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Properties of quasi-invariant measures on topological groups and associated algebras.

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Abstract

Properties of quasi-invariant measures relative to dense subgroups are considered on topological groups. Mainly non-locally compact groups are considered such as

(i) a group of diffeomorphisms $Diff(t, M)$ of real or non-Archimedean manifold M in cases of locally compact and nonlocally compact M, where t is a class of smoothness,

(ii) a Banach-Lie group over a classical or non-Archimedean field,

(iii) loop groups of real and non-Archimedean manifolds.

Recently quasi-invariant measures on a group of diffeomorphisms were constructed for real locally compact M in [8, 24] and for non-locally compact real or non-Archimedean manifolds M in [10, 12, 14, 18, 20, 21]. Such groups are also Banach manifolds or strict inductive limits of their sequences. Then on a real and non-Archimedean loop groups and semigroups of families of mappings from one manifold into another they were elaborated in [13, 15, 16, 17]. On real Banach-Lie groups quasi-invaraint measures were constructed in [3].

This article is devoted to the investigation of properties of quasi-invariant measures that are important for analysis on topological groups and for construction irreducible representations [8, 23]. The following properties are investigated:

(1) convolutions of measures and functions,

(2) continuity of functions of measures,

(3) non-associative algebras generated with the help of quasi-invariant measures. The theorems given below show that many differences appear to be between locally compact and non-locally compact groups. The groups considered below are supposed to have structure of Banach manifolds over the corresponding fields.

1. Definitions. (a). Let G be a Hausdorff separable topological group. A real (or complex) Radon measure μ on $Af(G,\mu)$ is called left-quasi-invariant (or right) relative to a dense subgroup H of G, if μ_h (or μ^h) is equivalent to μ for each $h \in H$, where $Bf(G)$ is the Borel σ-field of G, $Af(G,\mu)$ is its completion by μ, $\mu_h(A) := \mu(h^{-1}A)$, $\mu^h(A) := \mu(Ah^{-1})$ for each $A \in Af(G,\mu)$, $d_\mu(h,g) := \mu_h(dg)/\mu(dg)$ (or $\tilde{d}_\mu(h,g) := \mu^h(dg)/\mu(dg)$) denote a left (or right) quasi-invariance factor. We assume that the uniformity τ_G on G is such that $\tau_G|H \subset \tau_H$, (G,τ_G) and (H,τ_H) are complete. We suppose also that there exists an open base in $e \in H$ such that their closures in G are compact (such pairs exist for loop groups and groups of diffeomorphisms and Banach-Lie groups). We denote by $M_l(G,H)$ (or $M_r(G,H)$) the set of left-(or right) quasi-invariant measures on G relative to H with a finite norm $\|\mu\| := \sup_{A\in Af(G,\mu)} |\mu(A)| < \infty$.

(b). Let $L^p_H(G,\mu,\mathbf{C})$ for $1 \le p \le \infty$ denotes the Banach space of functions $f : G \to \mathbf{C}$ such that $f_h(g) \in L^p(G,\mu,\mathbf{C})$ for each $h \in H$ and

$$\|f\|_{L^p_H(G,\mu,\mathbf{C})} := \sup_{h\in H} \|f_h\|_{L^p(G,\mu,\mathbf{C})} < \infty,$$

where $f_h(g) := f(h^{-1}g)$ for each $g \in G$. For $\mu \in M_l(G,H)$ and $\nu \in M(H)$ let

$$(\nu * \mu)(A) := \int_H \mu_h(A)\nu(dh) \text{ and } (q\tilde{*}f)(g) := \int_H f(hg)q(h)\nu(dh)$$

be convolutions of measures and functions, where $M(H)$ is the space of Radon measures on H with a finite norm, $\nu \in M(H)$ and $q \in L^s(H,\nu,\mathbf{C})$, that is

$$(\int_H |q(h)|^s|\nu|(dh))^{1/s} =: \|q\|_{L^s(H,\nu,\mathbf{C})} < \infty \text{ for } 1 \le s < \infty.$$

2. Lemma. *The convolutions*

$$* : M(H) \times M_l(G,H) \to M_l(G,H) \textit{ and}$$

$$\tilde{*} : L^1(H,\nu,\mathbf{C}) \times L^1_H(G,\mu,\mathbf{C}) \to L^1_H(G,\mu,\mathbf{C}).$$

are continuous **C**-*bilinear mappings*

Proof. It follows immediately from the definitions, Fubini theorem and because $d_\mu(h,g) \in L^1(H \times G, \nu \times \mu, \mathbf{C})$. In fact one has,

$$\|\nu * \mu\| \le \|\nu\| \times \|\mu\|,\ \|q\tilde{*}f\|_{L^1_H(G,\mu,\mathbf{C})} \le \|q\|_{L^1(H,\nu,\mathbf{C})} \times \|f\|_{L^1_H(G,\mu,\mathbf{C})}.$$

3. Definition. For $\mu \in M(G)$ its involution is given by the following formula: $\mu^*(A) := \overline{\mu(A^{-1})}$, where $\bar{b}$ denotes complex conjugated $b \in \mathbf{C}$, $A \in Af(G,\mu)$.

4. Lemma. *Let $\mu \in M_l(G,H)$ and G and H be non-locally compact with structures of Banach manifolds. Then μ^* is not equivalent to μ.*

Proof. Let $T : G \to TG$ be the tangent mapping. Then μ induces quasi-invariant measure λ from an open neighbourhood W of the unit $e \in G$ on a neighbourhood of the zero section V in T_eG and then it has an extension onto the entire T_eG. Let at first T_eG be a Hilbert space. Put $Inv(g) = g^{-1}$ then $T \circ Inv \circ T^{-1} =: K$ on V is such that there is not any operator B of trace class on T_eG such that $\tilde{M}_\lambda \subset B^{1/2}T_eG$ and $KT_eG \subset \tilde{M}_\lambda$, where $Re(1-\theta(z)) \to 0$ for $(Bz,z) \to 0$ and $z \in T_eG$, $\theta(z)$ is the characteristic functional of λ, $\tilde{M}_\lambda$ is the set of all $x \in T_eG$ such that λ_x is equivalent to λ (see theorem 19.1 [25]). Then using theorems for induced measures from a Hilbert space on a Banach space [2, 9], we get the statement of lemma 4.

5. Lemma. *For $\mu \in M_l(G,H)$ and $1 \le p < \infty$ the translation map $(q,f) \to f_q(g)$ is continuous from $H \times L^p_H(G,\mu,\mathbf{C})$ into $L^p_H(G,\mu,\mathbf{C})$.*

Proof. For metrizable G in view of the Lusin theorem (2.3.5 in [5]) and definitions of τ_G and τ_H for each $\epsilon > 0$ there are a neighbourhood $V \ni e$ in H and compacts K_1 and K in G such that the closure $cl_G V K_1 =: K_2$ is compact in G with $K_2 \subset K$, the restriction $f|_{K_2}$ is continuous and $(|\tilde{\mu}|+|\mu|)(G \setminus K_2) < \epsilon$, where $\tilde{\mu}(dg) := f(g)\mu(dg)$.

6. Proposition. *For a probability measure $\mu \in M(G)$ there exists an approximate unit, that is a sequence of non-negative continuous functions $\psi_i : G \to \mathbf{R}$ such that $\int_G \psi_i(g)\mu(dg) = 1$ and for each neighbourhood $U \ni e$ in G there exists i_0 such that $supp(\psi_i) \subset U$ for each $i > i_0$.*

Proof follows from the Radon property of μ and the existence of countable base of neighbourhoods in $e \in G$.

7. Proposition. *If $(\psi_i : i \in \mathbf{N})$ is an approximate unit in H relative to a probability measure $\nu \in M(H)$, then $\lim_{i\to\infty} \psi_i * f = f$ in the $L^1_H(G,\mu,\mathbf{C})$ norm, where $\mu \in M_l(G,H)$, $f \in L^1_H(G,\mu,\mathbf{C})$.*

Proof follows from lemma III.11.18 [6] and lemmas 2, 5.

8. Lemma. *Suppose $g \in L^q_H(G,\mu,\mathbf{C})$ and $(g^x|_H) \in L^q(H,\nu,\mathbf{C})$ for each $x \in G$, $f \in L^p(H,\nu,\mathbf{C})$ with $1 < p < \infty$, $1/p+1/q=1$, where $g^x(y) := g(yx)$ for each x and $y \in G$. Let μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then $f\tilde{*}g \in L^1_H(G,\mu,\mathbf{C})$ and there exists a function $h : G \to \mathbf{C}$ such that $h|_H$ is continuous, $h = f\tilde{*}g$ μ-a.e. on G and h vanishes at ∞ on G.*

Proof. In view of Fubini theorem and Hölder inequality we have

$$\|f\tilde{*}g\|_{L^1_H(G,\mu,\mathbf{C})} = \sup_{s\in H}\int_G\int_H |f(y)| \times |g(z)|\nu(dy)\mu((ys)^{-1}dz) \le$$

$$\sup_{s\in H}(\int_G\int_H |g(z)|^q\nu(dy)\mu((ys)^{-1}dz))^{1/q} \times (\int_G\int_H |f(y)|^p\nu(dy)\mu((ys)^{-1}dz)^{1/p} \le$$

$$\|f\|_{L^p(H,\nu,\mathbf{C})} \times \|g\|_{L^q_H(G,\mu,\mathbf{C})} \times \nu(H)\mu(G).$$

The equation $\alpha_f(\phi) := \int_H f(y)\overline{\phi(y)}\nu(dy)$ defines a continuous linear functional on $L^q(H,\nu,\mathbf{C})$. In view of lemma 5 the function $\alpha_f(g^{(sx)^{-1}}) =: \tilde{h}((sx)^{-1}) =: w(s,x)$ of two variables s and x is continuous on $H \times H$ for $s, x \in H$, since the mapping $(s,x) \mapsto (sx)^{-1}$ is continuous from $H \times H$ into H. By Fubini theorem (see §2.6.2 in [5])

$$\int_G h(y)\psi(y)\mu(dy) = \int_G\int_H f(y)g(yx)\psi(x)\nu(dy)\mu(dx) =$$

$$\int_H f(y)[\int_G g(yx)\psi(x)\mu(dx)]\nu(dy)$$

for each $\psi \in L^p(G,\mu,\mathbf{C})$, since

$$\int_G\int_H |f(y)g(yx)\psi(x)|\ |\nu|(dy)\ |\mu|(dx) < \infty,$$

where $|\nu|$ denotes the variation of the real-valued measure ν, $h(y) := \tilde{h}(y^{-1})$. Here ψ is arbitrary in $L^p(G,\mu,\mathbf{C})$, from this it follows, that $\mu(\{y : h(y) \ne (f\tilde{*}g)(y), y \in G\}) = 0$, since h and $(f\tilde{*}g)$ are μ-measurable functions due to Fubini theorem and the continuity of the composition and the inversion in a topological group. In view of Lusin theorem (see §2.3.5 in [5]) for each $\epsilon > 0$ there are compact subsets $C \subset H$ and $D \subset G$ and functions $f' \in L^p(H,\nu,\mathbf{C})$

and $g' \in L^q_H(G,\mu,\mathbf{C})$ with closed supports $supp(f') \subset C$, $supp(g') \subset D$ such that $cl_G CD$ is compact in G,

$$\|f' - f\|_{L^p(H,\nu,\mathbf{C})} < \epsilon \text{ and } \|g' - g\|_{L^q_H(G,\mu,\mathbf{C})} < \epsilon,$$

since by the supposition of §1 the group H has the base B_H of its topology τ_H, such that the closures $cl_G V$ are compact in G for each $V \in \mathsf{B}_H$. From the inequality

$$|h'(x) - h(x)| \leq (\|f\|_{L^p(H,\nu,\mathbf{C})} + \epsilon)\epsilon + \epsilon\|g\|_{L^q_H(G,\mu,\mathbf{C})}$$

it follows that for each $\delta > 0$ there exists a compact subset $K \subset G$ with $|h(x)| < \delta$ for each $x \in G \setminus K$, where $h'(x^{-1}) := \alpha_{f'}(g'^x)$.

9. Proposition. *Let $A, B \in Af(G,\mu)$, μ and ν be probability measures, $\mu \in M_l(G,H)$, $\nu \in M(H)$. Then the function $\zeta(x) := \mu(A \cap xB)$ is continuous on H and $\nu(yB^{-1} \cap H) \in L^1(H,\nu,\mathbf{C})$. Moreover, if $\mu(A)\mu(B) > 0$, $\mu(\{y \in G : yB^{-1} \cap H \in Af(H,\nu)$ and $\nu(yB^{-1} \cap H) > 0\}) > 0$, then $\zeta(x) \neq 0$ on H.*

Proof. Let $g_x(y) := ch_A(y)ch_B(x^{-1}y)$, then $g_x(y) \in L^q_H(G,\mu,\mathbf{C})$ for $1 < q < \infty$, where $ch_A(y)$ is the characteristic function of A. In view of propositions 6 and 7 there exists $\lim_{i\to\infty} \psi_i * g_x = g_x$ in $L^1_H(G,\mu,\mathbf{C})$. In view of lemma III.11.18 [6] and lemma 8, $\zeta(x)|_H$ is continous. There is the following inequality:

$$1 \geq \int_H \mu(A \cap xB)\nu(dx) = \int_H \int_G ch_A(y)ch_B(x^{-1}y)\mu(dy)\nu(dx).$$

In view of Fubini theorem there exists

$$\int_H ch_B(x^{-1}y)\nu(dy) = \nu((yB^{-1}) \cap H) \in L^1(G,\mu,\mathbf{C}), \text{ hence}$$

$$\int_H \mu(A \cap xB)\nu(dx) = \int_G \nu(yB^{-1} \cap H)ch_A(y)\mu(dy).$$

10. Corollary. *Let $A, B \in Af(G,\mu)$, $\nu \in M(H)$ and $\mu \in M_l(G,H)$ be probability measures. Then denoting $Int_H V$ the interior of a subset V of H with respect to τ_H, one has*

(i) $Int_H(AB) \cap H \neq \emptyset$, when

$$\mu(\{y \in G : \nu(yB \cap H) > 0\}) > 0;$$

(ii) $Int_H(AA^{-1}) \ni e$, *when*

$$\mu(\{y \in G : \nu(yA^{-1} \cap H) > 0\}) > 0.$$

Proof. $AB \cap H \supset \{x \in H : \mu(A \cap xB^{-1}) > 0\}$.

11. Corollary. *Let $G = H$. If $\mu \in M_l(G, H)$ is a probability measure, then G is a locally compact topological group.*

Proof. Let us take $\nu = \mu$ and $A = C \cup C^{-1}$, where C is a compact subset of G with $\mu(C) > 0$, whence $\mu(yA) > 0$ for each $y \in G$ and inevitably $Int_G(AA^{-1}) \ni e$.

12. Lemma. *Let $\mu \in M_l(G, H)$ be a probability measure and G be non-locally compact. Then $\mu(H) = 0$.*

Proof. This follows from theorem 19.2 [25] and theorem 3.21 and lemma 3.26 [19] and the proof of lemma 4, since the embedding $T_eH \hookrightarrow T_eG$ is a compact operator in the non-Archimedean case and of trace class in the real case (see also the papers about construction of quasi-invariant measures on the groups considered here [3, 8, 10, 12, 13, 14, 16, 17, 18], [20, 21, 24]). Indeed, the measure μ on G is induced by the corresponding measure ν on a Banach space Z for which there exists a local diffeomorphism $A : W \to V$, where W is a neighbourhood of e in G and V is a neighbourhood of 0 in Z. The measure μ on G is quasi-invariant relative to H. Therefore, the measure ν on U is quasi-invariant relative to the action of elements $\psi \in W' \subset W \cap H$ due to the local diffeomorphism A, that is, ν_ϕ is equivalent to ν for each $\phi := A\psi A^{-1}$, where $AW'A^{-1}U \subset V$, W' is an open neighbourhood of e in H and U is an open neighbourhood of 0 in Z, $\nu_o(E) := \nu(\phi^{-1}E)$, ϕ is an operator on Z such that it may be non-linear. The quasi-invariance factor $\rho_\nu(\phi, v)$ has expression through $|det(\phi')|$ and the quasi-invariance factor $q_\nu(z, x)$ relative to linear shifts $z \in Z'$ given by theorems from §26 [25] in the real case and theorem 3.28 [19] in the non-Archimedean case:

$$\nu_\phi(dx)/\nu(dx) = |det\{\phi'(\phi^{-1}(x))\}| q_\nu(x - \phi^{-1}(x), x),$$

where $x \in U$, $\phi = A\psi A^{-1}$, $\psi \in W'$. Then $(A\psi A^{-1}v - v) \in Z'$ for each $v \in V$ and $\psi \in W'$, where ν on Z is quasi-invariant relative to shifts on vectors $z \in Z'$ and there exists a compact operator in the non-Archimedean case and an operator of trace class in the real case of embedding $\theta : Z' \hookrightarrow Z$ such that $\nu(Z') = 0$.

13. Theorem. *Let (G,τ_G) and (H,τ_H) be a pair of topological non-locally compact groups G, H (Banach-Lie, Frechet-Lie or groups of diffeomorphisms or loop groups) with uniformities τ_G,τ_H such that H is dense in (G,τ_G) and there is a probability measure $\mu \in M_l(G,H)$ with continuous $d_\mu(z,g)$ on $H\times G$. Also let X be a Hilbert space over $\mathbf{C}$ and $U(X)$ be the unitary group. Then (1) if $T: G\to U(X)$ is a weakly continuous representation, then there exists $T': G\to U(X)$ equal μ-a.e. to T and $T'|_{(H,\tau_H)}$ is strongly continuous;*

(2) if $T: G\to U(X)$ is a weakly measurable representation and X is separable, then there exists $T': G\to U(X)$ equal to T μ-a.e. and $T'|_{(H,\tau_H)}$ is strongly continuous.

Proof. Let $R(G) := (I)\cup L^1(G,\mu)$, where I is the unit operator on L^1. Then we can define

$$A_{(\lambda e+a)_h} := \lambda I + \int_G a_h(g)[d(h^{-1},g)]T_g\mu(dg),$$

where $a_h(g) := a(h^{-1}g)$. Then

$$|(A_{(\lambda e+a)_h} - A_{\lambda e+a}\xi,\eta)| \le \int_G |a_h(g)d_\mu(h,g)-a(g)|\ |(T_g\xi,\eta)|\mu(dg),$$

hence A_{a_h} is strongly continuous with respect to $h\in H$, that is,

$$\lim_{h\to e}|A_{a_h}\xi - A_a\xi| = 0.$$

Denote $A_{a_h} = T'_hA_a$ as in §29 [22], so $T'_h\xi = A_{a_h}\xi$, where $\xi = A_a\xi_0$, $a\in L^1$. Whence

$$(T'_h\xi,T'_h\xi) = (A_{a_h}\xi_0,A_{a_h}\xi_0) =$$

$$\int_G \bar{a}_h(g)(T_g\xi_0,T_{g'}\xi_0)d_\mu(h^{-1},g)d_\mu(h^{-1},g')a_h(g')\mu(dg)\mu(dg')$$

$$= \int_G \bar{a}(z)a(z')(U_z\xi_0,U_{z'}\xi_0)\mu(dz)\mu(dz') = (\xi,\xi).$$

Therefore, T'_h is uniquely extended to a unitary operator in the Hilbert space $X'\subset X$. In view of lemma 12, $\mu(H)=0$. Hence T' may be considered equal to T μ-a.e. Then the space $span_{\mathbf{C}}[A_{a_h}: h\in H]$ is evidently dense in X, since

$$(A_{a_h}\xi_1,A_{x_q}\xi_0) = (\int_G a_h(g)T_gd(h^{-1},g)\mu(dg)\xi_1, \int_G x_q(g')T_{g'}d(q^{-1},g')\mu(dg')\xi_0) =$$

$$(T_h\int_G a(g)T_g\mu(dg)\xi_1, T_q\int_G x(g')T_{g'}\mu(dg')) = (T_{q^{-1}h}A_a\xi_1,A_x\xi_0).$$

For proving the second statement let
$\mathsf{R} := [\xi : A_a\xi = 0$ for each $a \in L^1(G,\mu)]$. If

$$(A_a\xi,\eta) = \int_G a(g)(T_g\xi,\eta)\mu(dg) = \int_G a(g)(T'_g\xi,\eta)\mu(dg)$$

for each $a(g) \in L^1(G,\mu,\mathbf{C})$, then $(T_g\xi,\eta) = (T'_g\xi,\eta)$ for μ-almost all $g \in G$. Suppose that $\{\xi_n : n \in \mathbf{N}\}$ is a complete orthonormal system in X. If $\xi \in X$, then

$$\int_G a(g)(T_g\xi,\xi_m)\mu(dg) = 0$$

for each $g \in G \setminus S_m$, where $\mu(S_m) = 0$. Therefore, $(T_g\xi,\xi_m) = 0$ for each $m \in \mathbf{N}$, if $g \in G \setminus S$, where $S := \bigcup_{m=1}^{\infty} S_m$. Hence $T_g\xi = 0$ for each $g \in G \setminus S$, consequently, $\xi = 0$. Then $(T_g\xi_n,\xi_m) = (T'_g\xi_n,\xi_m)$ for each $g \in G \setminus \gamma_{n,m}$, where $\mu(\gamma_{n,m}) = 0$. Hence $(T_g\xi_n,\xi_m) = (T'_g\xi_n,\xi_m)$ for each $n,m \in \mathbf{N}$ and each $g \in G \setminus \gamma$, where $\gamma := \bigcup_{n,m}\gamma_{n,m}$ and $\mu(\gamma) = 0$. Therefeore, $\mathsf{R} = 0$.

14. Definition and note. Let $\{G_i : i \in \mathbf{N_o}\}$ be a sequence of topological groups such that $G = G_0$, $G_{i+1} \subset G_i$ and G_{i+1} is dense in G_i for each $i \in \mathbf{N_o}$ and their topologies are denoted τ_i, $\tau_i|_{G_{i+1}} \subset \tau_{i+1}$ for each i, where $N_o := \{0,1,2,...\}$. Suppose that these groups are supplied with real probability quasi-invariant measures μ^i on G_i relative to G_{i+1}. For example, such sequences exist for groups of diffeomorphisms or loop groups considered in previous papers [10, 12, 13, 15, 16, 17, 18], [20, 21]. Let $L^2_{G_{i+1}}(G_i,\mu^i,\mathbf{C})$ denotes a subspace of $L^2(G_i,\mu^i,\mathbf{C})$ as in §1(b). Such spaces are Banach and not Hilbert in general. Let $\tilde{L}^2(G_{i+1},\mu^{i+1},L^2(G_i,\mu^i,\mathbf{C})) := H_i$ denotes the subspace of $L^2(G_i,\mu^i,\mathbf{C})$ of elements f such that

$$\|f\|_i^2 := [\|f\|^2_{L^2(G_i,\mu^i,\mathbf{C})} + \|f\|'^2_i]/2 < \infty, \text{ where}$$

$$\|f\|'^2_i := \int_{G_{i+1}}\int_{G_i} |f(y^{-1}x)|^2\mu^i(dx)\mu^{i+1}(dy).$$

Evidently H_i are Hilbert spaces due to the parallelogram identity. Let

$$f^{i+1} * f^i(x) := \int_{G_{i+1}} f^{i+1}(y)f^i(y^{-1}x)\mu^{i+1}(dy)$$

denotes the convolution of $f^i \in H_i$.

15. Lemma. *The convolution* $* : H_{i+1} \times H_i \to H_i$ *is a continuous bilinear mapping.*

Proof. In view of Fubini theorem and Cauchy inequality:

$$\int_{G_{i+1}}\int_{G_i}|f^{i+1}*f^i(z^{-1}x)|^2\mu^i(dx)\mu^{i+1}(dz)=$$

$$\int_{G_{i+1}}\int_{G_i}\int_{G_{i+1}}f^{i+1}(y)f^i(y^{-1}z^{-1}x)\mu^{i+1}(dy)\int_{G_{i+1}}\bar{f}^{i+1}(q)\bar{f}^i(q^{-1}z^{-1}x)\mu^{i+1}(dq)\mu^i(dx)\mu^{i+1}(dz)$$

$$\leq\int_{G_i}\int_{G_{i+1}}(\int_{G_{i+1}}|f^{i+1}(y)|^2\mu^{i+1}(dy))^{1/2}(\int_{G_{i+1}}|f^{i+1}(q)|^2\mu^{i+1}(dq))^{1/2}$$

$$(\int_{G_{i+1}}|f^i(y^{-1}z^{-1}x)|^2\mu^{i+1}(dy))^{1/2}(\int_{G_{i+1}}|f^i(q^{-1}z^{-1}x)|^2\mu^{i+1}(dq))^{1/2}\mu^i(dx)\mu^{i+1}(dz)\leq$$

$$\|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\int_{G_i}[\int_{G_{i+1}}\int_{G_{i+1}}|f^i(y^{-1}z^{-1}x)|^2\mu^{i+1}(dy)\mu^{i+1}(dz)]^{1/2}$$

$$[\int_{G_{i+1}}\int_{G_{i+1}}|f^i(q^{-1}z^{-1}x)|^2\mu^{i+1}(dq)\mu^{i+1}(dz)]^{1/2}\mu^i(dx)\leq$$

$$\|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\int_{G_{i+1}}\int_{G_i}\int_{G_{i+1}}|f^i(y^{-1}z^{-1}x)|^2\mu^{i+1}(dy)\mu^{i+1}(dz)\mu^i(dx)$$

$$=\|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}(\int_{G_i}\int_{G_{i+1}}\int_{G_{i+1}}|f^i(y^{-1}\gamma)|^2\mu^{i+1}(dy)\mu^{i+1}(dz)d_{\mu^i}(z^{-1},\gamma)\mu^i(d\gamma))\leq$$

$$\|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\int_{G_i}\int_{G_{i+1}}|f^i(z^{-1}x)|^2\mu^{i+1}(dz)\mu^i(dx),\text{ since}$$

$$\int_{G_i}\int_{G_{i+1}}d_{\mu^i}(z^{-1},\gamma)\mu^i(d\gamma)\mu^{i+1}(dz)=\int_{G_{i+1}}\mu^{i+1}(dz)\int_{G_i}\mu^i(zd\gamma)=1.\text{ Then}$$

$$\|f^{i+1}*f^i\|^2_{L^2(G_i,\mu^i,\mathbf{C})}=\int_{G_i}|\int_{G_{i+1}}f^{i+1}(y)f^i(y^{-1}x)\mu^{i+1}(dy)|^2\mu^i(dx)$$

$$\leq\|f^{i+1}\|^2_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\int_{G_i}\int_{G_{i+1}}|f^i(z^{-1}x)|^2\mu^{i+1}(dz)\mu^i(dx).\text{ Therefore,}$$

$$\|f^{i+1}*f^i\|_i\leq\|f^{i+1}\|_{L^2(G_{i+1},\mu^{i+1},\mathbf{C})}\|f^i\|_i.$$

16. Definition. Let $l_2(\{H_i : i\in\mathbf{N_o}\})=: H$ be the Hilbert space consisting of elements $f=(f^i : f^i\in H_i, i\in\mathbf{N_o})$. for which

$$\|f\|^2:=\sum_{i=0}^{\infty}\|f^i\|_i^2<\infty.$$

For elements f and $g\in H$ their convolution is defined by the formula: $f\star g:=h$ with $h^i:=f^{i+1}*g^i$ for each $i\in\mathbf{N_o}$. Let $*: H\to H$ be an involution such that $f^* := (\bar{f}^{j\wedge} : j \in \mathbf{N_o})$, where $f^{j\wedge}(y_j) := f^j(y_j^{-1})$ for each $y_j \in G_j$, $f := (f^j : j \in \mathbf{N_o})$, $\bar{z}$ denotes the complex conjugated $z \in \mathbf{C}$.

17. Lemma. *H is a non-associative non-commutative Hilbert algebra with involution $*$, that is $*$ is conjugate-linear and $f^{**} = f$ for each $f \in H$.*

Proof. In view of Lemma 15 the convolution $h = f \star g$ in the Hilbert space H has the norm $\|h\| \le \|f\|\ \|g\|$, hence is a continuous mapping from $H \times H$ into H. From its definition it follows that the convolution is bilinear. It is non-associative as follows from the computation of i-th terms of $(f \star g) \star q$ and $f \star (g \star q)$, which are $(f^{i+2} * g^{i+1}) * q^i$ and $f^{i+1} * (g^{i+1} * q^i)$ respectively, where f, g and $q \in H$. It is non-commutative, since there are f and $g \in H$ for which $f^{i+1} * g^i$ are not equal to $g^{i+1} * f^i$. Since $f^{j\wedge\wedge}(y_j) = f^j(y_j)$ and $\bar{\bar{z}} = z$. one has $f^{**} = (f^*)^* = f$.

18. Note. In general $(f \star g^*)^* \ne g \star f^*$ for f and $g \in H$, since there exist f^j and g^j such that $g^{j+1} * (f^j)^* \ne (f^{j+1} * (g^j)^*)^*$. If $f \in H$ is such that $f^j|_{G_{j-1}} = f^{j+1}$, then

$$((f^{j-1})^* * f^j)(e) = \int_{G_{j+1}} \bar{f}^{j+1}(y^{-1}) f^{j+1}(y) \mu^{j+1}(dy) = \|f^{j+1}\|^2_{L^2(G_{j+1}, \mu^{j+1}, \mathbf{C})},$$

where $j \in \mathbf{N_o}$.

19. Definition. Let $l_2(\mathbf{C})$ be the standard Hilbert space over the field $\mathbf{C}$ be considered as a Hilbert algebra with the convolution $\alpha \star \beta = \gamma$ such that $\gamma^i := \alpha^{i+1}\beta^i$, where $\alpha := (\alpha^i : \alpha^i \in \mathbf{C}, i \in \mathbf{N_o})$, α, β and $\gamma \in l_2(\mathbf{C})$.

20. Note. The algebra $l_2(\mathbf{C})$ has two-sided ideals $J_i := \{\alpha \in l_2(\mathbf{C}) : \alpha^j = 0 \text{ for each } j > i\}$, where $i \in \mathbf{N_o}$. That is, $J \star l_2(\mathbf{C}) \subset J$ and $l_2(\mathbf{C}) \star J = J$ and J is the $\mathbf{C}$-linear subspace of $l_2(\mathbf{C})$, but $J \star l_2(\mathbf{C}) \ne J$. There are also right ideals, which are not left ideals: $K_i := \{\alpha \in l_2(\mathbf{C}) : \alpha^j = 0 \text{ for each } j = 0, ..., i\}$, where $j \in \mathbf{N_o}$. That is, $l_2(\mathbf{C}) \star K_i = K_i$, but $K_i \star l_2(\mathbf{C}) = K_{i-1}$ for each $i \in \mathbf{N_o}$, where $K_{-1} := l_2(\mathbf{C})$. The algebra $l_2(\mathbf{C})$ is the particular case of H, when $G_j = \{e\}$ for each $j \in \mathbf{N_o}$. We consider further H for non-trivial topological groups outlined above.

21. Theorem. *If F is a maximal proper left or right ideal in H, then H/F is isomorphic as the nonassociative noncommutative algebra over $\mathbf{C}$ with $l_2(\mathbf{C})$.*

Proof. Since F is the ideal, it is the $\mathbf{C}$-linear subspace of H. Suppose, that there exists $j \in \mathbf{N_o}$ such that $f^j = 0$ for each $f \in F$, then $f^i = 0$ for each $i \in \mathbf{N_o}$, since the space of bounded complex-valued continuous functions $C^0_b(G_\infty, \mathbf{C})$ on $G_\infty := \bigcap_{j=0}^\infty G_j$ is dense in each $H_j := \{f^j : f \in H\}$

and $C_b^0(G_\infty, \mathbf{C}) \cap F_j = \{0\}$ and $C_b^0(G_j, \mathbf{C})|_{G_{j+1}} \supset C_b^0(G_{j+1}, \mathbf{C})$. Therefore, $F_j \neq \{0\}$ for each $j \in \mathbf{N_o}$, consequently, $\mathbf{C} \hookrightarrow F_j$ for each $j \in \mathbf{N_o}$. Since $\mathbf{C}$ is embeddable into each F_j, then there exists the embedding of $l_2(\mathbf{C})$ into F, where $H_j := \{f^j : f \in H\}$, $\pi_j : H \to H_j$ are the natural projections.

The subalgebra F is closed in H, since H is a topological algebra and F is a maximal proper subalgebra. The space $H_\infty := \bigcap_{j \in \mathbf{N_o}} H_j$ is dense in each H_j and the group $G_\infty := \bigcap_{j \in \mathbf{N_o}} G_j$ is dense in each G_j.

Suppose that $F_i = H_i$ for some $i \in \mathbf{N_o}$, then $F_j = H_j$ for each $j \in \mathbf{N_o}$, since $C_b^0(G_\infty, \mathbf{C})$ is dense in each H_j and $C_b^0(G_j, \mathbf{C})|_{G_{j+1}} \supset C_b^0(G_{j+1}, \mathbf{C})$. The ideal F is proper, consequently, $F_j \neq H_j$ as the $\mathbf{C}$-linear subspace for each $j \in \mathbf{N_o}$, where $F_j = \pi_j(F)$.

There are linear continuous operators from $l_2(\mathbf{C})$ into $l_2(\mathbf{C})$ given by the following formulas: $x \mapsto (0, ..., 0, x^0, x^1, x^2, ...)$ with 0 as n coordinates at the beginning, $x \mapsto (x^n, x^{n+1}, x^{n+2}, ...)$ for $n \in \mathbf{N}$; $x \mapsto (x^{kl+\sigma_k(i)} : k \in \mathbf{N_o}, i \in (0, 1, ..., l-1))$, where $\mathbf{N} \ni l \geq 2$, $\sigma_k \in S_l$ are elements of the symmetric group S_l of the set $(0, 1, ..., l-1)$. Then $f \star (g \star h) + l_2(\mathbf{C})$ and $(f \star g) \star h + l_2(\mathbf{C})$ are considered as the same class, also $f \star g + l_2(\mathbf{C}) = g \star f + l_2(\mathbf{C})$ in $H/l_2(\mathbf{C})$, since $(f + l_2(\mathbf{C})) \star (g + l_2(\mathbf{C})) = f \star g + l_2(\mathbf{C})$ for each f, g and $h \in H$. For each $f, g, h \in F$: $f \star (g \star h) + l_2(\mathbf{C})$ and $(f \star g) \star h + l_2(\mathbf{C})$ are considered as the same class, also $f \star g + l_2(\mathbf{C}) = g \star f + l_2(\mathbf{C})$ in $F/l_2(\mathbf{C})$, since $(f + l_2(\mathbf{C})) \star (g + l_2(\mathbf{C})) = f \star g + l_2(\mathbf{C}) \subset F$ for each f and $g \in F$. Therefore, the quotient algebras $H/l_2(\mathbf{C})$ and $F/l_2(\mathbf{C})$ are associative commutative Banach algebras.

Let us adjoint a unit to $H/l_2(\mathbf{C})$ and $F/l_2(\mathbf{C})$. As a consequence of the Gelfand and Mazur theorem we have, that $(H/l_2(\mathbf{C}))/(F/l_2(\mathbf{C}))$ is isomorphic with $\mathbf{C}$ (see theorem V.6.12 [6] and theorem III.11.1 [22]). On the other hand, as it was proved above $F_j \neq H_j$ for each $j \in \mathbf{N_o}$, hence there exists the following embedding $l_2(\mathbf{C}) \hookrightarrow (H/F)$ and $(H/F)/l_2(\mathbf{C})$ is isomorphic with $(H/l_2(\mathbf{C}))/(F/l_2(\mathbf{C}))$. Therefore, H/F is isomorphic with $l_2(\mathbf{C})$.

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