as in Example 1, $H(P)$ is equal to the space of functions $[0,\infty[\longrightarrow\mathbb{R}$ absolutely continuous, vanishing at zero, and whose derivatives belong to $L^2(\mathbb{R}_+, dt)$, with the natural norm

$$\|f\| = \|\dot{f}\|_{L^2(\mathbb{R}_+,dt)} \,.$$

Example 4 : $E = \mathcal{S}'(\mathbb{R}^n)$ (resp. $\mathcal{D}'(\mathbb{R}^n)$) with its usual topology.

It is a Lusin space (not Fréchet). Then $E' = \mathcal{S}(\mathbb{R}^n)$ (resp. $\mathcal{D}(\mathbb{R}^n)$).
By Minlos' theorem there exists a Gaussian probability P on E such that for every $\varphi \in E'$:

$$\langle \varphi, \bullet \rangle_{E',E} \text{ is Gaussian}$$

with variance

$$\|\varphi\|^2_{L^2(\mathbb{R}^n)} \,.$$

The Cameron-Martin space is the space $L^2(\mathbb{R}^n, dt)$, considered as a subspace of E' by the canonical injection

$$L^2(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)\,.$$

Example 5 : $(E_i, H_i, P_i)_{1\le i\le n}$ is a finite family of abstract Wiener spaces.

Then the following is a Wiener space :

$$E := \prod_{i=1}^n E_i\,; \quad H := \bigoplus_{i=1}^n H_i\,; \quad P := \bigotimes_{i=1}^n P_i\,.$$

(E is equipped with the product topology. It is a Lusin space).
P is clearly a Radon measure, since E is Lusin.

We have to prove that P is Gaussian, or what is the same, that for every $x' \in E'$, the random variable on E, $x'(\bullet)$ is Gaussian.
But we have (algebraically) :

$$E' = \prod_{i=1}^n E'_i\,.$$

If $x'_i \in E'_i$, $\quad (1 \leq i \leq n)$, let us denote by X_i the random variable :

$$x \rightsquigarrow X_i(x) = x'_i(x_i)\,,\ \text{if } x = (x_1, ..., x_n) \in E.$$

The (X_i) $(1 \leq i \leq n)$ are P-Gaussian and P independent, and if $x' = (x'_1, x'_2, ..., x'_n)$ we have :

$$x'(\bullet) = \sum_{i=1}^{n} X_i(\bullet).$$

This is a Gaussian random variable.
Moreover :

$$\begin{aligned} \|x'(\bullet)\|^2_{E'_2(P)} &= \sum_{i=1}^{n} \|X_i(\bullet)\|^2_{L^2(E,P)} \\ &= \sum_{i=1}^{n} \|x'_i(\bullet)\|^2_{(E_i)'_2(P_i)}. \end{aligned}$$

From this we conclude that

$$E'_2(P) = \bigoplus_{i=1}^{n} (E_i)'_2(P_i).$$

Considering each $(E_i)'_2(P_i)$ as a subspace of $E'_2(P)$ and each E_i as a subspace of E, and apply the barycenter isomorphism S of $E'_2(P)$ onto $H(P)$, we can see that

$$S \text{ maps } E'_2(P_i) \text{ onto } H_i(P_i)\,.$$

The assertion is proven.

Example 6 : (E_n, H_n, P_n) is a sequence of Wiener spaces.

Then :

$$\Big(\prod_{n\in\mathbb{N}} E_n,\ \bigoplus_{n\in\mathbb{N}} H_n,\ \bigotimes_{n\in\mathbb{N}} P_n\Big) \text{ is a Wiener space, too.}$$

The proof is almost identical to the preceding one. It suffices to notice that $(\prod E_n)'$ is algebraically isomorphic to the direct sum $\bigoplus_n E'_n$ and to complete $\bigoplus_n H'_n$ for the canonical prehilbertian structure.

Example 7 : (E, H, P) is a Wiener space and E_0 is a Borel vector subspace, carrying P.

If P_0 denotes the induced probability by P on E_0, then (E_0, H, P_0) is a Wiener space. Actually, E_0, being Borelian, is a Lusin space for the induced topology.
Moreover :

$$H \subset E_0$$

since $P(E_0) = 1$.
Now each linear continuous form on E_0, is the restriction to E_0 of a linear continuous form on E, therefore it is Gaussian (since $P(E_0) = 1$).

The announced result now follows.

Example 8 : Brownian motion with values in $\mathbb{R}^{\mathbb{N}}$ (or infinite dimensional Brownian motion)

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}^+}, P\right)$ a process basis and let (B_t^n) a sequence of independent real Brownian motions. We shall denote

$$\mathbb{B}_t := (B_t^n)_n \quad t \in [0, \infty[.$$

Each real Brownian motion determines the usual Wiener measure on $\mathcal{C}_0([0, \infty[, \mathbb{R})$.
Therefore $\mathbb{B}$ defines a Gaussian measure on $\left[\mathcal{C}_0([0, \infty[, \mathbb{R})\right]^{\mathbb{N}} \simeq \mathcal{C}_0\left([0, \infty[, \mathbb{R}^{\mathbb{N}}\right)$, namely the product measure. Let μ_1 denotes this probability ; $\mathcal{C}_0([0, \infty[, \mathbb{R})^{\mathbb{N}}$ will be equipped with the product topology : it is a Lusin space.
$\mathcal{H}(\mu_1)$, the Cameron-Martin space of μ_1 is the Hilbertian sum of the "ordinaries" Cameron-Martin on $\mathbb{R}$. Therefore :

$$f \in \mathcal{H}(\mu_1) \Longleftrightarrow f = (f_n)_n \quad \text{and} \quad \sum_n \int_0^\infty |\dot{f}_n(t)|^2 \, dt < \infty.$$

$\mathcal{H}(\mu_1)$ is then the subspace of $W^{1,2}([0, \infty[; \ell^2)$ constitued by the functions null at zero. (The notation $W^{1,2}$ comes from the theory of Sobolev spaces).

Let us notice that,
for every $x = (x_n) \in \ell^2$ and every $t \in [0, \infty[$:

$$\sum x_n \; B_n(t) \text{ exists (almost surely)}$$

and that

$$t \rightsquigarrow \sum x_n \ B_n(t) \text{ is a real Brownian motion with variance } t\|x\|^2_{\ell^2}.$$

Now we give an important property of the infinite dimensional Brownian motion which will permit us to define the Brownian motion of an abstract Wiener space.

Lemma : *Let on $\mathbb{R}^{\mathbb{N}}$ the canonical normal measure $\gamma_1^{\otimes\mathbb{N}}$ and let K be a compact, disked (= convex, balanced) of $\mathbb{R}^{\mathbb{N}}$ such that $\gamma_1^{\otimes\mathbb{N}}(K) > 0$ and let F be the vector subspace of $\mathbb{R}^{\mathbb{N}}$ generated by K. Then P-almost surely the paths of $\mathbb{B}$ are contained in F :*

$$P\{\omega; \mathbb{B}_t(\omega) \in F \quad \forall t\} = 1.$$

Before proving this result let us make the following remarks :

a) If $\gamma_1^{\otimes\mathbb{N}}(K) > 0$, then $\gamma_1^{\otimes\mathbb{N}}(F) = 1$. This results from the zero-one law. Therefore :

$$\forall t : \quad P\{\omega, \mathbb{B}_t(\omega) \in F\} = 1$$

(a result weaker than the conclusion of the lemma).

b) There exists disked compacts of $\mathbb{R}^{\mathbb{N}}$ with positive measure. Actually, if K is an arbitrary compact, its disked hull is compact (since $\mathbb{R}^{\mathbb{N}}$ is complete).

Proof :

Let us suppose $t \rightsquigarrow \mathbb{B}_t$ continuous (this does not loss generality) and let q_K be the gauge of K :

$$q_K(x) = \inf\{\lambda, \quad \lambda > 0 \quad x \in \lambda K\};$$

q_K is convex, lower semi-continuous, with values in $[0, \infty]$.

Then $t \rightsquigarrow q_K^2(\mathbb{B}_t)$ is a sub-martingale with values in $[0, \infty]$, l.s.c. (but not c.à d. l.à g. in general) : it is a composition of continuous function with l.s.c. function.

Let $T < \infty$ and $D \subset [0, T]$, denumerable, dense in $[0, T]$ and such that $T \in D$. Then :

$$\forall t \in [0, T] \ : \ q_K^2(\mathbb{B}_t) \leq \liminf_{\substack{t_n \in D \\ t_n \to t}} \ q_K^2(\mathbb{B}_{t_n}) \leq \sup_{t \in D} \ q_K^2(\mathbb{B}_t)$$

Therefore

$$\sup_{t \leq T} \ q_K^2 \ (\mathbb{B}_t) = \sup_{t \in D} \ q_K^2 \ (\mathbb{B}_t)$$

and

$$\mathbb{E}\Big\{\sup_{t\leq T}\, q_K^2\ (\mathbb{B}_t)\Big\} \leq \mathbb{E}\Big\{\sup_{t\in D}\, q_K^2(\mathbb{B}_t)\Big\} \leq 4\mathbb{E}\Big\{\ q_K^2(\mathbb{B}_T)\Big\}$$

(by Doob's inequality). Now :

$$4\mathbb{E}\Big\{q_K^2(\mathbb{B}_T)\Big\} = 4T\ \mathbb{E}\Big\{q_K^2(\mathbb{B}_1)\Big\} \leq 4T\int_{\mathbb{R}^{\mathbb{N}}} q_K^2\ d\gamma_1^{\otimes\mathbb{N}}.$$

But, by Fernique's theorem we have :

$$\int_{\mathbb{R}^{\mathbb{N}}} q_K^2\ d\gamma_1^{\otimes\mathbb{N}} = \int_F q_K^2\ d^F(\gamma_1^{\otimes N}) < \infty$$

where $d^F(\gamma_1^{\otimes\mathbb{N}})$ denotes the induced measure on F (which carries $\gamma_1^{\otimes\mathbb{N}}$).
Finally, for almost every ω, $(\mathbb{B}_t)$ stays in an homothetic of K when $t \in [0,T]$. T being arbitrary :

— *the lemma is proven.* —

Example 9 : Brownian motion in an abstract Wiener space

Let (E, H, P) a Wiener space. We shall suppose fulfilled the following condition :

"There exists a compact disk K of E such that $P(K) > 0$".

Now let $\pi : E \to \mathbb{R}^{\mathbb{N}}$ a linear continuous injection of E into $\mathbb{R}^{\mathbb{N}}$ mapping P on $\gamma_1^{\otimes\mathbb{N}}$, and let $L = \pi(K)$, where K is as above.

The vector subspace generated by L (v.s.(L)) carries $\gamma_1^{\otimes\mathbb{N}}$ as we just saw and it is clearly contained in $\pi(E)$. Moreover, we know that, almost surely the paths of the canonical Brownian motion with values in $\mathbb{R}^{\mathbb{N}}$ are contained in v.s.(L). Then $\pi^{-1}(\mathbb{B}_t) = W_t$ has sense, and defines a E-valued process.

Now we can see easily that if $f \in E'$, there exists an $x \in \ell^2$ associated to it such that

$$\|f\|_{E_2'(P)} = \|x\|_{\ell^2}$$

and that :

$$f \circ W_t \text{ is a Brownian motion (real)}$$

with variance

$$t\|f\|^2_{E_2'(P)}.$$

This result remains true if $f \in E_2'(P)$.

Therefore we proved the following theorem :

THEOREM 3 : *Let E be a Lusin l.c.s. and let P be a Gaussian probability on E satisfying the above condition. There exists a process $(W_t)_{t\in\mathbb{R}_+}$ with values in E such that, for every $f \in E_2'(P)$, $(f \circ W_t)_{t\in\mathbb{R}_+}$ is a real Brownian motion with variances $t\|f\|_{E_2'(P)}$.*

(W_t) is called the ***"canonical Brownian motion"*** associated to P, or to (E, P).

Remark : The condition of the theorem is automatically fulfilled when E is Banach, or complete, or more generally quasi-complete.

Now let us end by a characterization of a Gaussian probability which will be used in several places :

A Borelian probability P on a Lusin l.c.s. E is Gaussian if and only if it satisfies the following property :

"If X and Y are E-valued random variables (defined on a same probability space) independent and having the common law P, then for every $\theta \in \mathbb{R}$, the E-valued random variable $X \cos\theta + Y \sin\theta$ has the law P".

This property is immediately deduced from the corresponding characteristics of a real centered Gaussian law.

This property was used by Fernique to define a Gaussian probability on a general topological vector space (not necessarily locally convex).

CHAPTER TWO

Measurable linear operators

1 - Generalities

Let us recall some fundamental notions.

Let X be a Hausdorff completely regular topological space and let μ be a Borelian probability on X. Let Y be another Hausdorff topological space. Then we have three kinds of measurability of a linear application $f : X \to Y$:

a) the ***"Borel-measurability"*** (does not depend of μ),
i .e. the measurability with respect to the Borelian σ-fields $\mathcal{B}_X$ and $\mathcal{B}_Y$,

b) the ***"Lusin-measurability"*** :
$f : X \to Y$ is said μ-Lusin measurable if it satisfies the following condition :
for every $\varepsilon > 0$ there exits a compact $K_\varepsilon \subset X$ such that :

$$\mu(X \setminus K_\varepsilon) < \varepsilon \quad \text{and} \quad f_{|K_\varepsilon} \text{ is continuous,}$$

c) the ***"μ-measurability"***, or ***"Lebesgue-measurability"*** :
$f : X \to Y$ is said μ-measurable if it is measurable with respect to $\mathcal{B}_X^\mu$ (the μ-completed σ-field of $\mathcal{B}_X$) and $\mathcal{B}_Y$.

Clearly :

$$\text{a)} \implies \text{c)} \quad \text{and} \quad \text{b)} \implies \text{c)}.$$

Moreover :

$$\text{if } \mu \text{ is Radon, then c)} \implies \text{b)}.$$

Actually this fact is well known if X is locally compact.

In the general case we can do the following reasoning :
Let L_ε be a compact such that :

$$\mu(L_\varepsilon) \le \varepsilon/2.$$

By the Egoroff's theorem there exists a compact $K_\varepsilon \subset L_\varepsilon$ such that :

$$\mu(L_\varepsilon \setminus K_\varepsilon) < \varepsilon/2 \quad \text{and} \quad f_{|K_\varepsilon} = (f_{|L_\varepsilon})_{|K_\varepsilon} \text{ be continuous.}$$

Then

$$\mu(X \setminus K_\varepsilon) < \varepsilon ,$$

and the result is proven.

Let us suppose now that Y is a Hausdorff l.c.s.; we can then define three kinds of scalar measurability :

$f : X \to Y$ is said scalarly measurable (resp. μ-measurable, resp. Lusin-measurable) if : for every linear continuous form y': $y' \circ f$ is measurable (resp. μ-measurable, resp. Lusin-measurable).

Lemma 1 : *Let (X, μ) as above, with μ Radon and let Y be a Lusin l.c.s. (Hausdorff). If $f : X \to Y$ is scarlarly μ-measurable, then it is μ-measurable.*

(Therefore we have the equivalence : μ-measurability $\Longleftrightarrow$ scalar μ-measurability).

Proof :

Let j be a linear continuous injection of Y into $\mathbb{R}^{\mathbb{N}}$, then :

$$j(Y) \text{ is a Borel set of } \mathbb{R}^{\mathbb{N}}$$

and :

$$j : Y \to j(Y) \text{ is bi-measurable}$$

(with respect to the Borel σ-fields).
Let $g = j \circ f$, $g : X \to \mathbb{R}^{\mathbb{N}}$, then g is μ-measurable by the definition of the topology of $\mathbb{R}^{\mathbb{N}}$. Therefore :

$$j^{-1} \circ j \circ f \quad \text{is } \mu\text{-measurable}$$

since j^{-1} is Borelian.

— Lemma 1 is proven.—

We have a similar result for Lusin-measurability.

In what follows X and Y will be Hausdorff locally convex spaces ("l.c.s.") and f will be linear (often denoted T).

Lemma 2 : *Let E be a Hausdorff l.c.s., μ be a Radon probability on E and $T : E \to F$ linear Lusin-measurable (where F is a topological vector space). Then there exists a Borel subspace E_0 of E, carrying μ and such that $T_{|E_0}$ is Borelian.*

Proof :

Let $\varepsilon > 0$ and K_ε be a compact (not necessarily convex) such that :

$$\mu(K_\varepsilon) \geq 1 - \varepsilon \quad \text{and} \quad T_{|K_\varepsilon} \text{ be continuous,}$$

and let E^ε be the vector subspace generated by K_ε.

I assert that E^ε is Borelian and $T_{|E^\varepsilon}$ is a Borelian map. In fact,let us fix $n \in \mathbb{N}$ and let :

$$K_\varepsilon^n := \{x \in E, \quad x = \sum_{i=0}^{i=n} \lambda_i x_i, \quad |\lambda_i| \leq 1, \quad x_i \in K_\varepsilon, \quad \forall i\}.$$

First K_ε^n is compact because : if $(x^\alpha)_\alpha$ is an ultrafilter on K_ε^n, it converges to $x \in K_\varepsilon^n$. Actually :

$$x^\alpha = \sum_{i=0}^{i=n} \lambda_i^\alpha x_i^\alpha$$

then :

$$\lambda_i^\alpha \underset{\alpha}{\longrightarrow} \lambda_i, \quad (0 \leq i \leq n)$$

since $[-1, +1]$ is a compact of $\mathbb{R}$ and

$$x_i^\alpha \underset{\alpha}{\longrightarrow} x_i,$$

since K_ε is compact.

Moreover : $T_{|K_\varepsilon^n}$ is continuous, therefore :

$$T(x^\alpha) = \sum_{i=0}^{i=n} \lambda_i^\alpha T(x_i^\alpha) \longrightarrow \sum_{i=0}^{i=n} \lambda_i T(x_i).$$

Let us notice that K_ε^n is not necessarily convex, but it is balanced .
Then :

$$T_{|\bigcup_m mK_\varepsilon^n} \quad \text{is Borelian,}$$

and :

$$T_{|\bigcup_n \bigcup_m (mK_\varepsilon^n)} \quad \text{is Borelian too.}$$

We can write

$$E^\varepsilon = \bigcup_n \bigcup_m mK_\varepsilon^n \,;$$

then E_ε is Borelian and it is a vector subspace.

Now let $\varepsilon = 1/p, (p \in \mathbb{N}^*)$. We may choose the sequence $(K_{1/p})_p$ increasing. Then :

$$E_0 = \bigcup_{p \in \mathbb{N}^*} E^{1/p}$$

satisfies the condition of the lemma.

— *Lemma 2 is proven.*—

Remark 1 : The proof of Lemma 2 can be simplified if we know that μ is almost carried by compact convex sets. In this case :

$$\bigcup_m mK_\varepsilon \quad \text{is a vector space}$$

if K_ε is disked.

Remark 2 : If E and F are l.c.s. and μ a Radon-measure on E, the Lusin-measurable functions from E into F form a vector space (it is obvious by the definition). If E and F are Lusin, the Borel functions from E into F form a vector space. It is obvious since :

$$\mathcal{B}_E \otimes \mathcal{B}_F = \mathcal{B}_{E \times F}$$

and :

$$\mathcal{B}_{\mathbb{R} \times E} = \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_E.$$

In the sequel, unless the contrary is specified, we shall assume that the vector spaces we consider are Lusin.

Lemma 3 : *Let (E_1, μ_1) and (E_2, μ_2) be two (Lusin) l.c.s. with (Radon) probabilities. Let F be another Lusin l.c.s. and $T : E_1 \times E_2 \to F$ linear, $\mu_1 \otimes \mu_2$-measurable. Then for every $x_2 \in E_2$, the map $T(\bullet, x_2)$ is μ_1-measurable.*

Proof :

By Fubini's theorem :

$$\text{for } \mu_2\text{-almost every } x_2 \in E_2 : \quad T(\bullet, x_2) \text{ is } \mu_1\text{-measurable ,}$$

there exists $x_2^0 \in E_2$ such that :

$$T(\bullet, x_2^0) \text{ is } \mu_1\text{-measurable.}$$

By the remark preceding the lemma :

$$T(\bullet, 0) = \frac{1}{2}\left[T(\bullet, x_2^0) + T(\bullet, -x_2^0)\right] \quad \text{is } \mu_1\text{-measurable.}$$

Now for each $x_2 \in E_2$:

$$T(\bullet, x_2) = T(\bullet, 0) + T(0, x_2).$$

— *Lemma 3 is proven.*—

Remark : Let E be a l.c.s. with the Borel probability μ on it and let E_0 be a Borel subspace of E, carrying μ. Let F another l.c.s. and $T_0 : E_0 \to F$ linear, Borelian. Every linear extension of T_0 from E_0 to E (easily obtained by considering an Hamel basis of E which completes an Hamel basis of E_0) is clearly μ-measurable, but not necessarily Borelian.

This fact should be kept in mind.

2 - Measurable linear mappings from a Wiener space

In the sequel, E will denote a Lusin l.c.s. (and Hausdorff), μ a Gaussian centered probability on E whose reproducing kernel Hilbert-space will be denoted by H or $H(\mu)$. We shall suppose that Supp $(\mu) = E$, then H is dense in E. Otherwise there exists x_0' in E' such that

$$x_0' \neq 0, \; x_0'(h) = 0 \text{ for each } h \text{ in } H.$$

Therefore :

$$0 = x_0'(Sx_0') = \int_E |x_0'(x)|^2 \, d\mu(x).$$

Then :

$$\mu\{x; x_0'(x) = 0\} = 1$$

which contradicts Supp $(\mu) = E$.

The results that follow are true, but trivial, if:

$$E = H = \mathbb{R}^n,$$

so we shall exclude this case in our proofs and we shall suppose : $dim\, H = +\infty$.

Let F be another Lusin l.c.s., let $T : E \to F$, and $T' : E \to F$ be two μ-measurable linear maps. Let us suppose that : $T = T'$ μ-almost surely. Then :

$$\{x, \quad T(x) = T'(x)\},$$

is a subspace carrying μ and :

$$H \subset \{T = T'\}.$$

In other words:

Two linear measurable maps, equal almost surely, coincide on H.

(The converse is true, see below).
Then we can speak about the value at a point of H, of a μ-equivalence class of linear μ-measurable maps.

On the other hand, by using the caracterization of a Gaussian measure given at the end of chapter one : $T(\mu)$ is again a Gaussian measure if T is μ-measurable and linear.

Let us give a consequence from Lemma 2 .

THEOREM 1 : *Let F be a Banach and $T : E \to F$ linear μ-measurable. Then we have :* $\int_E \| T(x) \|^2 \mu(dx) < \infty$.

Proof :

By Lemma 2, there exists a Borelian subspace, E_0, carrying μ such that $T_{|E_0}$ is Borelian. Then E_0 is a Lusin-space for the induced topology and

$$\mathcal{B}_{E_0 \times E_0} = \mathcal{B}_{E_0} \otimes \mathcal{B}_{E_0}$$

Now let us consider the Borelian semi-norm on E_0 :

$$x \rightsquigarrow \| Tx \|_F$$

By the Fernique's theorem (see chapter one) we have :

$$\int_{E_0} \| Tx \|_F^2 \, \mu_0(dx) < \infty$$

where $\mu_0 = \mu_{|E_0}$ and therefore :

$$\int_E \| Tx \|_F^2 \, \mu(dx) < \infty$$

— *Q.E.D.*—

◇ **First we examine the case :** $F = \mathbb{R}$, i.e. we shall consider the μ-measurable linear forms.

Let us notice that we have met such linear forms : the linear forms defined by an element of $E_2'(\mu)$. In fact such an element is an almost sure limit of a sequence of linear continuous forms, whose the convergence set is a Borelian vector space carrying μ. Then we extend it outside of this vector subspace with preservation of the linearity, and we obtain a linear μ-measurable form .

Now we shall see that there are not other μ-measurable linear forms. Actually let $f : E \to \mathbb{R}$ linear μ-measurable, therefore defining an element of $L^2(E, \mu)$ and let us denote by $\widehat{v}$ its orthogonal projection on $E_2'(\mu)$.

I assert that :

$$f = \widehat{v},$$

because : $f - \widehat{v}$ is orthogonal to each $\widehat{u} \in E_2'(\mu)$.
Actually, let $(e_n')_n$ be a sequence of elements in E', which constitues an orthonormal basis of $E_2'(\mu)$, the family of random variables on E :

$$\{< e_n', \bullet >, \ n \in \mathbb{N}\} \bigcup \{f - \widehat{v}\} \text{ is Gaussian};$$

(it defines a μ-measurable linear mapping from E to $\mathbb{R}^{\mathbb{N}} \otimes \mathbb{R}$).
Since $f - \widehat{v}$ is orthogonal to the $< e_n', \bullet >$, $(n \in \mathbb{N})$, it is independent from the $< e_n', \bullet >$ and therefore it is independent from $\mathcal{B}_E = \sigma\{< e_n', \bullet >, \quad n \in \mathbb{N}\}$.
Therefore :

$$f - \widehat{v} \quad \text{is a constant (almost surely)}$$

and this constant is null, since $f - \widehat{v}$ is centered.

The announced result is established.

From what we saw in the chapter one, we can see that the restriction of f to H is continuous. This last fact can be seen otherwise as follows :

if E_0 is a Borel subspace of E carrying μ and such that $f_{|E_0}$ be Borelian then, since :

$$H \subset E_0 \text{ with continuous injection}$$

(E_0 being equipped with the induced topology, H with the Hilbertian topology) we have :

$$f_{|H} \text{ Borelian and therefore continuous}$$

(all Borelian linear forms on a Banach space being continuous).

Summing up that precedes, we have obtained the following theorem :

THEOREM 2 : *Let (E, H, μ) be an abstract Wiener space. Then there is an identity between :*

(i) : the μ-equivalence classes of linear μ-measurable forms,

(ii) : the almost sure limits of sequences of linear continuous forms.

Moreover all linear continuous forms on H (with its topology) admit a linear μ-measurable extension to E and two such extensions belong to the same class. If g is a linear continuous form, we shall call ***"μ-essential extension of g"*** whatever extension.

Let us notice the equality:

$$\| g \|_{H'}^2 \equiv \int_E | f(x) |^2 \mu(dx)$$

if f extends g.

Remark 1 : We have no more the μ-unicity of the extension if we do not suppose the μ-measurability of extensions, as we can see :

let f be a linear form on E, not μ-almost surely equal to zero and vanishing on H (we can build such a linear form by mean of a suitable Hamel basis).

Remark 2 : The μ-measurable linear forms, obviously separate the elements of E.

Now let us give a criterion for an element of E to belong to H.

Proposition 1 : *Let $a \in E$. The following properties are equivalent :*

(i) : $a \in H$

(ii) : there exists $C \in [0, \infty[$ such that : $| f(a) | \leq C \| f \|_{L^2(E,\mu)}$ for each μ-measurable linear form f.

Proof :

$(i) \Rightarrow (ii)$: it is trivial, we have $C = \| a \|_H$,

$(ii) \Rightarrow (i)$: let us notice that if $f = g$ almost surely (f and g linear μ-measurable) then, by condition (ii),

$$f(a) = g(a).$$

Therefore we can define a linear mapping $E'_2(\mu) \to \mathbb{R}$ by:

$$f \rightsquigarrow f(a).$$

This linear form is continuous (with norm $\leq C$). So there exists $u \in H$ such that :

$$\text{for every } f \text{ in } E'_2(\mu) \ : \quad f(a) = f(u) = (Sf, u)_H .$$

Therefore

$$\text{for every } f \in E'_2(\mu) \ : \quad f(u - a) = 0.$$

And $u = a$ since the f are separating.

— *Q.E.D.*—

Corollary : *Let $a \in E$, the following properties are equivalent :*

(i) : $a \in H$

(ii) : for every sequence (f_n) of μ-measurable linear forms converging almost surely to zero, we have bigl$(f_n(a)\big) \longrightarrow 0$.

Proof :

$(i) \Rightarrow (ii)$: if $f_n \longrightarrow 0$ almost surely then :

$$f_n \longrightarrow 0 \text{ in } L^2(E,\mu)$$

(since the $f_n(\bullet)$ are Gaussian random variables).
Therefore

$$f_n(a) \longrightarrow 0 \quad \text{by Proposition 1 .}$$

$(ii) \Rightarrow (i)$: suppose (ii) realized and suppose that $a \notin H$. Then there exists a sequence $(f_n)_n$ of μ-measurable linear forms converging to zero in L^2 such that $(f_n(a))$ does not converge to zero.
So there exists a subsequence (f_{n_k}) such that :

$$f_{n_k} \underset{k}{\longrightarrow} 0 \quad \text{in } L^2$$

and

$$\big(f_{n_k}(a)\big)_k \text{ converges in } \overline{\mathbb{R}} \text{ to } \alpha \neq 0.$$

From this we deduce the existence of a subsequence of $(f_{n_k})_k$ denoting by $(g_\ell)_\ell$ converging almost surely to zero, but such that :

$$\big(g_\ell(a)\big)_\ell \text{ does not converge in } \mathbb{R} \text{ to zero,}$$

— *The Corollary is proven.*—

Now let us fix some notations :

If $(f_n)_n$ is a sequence of linear μ-measurable forms converging almost surely to zero. we shall denote the set of convergence of $(f_n)_n$ by: $C\{(f_n)_n\}$. This is a vector subspace carrying μ. So :

$$H \subset C\{(f_n)_n\}.$$

Proposition 2 : *H is equal to the intersection of the* $C\{(f_n)_n\}$ *where* $(f_n)_n$ *is a sequence of elements in* $E_2'(\mu)$ *converging almost surely to zero.*

Proof :

Let us denote by D the intersection of the $C\{(f_n)_n\}$.
Clearly :

$$H \subset D.$$

Let us suppose that there exists $a \in D \backslash H$. Since $a \notin H$, by the preceding Corollary there exists $(f_n)_n \longrightarrow 0$ almost surely such that :

$$\big(f_n(a)\big)_n \text{ does not converge to zero}.$$

There is a contradiction.

—*Q.E.D.*—

As an immediate consequence we have the following theorem :

THEOREM 3 : *H is the intersection of the Borel vector subspaces carrying* μ.

◇ **Now let us come back to the general case** of a μ-measurable linear mapping between two Lusin spaces.

Let $T : E \to F$ linear and μ-measurable. As when T is continuous we have the relation

$$\text{for every } f \in L^2(E,\mu): \quad T\Big(\int_E f(x)x\mu(dx)\Big) = \int_E f(x)T(x)\mu(dx)$$

where the right member is a "weak integral":

$$\forall g \in F' \ : \ < g, \int_E f(x)T(x)\mu(dx) > := \int_E f(x) g \circ T(x)\mu(dx).$$

(Let us notice that the last integral has a sense since : $g \circ T$ is a μ-measurable linear functional on E). In fact :

$$< g, T\Big(\int_E f(x)x\mu(dx)\Big) >_{E',E} = g \circ T\Big(\int_E f(x)x\mu(dx)\Big)$$

$$= \int_E g \circ T(x) f(x)\mu(dx) \qquad (\text{since} \quad g \circ T \in E_2'(\mu))$$

$$=< g, \int_E f(x)T(x)\mu(dx) > \qquad (\text{ g being continuous }).$$

Proposition 3 : *Let $\nu = T(\mu)$ and let K be the Cameron-Martin space of ν. Then $T(H) = K$. Moreover, T induces a linear continuous function from H into K.*

(This result was already proven in chapter one, when T was continuous).

Proof :

$S_\nu : g \rightsquigarrow \int_F g(y) y d\nu$ is an isometry from $F'_2(\nu)$ onto K.

Let $G := \left\{ \int_E g \circ T(x)\, x\mu(dx) \,;\, g \in F'_2(\nu) \right\}$. Let $g \in F'_2(\nu)$, so $g \circ T \in E'_2(\mu)$. By the remark preceding this proposition :

$$T\Big(\int_E g \circ T(x) x\mu(dx)\Big) = \int_E g \circ T(x) T(x) \mu(dx)$$
$$= \int_F g(y) y \nu(dy)$$

So

$$K = T(G) \subset T(H).$$

Now we prove that $T(G) = T(H)$:

Let $\widehat{u} \in E'_2(\mu)$, orthogonal to the $g \circ T$, with $g \in F'_2(\nu)$ and let u be the barycenter of $\widehat{u}$. I claim that :

$$T(u) = 0.$$

In fact for $g \in F'$:

$$< g, T(u) >_{F',F} = g \circ T\Big(\int_E \widehat{u}(x) x\mu(dx)\Big)$$
$$= \int_E \widehat{u}(x) g \circ T(x) \mu(dx) = 0$$

This equality being true for every $g \in F'$, then $T(u) = 0$. So :

$$T(G) = T(H).$$

Now the continuity of $T_{|H}$ is obvious : $T_{|H}$ is the transpose of the isometry: $g \rightsquigarrow g \circ T$ from $F'_2(\nu)$ into $E'_2(\mu)$ as we can see :

let $\widehat{v} \in F_2'(\nu)$ with barycenter v, then

$$\begin{aligned}\Big(T\Big(\int_E g \circ T(x) x \mu(dx)\Big); v\Big)_K &= \Big(\int_E g \circ T(x) T(x) \mu(dx); v\Big)_K \\ &= \Big(\int_F g(y)\, y\, \nu(dy); v\Big)_K \\ &= \int_F g(y) \widehat{v}(y) \nu(dy) \\ &= \int_E g \circ T(x) \widehat{v} \circ T(x) \mu(dx) \\ &= (g \circ T; \widehat{v} \circ T)_{E_2'(\mu)}.\end{aligned}$$

— *Q.E.D.*—

We have seen that if $T_1 = T_2$ μ-almost surely, then their restrictions to H coincide. The converse is true.

Proposition 4 : *Let T_1 and T_2 be two μ-measurable linear maps from E to F such that $T_{1|H} = T_{2|H}$, then $T_1 = T_2$ μ-amost surely.*

Proof :

The result is true if $F = \mathbb{R}$ (linear measurable functionals) as we have seen above. Therefore the proposition remains true if $F = \mathbb{R}^{\mathbb{N}}$.

For the general case, let us consider a continuous linear injection j from F into $\mathbb{R}^{\mathbb{N}}$ and the functions $j \circ T_i$, $(i = 1, 2)$.

— *Q.E.D.*—

3 - Comparison of the Cameron-Martin spaces.

THEOREM 4 : *Let (E, H_1, μ_1) and (E, H_2, μ_2) be two Wiener spaces (with the same E); we suppose that every subspace carrying μ_2 carries μ_1, then :*

(i) : every linear form μ_2-measurable is μ_1-measurable and we can determine a linear continuous function from $E_2'(\mu_2)$ into $E_2'(\mu_1)$

(ii) : $H_1 \subset H_2$ with continuous injection.

Proof :

(i) : Let $f : E \to \mathbb{R}$ linear and μ_2-measurable. Then there exists a Borelian subspace F of E such that :

$$f_{|F} \text{ is Borelian} \quad \text{and} \quad \mu_2(F) = 1.$$

Since $\mu_1(F) = 1$, then

$$f \text{ is } \mu_1\text{-measurable.}$$

Now let us suppose that $f_1 = f_2$, μ_2-almost surely (f_1 and f_2 linear, μ_2-measurable). Then $\{f_1 = f_2\}$ is a subspace carrying μ_2 and therefore μ_1. This means

$$f_1 = f_2, \quad \mu_1\text{-almost surely.}$$

So we have defined a canonical injection

$$E_2'(\mu_2) \to E_2'(\mu_1).$$

The continuity of this injection follows from the closed-graph theorem :

if $f_n \longrightarrow f$ in $E_2'(\mu_2)$ and $f_n \longrightarrow g$ in $E_2'(\mu_1)$ then there exists a subsequence (f_{n_k}) converging μ_2-almost surely to f.

Therefore

$$g = f, \ \mu_1\text{-almost surely.}$$

There exists $C \in [0, \infty[$ such that :

$$\int (f(x))^2 \mu_1(dx) \leq C \int (f(x))^2 \mu_2(dx), \quad \forall f \in E_2'(\mu_2)$$

and *(i)* is proven.

(ii) : The inclusion $H_1 \subset H_2$ follows immediately from Theorem 3 and the continuity of this inclusion follows from the closed-graph theorem.

— *Q.E.D.*—

Remark : If μ_1 and μ_2 satisfy the condition of the Theorem 4, for every F Lusin l.c.s., we have :

$$T : E \to F, \text{ linear, } \mu_2\text{-measurable} \Rightarrow T \ \mu_1\text{-measurable}$$

(by Lemma 1).

Now we shall be given with two Wiener spaces (E_i, H_i, μ_i), $(i = 1, 2)$ and $T : E_1 \to E_2$ linear and μ_1-measurable. Then we can easily prove the following theorem :

THEOREM 5 : *Let us suppose that every vector subspace of E_2 carrying μ_2 carries $T(\mu_1)$. Then $T(H_1) \subset H_2$ and $T_{|H_1} : H_1 \to H_2$ is continuous.*

Proof :

Let $\nu_1 = T(\mu_1)$. We have noticed that ν_1 is Gaussian. Let K_1 be the Cameron-Martin space of ν_1. We know that T maps continuously H_1 into K_1 (with their Hilbertian topology).

By the hypothesis every subspace carrying μ_2 carries ν_1, therefore $K_1 \subset H_2$ with continuous injection. So

$$T : H_1 \to H_2 \text{ is continuous.}$$

— *Q.E.D.*—

Corollary 1 : *Under the hypothesis of the Theorem 5, the restriction of T to H_1 is continuous from H_1 into E_2.*

Next we shall be concerned with the inverse problem : let $T : H_1 \to H_2$ linear and continuous, does there exist an extension $\widetilde{T} : E_1 \to E_2$ linear and μ_1-measurable ? The answer is affirmative.
(Let us notice that if such an extension exists, it is unique up-to a μ_1-equivalence by Proposition 4).

THEOREM 6 : *Let $T : H_1 \to H_2$ linear and continuous. Then there exists an extension $\widetilde{T} : E_1 \to E_2$ of T, linear and μ_1-measurable.*

Proof :

Let us suppose first that T is unitary and surjective.

Let j_{E_2} a continuous linear injection of E_2 into $\mathbb{R}^{\mathbb{N}}$ mapping μ_2 onto $\gamma_1^{\otimes \mathbb{N}}$. To this injection we associate an orthonormal basis of H_2, namely f_n : $f_n = j_{E_2}^{-1}(e_n)$, where $(e_n)_n$ denotes the canonical basis of ℓ^2.
Let $g_n = T^{-1}(f_n)$: by the hypothesis, (g_n) is an orthonormal basis of H_1 which defines a (equivalence class of) linear μ_1-measurable injection of E_1 into $\mathbb{R}^{\mathbb{N}}$, let j_{E_1} , and j_{E_1} maps μ_1 onto $\gamma_1^{\otimes \mathbb{N}}$ but it is neither continuous, nor injective in general.

Let $G = j_{E_1}(E_1) \cap j_{E_2}(E_2) \subset \mathbb{R}^{\mathbb{N}}$. G is a vector subspace of $\gamma_1^{\otimes \mathbb{N}}$-probability equal to one. Now let us consider

$$j_{E_2}^{-1} \circ j_{E_1} : \quad j_{E_1}^{-1}(G) \to E_2.$$

It is a linear map, extending T.
Now we can extend

$$j_{E_2}^{-1} \circ j_{E_1} \text{ to } E_1.$$

and obtain a linear μ_1-measurable extension of T to E_1. It is clear that :

$$\widetilde{T}(\mu_1) = \mu_2 \, .$$

In the general case :

We may clearly suppose that $\| T \| \leq 1$.
Let $E_1 \times E_2$ with the Gaussian probability $\mu_1 \otimes \mu_2$, whose Cameron-Martin space is equal to $H_1 \otimes H_2$.
Let

$$A = \sqrt{I_{H_2} - TT^*} \, , \qquad A : H_2 \to H_2$$

$$B = \sqrt{I_{H_1} - T^*T} \, , \qquad B : H_1 \to H_1$$

I claim that :

$$TB = AT \quad \text{and} \quad T^*A = BT^* \, .$$

To see this fact, it is sufficient to consider the binomial expansion of

$$(I_{H_2} - TT^*)^{\frac{1}{2}} \quad \text{and} \quad (I_{H_1} - T^*T)^{\frac{1}{2}}.$$

Moreover : let $U : H_1 \otimes H_2 \to H_1 \otimes H_2$ defined by :

$$U(x, y) := (-Bx + T^*y, Tx + Ay),$$

taking account of the commutation relations above, we can easily verify that U is unitary surjective, with the inverse equal to U.
By the first part of the proof there exists an extension

$$\widetilde{U} : E_1 \times E_2 \to E_1 \times E_2$$

linear and $\mu_1 \otimes \mu_2$-measurable.

Next define :

$$v(x,y) = proj_{E_2}\left[\frac{\widetilde{U}(x,y)+\widetilde{U}(x,-y)}{2}\right] ;$$

v is linear and $\mu_1 \otimes \mu_2$-measurable from $E_1 \times E_2$ into E_2 extending T. Moreover :

$$v(x,y) = v(x,0), \quad \forall x.$$

By Lemma 3, $v(\bullet, 0)$ is μ_1-measurable and : $\widetilde{T}(\bullet) = v(\bullet, 0)$ satisfies the property of the theorem.

— *Q.E.D.*—

$\widetilde{T}$ is called a "*μ_1-**essential extension of** T*".

Now we give a particular case which will be useful in the sequel.

THEOREM 7 : *Let us suppose that, in addition with the hypothesis of Theorem 6, T is Hilbert-Schmidt. Then $\widetilde{T}$ has μ_1-almost surely its values in H_2 and we have :*

$$\int_{E_1} \| \widetilde{T}x \|^2_{H_2} \, \mu_1(dx) < \infty .$$

Conversely, if $\widetilde{T}$ is a linear μ_1-measurable map from E_1 into H_2 satisfying the above condition, its restriction T to H_1 is Hilbert-Schmidt. We have the equality :

$$\| T \|^2_{H,S} = \int_{E_1} \| \widetilde{T}x \|^2_{H_2} \, \mu_1(dx).$$

Proof :

Let $Tu = \sum \lambda_n (u, e_n)_{H_1} f_n$ be a spectral decomposition of T, $\sum \lambda_n^2 < \infty$, (e_n) and (f_n) be orthonormal basis in H_1 and H_2 respectively. Let us define :

$$\widetilde{T}(x) = \sum \lambda_n \widetilde{e}_n(x) f_n$$

($\widetilde{e}_n \in (E_1')_2$ is such that $S(\widetilde{e}_n) = e_n$).

This series converges in $L^2(E_1, \mu_1, H_2)$ and $\widetilde{T}$ is an extension of T.

Moreover :

$$\int_{E_1} \| \widetilde{T}x \|_{H_2}^2 \, \mu_1(dx) = \sum \lambda_n^2$$

$$= \| T \|_{H,S}^2 \ .$$

Conversely let $\widetilde{T} : E_1 \to H_2$ linear, μ_1-measurable with

$$\int_{E_1} \| \widetilde{T}x \|_{H_2}^2 \, \mu_1(dx) < \infty \ ;$$

$\mu_2 = \widetilde{T}(\mu_1)$ is a Gaussian measure on H_2 with reproducing kernel Hilbert space $T(H_1)$ where $T = \widetilde{T}_{|H_1}$.
By Minlos' theorem :

$$T : H_1 \to H_2 \qquad \text{is Hilbert-Schmidt.}$$

— *Q.E.D.*—

Remark : every μ_1-measurable linear function from E_1 into the Hilbert H_2 is an almost sure limit of linear continuous mappings from E_1 into H_2.

Actually if H_1 denotes the Cameron-Martin space of μ_1 and if $i : H_1 \to E_1$ is the canonical injection, then $\widetilde{T} \circ i : H_1 \to H_2$ is Hilbert-Schmidt :

$$\widetilde{T} \circ i(u) = \lim_{n \to \infty} \sum_{k=0}^{n} \lambda_k (e_k, u) f_k \, .$$

Therefore $\widetilde{T}(\bullet)$ is a L^2-limit of mappings :

$$\sum_{k=0}^{n} \lambda_k \widetilde{e}_k(\bullet) f_k \ .$$

But such a mapping is a L^2-limit of continuous mappings with finite rank, since the continuous linear functionals are dense in $(E_1)_2^{'}$.
Therefore each linear measurable function from E_1 into H_2 is a L^2-limit of continuous linear mappings.

By extracting some subsequence : we have the announced result.

4 - Some applications

a) . Quantization of a linear measurable operator

Let (E, H, μ) be a Wiener space, let $T : H \to H$ linear, continuous with $\| T \| \leq 1$ and let $B = (I - T^*T)^{\frac{1}{2}}$. The μ-essential extensions of T^* and B will be denoted by $\widetilde{T}^*$ and by $\widetilde{B}$ respectively (T^* is the adjoint of T). Let us set

$$\Gamma(T)f(x) = \int_E f(T^*x + \widetilde{B}y)\ \mu(dy)$$

where f is a Borelian (or μ-measurable) positive function.

Then $\Gamma(T)f$ does not depend from the extensions chosen. Moreover :

$$\forall p \geq 1, \quad \Gamma(T)\ :\ L^p(E, \mu) \to L^p(E, \mu) \quad \text{is contracting .}$$

Actually if f is Borelian, $\Gamma(T)f$ is well defined since

$$(x, y) \rightsquigarrow \widetilde{T}^*x + \widetilde{B}y \qquad \text{is } \ \mu \otimes \mu\text{-measurable}$$

and we have seen that :

$$\text{for every x :} \quad y \rightsquigarrow \widetilde{T}^*x + \widetilde{B}y \ \text{ is } \ \mu\text{-measurable.}$$

Moreover the functions

$$(x, y) \rightsquigarrow f(\widetilde{T}^*x + \widetilde{B}y)$$

and

$$x \rightsquigarrow f(x)$$

have the same law and therefore :

$$N_p\big(\Gamma(T)f\big) \leq N_p(f), \quad 1 \leq p \leq \infty.$$

b) . Linear measurable functions on $E = \mathcal{C}_0([0,1])$ with the Wiener-measure, denoted P

The dual of E is isomorphic to the space $\mathcal{M}(]0,1])$ of real Borel measures on $]0,1]$. If $\varphi \in L^2(]0,1], dt)$, the random variable :

$$\int_0^1 \varphi(t)\, d\omega(t) \quad \text{(Wiener-Itô integral)}$$

is a P-measurable linear form. Actually it is a L^2-limit of random variables from the form

$$\sum \varphi(t_i)[\omega_{t_{i+1}} - \omega_{t_i}]$$

which are linear and continuous.

We shall see that all P-measurable functionals L are of this form.

First let L be a continuous linear form :

Then there exists an unique (signed) measure μ on $]0,1]$ such that :

$$\forall\omega \in \mathcal{C}_0(]0,1]), \quad L(\omega) = \int_{]0,1]} \omega(t)\ \mu(dt)$$
$$= \int_{]0,1]} \omega(t)\ dF_\mu(t).$$

But by the ***stochastic*** integration by parts formula, we obtain :

$$\int_{]0,1]} \omega(T)\ dF_\mu(t) = -\int_0^1 F_\mu(t)\ d\omega(t) + [F_\mu(t)\ \omega(t)]_0^1$$

(The integral in the right member is a Wiener-Itô integral).

Therefore

$$L(\omega) = \int_0^1 \mu(]t,1])\ d\omega(t).$$

In the general case :

We approach L (in the $E_2'(P)$-topology) by linear continuous forms and it suffices to notice that the functions

$$s \rightsquigarrow \mu(]s,1])\,, \quad (\mu : \text{ real measure on }]0,1])$$

are dense in $L^2([0,1], dt])$.

Remark : In the case of the Brownian motion indexed by $\mathbb{R}_+$
we can prove by absolutely the same way that every linear measurable form on $\mathcal{C}_0([0,\infty[,\mathbb{R})$, equipped with the Wiener measure is the Itô integral of an (unique) element of $L^2([0,\infty[, dt)$. The converse is also true .

c) . Infinite dimensional Brownian motion

Let $(\ \mathbb{B}_t\ )_{t\in\mathbb{R}^+} = \Big(\big(\ \mathbb{B}_n(t)\ \big)_{n\in\mathbb{N}}\Big)_{t\in\mathbb{R}^+}$ be an infinite-dimensional Brownian motion with parameter set $\mathbb{R}_+$ or $[0,1]$.

Then it corresponds to it the following (abstract) Wiener space :

$$E = \mathcal{C}_0([0,\infty[, \mathbb{R}^{\mathbb{N}}), \quad H = W^{1,2}([0,\infty[, l^2), \quad P = (\mu_1)^{\otimes \mathbb{N}}$$

where μ_1 is the ordinary Wiener measure.
As above we see that every linear measurable functional is a stochastic integral with respect to $(\mathbb{B}_t)$ of an element of $L^2([0,\infty[, l^2)$. This means :

$$L(\bullet) = \int < f(t), d\mathbb{B}_t > := \sum_n \int f_n(t) d\mathbb{B}_n(t).$$

The conclusion remains true for a Brownian motion with values in the abstract Wiener space (F, H, μ) (see chapter one) : every linear measurable functional is the stochastic integral of an element of $L^2([0,\infty[, H)$.

d) . Expansions in "Fourier series"

Let (E, H, μ) be an abstract Wiener space and $j : E \to \mathbb{R}^{\mathbb{N}}$ a continuous linear injection from E into $\mathbb{R}^{\mathbb{N}}$ such that $j(\mu) = \gamma_1^{\otimes \mathbb{N}}$.
Let $\widetilde{e}_n$ the (continuous) linear form on E defined as

$$\widetilde{e}_n(x) := \big(j(x)\big)_n$$

and let $e_n = S(\widetilde{e}_n)$, the barycenter ;we have :

$$e_n \in H.$$

I claim that :

$$x = \sum_{n=0}^{\infty} \widetilde{e}_n(x) \cdot e_n\,, \qquad \text{almost surely.}$$

Actually let (f_n) the canonical *"basis"* of $\mathbb{R}^{\mathbb{N}}$.
Then

$$f_n = j(e_n).$$

In $\mathbb{R}^{\mathbb{N}}$ we have the expansion :

$$y = \sum y_n f_n$$

(if $y = (y_n)_n$)

Now let K_ε be a compact set in $\mathbb{R}^{\mathbb{N}}$, contained in $j(E)$ and such that :

$$\gamma_1^{\otimes \mathbb{N}}(K_\varepsilon) \geq 1 - \varepsilon$$

and let

$$L_\varepsilon = j^{-1}(K_\varepsilon).$$

Then j defines an homeomorphim from L_ε onto K_ε, and we have the equivalence :

$$\left(x \in K_\varepsilon, \quad \sum \widetilde{e}_n(x) e_n = x\right) \Longleftrightarrow \left(y \in L_\varepsilon, \quad y = \sum y_n f_n\right).$$

Therefore :

$$x = \sum \widetilde{e}_n(x) e_n, \qquad \text{for every } x \text{ in } K_\varepsilon.$$

Finally on $\bigcup_{m \geq 1} K_{\frac{1}{m}}$, (a Borel set carrying μ), we have :

$$x = \sum \widetilde{e}_n(x) e_n,$$

and the result is proven for $(e_n) = \left(j^{-1}(f_n)\right)$.

Now let $(\widetilde{e}_n^1)$ be an orthonormal basis of $E_2'(\mu)$, not necessarily contained in E'. We set as before

$$e_n^1 = S(\tilde{e}_n^1).$$

Let $\widetilde{e}_n$ and (e_n) as above, $(e_n = j^{-1}(f_n))$. To this change of basis, we associate an unitary operator $A \ : \ H \to H$. Let $\widetilde{A}$ a linear $\mu-$measurable extension of A, $\widetilde{A} \ : \ E \to E$; we have $\widetilde{A}(\mu) = \mu$.
Let $K_\varepsilon \subset E$ be a compact such that :

- $\mu(K_\varepsilon) \ \geq \ 1 - \varepsilon$
- $\sum \widetilde{e}_n(x) e_n = x, \quad \forall x \in K_\varepsilon$
- the restriction of $\widetilde{A}$ to K_ε is continuous.

Then :

$$x \in K_\varepsilon \ \Rightarrow \ \sum \widetilde{e}_n(x) \widetilde{A}(e_n) = \widetilde{A}(x).$$

Therefore

$$\sum \widetilde{e}_n(x) e_n^1 = \widetilde{A}(x) \quad \text{for every } x \in K_\varepsilon.$$

But

$$\widetilde{e}_n(x) = \widetilde{e}^1_n(\widetilde{A}x) \quad \text{almost surely.}$$

In fact the two measurable forms on E :

$$x \rightsquigarrow \widetilde{e}_n(x) \quad \text{and} \quad x \rightsquigarrow (Ae_n, \widetilde{A}x) = \widetilde{Ae_n}(\widetilde{A}x)$$

coincide on H since A is unitary, therefore coincide almost everywhere on E. We can therefore suppose that :

$$\widetilde{e}_n(x) = \widetilde{e}^1_n(\widetilde{A}x), \quad \forall x, \forall n$$

and this means that :

$$\sum \widetilde{e}^1_n(y)\, e^1_n \longrightarrow y \quad \text{for } y \in \widetilde{A}(K_\varepsilon).$$

Since $\widetilde{A}(K_\varepsilon)$ carries μ up to ε , we have the announced result.