Annales mathématiques Blaise Pascal

João B. Prolla
Samuel Navarro

Approximation results in the strict topology

Annales mathématiques Blaise Pascal, tome 4, nº 2 (1997), p. 61-82
<http://www.numdam.org/item?id=AMBP_1997__4_2_61_0>

Ann. Math. Blaise Pascal, Vol. 4, N° 2, 1997, pp.61-82

Approximation Results in the Strict Topology

João B. Prolla and Samuel Navarro*

Abstract: In this paper we prove results of the Weierstrass-Stone type for subsets W of the vector space V of all continuous and bounded functions from a topological space X into a real normed space E, when V is equipped with the strict topology β. Our main results characterize the β-closure of W when (1) W is β-truncation stable; (2) $E = \mathbb{R}$ and W is a subalgebra; (3) $E = \mathbb{R}$ and W is the convex cone of all positive elements of some algebra; (4) W is uniformly bounded; (5) X is a completely regular Hausdorff space and W is convex.

§1. Introduction and definitions

Let X be a topological space and let E be a real normed space. We denote by $B(X;E)$ the normed space of all bounded E-valued functions on X, equipped with the supremum norm

$$||f||_X = \sup\{||f(x)||; x \in X\}$$

*Partially supported by FONDECYT grant 1950546 and DICYT-USACH

for each $f \in B(X;E)$. We denote by $B_0(X;E)$ the subset of all $f \in B(X;E)$ that vanish at infinity, i.e., those f such that for every $\varepsilon > 0$, the set $K = \{t \in X; ||f(t)|| \geq \varepsilon\}$ is compact (or empty). And we denote by $B_{00}(X;E)$ the subset of all $f \in B(X;E)$ which have compact support. We denote by $C(X;E)$ the vector space of all continuous E-valued functions on X, and set

$$\begin{aligned} C_b(X;E) &= C(X;E) \cap B(X;E), \\ C_0(X;E) &= C(X;E) \cap B_0(X;E), \\ C_{00}(X;E) &= C(X;E) \cap B_{00}(X;E) \end{aligned}$$

We denote by $I(X)$ the set of all $\varphi \in B(X;\mathbb{R})$ such that $0 \leq \varphi(x) \leq 1$, for all $x \in X$. We then define

$$\begin{aligned} D(X) &= C_b(X;\mathbb{R}) \cap I(X), \\ D_0(X) &= B_0(X;\mathbb{R}) \cap I(X). \end{aligned}$$

The **strict topology** β on $C_b(X;E)$ is the locally convex topology determined by the family of seminorms

$$p_\varphi(f) = \sup\{\varphi(x)||f(x)||; x \in X\}$$

for $f \in C_b(X;E)$, when φ ranges over $D_0(X)$. Clearly, given $\varphi \in D_0(X)$ there is a compact subset K such that $\varphi(x) < \varepsilon$ for all $x \notin K$. Therefore, our strict topology is coarser than the strict topology introduced by R. Giles [3]. To see that they actually coincide, let $\psi \in B(X;\mathbb{R})$ be such that, for each $\varepsilon > 0$ there is a compact subset K such that $\psi(x) < \varepsilon$ for all $x \notin K$. We may assume $||\psi||_X < 1$. Choose compact sets K_n with $\phi = K_0 \subset K_1 \subset K_2 \subset \ldots$ such that $|\psi(x)| < 2^{-n}$, for all $x \notin K_n$.

Let $\psi_n \in B_0(X;\mathbb{R})$ be the characteristic function of K_n multiplied by 2^{-n}, i.e., $\psi_n(x) = 2^{-n}$, if $x \in K_n$; and $\psi_n(x) = 0$ if $x \notin K_n$. Let $\varphi = \sum_{n=1}^{\infty} \psi_n$. For each $\varepsilon > 0$, we claim that the set $K = \{x \in X; \varphi(x) \geq \varepsilon\}$ is compact (or empty). If $\varepsilon > 1$, then $K = \phi$. If $\varepsilon = 1$, then $K = K_1$, because $\varphi(t) = 1$ precisely for $t \in K_1$. If $\varepsilon < 1$, let $n \geq 0$ be such that $2^{-(n+1)} \leq \varepsilon < 2^{-n}$. Then $K = K_{n+1}$. Hence $\varphi \in D_0(X)$. We claim now that $\psi(x) \leq \varphi(x)$ for all $x \in X$. We first notice that $\varphi(x) = 0$ if, and only if $x \notin \bigcup_{n=1}^{\infty} K_n$. Indeed, if the point $x \notin \bigcup_{n=1}^{\infty} K_n$, then $\psi_k(x) = 0$ for all $n = 1, 2, 3, \ldots$, and so $\varphi(x) = 0$. Conversely, if $\varphi(x) = 0$, then $\psi_n(x) = 0$ for all $n = 1, 2, 3, \ldots$ and therefore $x \notin K_n$ for all $n = 1, 2, 3, \ldots$ Hence $x \notin \bigcup_{n=1}^{\infty} K_n$. Let now $x \in X$. If $\varphi(x) = 0$, then $x \notin K_n$ for all $n = 1, 2, 3, \ldots$ and so $|\psi(x)| < 2^{-n}$ for all $n = 1, 2, 3, \ldots$. Hence $\psi(x) = 0$ and so $\psi(x) = \varphi(x)$. Suppose now $\varphi(x) > 0$. Then $x \in \bigcup_{n=1}^{\infty} K_n$. Let N be the smallest positive integer $n \geq 1$ such that $x \in K_n$. If $N = 1$, then $x \in K_1$ and so $\varphi(x) = 1 > \psi(x)$. If $N > 1$, then $x \in K_N$ and $x \notin K_{N-1}$. Hence

$$\varphi(x) = \sum_{n=N}^{\infty} 2^{-n} = 2^{-(N-1)}$$

and $\psi(x) < 2^{-(N-1)}$, since $x \notin K_{N-1}$. Therefore $\psi(x) < \varphi(x)$, whenever $\varphi(x) > 0$.

Given any non-empty subset $S \subset C(X; E)$ we denote by $x \equiv y$ (mod. S) the equivalence relation defined by $f(x) = f(y)$ for all $f \in S$. For each $x \in X$, the equivalence class of x (mod. S) is denoted by $[x]_S$, i.e.,

$$[x]_S = \{t \in X \ ; \ x \equiv t(\text{mod. S})\}$$

For any non-empty subset $K \subset X$ and any $f : X \to E$, we denote by f_K its restriction to K. If $S \subset C(X; E)$ and $K \subset X$, then for each $x \in K$ one has

$$[x]_{S_K} = K \cap [x]_S.$$

If $S \subset C_b(X; \mathbb{R})$, we define S^+ by

$$S^+ = \{f \in S \ ; \ f \geq 0\}.$$

If $S = C_b(X; \mathbb{R})$, we write $S^+ = C_b^+(X; \mathbb{R})$.

Definition 1. Let $S \subset C_b(X;\mathbb{R})$ and let $W \subset C_b(X;E)$ be given. We say that W is **β-localizable under** S if, for every $f \in C_b(X;E)$, the following are equivalent:

(1) f belongs to the β-closure of W;
(2) for every $\varphi \in D_0(X)$, every $\varepsilon > 0$ and every $x \in X$,there is some $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_S$.

Remark. Clearly, (1) $\Rightarrow$ (2) in any case. Hence a set W is β-localizable under S if, and only if, (2) $\Rightarrow$ (1). Notice also that if W is β-localizable under S and $T \subset S$, then W is β-localizable under T. Indeed, $T \subset S$ implies $[x]_S \subset [x]_T$.

Definition 2. We say that a set $W \subset C_b(X,E)$ is **β-truncation stable** if, for every $f \in W$ and every $M > 0$, the function $T_M \circ f$ belongs to the β-closure of W, where $T_M : E \to E$ is the mapping defined by

$$\begin{aligned} T_M(v) &= v, \text{ if } ||v|| < 2M; \\ T_M(v) &= \frac{v}{||v||}.2M, \text{ if } ||v|| \geq 2M \end{aligned}$$

Notice that, when $E = \mathbb{R}$, the mapping $T_M : \mathbb{R} \to \mathbb{R}$ is given by

$$\begin{aligned} T_M(r) &= r, \text{ if } ||r|| < 2M; \\ T_M(r) &= 2M, \text{ if } r > 2M; \\ T_M(r) &= -2M, \text{ if } r > -2M. \end{aligned}$$

Remark that, for every $f \in C_b(X;E)$, one has $||T_M \circ f||_X \leq 2M$.

Notice that when $W \subset C_b^+(X;\mathbb{R})$, then W is β-truncation stable if, for every $f \in W$ and every constant $M > 0$, the function $P_M \circ f$ belongs to the β-closure of W, where $P_M : \mathbb{R} \to \mathbb{R}_+$ is the mapping defined by $P_M = \max(0, T_M)$, i.e.,

$$\begin{aligned} P_M(r) &= 0, \text{ if } r < 0; \\ P_M(r) &= r, \text{ if } 0 \leq r \leq 2M; \\ P_M(r) &= 2M, \text{ if } r > 2M. \end{aligned}$$

Definition 3. Let $W \subset C_b(X;E)$ be a non-empty subset. A function $\psi \in D(X)$ is called a **multiplier of** W if $\psi f + (1-\psi)g$ belongs to W, for each pair, f and g, of elements of W.

Definition 4. A subset $S \subset D(X)$ is said to have **property** V if

(a) $\psi \in S$ implies $(1-\psi) \in S$;
(b) the product $\varphi\psi$ belongs to S, for any pair, φ and ψ, of elements of S.

Notice that the set of all multipliers of a subset $W \subset C_b(X;E)$ has property V. Indeed, condition (a) is clear and the equation

$$(\varphi\psi)f + (1-\varphi\psi)g = \varphi[\psi f + (1-\psi)g] + (1-\varphi)g$$

show that (b) holds as well.

When X is locally compact, R.C. Buck [1] proved a Weierstrass-Stone Theorem for subalgebras of $C_b(X;\mathbb{R})$ equipped with the strict topology. This result was extended and generalized by Glicksberg [4], Todd [7], Wells [8] and Giles [3]. See also Buck [2], where modules are dealt with, and Prolla [5], where the strict topology is considered as an example of weighted spaces.

Our versions of the Weierstrass-Stone Theorem are analogues of Chapter 4 of Prolla [6] for arbitrary subsets of $C(X;E)$ equipped with the uniform convergence topology, X compact. Whereas the previous results dealt only with algebras or vector spaces which are modules over an algebra, our results now go much further: we are able to cover the case of convex sets (when X is completely regular) or β-truncation stable sets (when X is just a topological space). The latter case cover both algebras and the convex cones obtained by taking the set of positive elements of an algebra.

§2. β-truncation stable subsets

Theorem 1. *Let $W \subset C_b(X;E)$ be a β-truncation stable non-empty subset, and let A be the set of all multipliers of W. Then W is β-localizable under A.*

Proof. Let $f \in C_b(X;E)$ be given and assume condition (2) of Definition 1, with $S = A$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that φ is not identically zero. Choose $M > 0$ so big that $M > ||f||_X, M > \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \varepsilon/(6M)\}$ is non-empty. Consider the non-empty subset $W_K \subset C(K;E)$. Clearly, the set $A_K \subset D(K)$ is a set of multipliers of W_K. Take a point $x \in K$. By condition (2) applied to $\varepsilon^2/(12M)$, there exists $g_x \in W$ such that $\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(12M)$ for all $t \in [x]_A$. Let $M \subset D(K)$ be the set of all multipliers of $W_K \subset C(K;E)$. Then M has property V. Now $A_K \subset M$ implies

$$[x]_M \subset [x]_{A_K} = [x]_A \cap K.$$

Hence $\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(12M)$ holds for all $t \in K$ such that $t \in [x]_M$. Now $\varphi(t) \geq \varepsilon/(6M)$ for all $t \in K$ and therefore

$$||f(t) - g_x(t)|| < \varepsilon/2$$

for all $t \in [x]_M$. By Theorem 1, Chapter 4, of Prolla [6] applied to $W_K \subset C(K;E)$ and to the set $M \subset D(K)$, there is $g_1 \in W$ such that

$$||f(t) - g_1(t)|| < \varepsilon/2$$

for all $t \in K$. Let $h = T_M \circ g_1$. By hypothesis, h belongs to the β-closure of W, and there is $g \in W$ such that $p_\varphi(h - g) < \varepsilon/2$. We claim that $p_\varphi(f - h) < \varepsilon/2$. Let

$t \in K$. Then

$$||g_1(t)|| \leq ||f(t) - g_1(t)|| + ||f(t)|| < \varepsilon/2 + M < 2M$$

and so $h(t) = T_M(g_1(t)) = g_1(t)$. Hence

$$\begin{aligned} \varphi(t)||f(t) - h(t)|| &= \varphi(t)||f(t) - g_1(t)|| \\ &\leq ||f(t) - g_1(t)|| < \varepsilon/2. \end{aligned}$$

Suppose now $t \notin K$. Then

$$\begin{aligned} &\varphi(t)||f(t) - h(t)|| < \frac{\varepsilon}{6M}||f(t) - h(t)|| \\ &\leq \frac{\varepsilon}{6M}\left(||f||_X + ||h||_X\right) < \frac{\varepsilon}{6M}.3M = \frac{\varepsilon}{2}, \end{aligned}$$

because $||h||_X \leq 2M$, and $||f||_X < M$.

This establishes our claim that $p_\varphi(f - h) < \frac{\varepsilon}{2}$. Hence $p_\varphi(f - g) < \varepsilon$, and f belongs to the β-closure of W. □

Theorem 2. *Let $W \subset C_b(X; E)$ be a β-truncation stable non-empty subset, and let B be any non-empty set of multipliers of W. Then W is β-localizable under B.*

Proof. Let A be the set of all multipliers of W. By Theorem 1 the set W is β-localizable under A. Now $B \subset A$, so W is also β-localizable under B. □

§3. The case of subalgebras

Lemma 1. *If $B \subset C_b(X; \mathbb{R})$ is a uniformly closed subalgebra, and $T : \mathbb{R} \to \mathbb{R}$ is a continuous mapping, with $T(0) = 0$, then $T \circ f$ belongs to B, for every $f \in B$.*

Proof. Let $f \in B$ and $\varepsilon > 0$ be given. Choose $k \geq ||f||_X$. By Weierstrass' Theorem, there exists an algebraic polynomial p such that $|T(t) - p(t)| < \varepsilon$ for all $t \in \mathbb{R}$ with $|t| \leq k$, and we may assume $p(0) = T(0) = 0$. Hence, for every $x \in X$, we have $|T(f(x)) - p(f(x))| < \varepsilon$, because $|f(x)| \leq k$. Now $p \circ f$ belongs to B, and therefore $T \circ f$ belongs to the uniform closure of B, that is B itself. □

Corollary 1. *Every subalgebra $W \subset C_b(X;\mathbb{R})$ is β-truncation stable.*

Proof. Let $f \in W$ and $M > 0$ be given. Let B be the β-closure of W in $C_b(X;\mathbb{R})$. We know that B is then a uniformly closed subalgebra. By Lemma 1 applied to $T = T_M$, we see that $T_M \circ f$ belongs to the β-closure of W as claimed. □

Corollary 2. *Every uniformly closed subalgebra of $C_b(X;\mathbb{R})$ is a lattice.*

Proof. Since

$$\begin{aligned} \max(f,g) &= \frac{1}{2}\Big[f + g + |f - g|\Big] \\ \min(f,g) &= \frac{1}{2}\Big[f + g - |f - g|\Big] \end{aligned}$$

it suffices to show that $|f| \in B$, for every $f \in B$. This follows from Lemma 1, by taking $T : \mathbb{R} \to \mathbb{R}$ to be the mapping $T(t) = |t|$, for $t \in \mathbb{R}$. □

Theorem 3. *Every subalgebra $W \subset C_b(X;\mathbb{R})$ is β-localizable under itself.*

Proof. Let $f \in C_b(X;\mathbb{R})$ and assume that condition (2) of Definition 1 holds with $S = W$. Notice that for every $x \in X$ one has

$$[x]_W = [x]_B$$

where B is the β-closure of W. Let now

$$V = \{\psi \in B;\ ||\psi||_X \leq 1\} \ \text{ and } \ A = \{\psi \in B;\ 0 \leq \psi \leq 1\}.$$

It is easy to see that

$$[x]_B = [x]_V \subset [x]_A \ ,$$

for each $x \in X$. Notice that, by Corollary 2, every $\psi \in V$ can be written in the form $\psi = \psi^+ - \psi^-$, with ψ^+ and ψ^- in A. Hence $[x]_A \subset [x]_V$ is also true. Hence f satisfies condition (2) of Definition 1 with respect to $S = A$. Now A is a set of multipliers of B, and the algebra B, by Corollary 1, is β-truncation stable. Hence, by Theorem 3, the function f belongs to the β-closure of B, that is B itself. We have proved that f belongs to the β-closure of W. Hence W is β-localizable under $S = W$. □

Corollary 3. *Let $W \subset C_b(X;\mathbb{R})$ be a subalgebra, and let $f \in C_b(X;\mathbb{R})$ be given. Then f belongs to the β-closure of W if, and only if, the following conditions are satisfied:*

(1) *for each pair, x and y, of elements of X such that $f(x) \neq f(y)$, there is some $g \in W$ such that $g(x) \neq g(y)$;*
(2) *for each $x \in X$ such that $f(x) \neq 0$ there is some $g \in W$ such that $g(x) \neq 0$.*

Proof. Clearly, if $f \in \overline{W}^{\beta}$, then (1) and (2) are satisfied. Conversely, assume that conditions (1) and (2) are verified.

Let $x \in X$ be given. By condition (1) the function f is constant on $[x]_W$. Let $f(x)$ be its value. If $f(x) = 0$, then $g_x = 0$ belongs to W and $f(t) = f(x) = 0 = g_x(t)$ for all $t \in [x]_W$. If $f(x) \neq 0$, by condition (2) there is $g \in W$ such that $g(x) \neq 0$. Define $g_x = [f(x)/g(x)]g$. Then $g_x \in W$ and $g_x(t) = f(x) = f(t)$ for all $t \in [x]_W$. Hence f satisfies condition (2) of Definition 1 with respect to $S = W$. By Theorem 3, we conclude that f belongs to the β-closure of W. □

Corollary 3 implies the following results.

Corollary 4. *Let A be a subalgebra of $C_b(X; \mathbb{R})$ which for each $x \in X$ contains a function g with $g(x) \neq 0$, and let $f \in C_b(X; \mathbb{R})$ be given. Then f belongs to the β-closure of A if, and only if, for each pair, x and y, of elements of X such that $f(x) \neq f(y)$, there is some $g \in A$ such that $g(x) \neq g(y)$.*

Corollary 5. *Let A be a subalgebra of $C_b(X; \mathbb{R})$ which separates the points of X and for each $x \in X$ contains a function g with $g(x) \neq 0$. Then A is β-dense in $C_b(X; \mathbb{R})$.*

Corollary 6. *If X is a locally compact Hausdorff space, then $C_{00}(X; \mathbb{R})$ is β-dense in $C_b(X; \mathbb{R})$.*

Lemma 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(t) \geq 0$ for all $t \in \mathbb{R}$ and $f(0) = 0$. If $k > 0$ and $\varepsilon > 0$ are given, there is a real algebraic polynomial p such that $p(t) \geq 0$ for all $0 \leq t \leq k, p(0) = 0$ and $|p(t) - f(t)| \leq \varepsilon$ for all $0 \leq t \leq k$.*

Proof. Define $g : [0,1] \to \mathbb{R}$ by setting $g(u) = f(ku)$, for each $u \in [0,1]$. Clearly, $g(u) \geq 0$, for all $0 \leq u \leq 1$ and $g(0) = 0$. Now, given $\varepsilon > 0$, choose n so that the n-th Bernstein polynomial of g, written $B_n g$, is such that

$$|(B_n g)(u) - g(u)| < \varepsilon$$

for all $0 \leq u \leq 1$. For $t \in \mathbb{R}$, define $p(t) = (B_n g)(t/k)$. Since $B_n g \geq 0$ in $[0,1]$, it follows that $p(t) \geq 0$, for $t \in [0,k]$. Since $(B_n g)(0) = g(0) = f(0) = 0$, we see that $p(0) = 0$. It remains to notice that, for any $0 \leq t \leq k$ we have $0 \leq t/k \leq 1$ and

$$|p(t) - f(t)| = |(B_n g)(t/k) - g(t/k)| < \varepsilon$$ □

Lemma 3. *If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then A^+ is β-truncation stable.*

Proof. Let $f \in A^+$ and $M > 0$ be given. We claim that $P_M \circ f$ belongs to the β-closure of A^+. Let $k > 0$ be such that $0 \leq f(x) \leq k$ for all $x \in X$. Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. By Lemma 2 above there exists a polynomial $p : \mathbb{R} \to \mathbb{R}$ such that $p(t) \geq 0$ for all $0 \leq t \leq k, p(0) = 0$ and $|p(t) - P_M(t)| < \varepsilon$ for all $0 \leq t \leq k$. Let $x \in X$. Then $\varphi(x) \leq 1$ and so $\varphi(x)|p(f(x)) - P_M(f(x))| < \varepsilon$. Now $p \circ f$ belongs to A (since $p(0) = 0$) and $p(f(x)) \geq 0$ for all $x \in X$, since $0 \leq f(x) \leq k$. Hence $p \circ f \in A^+$. This ends the proof that $P_M \circ f$ belongs to the β-closure of A^+ as claimed. □

Theorem 4. *If $A \subset C_b(X; \mathbb{R})$ is a subalgebra, then A^+ is localizable under itself.*

Proof. Let $f \in C_b(X; \mathbb{R})$ be given satisfying condition (2) of Definition 1 with respect to $S = A^+$. Define $B = \{f \in A; 0 \leq f \leq 1\}$. It is easy to see that $[x]_S = [x]_B$, for every $x \in X$. Hence f satisfies condition (2) of Definition 1 with respect to B, which is a set of multipliers of A^+. By Lemma 3, the set A^+ is β-truncation stable. Therefore A^+ is β-localizable under B, by Theorem 2. Hence f belongs to the β-closure of A^+.

Theorem 4. *Let $A \subset C_b(X; \mathbb{R})$ be a subalgebra and let $f \in C_b^+(X; \mathbb{R})$ be given. Then f belongs to the β-closure of A^+ if, and only if, the following two conditions hold:*

(1) *for each pair, x and y, of elements of X such that $f(x) \neq f(y)$, there is some $g \in A^+$ such that $g(x) \neq g(y)$;*
(2) *for each $x \in X$ such that $f(x) > 0$ there is some $g \in A^+$ such that $g(x) > 0$.*

Proof. If f belongs to the β-closure of A^+ the two conditions (1) and (2) above are easily seen to hold. Conversely, assume that conditions (1) and (2) above hold. Let $x \in X$ be given. By condition (1), the function f is constant on $[x]_S$ where $S = A^+$. Let $f(x) \geq 0$ be its constant value. If $f(x) = 0$, then $g_x = 0$ belongs to

A^+ and $f(t) = f(x) = 0 = g_x(t)$ for all $t \in [x]_S$. If $f(x) > 0$, then by condition (2) there is $g_x \in A^+$ such that $g(x) > 0$. Let $g_x = [f(x)/g(x)]g$. Then $g_x \in A^+$ and $g_x(t) = f(x) = f(t)$ for all $t \in [x]_S$. Hence f satisfies condition (2) of Definition 1 with respect to $W = A^+$ and $S = A^+$. By Theorem 4, we conclude that f belongs to the β-closure of A^+. □

§4. The case of uniformly bounded subsets

Theorem 5. *Let W be a uniformly bounded subset of $C_b(X;E)$ and let A be the set of all multipliers of W. Then W is β-localizable under A.*

Proof. Let $f \in C_b(X;E)$ be given and assume that condition (2) of Definition 1 holds with $S = A$. Let $\varepsilon > 0$ and $\varphi \in D_0(X)$ be given. Choose $M > 0$ so big that $M > ||f||_X$ and $M > k = \sup\{||g||_X; g \in W\}$, and the compact set $K = \{t \in X; \varphi(t) \geq \varepsilon/(2M)\}$ is non-empty. (Without loss of generality we may assume that φ is not identically zero). Consider the non-empty set $W_K \subset C(K;E)$. Clearly, the set A_K is a set of multipliers of W_K. Take a point $x \in K$. By condition (2) applied to $\varepsilon^2/(2M)$, there exists some $g_x \in W$ such that

$$\varphi(t)||f(t) - g_x(t)|| < \varepsilon^2/(2M)$$

for all $t \in [x]_A$. Hence $||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_{A_K}$, since $\varphi(t) \geq \varepsilon/(2M)$ for all $t \in K$. Let now M be the set of all multipliers of $W_K \subset C(K;E)$. Since $A_K \subset M$, it follows that $[x]_M \subset [x]_{A_K}$ and so $||f(t) - g_x(t)|| < \varepsilon$ for all $t \in [x]_M$. By Theorem 1, Chapter 4 of Prolla [6] there is $g \in W$ such that $||f(t) - g(t)|| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(t - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and

$$\varphi(x)||f(x) - g(x)|| \leq ||f(x) - g(x)|| < \varepsilon.$$

If $x \notin K$, then

$$\varphi(x)||f(x) - g(x)|| \leq \frac{\varepsilon}{2M}[||f||_X + ||g||_X] < \varepsilon.$$

Hence f belongs to the β-closure of W and so W is β-localizable under A. □

Theorem 6. *Let W be a uniformly bounded subset of $C_b(X;E)$ and let B be any non-empty set of multipliers of W. Then W is β-localizable under B.*

Proof. Let A be the set of all multipliers of W. Since $B \subset A$ and by Theorem 5 the set W is β-localizable under A, it follows that W is also β-localizable under B. □

Theorem 7. *Let A be a non-empty subset of $D(X)$ with property V and let $f \in D(X)$. Then f belongs to the β-closure of A if, and only if, the following two conditions hold:*

(1) *for every pair of points, x and y, of X such that $f(x) \neq f(y)$, there exists $g \in A$ such that $g(x) \neq g(y)$;*

(2) *for every $x \in X$ such that $0 < f(x) < 1$, there exists $g \in A$ such that $0 < g(x) < 1$.*

Proof. It is easy to see that conditions (1) and (2) are necessary for f to belong to the β-closure of A. Conversely, assume that f satisifes conditions (1) and (2).

Let $\varphi \in D_0(X)$ and $\varepsilon > 0$ be given. Without loss of generality we may assume that φ is not identically zero. Choose $\delta > 0$ so small that $2\delta < \varepsilon$ and the compact set $K = \{t \in X; \varphi(t) \geq \delta\}$ is non-empty. Clearly, A_K has property V. Since conditions (1) and (2) hold, we may apply Theorem 1, Chapter 8, Prolla [6] to conclude that f_K belongs to the uniform closure of A_K. Hence there is some $g \in A$ such that $|f(t) - g(t)| < \varepsilon$ for all $t \in K$. We claim that $p_\varphi(f - g) < \varepsilon$. Let $x \in X$. If $x \in K$, then $\varphi(x) \leq 1$ and $\varphi(x)|f(x) - g(x)| \leq |f(x) - g(x)| < \varepsilon$.

If $x \notin K$, then $\varphi(x) < \delta$ and

$$\varphi(x)|f(x) - g(x)| \leq \delta[||f||_X + ||g||_X] \leq 2\delta < \varepsilon.$$

Hence f belongs to the β-closure of A. □

Remark. We say that a subset $A \subset D(X)$ has property VN if $fg + (1-f)h \in A$

for all $f, g, h \in A$. Clearly, if A has property VN and contains 0 and 1, then A has property V.

Corollary 6. *Let A be a non-empty subset of $D(X)$ with property V, and let W be its β-closure. Then W has property VN and W is a lattice.*

Proof. (a) *W has property VN*: Let f, g, φ belong to W, and let $h = \varphi f + (1-\varphi)g$. Assume $h(x) \neq h(y)$. Then at least one of the following three equalities is necessarily false: $\varphi(x) = \varphi(y), f(x) = f(y)$ and $g(x) = g(y)$. Since φ, f and g belong all three to W, there exists $a \in A$ such that $a(x) \neq a(y)$. Hence h satisfies condition (1) of Theorem 7. Suppose now that $0 < h(x) < 1$. If $0 < \varphi(x) < 1$, then $0 < a(x) < 1$ for some $a \in A$, because φ belongs to the β-closure of A. Assume that $\varphi(x) = 0$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, it follows that $0 < a(x) < 1$ for some $a \in A$. Similarly, if $\varphi(x) = 1$ then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there is $a \in A$ such that $0 < a(x) < 1$. Hence h satisfies condition (2) of Theorem 7. By Theorem 7 above, the function h belongs to W.

(b) *W is lattice*: Let f and g belong to W. Let $h = \max(f, g)$. Let x and y be a pair of points of X such that $h(x) \neq h(y)$. Then at least one of the two equalities $f(x) = f(y), g(x) = g(y)$ must be false. Since f and g both belong to the β-closure of A, there exists $a \in A$ such that $a(x) \neq a(y)$. On the other hand, let $x \in X$ be such that $0 < h(x) < 1$. If $f(x) \geq g(x)$, then $h(x) = f(x)$ and so $0 < f(x) < 1$. Since $f \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. Assume now $f(x) < g(x)$. Then $h(x) = g(x)$ and so $0 < g(x) < 1$. Since $g \in W$, there exists $a \in A$ such that $0 < a(x) < 1$. By Theorem 7 above, the function h belongs to W. Similarly, one shows that the function $\min(f, g)$ belongs to W. □

Corollary 7. *Let A be a β-closed non-empty subset of $D(X)$ with property V. Then A has property VN and A is a lattice.*

Proof. Immediate from Corollary 6. □

§5. The case of convex subsets

In this section we suppose that X is a completely regular Hausdorff space. We denote its Stone-Čech compactification by βX, and by $\beta : C_b(X;\mathbb{R}) \to C(\beta X;\mathbb{R})$ the linear isometry which to each $f \in C_b(X;\mathbb{R})$ assigns its (unique) continuous extension to βX. Since β is an algebra (and lattice) isomorphism, the image $\beta(A)$ of any subset $A \subset C_b(X,\mathbb{R})$ with property V is contained in $D(\beta X)$ and has property V. If $B = \beta(A)$, then for each $x \in X$ one has

$$[x]_A = [x]_B \cap X.$$

If Y denotes the quotient space of βX by the equivalence relation $x \equiv y$ if and only if $\varphi(x) = \varphi(y)$, for all $\varphi \in B$, then Y is a compact Hausdorff space.

If $x \in X$ and $K_x \subset X$ is a compact subset disjoint from $[x]_A$, then $\pi(K_x)$ is a compact subset in Y which does not contain the point $\pi(x)$. (Here we have denoted by π the canonical projection $\pi : \beta X \to Y$. Indeed, if $\pi(x) \in \pi(K_x)$, then $\pi(x) = \pi(y)$ for some $y \in K_x$. Now $y \in [x]_B$ because that $y \in [x]_A$. But $K_x \cap [x]_A = \phi$, and we have reached a contradiction. Hence $\pi(x) \notin \pi(K_x)$. We will apply these remarks in the proof of the following lemma.

Lemma 4. *Let $A \subset D(X)$ be a subset with property V and containing some constant $0 < c < 1$. Let $x \in X$ and let $K_x \subset X$ be a compact subset, disjoint from $[x]_A$. Then, there exists an open neighborhood $W(x)$ of $[x]_A$ in X, disjoint from K_x and such that given $0 < \delta < 1$ there is $\varphi \in A$ such that*

(1) $\varphi(t) < \delta$, *for all* $t \in K_x$;
(2) $\varphi(t) > 1 - \delta$, *for all* $t \in W(x)$.

Proof. Let $N(x)$ be the complement of K_x in βX. Then $N(x)$ is an open neighborhood of $[x]_A$ in βX. We know that $\pi(K_x)$ is a compact subset of Y which does not contain the point $y = \pi(x)$. Let $f \in C(Y;\mathbb{R})$ be a mapping such that $0 \leq f \leq 1, f(y) = 0$ and $f(t) = 1$ for all $t \in \pi(K_x)$. Let $g = f \circ \pi$. By Theorem 1, Chapter 8, Prolla [6], the function g belongs to the uniform closure of B in $D(\beta X)$. Notice that $a(x) = 0$ and $g(u) = 1$, for all $u \in K_x$. Define $N(x) = \{t \in \beta X; g(t) < 1/4\}$. Clearly, $[x]_B \subset N(x)$, since $g(t) = 0$ for all $t \in [x]_B$. It is also clear that $N(x)$ is disjoint from K_x. Let us define $W(x) = N(x) \cap X$. Then $W(x)$ is an open neighborhood of $[x]_A$ in X, which is disjoint from K_x.

Given $0 < \delta < 1$, let p be a polynomial determined by Lemma 1, Chapter 1, Prolla [6], applied to $a = 1/4$ and $b = 3/4$, and $\varepsilon = \delta/2$. Let $h(t) = p(g(t))$, for all $t \in \beta X$. Since $\overline{B}$ has property V, it follows that $h \in \overline{B}$. If $t \in K_x$, then $g(t) = 1$ and so $h(t) < \delta/2$. If $t \in W(x)$, then $g(t) < 1/4$ and so $h(t) > 1 - \delta/2$. Choose now $\psi \in B$ with $||\psi - h||_X < \delta/2$, and let $\varphi \in A$ be such that $\beta(\varphi) = \psi$. Then $\varphi \in A$ satisfies conditions (1) and (2). □

Theorem 8. *Let $W \subset C_b(X;E)$ be a non-empty subset and let A be a set of multipliers of W which has property V and contains some constant $0 < c < 1$. Then W is β-localizable under A.*

Proof. Assume that condition (2) of Definition 1 is true with $S = A$. For each $x \in X$, there is some $g_x \in W$ such that, for all $t \in [x]_A$, one has $\varphi(t)||f(t) - g_x(t)|| < \varepsilon/2$. Consider the compact subset K_x of X defined by

$$K_x = \{t \in X; \varphi(t)||f(t) - g_x(t)|| \geq \frac{\varepsilon}{2}\}.$$

Clearly, K_x is disjoint from $[x]_A$. Now for each $x \in X$, select an open neighborhood $W(x)$ of $[x]_A$, disjoint from K_x, according to Lemma 4.

Select and fix a point $x_1 \in X$. Let $K = K_{x_1}$. By compactness of K, there exists a finite set $\{x_2, \ldots, x_m\} \subset K$ such that

$$K \subset W(x_2) \cup W(x_3) \cup \ldots \cup W(x_m)$$

Let $k = \sum_{i=1}^{m} p_\varphi(f - g_{x_i})$ and let $0 < \delta < 1$ be so small that $\delta k < \varepsilon/2$.

By Lemma 4, there are $\varphi_2, \ldots, \varphi_m \in A$ such that

(a) $\varphi_i(t) < \delta,$ *for all* $t \in K_{x_i}$;

(b) $\varphi_i(t) > 1 - \delta,$ *for all* $t \in W(x_i)$

for $i = 2, \ldots, m$. Define

$$\begin{aligned}
&\psi_2 = \varphi_2 \\
&\psi_3 = (1 - \varphi_2)\varphi_3 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&\psi_m = (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_{m-1})\varphi_m.
\end{aligned}$$

Clearly, $\psi_i \in A$ for all $i = 2, \ldots, m$. Now

$$\psi_2 + \ldots + \psi_j = 1 - (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_j)$$

for all $j \in \{2, \ldots, m\}$, can be easily seen by induction. Define

$$\psi_1 = (1 - \varphi_2)(1 - \varphi_3)\ldots(1 - \varphi_m)$$

then $\psi_1 \in A$ and $\psi_1 + \psi_2 + \ldots + \psi_m = 1$.

Notice that

(c) $\psi_i(t) < \delta$ for all $t \in K_{x_i}$

for each $i = 1, 2, \ldots, m$. Indeed, if $i \geq 2$ then (c) follows from (a). If $i = 1$, then for $t \in K$, we have $t \in W(x_j)$ for some $j = 2, \ldots, m$. By (b), one has $1 - \varphi_j(t) < \delta$ and so

$$\psi_1(t) = (1 - \varphi_j(t))\prod_{i \neq j}(1 - \varphi_i(t)) < \delta.$$

Let us write $g_i = g_{x_i}$ for $i = 1, 2, \ldots, m$.

Define $g = \psi_1 g_1 + \psi_2 g_2 + \ldots + \psi_m g_m$.

Notice that

$$\begin{aligned}
g \;=\; & \varphi_2 g_2 + (1 - \varphi_2)[\varphi_3 g_3 + (1 - \varphi_3)[\varphi_4 g_4 + \ldots + \\
& +(1 - \varphi_{m-1})[\varphi_m g_m + (1 - \varphi_m)g_1]\ldots]].
\end{aligned}$$

Hence $g \in W$. Let $x \in X$ be given. Then

$$\begin{aligned}\varphi(x)||f(x)-g(x)|| &= \varphi(x)||\sum_{i=1}^{m}\psi_i(x)(f(x)-g_i(x))|| \\ &\leq \varphi(x)||\sum_{i=1}^{m}\psi_i(x)||(f(x)-g_i(x))||\end{aligned}$$

Define $I = \{1 \leq \tau \leq m; x \notin K_{x_i}\}$; $J = \{1 \leq i \leq m; x \in K_{x_i}\}$.
If $i \in I$, then $x \notin K_{x_i}$ and

$$\varphi(x)||f(x)-g_i(x)|| < \frac{\varepsilon}{2}$$

and therefore

$$(*)\sum_{i\in I}\varphi(x)\psi_i(x)||f(x)-g_i(x)|| \leq \frac{\varepsilon}{2}\sum_{i\in I}\psi_i(x) \leq \frac{\varepsilon}{2}.$$

If $i \in J$, then by (c), $\psi_i(x) < \delta$ and so

$$(**)\sum_{i\in J}\varphi(x)\psi_i(x)||f(x)-g_i(x)|| \leq \delta k < \frac{\varepsilon}{2}.$$

From $(*)$ and $(**)$ we get $\varphi(x)||f(x)-q(x)|| < \varepsilon$. □

Theorem 9. *Let $W \subset C_b(X;E)$ be a non-empty convex subset and let A be the set of all multipliers of W. Then W is β-localizable under A.*

Proof. The set A has property V and, since W is convex, every constant $0 < c < 1$ belongs to A. □

Theorem 10. *Let $W \subset C_b(X;E)$ b a non-empty convex subset and let B be any non-empty set of multipliers of W. Then W is β-localizable under B.*

Proof. Similar to that of Theorem 6, using now Theorem 9 instead of Theorem 5.

□

Corollary 8. *Let $W \subset C_b(X;E)$ be a non-empty convex subset such that the set of all multipliers of W separates the points of X. Then, for each $f \in C_b(X;\mathbb{R})$ the following are equivalent:*

(1) *f belongs to the β-closure of W;*
(2) *for each $\varepsilon > 0$ and each $x \in X$, there is some $g \in W$ such that $\|f(x) - g(x)\| < \varepsilon$.*

Proof. Clearly, (1) $\Rightarrow$ (2). Suppose now that (2) holds. Let $\varphi \in D_0(X), \varepsilon > 0$ and $x \in X$ be given. Notice that $[x]_W = \{x\}$. If $\varphi(x) = 0$, for any $g \in W$ one has $\varphi(x)\|f(x) - q(x)\| = 0 < \varepsilon$. If $\varphi(x) > 0$, by (2) there is $g \in W$ such that $\|f(x) - g(x)\| < \varepsilon/\varphi(x)$. Hence $\varphi(x)\|f(x) - q(x)\| < \varepsilon$, and by Theorem 9, (1) is true. □

Corollary 9. *Let $S \subset X$ be a non-empty closed subset and let $V \subset E$ be a non-empty convex subset. Let $W = \{g \in C_b(X;E); g(S) \subset V\}$. Then, for each $f \in C_b(X;E)$ the following are equivalent:*

(1) *f belongs to the β-closure of W;*
(2) *for each $x \in S, f(x)$ belongs to the closure of V in E*

Hence, $\overline{W}^{\beta} = \{f \in C_b(X;E); f(S) \subset \overline{V}\}$, where $\overline{V}$ is the closure of V in E.

Proof. Clearly, (1) $\Rightarrow$ (2). Conversely, assume that (2) holds. Clearly, W is a convex set such that $D(X)$ is the set of all multipliers of W. Since X is a completely regular Hausdorff space, $D(X)$ separates the points of X. Let $\varepsilon > 0$ and $x \in X$ be given. If $x \in S$ there is $v \in V$ such that $\|f(x) - v\| < \varepsilon$, and the constant mapping on X whose value is v belongs to W and $g(x) = v$. If $x \notin S$, choose $\varphi \in C_b(X;\mathbb{R}), 0 \leq \varphi \leq 1, \varphi(t) = 1$ for all $t \in S$ and $\varphi(x) = 0$; and let $g \in C_b(X;E)$

be defined by $g = \varphi \otimes v_0 + (1-\varphi) \otimes f(x)$, where $v_0 \in V$ is chosen arbitrarily. Then $g(t) = v_0$ for all $t \in S$, and therefore $g \in W$, and $g(x) = f(x)$. Hence (2) of Corollary 8 is verified and so f belongs to the β-closure of W. □

Corollary 10. *Let $W \subset C_b(X;E)$ be a non-empty convex subset such that the set of all multipliers of W separates the points of X and, for each $x \in X$, the set $W(x) = \{g(x); g \in W\}$ is dense in E. Then W is β-dense in $C_b(X;E)$.*

Proof. Apply Corollary 8. □

Corollary 11. *The vector subspace $W = C_b(X;\mathbb{R}) \otimes E$ is β-dense in $C_b(X;E)$.*

Proof. The set A of all multipliers of W is $D(X)$, and $W(x) = E$, for each $x \in X$. It remains to apply Corollary 10. □

Corollary 12. *If X is a locally compact Hausdorff space, then $C_{00}(X;\mathbb{R}) \otimes E$ is β-dense in $C_b(X;E)$.*

Proof. Let $W = C_{00}(X;\mathbb{R}) \otimes E$. As in the previous corollary, the set A of all multipliers of W is $D(X)$, and for each $x \in X, W(x) = E$. □

Theorem 11. *Let $A \subset C_b(X;\mathbb{R})$ be a subalgebra and let $W \subset C_b(X;E)$ be a vector subspace which is an A-module, i.e., $AW \subset W$. Then W is β-localizable under A.*

Proof. Let $f \in C_b(X;E)$ be given . Assume that condition (2) of Definition 1 holds with $S = A$. Without loss of generality we may assume that A is β-closed and contains the constants. Let M be the set of all multipliers of W. We claim that, for each $x \in X$, one has $[x]_M \subset [x]_A$. Indeed, let $t \in [x]_M$ and let $\varphi \in A$. If $\varphi = 0$, then $\varphi \in M$ and $\varphi(t) = \varphi(x)$. Assume $\varphi \neq 0$. Write $\varphi = \varphi^+ - \varphi^-$, where $\varphi^+ = \max(\varphi, 0)$ and $\varphi^- = \max(-\varphi, 0)$. By Corollary 2, §3, both φ^+ and φ^- belong to A. If $\varphi^+ = 0$, then φ^+ belongs to M and $\varphi^+(t) = \varphi^+(x)$. If $\varphi^+ \neq 0$, let $\psi = \varphi^+/||\varphi^+||_X$. Now ψ belongs to A and $0 \leq \psi \leq 1$. Hence $\psi \in M$ and therefore $\psi(t) = \psi(x)$. Consequently, one has $\varphi^+(t) = \varphi^+(x)$. Similarly, one proves that $\varphi^-(t) = \varphi^-(x)$. Hence $\varphi(t) = \varphi(x)$. This ends the proof that $[x]_M \subset [x]_A$ for all $x \in X$. Hence condition (2) of Definition 1 is verified with $S = M$. By Theorem 9, W is β-localizable under M. Hence f belongs to the β-closure of W. □

Corollary 13. *Let $W \subset C_b(X; E)$ be a vector subspace, and let*

$$A = \{\psi \in C_b(X; \mathbb{R}); \psi g \in W \text{ for all } g \in W\}.$$

Then W is β-localizable under A.

Proof. Clearly A is a subalgebra of $C_b(X; \mathbb{R})$ and W is an A-module. □

References

[1] R.C. Buck, Bounded continuous functions on a locally compact space, *Michigan Math. J.* **5** (1958), 95-104.

[2] R.C. Buck, Approximation properties of vector-valued functions, *Pacific J. Math.* **53** (1974), 85-94.

[3] R. Giles, A generalization of the strict topology, *Trans. Amer. Math. Soc.* **161** (1971), 467-474.

[4] I. Glicksberg, Bishop's generalized Stone-Weierstrass Theorem for the strict topology, *Proc. Amer. Math. Soc.* **14** (1963), 329-333.

[5] J.B. Prolla, *Approximation of Vector Valued Function.* North-Holland Publ. Co., Amsterdam, 1977.

[6] J.B. Prolla, *Weierstrass-Stone, the Theorem.* Verlag Peter Lang, Frankfurt am Main, 1993.

[7] C. Todd, Stone-Weirstrass theorems for the strict topology, *Proc. Amer. Math. Soc.* **16** (1965), 654-659.

[8] J. Wells, Bounded continuous vector-valued functions on a locally compact space, *Michigan Math. J.* **12** (1965), 119-126.

João B. Prolla
IMECC-UNICAMP
Caixa Postal 6065
13083-970 Campinas SP
Brasil

Samuel Navarro
Departamento de Matematicas
Universidad de Santiago
Casilla 5659 C-2
Santiago
Chile

Manuscrit reçu en Janvier 1997