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## Weakly Compact Operators and the Dunford-Pettis Property on Uniform Spaces

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**ABSTRACT.** Let  $(X, \mathcal{U})$  be a Hausdorff uniform space and  $C_b(X)$  the space of all bounded continuous real-valued functions on  $X$ . The subspace of  $C_b(X)$ , consisting of the all uniformly continuous functions with respect to  $\mathcal{U}$ , is denote by  $C_{ub}(X)$ . In this paper we give a characterization of weakly compact operators and  $\beta_u$  - *continuous* defined from  $C_{ub}(X)$  into a Banach space  $E$ , where  $\beta_u$  is the finest locally convex topology agreeing with the pointwise topology on each uniformly equicontinuous and bounded subsets of  $C_{ub}(X)$ . We also show that  $(C_{ub}(X), \beta_u)$  has the Strict Dunford-Pettis Property and the Dunford-Pettis Property, both under special conditions.

### 1. INTRODUCTION AND NOTATIONS

All uniform spaces  $(X, \mathcal{U})$  are assumed to be Hausdorff uniform spaces. Basic references for the measures theory on topological spaces are in Varadarajan [7]. We will denote by  $C_b(X)$  the space of all real-valued bounded continuous functions defined on  $X$ , and  $C_{ub}(X)$ , the subspace of  $C_b(X)$ , consists of those functions which are uniformly continuous.  $\mathcal{H}$  will denote the collection of all uniformly equicontinuous and bounded (U.E.B.) subsets of  $C_{ub}(X)$ .  $\beta_u$  will denote the finest locally convex topology agreeing with the pointwise topology on each  $H \in \mathcal{H}$ . A uniform measure on  $X$  is defined to be a bounded linear functional on  $C_{ub}(X)$  which is pointwise continuous on each  $H \in \mathcal{H}$  (see [1], [2], [4]) and the space of all uniform measures will be denoted by  $M_u(X)$ . It is well known that the dual of  $(C_{ub}(X), \beta_u)$  is  $M_u(X)$ .

Let  $M(X)$  be the dual of  $(C_{ub}(X), \|\cdot\|)$ , where the  $\|\cdot\|$  denotes the supremum norm. We denote by  $\mathcal{H}$ -top the locally convex topology on  $M(X)$  of uniform convergence on the U.E.B. sets. Let  $M_d(X)$  be the subspace of  $M(X)$  generated by the Dirac measures. It was proved in [4] that the  $\mathcal{H}$ -top closure of  $M_d(X)$  is the space  $M_u(X)$ .

We denote by  $\beta_t$  the locally convex topology on  $C_b(X)$  agreeing with the compact-open topology on each norm-bounded subset of  $C_b(X)$ . Sentilles [6] proved that the

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dual of  $(C_b(X), \beta_t)$  is the space  $M_t(X)$  of all tight measures on  $X$ . One of the Sentilles's results that we will use here is the following: "a subset  $A$  of  $M_t(X)$  is uniformly tight if, and only if,  $A$  is  $\beta_t$ -equicontinuous. It is also known that  $C_{ub}(X)$  is  $\beta_t$ -dense on  $C_b(X)$  (see [1]).

The proof of the following technical lemma is a simple verification and it will be omitted..

**Lemma 1.** *The canonical mapping  $\Phi : (C_{ub}(X), \beta_u) \rightarrow (C_{ub}(\widehat{X}), \beta_u)$  defined by  $f \rightarrow \widehat{f}$  is an isomorphism, where  $\widehat{X}$  denotes the completion of  $(X, \mathcal{U})$  and  $\widehat{f}$  is the unique uniform extension of  $f$  to  $\widehat{X}$ .*

## 2. WEAKLY COMPACT OPERATORS

Let  $E$  be a Banach space. In this section we will study  $E$ -valued linear norm-continuous operators on  $C_{ub}(X)$ , in particular, weakly compact operators.

**Definition 1.** *Let  $T$  be a  $E$ -valued linear continuous operator on  $C_{ub}(X)$ . We shall say that  $T$  is a tight additive operator if its restriction to the unit ball of  $C_{ub}(X)$  is continuous for the compact-open convergence topology.*

Note that  $T$  is tight additive if, and only if,  $T$  is  $\beta_t$ -continuous and so, by the density of  $C_{ub}(X)$  in  $C_b(X)$  via the topology  $\beta_t$ ,  $T$  has a unique continuous extension to  $C_b(X)$ .

From now on we will assume that  $T : C_{ub}(X) \rightarrow E$  is a weakly compact operator, that is,  $T$  transforms the unit ball of  $C_{ub}(X)$  into a relatively weakly compact subset of  $E$ . By the denseness,  $(C_{ub}(X), \beta_t)' = (C_b(X), \beta_t)' = M_t(X)$ , the space of all tight measures (see [6]), and by the weakly compactness of  $T$ ,  $T'''(C_{ub}(X)''') \subset E$ . On the other hand,  $T$  has a unique  $\beta_t$ -continuous extension  $\widetilde{T}$  to  $C_b(X)$  and, by the latter,

$$\widetilde{T}''(C_b(X)''') = \widetilde{T}''(M_t(X)') = T'''(C_{ub}(X)''') \subset E.$$

Then,  $\widetilde{T}$  is also a weakly compact operator.

The following theorem will give a characterization of tight operators and the proof is based on the well known result which says that  $\beta_u$  and the norm topology have the same bounded sets (see [1]).

**Theorem 2.** *Suppose that  $\mathcal{U}$  is metrizable and  $(X, \mathcal{U})$  is complete. Then, the following statements are equivalent:*

- (a)  $T$  is  $\beta_u$ -continuous
- (b)  $T$  is tight additive

**Proof.** Since  $(X, \mathcal{U})$  is metrizable and complete, we have that the pointwise topology and the compact-open topology coincide on each U.E.B. subset of  $C_{ub}(X)$ .  
 (a)  $\Rightarrow$  (b) Since  $T$  is  $\beta_u$ -continuous, we have that  $\{x' \circ T : \|x'\| \leq 1\} \subset M_u(X)$ .  
 On the other hand, since  $T$  is also weakly compact operator,

$$T'(\{x' : \|x'\| \leq 1\}) = \{x' \circ T : \|x'\| \leq 1\}$$

is relatively  $\sigma(M(X), M(X)')$ -compact. By this fact and since  $M(X)$  is an AL-space, the closure of solid hull of  $\{x' \circ T : \|x'\| \leq 1\}$  is  $\sigma(M(X), M(X)')$ -compact (see Cor. 8.8, p. 119, [5]). Since  $\sigma(M(X), M(X)')$  is finer than  $\sigma(M(X), C_{ub}(X))$ , we have that  $\{x' \circ T : \|x'\| \leq 1\}$  is relatively  $\sigma(M(X), C_{ub}(X))$ -compact. Then, by [1], p. 239,  $\{x' \circ T : \|x'\| \leq 1\}$  is a tight set.

Now, take a net  $(f_\alpha)_{\alpha \in I}$  in the unit ball of  $C_{ub}(X)$  converging to 0 in the topology of compact-open convergence; hence by the above and the tightness of

$$\{x' \circ T(f_\alpha) : \|x'\| \leq 1\},$$

we have  $|x' \circ T(f_\alpha)| \leq |x' \circ T(f_\alpha)| \rightarrow 0$  uniformly for  $\|x'\| \leq 1$ . This argument shows that  $\|Tf_\alpha\| \rightarrow 0$ .

(b)  $\Rightarrow$  (a) By the tightness of  $T$ , its restriction to each U.E.B. set is continuous for the topology of compact-open convergence. But, this topology coincides with the topology of pointwise convergence on each U.E.B. set; therefore  $T$  is  $\beta_u$ -continuous.

In the next theorem we shall use the following notations: If  $(X, \mathcal{U})$  is a uniform space and  $d$  is uniformly continuous pseudometrics (*u.c.p.*) on  $X$ , then  $\hat{X}_d$  denotes the completion of the metric space which comes from  $X, d$  and the corresponding projection,  $\pi_d$ .

**Theorem 3.** Let  $T$  be a weakly compact  $E$ -valued operator defined on  $C_{ub}(X)$ . Then, the following statements are equivalent:

1.  $T$  is  $\beta_u$ -continuous
2.  $T|_H$  is pointwise continuous for each U.E.B. set  $H$ .
3.  $\{x' \circ T : x' \in E'; \|x'\| \leq 1\}$  is  $\beta_u$ -equicontinuous.
4.  $\{x' \circ T : x' \in E'; \|x'\| \leq 1\}$  is relatively  $\sigma(M_u(X), C_{ub}(X))$ -compact
5. For each *u.c.p.*  $d$  on  $X$ ,  $\pi_d \circ T$  (natural definition) is a tight additive operator.

**Proof.** The equivalences (3)  $\Leftrightarrow$  (1)  $\Leftrightarrow$  (2) are clear. The equivalence (3)  $\Leftrightarrow$  (4) follows from [1, p. 228 and 241].

(1)  $\Rightarrow$  (5) Let  $d$  be a *u.c.p.* on  $X$ . Since  $\pi_d$  is uniformly continuous, we have that  $\pi_d \circ T : C_{ub}(X) \rightarrow E$ , defined by  $(\pi_d \circ T)(\hat{f}) = T\left(\hat{f} \circ \pi_d\right)$ , is  $\beta_u$ -continuous and a weakly compact operator. Therefore, by Th. 2.2,  $\pi_d \circ T$  is a tight additive operator.

(5)  $\Rightarrow$  (1) Let  $H \in \mathcal{H}$ ; hence  $d_H(x, y) = \sup \{|f(x) - f(y)| : f \in H\}$  is a *u.c.p.* on  $X$ . Denote by  $\pi_H$  the corresponding projection of  $d_H$  and defined by  $\hat{f}(\pi_H(x)) = f(x)$ , for any  $f \in H$ . It is not difficult to see that the function  $\hat{f}$  is well defined, it belongs to  $C_{ub}(X_{d_H})$  and the  $\hat{H} = \{\hat{f} : f \in H\}$  is a U.E.B. subset of  $C_{ub}(X_{d_H})$ .

Take a net  $(f_\alpha)_{\alpha \in I}$  in  $H$  such that  $f_\alpha \rightarrow 0$  pointwise. It easily follows that  $\hat{f}_\alpha \rightarrow 0$  pointwise and, from the hypothesis,  $\|(\pi_d \circ T)(\hat{f}_\alpha)\| \rightarrow 0$ . Therefore, since  $\|Tf\| = \|(\pi_d \circ T)(\hat{f})\|$ , for any  $f \in H$ , we get that  $T$  is  $\beta_u$ -continuous.

### 3. DUNFORD-PETTIS AND STRICT DUNFORD-PETTIS PROPERTY

In this section we will analyze the Strict Dunford-Pettis and the Dunford-Pettis Property of the locally convex space  $(C_{ub}(X), \beta_u)$ . We begin with the definition of these properties which were given by Grothendieck in his well known paper "Sur les applications linéaires faiblement compact d'espace du type  $C(K)$ ", *Canad. J. Math.* 5(1974), 183-201.

**Definition 2.** We shall say that a Hausdorff locally convex space  $E$  has the Dunford-Pettis Property (resp. Strict Dunford-Pettis Property) if for any Banach space  $F$  and every linear continuous and weakly compact operator  $T : E \rightarrow F$ ,  $T(C)$  is relatively compact (resp.  $\{T(x_n)\}$  is Cauchy) in  $F$  for any absolutely convex weakly compact subset  $C$  (resp. weak Cauchy sequence  $\{x_n\}$ ) in  $E$ .

**Theorem 4.** If  $\mathcal{U}$  is metrizable, then  $(C_{ub}(X), \beta_u)$  has the Dunford-Pettis and Strict Dunford-Pettis Properties.

**Proof.** First we shall assume that  $(X, \mathcal{U})$  is a complete metrizable uniform space. Let  $T$  be a weakly compact and  $\beta_u$ -continuous linear operator defined from  $C_{ub}(X)$  into an arbitrary Banach space  $F$ . By Th. 2.2,  $T$  is a tight operator and then  $T$  admits a unique extension  $\tilde{T}$  to  $C_b(X)$  which is  $\beta_t$ -continuous.

We shall first prove that  $(C_{ub}(X), \beta_u)$  has the Strict Dunford-Pettis Property. Let  $\{f_n\}_{n \in \mathbb{N}}$  be a  $\sigma(C_{ub}(X), M_u(X))$ -Cauchy sequence in  $C_{ub}(X)$ . Since  $M_u(X) =$

$M_t(X)$  and  $\{f_n\}_{n \in \mathbb{N}}$  is in  $C_b(X)$ , we have that this sequence is  $\sigma(C_b(X), M_t(X))$  – Cauchy. Now, by [3], we know that  $(C_b(X), \beta_t)$  has the Strict Dunford-Pettis Property, therefore  $\{Tf_n\}_{n \in \mathbb{N}} = \left\{ \tilde{T} f_n \right\}_{n \in \mathbb{N}}$  is convergent in  $F$ .

If  $\mathcal{U}$  is metrizable, then the conclusion follows from Lemma 1.1, since

$$T \circ \Phi : C_{ub}(\hat{X}) \rightarrow F$$

is  $\beta_u$  – continuous and weakly compact operator.

Now, to prove that  $(C_{ub}(X), \beta_u)$  has the Dunford-Pettis Property, we again assume that  $(X, \mathcal{U})$  is complete metrizable uniform space and follows the similar argument given above.

One of the open problems that we still face, is whether or not  $(C_{ub}(X), \beta_u)$  has the Strict Dunford-Pettis Property. We already proved that the answer is yes if  $\mathcal{U}$  is metrizable. In the next theorem we will assume that  $(C_{ub}(X), \beta_u)$  has the Strict Dunford-Pettis Property and we will prove that it has the Dunford-Pettis Property under the condition that  $X$  is  $\sigma$ –compact.

**Theorem 5.** *If  $(C_{ub}(X), \beta_u)$  possesses the Strict Dunford-Pettis Property and  $X$  is  $\sigma$ –compact, then  $(C_{ub}(X), \beta_u)$  has the Dunford-Pettis Property.*

**Proof.** Let  $\{K_n\}_{n \in \mathbb{N}}$  be an increasing sequence of compact subsets of  $X$  such that  $\bigcup_{n=1}^{\infty} K_n$  is dense on  $X$ . We will denote by  $L_n$  the closed absolutely convex hull of  $K_n$  in  $M_u(X)$  ( $X$  is a uniform subspace of  $M_u(X)$ ). Since  $K_n$  is a compact subset of  $M_u(X)$  in the  $\mathcal{H}$ -top and  $(M_u(X), \mathcal{H}$  – top) is complete, we have that  $L_n$  is a compact subset of  $M_u(X)$ . Moreover,  $\{L_n\}_{n \in \mathbb{N}}$  is an increasing sequence.

We claim that  $\bigcup_{n=1}^{\infty} L_n$  is  $\mathcal{H}$ -top dense in  $M_u(X)$ . In fact, take  $\mu \in M_u(X)$  and a balanced neighborhood  $V$  of  $\mu$ . Since  $M_d(X)$  is  $\mathcal{H}$ -top dense in  $M_u(X)$ ,  $V$  contains some element  $\nu = \sum_{i=1}^p \alpha_i \delta_{x_i}$  of  $M_d(X)$ , with  $x_1, x_2, \dots, x_p \in X$ . Suppose that  $0 < \alpha = \sum_{i=1}^p |\alpha_i| \leq 1$  (if  $\alpha = 0$ ,  $V \cap \left( \bigcup_{n=1}^{\infty} L_n \right) \neq \emptyset$  and we are done) and take neighborhoods  $W_i$

of  $\delta_{x_i}$ ,  $i = 1, 2, \dots, p$ , such that  $\sum_{i=1}^p \alpha_i W_i \subset V$ . Since  $W_i \cap X$  is a neighborhood of  $\delta_{x_i}$

in  $X$ , we get  $\delta_{y_i} \in K_{n_i}$  such that  $\delta_{y_i} \in W_i \cap X$ . Thus,  $\sum_{i=1}^p \alpha_i \delta_{y_i} \in \sum_{i=1}^p \alpha_i W_i \subset V$  and  $\sum_{i=1}^p \alpha_i \delta_{y_i} \in L_N$ , where  $N = \max \{n_i : i = 1, 2, \dots, p\}$ . Therefore,  $V \cap \left( \bigcup_{n=1}^{\infty} L_n \right) \neq \emptyset$ .

Suppose now that  $\alpha > 1$ ; hence  $\alpha \sum_{i=1}^p \frac{\alpha_i}{\alpha} \delta_{x_i} \in V$  and so  $\sum_{i=1}^p \frac{\alpha_i}{\alpha} \delta_{x_i} \in \frac{1}{\alpha} V \subset V$ . Applying

the above argument to  $\frac{1}{\alpha}V$ , we get  $V \cap \left(\bigcup_{n=1}^{\infty} L_n\right) \neq \emptyset$ .

From this, we have that  $(M_u(X), \mathcal{H} - \text{top})$  is a  $\sigma$ -compact space, which implies that  $(M_u(X), \sigma(M_u(X), C_{ub}(X)))$  is also a  $\sigma$ -compact space. The conclusion of the theorem follows from [3], Th. 1

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