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Abstract

A theorem due to G. Godefroy and J. Shapiro states that every continuous convolution operator, that is not just multiplication by a scalar (non-trivial), is hypercyclic on the space of entire functions in n variables endowed with the compact-open topology. We study the space of entire functions of Hilbert-Schmidt type $\mathcal{H}_H(E)$ on a Hilbert space E. We characterize its continuous convolution operators and prove the following: Every continuous non-trivial convolution operator is hypercyclic on $\mathcal{H}_H(E)$.

Key words: Hypercyclic, Hilbert-Schmidt, Holomorphic, Convolution operator, Exponential type.

1 Introduction

A cyclic (hypercyclic) vector for an operator $T : X \to X$ is a vector x such that the closed linear hull (closed hull) of the orbit $\mathcal{O}(T,x) \equiv \{x, Tx, T^2x, ...\}$ under the operator is the entire space. An operator T is cyclic (hypercyclic) whenever there exists a cyclic (hypercyclic) vector. Recall that an invariant subset for an operator $T : X \to X$ is a subset $S \subseteq X$ such that $TS \subseteq S$. Thus every orbit constitutes an invariant set and the invariant sets $\{0\}, X$ are called trivial. Note that the closed linear hull of an orbit under a continuous operator is the smallest closed invariant subspace that contains the vector under consideration. Consequently, a continuous operator lacks non-trivial invariant closed subspaces (subsets) if and only if every non-zero vector is cyclic (hypercyclic).

The theory of cyclic and hypercyclic operators is a natural part of the study of invariant subspaces and the approximation theory. An overview of the theory is exposed in [7]. The most natural problems are maybe (1): given an operator $T : X \to X$, is it hypercyclic and (2): given a space X, does it admit a hypercyclic operator $T : X \to X$. For example, it is known that no linear operator on a finite dimensional space is hypercyclic but every separable infinite-dimensional Fréchet space carries a hypercyclic operator (see [7] for more on this).

Godefroy and Shapiro show in [6] that every continuous non-trivial convolution operator is hypercyclic on the (Fréchet-) space of entire functions in n-variables (a convolution operator is an operator that commutes with all translations and it is called trivial when it is given by $x \mapsto \alpha x$ for some scalar α). It is known that the continuous convolution operators are the operators of the form $\varphi(D)$, $\varphi(D)f \equiv \sum_{\alpha \in N^n} \varphi_\alpha D^\alpha f$ where $\varphi = \sum_{\alpha \in N^n} \varphi_\alpha y^\alpha$ is an entire exponential type function in n variables. Thus, in particular, every operator of translation is hypercyclic and the one variable version of this particular result was obtained by Birkhoff already in the twenties [2]. Before Godefroy and Shapiro obtained their general result, MacLane [11] had established the hypercyclicity of differentiation D on the one variable entire functions. Hypercyclic properties of exponential type differential operators on spaces of holomorphic functions with infinite dimensional domains, have

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also been studied (see for example [1]). In this note we prove the analogue of Godefroy and Shapiro's result for entire functions of Hilbert-Schmidt type $\mathcal{H}_H(E)$ on a (separable) Hilbert space E (Theorem 3.1). $\mathcal{H}_H(E)$ is a separable Fréchet space and is built up of homogenous Hilbert-Schmidt polynomials. A similar, but different, type of holomorphy is studied in [4]. In fact, we prove that every continuous non-trivial convolution operator has a dense set of hypercyclic vectors but that there is a certain dense subspace for which every such type of hypercyclic vector must be outside. This result is interesting in view of a result of the following type: There exists a continuous linear operator on ℓ_1 for which every non-zero vector is hypercyclic (due to Read [14] and it is not known whether we can replace ℓ_1 with an infinite-dimensional separable Hilbert space (see [7] page 359)).

For our purpose we make use of the following well-known theorem due to Gethner, Godefroy, Shapiro, Kitai ([5], [6], [10]). The theorem is based on the Baire Category Theorem and gives a criterion, known as the Hypercyclicity Criterion, for an operator to be hypercyclic.

Theorem 1.1 (Hypercyclicity Criterion) *Let X be a separable Fréchet space and let $T : X \to X$ be a continuous linear operator. Assume that T satisfies the following (hypercyclicity) criterion (HC): there are dense subsets $Z, Y \subseteq X$ and a map $S : Y \to Y$ such that*

1. $T^n z \to 0 \quad \forall z \in Z,$
2. $S^n y \to 0 \quad \forall y \in Y,$
3. $TSy = y \quad \forall y \in Y.$

Then T is hypercyclic.

We emphasize that the subsets Z, Y and the operator S in the hypothesis need not to be linear. Moreover, it is not necessary that the map S is continuous. It is known that (HC) is not a necessary condition for an operator to be hypercyclic. We shall say that an operator T (on an arbitrary locally convex Hausdorff space X) satisfies the Strong Hypercyclicity Criterion (SHC) when it satisfies the condition (HC) in such a way that the set Z can be chosen as an invariant set for T.

2 Hilbert-Schmidt entire functions and convolution operators

In this section we introduce the space of entire functions of Hilbert-Schmidt type and characterize its continuous convolution operators.

If X is a complex vector space, we denote by $\mathcal{H}_G(X)$ the complex valued Gateaux holomorphic functions on X. If $f \in \mathcal{H}_G(X)$, we denote by $D^n_y f(x)$ the n:th directional derivative at x along y. Let E be a separable complex Hilbert space (we shall tacitly assume everywhere below that all vector spaces are complex and that all Hilbert spaces are separable). We denote by $\mathcal{P}_F(^nE) \subseteq \mathcal{H}_G(E)$ the space of n-homogenous polynomials on E of finite type. That is, $\mathcal{P}_F(^nE)$ is the subspace of the n-homogenous polynomials $\mathcal{P}(^nE)$ on E, spanned by the elements $(\cdot, y)^n$, $y \in E$, where $(\cdot,\cdot)$ denotes the inner product on E. We endow $\mathcal{P}_F(^nE)$ with the inner product defined by $((\cdot, y)^n, (\cdot, z)^n)_n \equiv n!(z, y)^n$ (More precisely, by the assumption on E we can identify the symmetric tensors $\otimes_{n,s}E$ with $\mathcal{P}_F(^nE)$ and $(\cdot,\cdot)_n$ is the inner product is induced from the inner product space $\otimes_{n,s}E$ in this way). The n-homogenous Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H(^nE)$, is the completion of $\mathcal{P}_F(^nE)$ w.r.t. the inner product $(\cdot,\cdot)_n$. We use the symbol $\|\cdot\|_n$ for the corresponding norm. In view of our purposes, it is convenient to note that

$$(P, (\cdot, y)^n/n!)_n = P(y), \quad y \in E, \ P \in \mathcal{P}_H(^nE). \tag{2.1}$$

Let (e_j) be an orthonormal basis in E. For a given multi-index $\alpha \in N_\infty \equiv \oplus_{k=1}^{\infty} N$, let $e_\alpha \equiv \prod_{\mathrm{supp}\,\alpha}(\cdot, e_j)^{\alpha_j} \in \mathcal{P}_H({}^{|\alpha|}E)$. Here $\mathrm{supp}\,\alpha \equiv \{j : \alpha_j \neq 0\}$ and $|\alpha| \equiv \sum \alpha_j$. The elements e_α, $|\alpha| = n$, form an orthogonal basis for $\mathcal{P}_H({}^nE)$ and $\|e_\alpha\|_n^2 = \alpha! \equiv \alpha_1!...$ (this follows from Lemma 1 in [4]). Thus $\mathcal{P}_H({}^nE)$ can be identified with the space of all sequences (P_α) such that $\sum_{|\alpha|=n} |P_\alpha|^2 \alpha! < \infty$ and in this way we have that

$$\|P\|_n^2 = \sum_{|\alpha|=n} |P_\alpha|^2 \alpha!, \qquad P \in \mathcal{P}_H({}^nE). \tag{2.2}$$

Let us note the following. The n-homogenous nuclear polynomials $\mathcal{P}_N({}^nE)$ and the continuous polynomials $\mathcal{P}_C({}^nE)$ can be put in duality by passing to the limit out of the inner product $(\cdot,\cdot)_n$ on $\mathcal{P}_F({}^nE)$. In this way we have that $\mathcal{P}_C({}^nE)$ is the topological dual of $\mathcal{P}_N({}^nE)$ (see Dineen [3] or Gupta [8] for further details). Recall that $\mathcal{P}_N({}^nE)$ is the Banach space obtained from the completion of $\mathcal{P}_F({}^nE)$ w.r.t. the nuclear norm. We have the following (continuous) injections

$$\mathcal{P}_N({}^nE) \to \mathcal{P}_H({}^nE) \to \mathcal{P}_C({}^nE). \tag{2.3}$$

The following lemma is crucial for our investigation and can, at this stage, only be found in a preprint [13]. Therefore we include here a proof.

Lemma 2.1 *Let E be a Hilbert space and let $P \in \mathcal{P}_H({}^mE)$, $Q \in \mathcal{P}_H({}^nE)$. Then $PQ \in \mathcal{P}_H({}^{n+m}E)$ and*

$$\|PQ\|_{n+m} \leq 2^{n+m}\|P\|_m\|Q\|_n. \tag{2.4}$$

Thus, multiplication by P defines a continuous operator between $\mathcal{P}_H({}^nE)$ and $\mathcal{P}_H({}^{n+m}E)$.

PROOF: Let (e_j) be an orthonormal basis in E and let $P = \sum_{|\alpha|=m} P_\alpha e_\alpha$, $Q = \sum_{|\alpha|=n} Q_\alpha e_\alpha$. Formally we have that $PQ = \sum_{|\gamma|=n+m} R_\gamma e_\gamma$, where

$$R_\gamma \equiv \sum_{\alpha \leq \gamma,\ |\alpha|=m} P_\alpha Q_{\gamma-\alpha}, \quad \gamma \in N_\infty. \tag{2.5}$$

It suffices to prove that the right hand side defines an element R in $\mathcal{P}_H({}^{n+m}E)$, i.e. that $\sum_{|\gamma|=n+m} |R_\gamma|^2 \gamma! < \infty$. Indeed, then both PQ and R define continuous polynomials and since they coincide on $E_j \equiv \mathrm{span}\{e_1, ..., e_j\}$ for all j, we deduce that $PQ = R$.

We have that

$$|R_\gamma|^2\gamma! \leq \left(\sum_{J_\gamma(m)} |P_\alpha||Q_{\gamma-\alpha}|\right)^2 \gamma! \leq$$
$$\leq N_\gamma(m)\gamma! \sum_{J_\gamma(m)} |P_\alpha|^2|Q_{\gamma-\alpha}|^2 \leq 2^{n+m}N_\gamma(m) \sum_{J_\gamma(m)} |P_\alpha|^2\alpha!|Q_{\gamma-\alpha}|^2(\gamma-\alpha)!,$$

where $J_\gamma(m) \subseteq N_\infty$ is the index set in the sum in (2.5) and $N_\gamma(m)$ denotes the number of elements $\#J_\gamma(m)$ in $J_\gamma(m)$. We derive an estimate for $N_\gamma(m)$ by using arguments from the probability theory. Consider a bowl with $|\gamma|$ objects of $\#\,\mathrm{supp}\,\gamma$ different kinds and of γ_j of sort $j \in \mathrm{supp}\,\gamma$ respectively. Assume that we pick m objects from the bowl. Given $\alpha \in J_\gamma(m)$, the probability of obtaining precisely α_j elements of each respective sort $j \in \mathrm{supp}\,\gamma$ is known to be

$$\binom{\gamma}{\alpha} \Big/ \binom{|\gamma|}{m}, \quad \binom{\gamma}{\alpha} \equiv \prod \binom{\gamma_i}{\alpha_i}, \quad \binom{0}{0} \equiv 1.$$

The number $N_\gamma(m)$ is now nothing but the number of elementary events and hence

$$N_\gamma(m) \le \binom{|\gamma|}{m} / \min_{\alpha\in J_\gamma(m)} \binom{\gamma}{\alpha} \le \binom{|\gamma|}{m} \le 2^{n+m}.$$

Thus

$$\sum_{|\gamma|=n+m} |R_\gamma|^2\gamma! \le 4^{n+m} \sum_{|\gamma|=n+m} \sum_{J_\gamma(m)} |P_\alpha|^2\alpha!|Q_{\gamma-\alpha}|^2(\gamma-\alpha)! = 4^{n+m}\|P\|_n^2\|Q\|_m^2$$

and the proof is complete. □

We denote by $\mathfrak{A}_H(E)$ the space of all formal expansions $f = \sum f_n$, $f_n \in \mathcal{P}_H(^nE)$, i.e. $\mathfrak{A}_H(E) \equiv \prod_n \mathcal{P}_H(^nE)$ $(\mathcal{P}_H(^0E) \equiv C)$. $\mathfrak{A}_H(E)$ is a ring by virtue of Lemma 2.1. The Hilbert-Schmidt polynomials, denoted by $\mathcal{P}_H(E)$, is the subring $\oplus_n\mathcal{P}_H(^nE)$, or alternatively, the space spanned by $\cup_n\mathcal{P}_H(^nE)$ in $\mathcal{H}_G(E)$.

If E is a Hilbert space, the space of entire functions of Hilbert-Schmidt type on E, denoted by $\mathcal{H}_H(E)$, is the space defined as follows. $\mathcal{H}_H(E)$ is the space of all $f = \sum f_n \in \mathfrak{A}_H(E)$ such that

$$\|f\|_{H:r} \equiv \sum r^n\|f_n\|_n/\sqrt{n!} < \infty, \quad r > 0, \tag{2.6}$$

endowed with the semi-norms thus defined. $\mathcal{H}_H(E)$ is a Fréchet space and, in particular, $\mathcal{H}_H(C^n)$ is the space of entire functions endowed with the compact-open topology. The series $\sum f_n$ converges absolutely in $\mathcal{H}_H(E)$ and uniformly on bounded sets for every $f = \sum f_n \in \mathcal{H}_H(E)$. Indeed, we have that $|f_n(y)| \le r^n\|f_n\|_n/\sqrt{n!}$, $n \ge 0$, if $\|y\| \le r$. Thus, $\mathcal{H}_H(E)$ is separable and every element in $\mathcal{H}_H(E)$ defines an entire function of bounded type so $\mathcal{H}_H(E)$ can also be described as the space of all $f \in \mathcal{H}_G(E)$ such that $f_n \equiv D^n_{(\cdot)}f(0)/n! \in \mathcal{P}_H(^nE)$, $n = 0, \ldots$, and such that (2.6) holds.

By Lemma 2.1 we obtain:

Theorem 2.1 *Let E be a Hilbert space. Then $fg \in \mathcal{H}_H(E)$ and $\|fg\|_{H:r} \le \|f\|_{H:2r}\|g\|_{H:2r}$ for all $f, g \in \mathcal{H}_H(E)$. Thus $\mathcal{H}_H(E)$ is a subring of $\mathfrak{A}_H(E)$ and multiplication by $f \in \mathcal{H}_H(E)$ defines an everywhere defined continuous operator on $\mathcal{H}_H(E)$.*

PROOF: Let $f, g \in \mathcal{H}_H(E)$. Then $fg = \sum h_n \in \mathfrak{A}_H(E)$ where $h_n \in \sum_{i+j=n} f_ig_j$. By Lemma (2.1) we obtain

$$\frac{r^n\|h_n\|_n}{\sqrt{n!}} \le \sum_{i+j=n} \frac{r^{i+j}\|f_ig_j\|_n}{\sqrt{i!}\sqrt{j!}} \le \sum_{i+j=n} \frac{(2r)^i\|f_i\|_i}{\sqrt{i!}}\frac{(2r)^j\|g_j\|_j}{\sqrt{j!}}. \tag{2.7}$$

This estimate completes the proof. □

Given $r > 0$ we denote by $\mathrm{EXP}_r(E)$ the (Banach-) space of all $\varphi = \sum \varphi_n \in \mathfrak{A}_H(E)$ such that for some $M > 0$, $\|\varphi_n\|_n \le Mr^n/\sqrt{n!}$, $n = 0, \ldots$ equipped with the norm $|\!|\!|\varphi|\!|\!|_{H:r} \equiv \sup_n \sqrt{n!}r^{-n}\|\varphi_n\|_n$. The symbol $\mathrm{EXP}_H(E)$ denotes the union $\cup_{r>0}\mathrm{EXP}_r(E)$ equipped with the corresponding inductive locally convex topology. Thus $\mathrm{EXP}_H(E)$ is given by all $\varphi = \sum \varphi_n \in \mathfrak{A}_H(E)$ such that $\overline{\lim}(\sqrt{n!}\|\varphi_n\|_n)^{1/n} < \infty$. Every $\varphi \in \mathrm{EXP}_H(E)$ defines an exponential type function. i.e. a Gateaux holomorphic function with $|\varphi(y)| \le Me^{r\|y\|}$ for some $M, r \ge 0$, and its power series converges in $\mathrm{EXP}_H(E)$. A proof of the "finite-dimensional" analogue of the following proposition can be found in [15] (see also [12] page 320).

Proposition 2.1 *Let E be a Hilbert space. Then $\mathcal{H}_H(E)$ is reflexive and the map $\mathcal{F} : \lambda \mapsto \sum \lambda_n$. $\lambda_n(y) \equiv \lambda((\cdot, y)^n/n!)$. defines an anti-linear isomorphism between $\mathcal{H}'_H(E)$ (strong topology) and $EXP_H(E)$.*

PROOF: Let $\varphi = \sum \varphi_n \in \mathrm{EXP}_r(E)$. Then $\|\varphi_n\|_n \leq \|\!|\varphi|\!\|_{H:r} r^n/\sqrt{n!}$ and we can define a functional $\lambda = \lambda_\varphi$ on $\mathcal{H}_H(E)$ by $\lambda(f) \equiv \sum (f_n, \varphi_n)_n$. Indeed, the following estimates show that λ is well-defined and is a continuous linear functional

$$|\lambda(f)| \leq \sum \|f_n\|_n \|\varphi_n\|_n \leq \|\!|\varphi|\!\|_{H:r} \sum \|f_n\|_n r^n/\sqrt{n!} = \|\!|\varphi|\!\|_{H:r} \|f\|_{H:r}. \tag{2.8}$$

Moreover, in view of (2.1) it follows that $\mathcal{F}\lambda = \varphi$.

Next we prove that $\mathcal{F}\mathcal{H}'_H(E) \subseteq \mathrm{EXP}_H(E)$. Let $\lambda \in \mathcal{H}'_H(E)$ be arbitrary. Every $\mathcal{P}_H(^nE)$ has the topology induced by $\mathcal{H}_H(E)$. Consequently, the restriction $\lambda|_n$ to $\mathcal{P}_H(^nE)$ belongs to $\mathcal{P}'_H(^nE)$ for all n. From this we conclude that $\lambda_n \in \mathcal{P}_H(^nE)$ for all n, i.e. $\mathcal{F}\lambda = \sum \lambda_n \in \mathfrak{A}_H(E)$, and $\lambda|_n = (\cdot, \lambda_n)_n$. Now there is an $r > 0$ such that $|\lambda(f)| \leq M\|f\|_{H:r}$ for all $f \in \mathcal{H}_H(E)$. Hence

$$\|\lambda_n\|_n^2 = |\lambda|_n(\lambda_n)| = |\lambda(\lambda_n)| \leq M\|\lambda_n\|_{H:r} \leq Mr^n \|\lambda_n\|_n/\sqrt{n!} \tag{2.9}$$

and thus $\mathcal{F}\lambda = \sum \lambda_n \in \mathrm{EXP}_H(E)$. $\mathcal{F}$ is one to one and thus $\mathcal{F}$ is a vector space isomorphism.

We prove that $\mathcal{F}^{-1}$ is continuous. Let $U = B^\circ$, $B = \{f \in \mathcal{H}_H(E) : \|f\|_r \leq M_r,\ r > 0\}$ be a neighbourhood of the origin in $\mathcal{H}'_H(E)$. Let $r_0 > 0$ be arbitrary and consider the neighbourhood of the origin $V_0 \equiv \{\varphi \in \mathrm{EXP}_{r_0}(E) : \|\!|\varphi|\!\|_{H:r_0} \leq M_{r_0}^{-1}\}$ in $\mathrm{EXP}_{r_0}(E)$. From (2.8) it follws that $\mathcal{F}^{-1}V_0 \subseteq U$ and thus $\mathcal{F}^{-1}$ is continuous since r_0 was arbitrary.

In order to complete the proof of that $\mathcal{F}$ is an isomorphism, we must prove that $\mathcal{F}$ is continuous. It suffices to prove that $\mathcal{F}$ is continuous for the weak topologies $\sigma(\mathcal{H}'_H, \mathcal{H}_H)$, $\sigma(\mathrm{EXP}_H, \mathrm{EXP}'_H)$. Let $\mu \in \mathrm{EXP}'_H(E)$ be arbitrary. Then $\mu \in \mathrm{EXP}'_r(E)$ for every r. For any n and r, $\mathcal{P}_H(^nE)$ has the topology induced by $\mathrm{EXP}_r(E)$. In view of this it follows that $\mu_n(y) \equiv \overline{\mu((\cdot, y)^n/n!)}$ belongs to $\mathcal{P}_H(^nE)$ and $\mu = (\cdot, \mu_n)_n$ on $\mathcal{P}_H(^nE)$ for all n. If $r > 0$ there is an $M_r > 0$ such that $|\mu(\varphi)| \leq M_r \|\!|\varphi|\!\|_{H:r}$ for all $\varphi \in \mathrm{EXP}_r(E)$. Let $r > 0$ be arbitrary and choose $R > r$. Then we obtain

$$r^n \|\mu_n\|_n^2/\sqrt{n!} \leq r^n |\mu(\mu_n)|/\sqrt{n!} \leq r^n M_R \|\!|\mu_n|\!\|_{H:R}/\sqrt{n!} \leq M_R (r/R)^n \|\mu_n\|_n. \tag{2.10}$$

Hence $f = f_\mu \equiv \sum \mu_n \in \mathcal{H}_H(E)$. Further, we conclude that $\langle \lambda, f\rangle = \langle \mathcal{F}\lambda, \mu\rangle$ for all $\lambda \in \mathcal{H}'_H(E)$ so $\mathcal{F}$ is weakly continuous.

We have proved that $\mathcal{F}$ is an isomorphism which implies that $\mathcal{F}$ is an isomorphism for the weak topologies $\tau'' \equiv \sigma(\mathcal{H}'_H, \mathcal{H}''_H)$ and $\sigma(\mathrm{EXP}_H, \mathrm{EXP}'_H)$. But we also proved that $\mathcal{F}$ is continuous for the dual pairs $\tau' \equiv \sigma(\mathcal{H}'_H, \mathcal{H}_H)$ and $\sigma(\mathrm{EXP}_H, \mathrm{EXP}'_H)$. From this we deduce that the injection $(\mathcal{H}'_H, \tau) \to (\mathcal{H}'_H, \tau'')$ is continuous and hence $\mathcal{H}_H(E) = \mathcal{H}''_H(E)$. Thus $\mathcal{H}_H(E)$ is semi-reflexive and therefore reflexive since $\mathcal{H}_H(E)$ is barreled. □

We put $\mathcal{H}_H(E)$ and $\mathrm{EXP}_H(E)$ into sesqui-linear duality by $\langle f, \varphi\rangle = \mathcal{F}^{-1}\varphi(f)$, i.e. by the formula $\sum (f_n, \varphi_n)_n$. In view of our purposes, it is convenient to note the following. Let $e_y \equiv \overline{e^{(\cdot,y)}} = \overline{\sum (\cdot, y)^n/n!} \in \mathrm{EXP}_H(E) \subseteq \mathcal{H}_H(E)$, $y \in E$. Then $\mathcal{F}$ is given by $\mathcal{F}\lambda(y) = \overline{\lambda(e_y)}$ and $\varphi(y) = \langle e_y, \varphi\rangle$, $f(y) = \langle f, e_y\rangle$ for all $\varphi \in \mathrm{EXP}_H(E)$ and $f \in \mathcal{H}_H(E)$.

Proposition 2.2 *Let E be a Hilbert space. Multiplication by $\varphi \in EXP_H(E)$ is a continuous operator on $EXP_H(E)$ and continuous for the duality between $EXP_H(E)$ and $\mathcal{H}_H(E)$. $\mathcal{H}_H(E)$ is stable under translations and the transpose $\bar\varphi(D) \equiv {}^t\varphi : \mathcal{H}_H(E) \to \mathcal{H}_H(E)$ is a continuous convolution operator on $\mathcal{H}_H(E)$. The family, $\{\bar\varphi(D) : \varphi \in EXP_H(E)\}$ is all the continuous convolution operators on $\mathcal{H}_H(E)$. (Compare [6] Prop. 5.2.)*

PROOF: Let $\varphi, \psi \in \mathrm{EXP}_H(E)$ and put $\phi \equiv \varphi\psi \in \mathfrak{A}_H(E)$. Then there are $M, r > 0$ such that $\|\varphi\|_n, \|\psi\|_n \leq Mr^n/\sqrt{n!}$ for all n. By Lemma 2.1, and since $i!j! \geq n!/2^n$ when

$i+j=n$, we obtain

$$\|\phi_n\|_n = \|\sum_{i+j=n} \varphi_i\psi_j\|_n \le \sum_{i+j=n} 2^{i+j}\|\varphi_i\|_i\|\varphi_j\|_j$$
$$\le M^2 2^n r^n \sum_{i+j=n} 1/\sqrt{i!j!} \le M^2 2^n r^n \frac{2^{n/2}(n+1)}{\sqrt{n!}} \le \frac{M^2(R)^n}{\sqrt{n!}},$$

for some $R = R(r) > 0$. Hence $\phi \in \mathrm{EXP}_H(E)$ and our estimates show that $\psi \mapsto \psi\varphi$ is continuous on $\mathrm{EXP}_H(E)$. By Proposition 2.1 this implies that this map is continuous for the duality between $\mathrm{EXP}_H(E)$ and $\mathcal{H}_H(E)$.

Since $\psi \mapsto \psi\varphi$ is weakly continuous its transpose $\bar{\varphi}(D) \equiv {}^t\varphi$ is continuous on $\mathcal{H}_H(E)$. Indeed, $\bar{\varphi}(D)$ is continuous for $\sigma(\mathcal{H}_H, \mathcal{H}'_H) = \sigma(\mathcal{H}_H, \mathrm{EXP}_H)$ and thus for the strong topology, which is the (Frechet-) topology on $\mathcal{H}_H(E)$ (see [9], Prop. 8 page 218 & Prop. 5 page 256, for details).

The transpose of multiplication by e_y on $\mathrm{EXP}_H(E)$ is the translation operator τ_y, $[\tau_y f](x) \equiv f(y+x)$. Thus $\mathcal{H}_H(E)$ is ("continuously") stable under translations. Further, it is easily checked that every operator $\bar{\varphi}(D)$ commutes with every translation operator on the total set $\{e_y : y \in E\}$ in $\mathcal{H}_H(E)$. From this we deduce that $\bar{\varphi}(D)$, $\varphi \in \mathrm{EXP}_H(E)$ are convolution operators.

Let T be a continuous convolution operator on $\mathcal{H}_H(E)$. Then the composition $\lambda_T \equiv \delta_0 \circ T$, where $\delta_0(f) \equiv f(0)$, belongs to $\overline{\mathcal{H}'_H(E)}$. Thus, by Proposition 2.1, there is a $\varphi \in \mathrm{EXP}_H(E)$ such that $\mathcal{F}\lambda_T = \varphi$, i.e. $\overline{\lambda_T(e_y)} = \overline{[Te_y](0)} = \varphi(y)$, $y \in E$. Hence if $y_0 \in E$

$$[Te_{y_0}](y) = [\tau_y(Te_{y_0})](0) = [T(\tau_y e_{y_0})](0) = e^{(y,y_0)}[Te_{y_0}](0) = e^{(y,y_0)}\overline{\varphi(y_0)}, \quad y \in E.$$

On the other hand

$$[\bar{\varphi}(D)e_{y_0}](y) = \langle e_{y_0}, \varphi e_y\rangle = \langle \tau_y e_{y_0}, \varphi\rangle = e^{(y,y_0)}\langle e_{y_0}, \varphi\rangle = e^{(y,y_0)}\overline{\varphi(y_0)}, \quad y \in E.$$

Hence, T and $\bar{\varphi}(D)$ coincide on the total set formed by the elements e_y, $y = E$, and thus, by continuity, on all of $\mathcal{H}_H(E)$. □

Remark: If $\varphi = \sum \varphi_n \in \mathrm{EXP}_H(E)$ and $f \in \mathcal{H}_H(E)$, $\bar{\varphi}(D)f = \sum \bar{\varphi}_n(D)f$ with absolute convergence in $\mathcal{H}_H(E)$. Moreover, if $\varphi_n = \sum_j \lambda_j(\cdot, y_j)^n \in \mathcal{P}_F({}^nE)$, $\bar{\varphi}_n(D) = \sum_j \overline{\lambda_j} D^n_{y_j}$. This motivates our notation.

3 An infinite-dimensional analogue of the Godefroy-Shapiro Theorem

We have characterized the continuous convolution operators on $\mathcal{H}_H(E)$ and in this section we prove our main result - the analogue of Godefroy & Shapiro's result for $\mathcal{H}_H(E)$. We start with a short discussion.

We have that $\bar{\varphi}(D)\circ\bar{\psi}(D) = \overline{\varphi\psi}(D)$ for all $\varphi, \psi \in \mathrm{EXP}_H(E)$. From this we deduce that $\mathcal{O}(\bar{\varphi}(D), \bar{\psi}(D)f) = \bar{\psi}(D)\mathcal{O}(\bar{\varphi}(D), f)$. Since every convolution operator $\bar{\varphi}(D)$, $\varphi \neq 0$ on $\mathcal{H}_H(E)$ has a dense range (its transpose is one to one) we conclude that if f is a hypercyclic vector for $\bar{\varphi}(D)$, then so is $\bar{\psi}(D)f$ for every $0 \neq \psi \in \mathrm{EXP}_H(E)$ (it is not known if every non-zero convolution operator is surjective, i.e. if the analogue of Malgrange's classical theorem holds [12]. However by virtue of Lemma 2.1 it is not difficult to prove that every homogenous convolution operator $\bar{P}(D)$, $0 \neq P \in \mathcal{P}_H({}^nE)$) is surjective). Thus a hypercyclic vector for a convolution operator must be outside the set $\mathcal{H}_0 \equiv \cup_{\psi\neq 0} \ker \bar{\psi}(D)$. $\mathcal{H}_0$ is a dense subspace of $\mathcal{H}_H(E)$. Indeed, since $\ker\bar{\varphi}(D) \cup \ker\bar{\psi}(D) \subseteq \overline{\varphi\psi}(D)$, $\mathcal{H}_0$ is a vector space. Further, assume that $0 \neq \varphi \in \mathcal{H}_0^\perp$. Since $\ker\bar{\psi}(D)^\perp = \overline{\mathrm{Im}\,\psi}$, we have that $\mathcal{H}_0^\perp = \cap_{\psi\neq 0}\overline{\mathrm{Im}\,\psi}$. Choose y_0 so that $\varphi(y_0) \neq 0$ and let y_1 be a vector orthogonal to y_0. We deduce that φ does not belong to $\overline{\mathrm{Im}\,\psi}$ where $\psi = (\cdot, y_1)\varphi$. Thus $\mathcal{H}_0^\perp$ contains no non-zero vectors hence $\mathcal{H}_0$ is dense in $\mathcal{H}_H(E)$.

Theorem 3.1 *Let E be a Hilbert space and let $\varphi \in EXP_H(E)$ be non-constant. Then $\bar{\varphi}(D) : \mathcal{H}_H(E) \to \mathcal{H}_H(E)$ has the property (SHC) and is thus hypercyclic. Thus there exists a hypercyclic vector $f \in \mathcal{H}_H(E) \setminus \mathcal{H}_0$ such that the (dense) subspace $\mathcal{M} = \{\bar{\psi}(D)f : \psi \in EXP_H(E)\}$ is invariant for $\bar{\varphi}(D)$ and every non-zero vector in $\mathcal{M}$ is hypercyclic for $\bar{\varphi}(D)$.*

PROOF: We shall prove that $T = \bar{\varphi}(D)$ has the property (SHC). Consider the subsets

$$V = \{y \in Y : |\varphi(y)| < 1\}, \quad W = \{y \in Y : |\varphi(y)| > 1\}.$$

By the assumption on φ, V and W are both non-empty and open. Let

$$\mathcal{H}_V(E) \equiv \text{span}\{e_y : y \in V\}$$

and define $\mathcal{H}_W(E)$ similarly. We claim that $\mathcal{H}_V(E)$ and $\mathcal{H}_W(E)$ both are dense in $\mathcal{H}_H(E)$. Assume that $\mathcal{H}_V(E)$ is not dense. By the Hahn-Banach theorem and Proposition 2.1 there is a $0 \neq \psi \in \text{EXP}_H(E)$ such that

$$0 = \langle e_y, \psi \rangle = \overline{\psi(y)}, \quad y \in V.$$

Thus ψ vanishes in a neighbourhood of the origin and hence $\psi = 0$. This is a contradiction which proves our claim for $\mathcal{H}_V(E)$ and the assertion concerning $\mathcal{H}_W(E)$ follows analogously. Next, let $y \in V$ be arbitrary. Then $\bar{\varphi}(D)^n e_y = \overline{\varphi(y)}^n e_y$ for all $n \geq 0$. This shows that $\bar{\varphi}(D)$ maps $\mathcal{H}_V(E)$ into $\mathcal{H}_V(E)$ and that $\bar{\varphi}(D)^n f \to 0$ for every $f \in \mathcal{H}_V(E)$. On $\mathcal{H}_W(E)$ we define the operator S by $Se_y \equiv e_y/\overline{\varphi(y)}$, $y \in W$. We conclude, in the same way as for T and $\mathcal{H}_V(E)$, that S maps $\mathcal{H}_W(E)$ into $\mathcal{H}_W(E)$ and that $S^n f \to 0$ for every $f \in \mathcal{H}_W(E)$. Finally we note that $TSe_y = \bar{\varphi}(D)e_y/\overline{\varphi(y)} = e_y$ for $y \in W$ and thus $TSf = f$ for all $f \in \mathcal{H}_W(E)$. This completes the proof. □

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