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Weak solutions for some Reaction-Diffusion Systems with Balance law and Critical growth with respect to the Gradient.*

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Abstract

This paper is concerned with the existence of weak solutions for 2×2 reaction-diffusion systems for which two main properties hold: the positivity of the solutions and the triangular structure. Moreover, the nonlinear terms have critical growth with respect to the gradient.

AMS subject classification :35J20, 35J25, 35J65, 45H15.

1 Introduction

This paper deals with existence results for the following Reaction-Diffusion system:

$$\begin{cases} -\Delta u &= f(x,u,v,\nabla u,\nabla v) + F(x) & \text{on } \Omega \\ -\Delta v &= g(x,u,v,\nabla u,\nabla v) + G(x) & \text{on } \Omega \\ u = v = 0 & & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $-\Delta$ denotes the Laplacian operator on Ω with Dirichlet boundary conditions. Since we are essentially concerned with systems frequently encountered in applications, we restrict ourself to the case of positive solutions satisfying the triangular structure. These two main properties are ensured (respectively) by the following hypotheses

$$\begin{cases} f(x,0,v,p,q),\ g(x,u,0,p,q) \geq 0,\ F(x),\ G(x) \geq 0, \\ \text{for all } (u,v,p,q) \in \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^N \times \mathbb{R}^N \text{ and a.e. } x \in \Omega. \end{cases} \tag{2}$$

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$$\begin{cases} f(x,u,v,p,q)+g(x,u,v,p,q)\leq 0 \\ f(x,u,v,p,q)\leq 0 \\ \text{for all } (u,v,p,q)\in \mathbb{R}^+\times\mathbb{R}^+\times\mathbb{R}^N\times\mathbb{R}^N \text{ and a.e. } x\in\Omega. \end{cases} \tag{3}$$

When f and g does not depend on the gradient, an extensive literature has dealt with this kind of problems (especially for the parabolic version), existence results have been given in [9], [10], [13], [14]. Excellant surveys treating reaction-diffusion systems include Rothe [16] and Smoller[17]. If f and g does depend on the gradient, an existence theorem has been proved in [12], by means of L^1 method introduced in [13], when the growth of the nonlinearities with respect to the gradient is subquadratic, namely

$$\begin{aligned} |f|+|g| &\leq C(|u|,|v|)\,(|\nabla u|^\alpha+|\nabla v|^\alpha+K) \\ C &\geq 0, C \text{ is nondecreasing; } K\in L^1(\Omega) \text{ and } 1\leq\alpha<2. \end{aligned}$$

Our objective is to investigate the case $\alpha=2$. This critical growth with respect to the gradient creates some difficulties in the passage to the limit for the approximating problem and the L^1 method can not be applied in this case. We adopt a different approach based on techniques introduced in [7] for the case of elliptic equations to deal with exponential test function of the truncations. We refer the reader to [4], [6], [7], [15] for a general survey of this method. Let us point out here that the parabolic version of such systems with L^2- data has been recently treated by the same authors, see [2]. Typical model where the results of this paper can be applied is the following

$$\begin{cases} -\Delta u &= -\rho_1(u,v)\,|\nabla u|^2+F \quad \text{on } \Omega \\ -\Delta v &= \rho_2(u,v)\,|\nabla u|^2+G \quad \text{on } \Omega \\ u=v=0 & \qquad\qquad\qquad\qquad\quad \text{on } \partial\Omega, \end{cases}$$

where the functions $\rho_i, i=1,2$, are nonnegative continuous bounded with respect to v such that $\rho_2\leq\rho_1$ and where $|.|$ denotes the euclidian norm in $\mathbb{R}^N$.
We have organised this paper in the following manner. In section 2, we give the precise setting of the problem and state our main result. In section 3, we truncate the system and establish suitable a priori estimates. Finally, we prove the convergence of the truncated problem to some solution of our system. The difficulties in this section are similar to those in [7], [15] and the techniques are of the same spirit. But specific new difficulties owed to the nature of the system must be handled.

2 Statement of the result

2.1 Assumptions

We first introduce the notion of solution to the problem (1) used here.

Definition 1 *We say that (u,v) is a weak solution of (1) if*

$$\begin{cases} u, v \in W_0^{1,1}(\Omega), \\ f(.,u,v,\nabla u,\nabla v),\ g(.,u,v,\nabla u,\nabla v) \in L^1(\Omega), \\ -\Delta u = f(.,u,v,\nabla u,\nabla v) + F & \textit{in } D'(\Omega) \\ -\Delta v = g(.,u,v,\nabla u,\nabla v) + G & \textit{in } D'(\Omega). \end{cases}$$

Throughout this note we will assume that:
$H1/$ $f,\, g : \Omega \times \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are measurable
$H2/$ $f,\, g : \mathbb{R}^2 \times \mathbb{R}^{2N} \to \mathbb{R}$ are continuous for almost every x in Ω.
$H3/$ $|f(x,u,v,p,q)| \leq C_1(|u|)\left(L(x) + \|p\|^2 + \|q\|^\alpha\right)$,
where $C_1 : [0,\infty) \to [0,\infty)$ is non-decreasing, $L \in L^1(\Omega)$, $1 \leq \alpha < 2$.
$H4/$ $|g(x,u,v,p,q)| \leq C_2(|u|,|v|)\left(K(x) + \|p\|^2 + \|q\|^2\right)$,
where $C_2 : [0,\infty)^2 \to [0,\infty)$ is non-decreasing, and $K \in L^1(\Omega)$.
$H5/$ $F,\, G \in L^1(\Omega)$.

Remark 1 *One can deduce from assumptions* (2) *and* (3) *that*

$$f(x,0,0,p,q) = g(x,0,0,p,q) \textit{ for all } p, q \in \mathbb{R}^N.$$

2.2 The main result

Theorem 2 *Assume that (2), (3) and $H1/ - H5/$ hold. Then there exists a positive weak solution of (1).*

Before giving the proof of this theorem, let us denote by T_k the truncation function

$$T_k(s) = \max(-k, \min(s,k)) \qquad k \in \mathbb{R}^+.$$

3 Proof of Theorem 2

3.1 Approximating scheme

Let us define ψ_n a truncation function by $\psi_n \in C_c^\infty(\mathbb{R})$, $0 \leq \psi_n \leq 1$, and

$$\psi_n(r) = \begin{cases} 1 & \text{si } |r| \leq n \\ 0 & \text{si } |r| \geq n+1. \end{cases}$$

Consider the following two functions

$$f_n(x,u,v,p,q) = \psi_n(|u| + |v| + \|p\| + \|q\|) f(x,u,v,p,q),$$

$$g_n(x,u,v,p,q) = \psi_n(|u| + |v| + \|p\| + \|q\|)g(x,u,v,p,q).$$

It is easily seen that f_n, g_n satisfy the same properties as f and g. Moreover $|f_n| + |g_n| \le \eta_n(x) \in L^1(\Omega)$. By a direct application of the Leray-Schauder fixed point theorem one can prove that the system

$$\begin{cases} -\Delta u_n &= f_n(x,u_n,v_n,\nabla u_n,\nabla v_n) + F(x) \quad \text{in } D'(\Omega) \\ -\Delta v_n &= g_n(x,u_n,v_n,\nabla u_n,\nabla v_n) + G(x) \quad \text{in } D'(\Omega) \\ u_n, v_n &\in W_0^{1,1}(\Omega) \end{cases} \tag{4}$$

has a weak positive solution (u_n, v_n) in $W_0^{1,q}(\Omega)$ with $1 \le q < \dfrac{N}{N-1}$ (see [12] for more details).

3.2 A priori estimates

Lemma 3 *let $u_n, v_n \in W_0^{1,q}(\Omega)$ be nonnegative sequences such that*

$$\begin{cases} -\Delta u_n &= f_n + F \quad \text{in } D'(\Omega) \\ -\Delta v_n &= g_n + G \quad \text{in } D'(\Omega) \end{cases} \tag{5}$$

$f_n \le 0$, $f_n + g_n \le 0$; F, $G \in L^1(\Omega)$; $F, G \ge 0$. Then
i/ There exists a constant R_1 depending on $\|F\|_{L^1(\Omega)}$, $\|G\|_{L^1(\Omega)}$ such that:

$$\int_\Omega |f_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| + |g_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| \le R_1.$$

ii/ The sequence (u_n, v_n) is relatively compact in $W_0^{1,q}(\Omega) \times W_0^{1,q}(\Omega)$ for all $1 \le q < \dfrac{N}{N-1}$.

Proof. *i/* Consider the equations satisfied by u_n and v_n, we can write

$$\begin{cases} u_n, v_n \in W_0^{1,q}(\Omega) \\ -f_n = \Delta u_n + F \quad \text{in } D'(\Omega) \\ -g_n = \Delta v_n + G \quad \text{in } D'(\Omega). \end{cases} \tag{6}$$

Since $u_n, v_n \ge 0$ and $-\Delta$ is dissipative in $L^1(\Omega)$ (see [8]), then

$$\int_\Omega \Delta u_n \le 0, \int_\Omega \Delta v_n \le 0.$$

Iintegrating (6) on Ω yields

$$-\int_\Omega f_n \le \int_\Omega F.$$

The fact that $f_n \leq 0$ and $F \in L^1(\Omega)$ allows us to conclude

$$\int_\Omega |f_n| = -\int_\Omega f_n \leq \|F\|_{L^1(\Omega)}.$$

Similarly, we get

$$\int_\Omega |f_n + g_n| = -\int_\Omega f_n + g_n \leq \int_\Omega F + G \leq \|F\|_{L^1(\Omega)} + \|G\|_{L^1(\Omega)}.$$

Therefore

$$\int_\Omega |g_n| \leq \int_\Omega |f_n + g_n| + \int_\Omega |f_n| \leq 2\|F\|_{L^1(\Omega)} + \|G\|_{L^1(\Omega)}.$$

ii/ This assertion follows by a direct application of a result in [8]. Indeed the applications

$$f_n \to u_n, \text{ and } g_n \to v_n$$

are compact from $L^1(\Omega)$ into $W_0^{1,q}(\Omega)$, with $1 \leq q < \dfrac{N}{N-1}$. Therefore, since by *i/* the nonlinearities f_n and g_n are uniformly bounded in $L^1(\Omega)$, we obtain the required result. ■

Lemma 4 *Let (u_n, v_n) be as in the previous lemma. Then*

i/ a/ $\lim\limits_{h\to\infty} \int\limits_{[u_n \geq h]} |f_n| = 0$ *uniformly on n.*

b/ $\lim\limits_{h\to\infty} \int\limits_{[2u_n+v_n \geq h]} |f_n| = 0$ *uniformly on n.*

ii/ There exists a constant R_2 depending on k, $\|F\|_{L^1(\Omega)}$ and $\|G\|_{L^1(\Omega)}$ such that

$$\int_\Omega |\nabla T_k(u_n)|^2 + \int_\Omega |\nabla T_k(v_n)|^2 \leq R_2.$$

iii/ There exists a constant R_3 depending on k, $\|F\|_{L^1(\Omega)}$, $\|G\|_{L^1(\Omega)}$ such that

$$\int_\Omega |\nabla T_k(u_n + v_n)|^2 \leq R_3.$$

iv/ Moreover

$$\lim_{h\to\infty} \frac{1}{h} \int_\Omega |\nabla T_h(u_n)|^2 = 0 \quad \textit{uniformly on } n.$$

Remark 2 *The same results can be found in [6] and [15].*

Proof. *i/ a/* We first define the following test-function for every $t, h > 0$

$$P_{t,h}(s) = \begin{cases} 0 & \text{if } 0 \leq s < h \\ \dfrac{s-h}{t} & \text{if } h \leq s \leq t+h \\ 1 & \text{if } s > t+h. \end{cases}$$

Writing

$$\int_\Omega \nabla u_n \nabla P_{t,h}(u_n) - \int_\Omega f_n P_{t,h}(u_n) = \int_\Omega F P_{t,h}(u_n),$$

and using the fact that $f_n \leq 0$ and $P_{t,h}(u_n) \geq 0$ yield

$$\frac{1}{t} \int_{[h \leq u_n \leq t+h]} |\nabla u_n|^2 - \int_{[u_n \geq t+h]} f_n \leq \int_{[u_n \geq t+h]} F P_{t,h}(u_n).$$

Since $\dfrac{1}{t} \displaystyle\int_{[h \leq u_n \leq t+h]} |\nabla u_n|^2 \geq 0$, we get

$$\int_{[u_n \geq t+h]} |f_n| \leq \int_{[u_n \geq t+h]} F P_{t,h}(u_n) \leq \int_{[u_n \geq h]} |F|.$$

Thanks to Lebesgue's theorem, we have by passing to the limit as t tends to 0

$$\int_{[u_n \geq h]} |f_n| \leq \int_{[u_n \geq h]} |F|.$$

But

$$|[u_n \geq h]| = \int_{[u_n \geq h]} dx \leq h^{-1} \|u_n\|_{L^1} \leq C h^{-1},$$

since u_n is bounded in $L^1(\Omega)$ by lemma 3.
On the other hand, since $F \in L^1(\Omega)$, we have

$$\int_A |F| \longrightarrow 0 \text{ as } |A| \to 0.$$

Therefore

$$\sup_n \int_{[u_n \geq h]} |f_n| \leq \sup\{\int_A |F| ; |A| \leq C h^{-1}\} \underset{h \to \infty}{\longrightarrow} 0$$

We conclude that

$$\lim_{h \to \infty} \int_{[u_n \geq h]} |f_n| = 0 \qquad \text{uniformly on } n.$$

$b/$ The main idea is to consider the equation satisfied by $2u_n + v_n$, and to take $P_{t,h}(2u_n + v_n)$ as a test function. We obtain

$$\frac{1}{t} \int\limits_{[h \le 2u_n + v_n \le t+h]} |\nabla(2u_n + v_n)|^2 - \int\limits_{[2u_n + v_n \ge t+h]} (2f_n + g_n) P_{t,h}(2u_n + v_n)$$

$$\le \int\limits_{[2u_n + v_n \ge t+h]} (2F + G) P_{t,h}(2u_n + v_n)$$

Since $f_n \le 0$, $f_n + g_n \le 0$ and $P_{t,h}(2u_n + v_n) \ge 0$, we obtain

$$\int\limits_{[2u_n + v_n \ge t+h]} |f_n| \, P_{t,h}(2u_n + v_n) \le \int\limits_{[2u_n + v_n \ge t+h]} (2F + G) P_{t,h}(2u_n + v_n)$$

and

$$\int\limits_{[2u_n + v_n \ge t+h]} |g_n| \, P_{t,h}(2u_n + v_n) \le \int\limits_{[2u_n + v_n \ge t+h]} (2F + G) P_{t,h}(2u_n + v_n)$$

The rest of the proof runs as in the previous step.
$ii/$ We multiply the first equation in (5) by $T_k(u_n)$ and we integrate on Ω, we obtain

$$\int\limits_{\Omega} |\nabla T_k(u_n)|^2 = \int\limits_{\Omega} f_n T_k(u_n) + \int\limits_{\Omega} F T_k(u_n) \le \int\limits_{\Omega} F T_k(u_n),$$

since $f_n T_k(u_n) \le 0$. We then have

$$\int\limits_{\Omega} |\nabla T_k(u_n)|^2 \le k \, \|F\|_{L^1(\Omega)} \, .$$

In the same way, we multiply the second equation in (5) by $T_k(v_n)$ and we integrate on Ω, we obtain

$$\int\limits_{\Omega} |\nabla T_k(v_n)|^2 = \int\limits_{\Omega} g_n T_k(v_n) + \int\limits_{\Omega} G T_k(v_n) \le \int\limits_{\Omega} (G + |g_n|) T_k(v_n).$$

We then have

$$\int\limits_{\Omega} |\nabla T_k(v_n)|^2 \le k(R_1 + \|G\|_{L^1}).$$

$iii/$ follows trivially from $ii/$.
$iv/$We first remark that u_n satisfies

$$-\Delta u_n \le F, \text{ in } D'(\Omega)$$

if we multiply this inequality by $T_h(u_n)$ and integrate on Ω, we obtain for every $M > 0$

$$\begin{aligned}\int_\Omega |\nabla T_h(u_n)|^2 &\leq \int_{\Omega\cap[u_n\leq M]} T_h(u_n)F + \int_{\Omega\cap[u_n>M]} T_h(u_n)F \\ &\leq M\int_\Omega F + h\int_\Omega F\chi_{[u_n>M]}.\end{aligned}$$

Hence

$$\frac{1}{h}\int_\Omega |\nabla T_h(u_n)|^2 \leq \frac{M}{h}\,\|F\|_{L^1} + \int_\Omega F\chi_{[u_n>M]}.$$

Fix $\varepsilon > 0$. Since u_n is bounded in $L^1(\Omega)$, we have $|[u_n > k]| \leq Ck^{-1}$. Therefore, there exists k_ε independent of n such that

$$\int_\Omega F\chi_{[u_n>k_\varepsilon]} \leq \frac{\varepsilon}{2}.$$

Taking $M = k_\varepsilon$ an letting h tend to infinity, we obtain the desired conclusion. ■

The last assertion in lemma 3 allows us to ensure the existence of a subsequence still denoted by (u_n, v_n) such that

$$\begin{array}{ll} u_n \longrightarrow u & \text{in } W_0^{1,q}(\Omega) \text{ strongly.} \\ u_n \longrightarrow u & a.e \text{ in } \Omega. \\ \nabla u_n \longrightarrow \nabla u & a.e \text{ in } \Omega. \\ v_n \longrightarrow v & \text{in } W_0^{1,q}(\Omega) \text{ strongly.} \\ v_n \longrightarrow v & a.e \text{ in } \Omega.. \\ \nabla v_n \longrightarrow \nabla v & a.e \text{ in } \Omega. \end{array}$$

In the next step, we will show that this subsequence (u_n, v_n) satisfies some useful properties.

Lemma 5 *Suppose that u_n, v_n, u and v are as above.*
i/ If

$$\begin{array}{l} |f_n| \leq C_1(|u_n|)(|\nabla u_n|^2 + |\nabla v_n|^\alpha + L) \\ C_1 \geq 0,\ C_1 \textit{ is nondecreasing; } L \in L^1(\Omega);\ 1 \leq \alpha < 2. \end{array}$$

Then for each fixed k

$$\lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^2\,\chi_{[u_n+v_n\leq k]} = 0.$$

ii/ If

$$\begin{array}{l} |g_n| \leq C_2(|u_n|, |v_n|)(|\nabla u_n|^2 + |\nabla v_n|^2 + K) \\ C_2 \geq 0,\ C_2 \textit{ is nondecreasing; } K \in L^1(\Omega). \end{array}$$

Then for each fixed k

$$\lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n+v_n) - \nabla(T_k(u+v)|^2 \chi_{[u_n+v_n\le k]} = 0.$$

Proof. *i/* Let

$$\varphi(s) = s\exp(\mu s^2),$$

where: $\mu \ge \max\left(\frac{C_1^2(k)}{4}, C_2^2(k,k)\right)$. An easy calculation allows us to write

$$\varphi'(s) - C_1(k)\,|\varphi(s)| > \frac{1}{2}$$
$$\varphi'(s) - 2C_2(k,k)\,|\varphi(s)| > \frac{1}{2}.$$

Let us also define the following function

$$H \in C^1(\mathbb{R}), 0 \le H(s) \le 1\ \forall s \in \mathbb{R}$$
$$H(s) = \begin{cases} 0 & \text{if } |s| \ge 1, \\ 1 & \text{if } |s| \le \frac{1}{2}. \end{cases}$$

Now let h and k be positive real numbers such that $k < h$ and take $\varphi(T_k(u_n) - T_k(u))H(\frac{u_n+v_n}{h})$ as a test function in the first equation of (5). We have

$$-\int_\Omega \Delta u_n \varphi(T_k(u_n) - T_k(u))H(\frac{u_n+v_n}{h}) = J_1 + J_2 \tag{7}$$

where

$$J_1 = \int_\Omega f_n\varphi(T_k(u_n) - T_k(u))H(\frac{u_n+v_n}{h})$$
$$J_2 = \int_\Omega F\varphi(T_k(u_n) - T_k(u))H(\frac{u_n+v_n}{h}).$$

For sake of brevity, we will denote by

$$\xi_{k,n} = (T_k(u_n) - T_k(u))$$
$$I = -\int_\Omega \Delta u_n \varphi(\xi_{k,n})H(\frac{u_n+v_n}{h}).$$

Integration by part yields

$$\begin{aligned} I &= \int_\Omega \nabla u_n(\nabla T_k(u_n) - \nabla T_k(u))\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h}) \\ &+ \frac{1}{h}\int_\Omega \nabla u_n \nabla(u_n+v_n)\varphi(\xi_{k,n})H'(\frac{u_n+v_n}{h}) \\ &= I_1 + I_2. \end{aligned}$$

For I_1, we have

$$\begin{aligned} I_1 &= -\int\limits_{[u_n>k]} \nabla u_n \nabla T_k(u)\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})+ \\ &+ \int\limits_{[u_n\leq k]} |\nabla(T_k(u_n)-T_k(u))|^2\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})+ \\ &\int\limits_{[u_n\leq k]} \nabla T_k(u)\nabla(T_k(u_n)-T_k(u))\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h}) \\ &= I_{1.1}+I_{1.2}+I_{1.3}. \end{aligned}$$

For the term J_1, we have

$$\begin{aligned} J_1 &= \int\limits_{[u_n\leq k]} f_n\varphi(\xi_{k,n})H(\frac{u_n+v_n}{h}) + \int\limits_{[u_n\geq k]} f_n\varphi(\xi_{k,n})H(\frac{u_n+v_n}{h}) \\ &\leq \int\limits_{[u_n\leq k]} f_n\varphi(\xi_{k,n})H(\frac{u_n+v_n}{h}), \end{aligned}$$

since $\varphi(\xi_{k,n})H(\frac{u_n+v_n}{h}) \geq 0$ on $[u_n>k]$ $(H\geq 0, \xi_{k,n}\geq 0$ on $[u_n>k])$ and $f_n\leq 0$ by hypotheses. Therefore

$$\begin{aligned} J_1 &\leq C_1(k)\int\limits_{[u_n\leq k]} L(x)\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}) \\ &+ C_1(k)\int\limits_{[u_n\leq k]} |\nabla u_n|^2\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}) \\ &+ C_1(k)\int\limits_{[u_n\leq k]} |\nabla T_h(v_n)|^\alpha\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}) \\ &= J_{1.1}+J_{1.2}+J_{1.3}. \end{aligned}$$

Consequently by equality (7), we have

$$I_{1.2}+I_2-J_{1.2} \leq J_{1.1}+J_{1.3}+J_2-I_{1.1}-I_{1.3}. \tag{8}$$

One can see that $I_{1.1}$ can be written as

$$\begin{aligned} I_{1.1} &= -\int\limits_\Omega \nabla T_h(u_n)\nabla T_k(u)\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})\chi_{[u_n\geq k]}\chi_{[u\geq k]} \\ &- \int\limits_\Omega \nabla T_h(u_n)\nabla T_k(u)\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})\chi_{[u_n\geq k]}\chi_{[u<k]}. \end{aligned}$$

Since $\nabla T_k(u)\chi_{[u>k]} = 0$ $a.e$ in Ω, and $\chi_{[u_n\geq k]}\chi_{[u<k]} \to 0$ $a.e$ in Ω as $n \to \infty$,

$$\lim_{n\to\infty} I_{1.1} = 0.$$

On the other hand, $\nabla T_k(u_n)$ is bounded in $L^2(\Omega)$ and converges to $\nabla T_k(u)$ a.e., and $\left|\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})\right| \leq C\,|\varphi'(2k)|$. Therefore, it follows from ([11] Lemme1.3 p12) that $\nabla(T_k(u_n) - T_k(u))\varphi'(\xi_{k,n})H(\frac{u_n+v_n}{h})$ converges weakly to 0 in$L^2(\Omega)$.

Then $\lim_{n\to\infty} I_{1.3} = 0$.

Now we investigate I_2. Since u_n and $u_n + v_n$ satisfy the hypotheses of the previous lemma, we get

$$\begin{aligned}
|I_2| &\leq \frac{1}{2h}\int_\Omega |\nabla T_h(u_n)|^2 \left|\varphi(\xi_{k,n})H'(\frac{u_n+v_n}{h})\right| \\
&+ \frac{1}{2h}\int_\Omega |\nabla T_h(u_n+v_n)|^2 \left|\varphi(\xi_{k,n})H'(\frac{u_n+v_n}{h})\right|.
\end{aligned}$$

Then $\lim_{h\to\infty} |I_2| = 0$ uniformly on n.

For the term $J_{1.1}$, we use Lebesgue's theorem to conclude that $\lim_{n\to\infty} J_{1.1} = 0$.

For $J_{1.2}$, we write

$$\begin{aligned}
J_{1.2} &= C_1(k)\int_{[u_n\leq k]} |\nabla(T_k(u_n) - T_k(u))|^2\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}) \\
&+ 2C_1(k)\int_{[u_n\leq k]} \nabla T_k(u_n)\nabla T_k(u)\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}) \\
&- C_1(k)\int_{[u_n\leq k]} |\nabla T_k(u)|^2\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h}).
\end{aligned}$$

Since $|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h})\chi_{[u_n\leq k]}$ converges to 0 a.e., and $|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h})\chi_{[u_n\leq k]} \leq \varphi(2k)$ and $\nabla(T_k(u_n)$ is bounded in $L^2(\Omega)$, it follows from ([11] lemme 1.3 p12) that $\nabla T_k(u_n)\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h})\chi_{[u_n\leq k]}$ converges weakly to 0 in $L^2(\Omega)$. This implies that the second term of this equality goes to zero as n tends to infinity. Concerning the last term, we remark that $|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h})\chi_{[u_n\leq k]} \underset{n\to\infty}{\longrightarrow} 0$ $a.e.$ on Ω and $|\nabla T_k(u)|^2\,|\varphi(\xi_{k,n})|\,H(\frac{u_n+v_n}{h})\chi_{[u_n\leq k]} \leq |\nabla T_k(u)|^2\,\varphi(2k) \in L^1(\Omega)$. Thanks to

Lebesgue's theorem, we then have

$$\lim_{n\to\infty} \int_{[u_n\le k]} |\nabla T_k(u)|^2 \, |\varphi(\xi_{k,n})| \, H(\frac{u_n+v_n}{h}) = 0.$$

To investigate the remaining term $J_{1.3}$, we apply Hölder's inequality as follows: choose β such that $\alpha - 1 < \beta < 1$, we have

$$\begin{aligned} J_{1.3} &\le C_1(k) \int_{[u_n\le k]} |\nabla T_h(v_n)|^\alpha \, |\xi_{k,n}| \exp(\mu\xi_{k,n}^2) H(\frac{u_n+v_n}{h}) \\ &\le C(k) \left(\int_\Omega |\nabla T_h(v_n)|^{\alpha\frac{2}{\alpha}} |\xi_{k,n}|^{\frac{2\beta}{\alpha}}\right)^{\frac{\alpha}{2}} \left(\int_\Omega |\xi_{k,n}|^{\frac{2(1-\beta)}{2-\alpha}}\right)^{\frac{2-\alpha}{2}} \\ &\le C(k) \left(\int_\Omega |\nabla T_h(v_n)|^2\right)^{\frac{\alpha}{2}} \left(\int_\Omega |\xi_{k,n}|^2\right)^{\frac{1-\beta}{2}} |\Omega|^{\frac{1+\beta-\alpha}{2}}. \end{aligned}$$

Now we use lemma 4-(ii) to obtain

$$J_{1.3} \le C(k)(h^2 \|F\|_{L^1} + h\|G\|_{L^1})^{\frac{\alpha}{2}} \left(\int_\Omega |\xi_{k,n}|^2\right)^{\frac{1-\beta}{2}} |\Omega|^{\frac{1+\beta-\alpha}{2}}$$

Passing to the limit as n tends to infinity (for fixed h, k), the strong convergence of $\xi_{k,n}$ to 0 in $L^2(\Omega)$ yields: $\limsup_{n\to\infty} J_{1.3} = 0$.
For the last term J_2, we have by a direct application of Lebesgue's theorem

$$\lim_{n\to\infty} J_2 = 0,$$

since $F \in L^1(\Omega)$, and $\left|\varphi(\xi_{k,n})H(\frac{u_n+v_n}{h})\right| \le C\,|\varphi(2k)|$.
In view of inequality (8), we have shown that for k, h fixed

$$\limsup_{n\to\infty}(I_{1.2} + I_2 - J_{1.2}) \le 0.$$

Then

$$\limsup_{h\to\infty} (\limsup_{n\to\infty} (I_{1.2} + I_2 - J_{1.2})) \le 0.$$

But $\lim_{h\to\infty} I_2 = 0$ uniformly on n, this should be $\limsup_{h\to\infty} (\limsup_{n\to\infty} |I_2|) = \lim_{h\to\infty} (\limsup_{n\to\infty} I_2) = 0$. It follows

$$\limsup_{h\to\infty} (\limsup_{n\to\infty}(I_{1.2} - J_{1.2})) \le 0.$$

Therefore

$$\limsup_{h\to\infty}(\limsup_{n\to\infty}\int_\Omega |\nabla(T_k(u_n)-T_k(u))|^2\left[\varphi'(\xi_{k,n})-C_1(k)\,|\varphi(\xi_{k,n})|\right]H(\frac{u_n+v_n}{h}))\le 0.$$

By the choice of μ we deduce that

$$\limsup_{h\to\infty}(\limsup_{n\to\infty}\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2\,H(\frac{u_n+v_n}{h}))=0.$$

On the other hand

$$\int_{[u_n+v_n\le k]} |\nabla T_k(u_n)-\nabla T_k(u)|^2\,H(\frac{u_n+v_n}{h})\le \int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2\,H(\frac{u_n+v_n}{h}).$$

Then for every fixed h,k such that $k<h$

$$\limsup_{h\to\infty}(\limsup_{n\to\infty}\int_{[u_n+v_n\le k]} |\nabla T_k(u_n)-\nabla T_k(u)|^2\,H(\frac{u_n+v_n}{h}))=0.$$

Choose $2k<h$. By the definition of H we get $H(\frac{u_n+v_n}{h})=1$ on $[u_n+v_n\le k]$.
Hence, this should be a limsup

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2\,\chi_{[u_n+v_n\le k]} &\le\\ \limsup_{n\to\infty}\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2\,\chi_{[u_n+v_n\le k]} &= 0.\end{aligned}$$

Therefore

$$\lim_{n\to\infty}\int_\Omega |\nabla T_k(u_n)-\nabla T_k(u)|^2\,\chi_{[u_n+v_n\le k]}=0.$$

ii/ Consider the equation satisfied by u_n+v_n

$$-\Delta(u_n+v_n)=f_n+g_n+F+G,$$

and use again $\varphi(T_k(u_n+v_n)-T_k(u+v))H(\frac{u_n+v_n}{h})$ as a test function where h and k such that $0<k<h$. We get

$$\begin{aligned}&\int_\Omega \nabla(u_n+v_n)\nabla(T_k(u_n+v_n)-T_k(u+v))\varphi'(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\ &+\frac{1}{h}\int_\Omega |\nabla(u_n+v_n)|^2\,\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h})=K_2+K_3,\end{aligned}\tag{9}$$

where we denote by

$$\begin{aligned}
\xi'_{k,n} &= T_k(u_n+v_n) - T_k(u+v)\\
K_1 &= \int_\Omega \nabla(u_n+v_n)\nabla(T_k(u_n+v_n) - T_k(u+v))\varphi'(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
K_2 &= \frac{1}{h}\int_\Omega |\nabla(u_n+v_n)|^2\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
K_3 &= \int_\Omega (f_n+g_n)\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
K_4 &= \int_\Omega (F+G)\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h}).
\end{aligned}$$

The proof of this assertion follows closely the steps used in the proof of the previous one, it suffies to replace u_n by u_n+v_n and u by $u+v$. Hence we obtain

$$\begin{aligned}
&\lim_{h\to\infty}\ (\limsup_{n\to\infty} K_2) = 0\\
&\lim_{n\to\infty}\ K_4 = 0,
\end{aligned}$$

for the term K_1 we have

$$\begin{aligned}
K_1 &= -\int_{[u_n+v_n>k]} \nabla(u_n+v_n)\nabla T_k(u+v)\varphi'(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
&+ \int_{[u_n+v_n\le k]} |T_k(u_n+v_n) - T_k(u+v)|^2\varphi'(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
&+ \int_{[u_n+v_n\le k]} \nabla T_k(u+v)(T_k(u_n+v_n) - T_k(u+v))\varphi'(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
&= K_{1.1} + K_{1.2} + K_{1.3}.
\end{aligned}$$

It is easily seen that K_1 can be treated in the same way as I_1.
The only difference is the investigation of the term K_3, indeed

$$\begin{aligned}
K_3 &= \int_{[u_n+v_n\le k]} (f_n+g_n)\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
&\quad + \int_{[u_n+v_n>k]} (f_n+g_n)\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h})\\
&\le \int_{[u_n+v_n\le k]} (f_n+g_n)\varphi(\xi'_{k,n})H(\frac{u_n+v_n}{h}),
\end{aligned}$$

since $\varphi(\xi'_{k,n}) \geq 0$ on $[u_n + v_n \leq k]$ and $H \geq 0$, $f_n + g_n \leq 0$ by hypotheses. Hence

$$\begin{aligned} |K_3| &\leq C_1(k) \int_{[u_n+v_n\leq k]} |\nabla u_n|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &+ C_1(k) \int_{[u_n+v_n\leq k]} (|\nabla v_n|^\alpha + L) \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &+ C_2(k,k) \int_{[u_n+v_n\leq k]} (|\nabla u_n|^2 + |\nabla v_n|^2 + K) \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &= K_{3.1} + K_{3.2} + K_{3.3}. \end{aligned}$$

Therefore equality (9) implies

$$K_{1.2} + K_2 - K_{3.3} \leq K_{3.1} + K_{3.2} + K_4 - K_{1.1} - K_{1.3}. \tag{10}$$

We have

$$\begin{aligned} K_{3.1} &= C_1(k) \int_{[u_n+v_n\leq k]} |\nabla u_n|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &\leq 2C_1(k) \int_{[u_n+v_n\leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &\quad +2C_1(k) \int_{[u_n+v_n\leq k]} |\nabla T_k(u)|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &\leq 2C_1(k)\varphi(2k) \int_\Omega |\nabla T_k(u_n) - \nabla T_k(u)|^2 H(\frac{u_n+v_n}{h}) \\ &\quad +2C_1(k) \int_\Omega |\nabla T_k(u)|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}). \end{aligned}$$

By using the first assertion (i) and Lebesgue's theorem for the second integral, we obtain

$$\limsup_{h\to\infty} \; (\lim_{n\to\infty} K_{3.1}) = 0.$$

As for the term $J_{1.3}$, we write

$$\lim_{n\to\infty} K_{3.2} \;\; = \;\; 0.$$

Let us now investigate $K_{3.3}$. We first remark that this term can be controlled as follows:

$$\begin{aligned} K_{3.3} &\leq C_2(k,k) \int\limits_{[u_n+v_n\leq k]} (3\,|\nabla u_n|^2 + 2\,|\nabla(u_n+v_n)|^2 + K) \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &\leq C_2(k,k) \int\limits_{[u_n+v_n\leq k]} (3\,|\nabla T_k(u_n)|^2 + K) \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}) \\ &+ 2C_2(k,k) \int\limits_{[u_n+v_n\leq k]} |\nabla T_k(u_n+v_n)|^2 \left|\varphi(\xi'_{k,n})\right| H(\frac{u_n+v_n}{h}). \end{aligned}$$

The first integral can be dropped by using the same arguments as before. In conclusion, we have

$$\limsup_{h\to\infty} (\limsup_{n\to\infty} \int\limits_{\Omega} |\nabla T_k(u_n+v_n) - \nabla T_k(u+v)|^2 \left[\varphi'(\xi'_{k,n}) - 2C_2(k,k)\left|\varphi(\xi'_{k,n})\right|\right] H(\frac{u_n+v_n}{h})) \leq 0.$$

As in the previous assertion, we obtain

$$\lim_{n\to\infty} \int\limits_{\Omega} |\nabla T_k(u_n+v_n) - \nabla T_k(u+v)|^2 \chi_{[u_n+v_n\leq k]} = 0.$$

3.3 Convergence

The aim of this paragraph is to show that (u,v) obtained before is in fact solution of the problem (1) in the sense of definition 1.
By the continuity of the functions f_n and g_n, we deduce

$$\begin{aligned} f_n(x,u_n,v_n,\nabla u_n,\nabla v_n) &\to f(x,u,v,\nabla u,\nabla v) \quad a.e \text{ in } \Omega. \\ g_n(x,u_n,v_n,\nabla u_n,\nabla v_n) &\to g(x,u,v,\nabla u,\nabla v) \quad a.e \text{ in } \Omega. \end{aligned}$$

These almost pointwise convergences are not sufficient to ensure that (u,v) is a solution of (1). In fact, we have to prove that the previous convergences are in $L^1(\Omega)$. In view of Vitali's theorem, we have to show that f_n and g_n are equi-integrable in $L^1(\Omega)$.

Lemma 6 *The sequences $(f_n(x,u_n,v_n,\nabla u_n,\nabla v_n))_n$ and $(g_n(x,u_n,v_n,\nabla u_n,\nabla v_n))_n$ are equi-integrable in $L^1(\Omega)$.*

Proof. Let A be a measurable subset of Ω, we have

$$\begin{aligned} \int\limits_A |f_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| &= \int\limits_{A\cap[u_n+v_n>k]} |f_n| + \int\limits_{A\cap[u_n+v_n\leq k]} |f_n| \\ &\leq \int\limits_{A\cap[2u_n+v_n>k]} |f_n| + \int\limits_{A\cap[u_n+v_n\leq k]} |f_n|\,. \end{aligned}$$

Thanks to lemma 4, we obtain $\forall\varepsilon > 0$ $\exists k_0$ such that if $k \geq k_0$ then

$$\int_{A\cap[2u_n+v_n>k]} |f_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| \leq \frac{\varepsilon}{4} \text{ for all } n.$$

Hypothesis (H_3) implies that for all $k > k_0$

$$\begin{aligned}\int_A |f_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| &\leq \frac{\varepsilon}{4} + C_1(k)\Big(\int_A L(x) + \int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(u_n)|^2\Big) \\ &+ C_1(k) \int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(v_n)|^\alpha .\end{aligned}$$

Using Hölder's inequality for $\alpha < 2$ and Lemma 4(ii), we obtain for the third integral

$$\begin{aligned}C_1(k) \int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(v_n)|^\alpha &\leq C_1(k) \left[\int_A |\nabla T_k(v_n)|^2\right]^{\frac{\alpha}{2}} |A|^{\frac{2-\alpha}{2}} \\ &\leq C_1(k) R_2^{\frac{\alpha}{2}} |A|^{\frac{2-\alpha}{2}} \\ &\leq \frac{\varepsilon}{4},\end{aligned}$$

whenever $|A| \leq \delta_1$, with $\delta_1 = (\frac{\varepsilon}{4} C_1(k)\ R_2^{-\frac{\alpha}{2}})^{\frac{2}{2-\alpha}}$.
To deal with the second integral we write

$$\int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(u_n)|^2 \leq 2 \int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 + 2 \int_A |\nabla T_k(u)|^2 .$$

According to lemma5, $|\nabla T_k(u_n) - \nabla T_k(u)|^2 \chi_{[u_n+v_n\leq k]}$ is equi-integrable in $L^1(\Omega)$ since it converges strongly to 0 in $L^1(\Omega)$. So, there exists δ_2 such that if $|A| \leq \delta_2$, then $2C_1(k) \displaystyle\int_{A\cap[u_n+v_n\leq k]} |\nabla T_k(u_n) - \nabla T_k(u)|^2 \leq \frac{\varepsilon}{4}$.
On the other hand L, $|\nabla T_k(u)|^2 \in L^1(\Omega)$, therefore there exists δ_3 such that

$$C_1(k) \left[2 \int_A |\nabla T_k(u)|^2 + \int_A L(x)\right] \leq \frac{\varepsilon}{4},$$

whenever $|A| \leq \delta_3$. Choose $\delta_0 = \inf(\delta_1,\delta_2,\delta_3)$, if $|A| \leq \delta_0$ we obtain

$$\int_A |f_n(x,u_n,v_n,\nabla u_n,\nabla v_n)| \leq \varepsilon.$$

Similarly, we get

$$\begin{aligned}\int_A |g_n| &= \int_{A\cap[u_n+v_n\le k]} |g_n| + \int_{A\cap[u_n+v_n>k]} |g_n|.\\ &\le \frac{\varepsilon}{4} + \int_{A\cap[u_n+v_n\le k]} |g_n|.\end{aligned}$$

By hypothesis $H4/$, we have

$$\int_A |g_n| \le \frac{\varepsilon}{4} + C_2(k,k) \int_{A\cap[u_n+v_n\le k]} \left(K(x) + |\nabla T_k(u_n)|^2 + |\nabla v_n|^2\right).$$

Therefore

$$\begin{aligned}\int_A |g_n| \le & \frac{\varepsilon}{4} + C_2(k,k) \int_{A\cap[u_n+v_n\le k]} K(x) + (C_2(k,k)+2) \int_{A\cap[u_n+v_n\le k]} |\nabla T_k(u_n)|^2 + \\ & +2C_2(k,k) \int_{A\cap[u_n+v_n\le k]} |\nabla T_k(u_n+v_n)|^2.\end{aligned}$$

Arguing in the same way as before, we obtain the required result.

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