ANNALES MATHÉMATIQUES BLAISE PASCAL

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Annales mathématiques Blaise Pascal, tome 9, nº 1 (2002), p. 85-100
<http://www.numdam.org/item?id=AMBP_2002__9_1_85_0>

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Abstract

In this paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K. In the present paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in K and apply it to Nörlund means for double sequences and series in K.

Throughout the present paper, K denotes a complete, non-trivially valued, non-archimedean field. The entries of sequences, series and infinite matrices are in K. In this paper, we prove the Silvermann-Toeplitz theorem for double sequences and series in K (see Theorem 2, proved in the sequel). We then introduce Nörlund means for double sequences and series in K and apply Silvermann-Toeplitz theorem for these means.

For analysis in the classical case a general reference is [2] while for analysis in non-archimedean fields a general reference is [1].

For a given infinite matrix $A = (a_{n,k})$ and a sequence $\{x_k\}$, the sequence $\{y_n\}$ is defined as follows:

$$y_n = \sum_{k=1}^{\infty} a_{n,k} x_k, \quad n = 1, 2, \cdots,$$

it being assumed that the series on the right converge. If $\lim_{n\to\infty} y_n = s$ whenever $\lim_{k\to\infty} x_k = s$, we say that A is regular. The criterion for A to be regular in terms of the entries of the matrix A are well-known (see [4], [6]).

Theorem 1. *$A = (a_{n,k})$ is regular if and only if*

(i) $\sup_{n,k} |a_{n,k}| < \infty$;

(ii) $\lim_{n\to\infty} a_{n,k} = 0,\ k = 1, 2, \cdots;$

and

(iii) $\lim_{n\to\infty} \sum_{k=1}^{\infty} a_{n,k} = 1.$

In the sequel, the following definitions are needed.

Definition 1. Let $\{x_{m,n}\}$ be a double sequence in K and $x \in K$. We say that $\lim_{m+n\to\infty} x_{m,n} = x$ if for each $\epsilon > 0$, the set $\{(m,n) \in \mathbb{N}^2 : |x - x_{m,n}| \geq \epsilon\}$ is finite. In such a case we say that x is the limit of $\{x_{m,n}\}$.

Definition 2. Let $\{x_{m,n}\}$ be a double sequence in K and $s \in K$. We say that

$$s = \sum_{m=1,n=1}^{\infty,\infty} x_{m,n},$$

if

$$s = \lim_{m+n\to\infty} s_{m,n},$$

where

$$s_{m,n} = \sum_{k=1,l=1}^{m,n} x_{k,l},\ m, n = 1, 2, \cdots.$$

Remark. If $\lim_{m+n\to\infty} x_{m,n} = x$, then the sequence $\{x_{m,n}\}$ is automatically bounded.

It is easy to prove the following results.

Lemma 1. $\lim_{m+n\to\infty} x_{m,n} = x$ if and only if

(i) $\lim_{n\to\infty} x_{m,n} = x,\ m = 1, 2, \cdots,$

(ii) $\lim_{m\to\infty} x_{m,n} = x,\ n = 1, 2, \cdots,$

and

(iii) for each $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|x - x_{m,n}| < \epsilon$, for all $m, n \geq N$, which we write as $\lim_{m,n\to\infty} x_{m,n} = x$.

Lemma 2. $\lim_{m+n\to\infty} s_{m,n}$ exists if and only if

$$\lim_{m+n\to\infty} x_{m,n} = 0 \,. \tag{1}$$

Given the matrix $A = (a_{m,n,k,l})$, we define

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}, \quad m,n = 1,2,\cdots, \tag{2}$$

assuming that the series on the right converge.

Necessary and sufficient conditions for $A = (a_{m,n,k,l})$ to be regular for the class of all double sequences and series in the classical case have been found by Kojima [3]. It has been found that convergence and boundedness play a vital role for double sequences and series, a role analogous to that of convergence for simple sequences and series. Robison [8] proved Silvermann-Toeplitz theorem for such a class of bounded and convergent double sequences in the classical case. We prove here its analogue in a complete, non-trivially valued, non-archimedean field.

In this context, the following definition is needed.

Definition 3. If whenever $\{x_{m,n}\}$ is a convergent sequence, $\{y_{m,n}\}$ converges to the same value, then the matrix $A = (a_{m,n,k,l})$ is said to be regular.

Theorem 2. *In order that whenever a sequence $\{x_{m,n}\}$ has a limit x, $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l}$ shall converge and $\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} x_{k,l} = x$, i.e., for $A = (a_{m,n,k,l})$ to be regular it is necessary and sufficient that*

(a) $\lim_{m+n\to\infty} a_{m,n,k,l} = 0, \quad k,l = 1,2,\cdots;$

(b) $\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} = 1;$

(c) $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l}| = 0, \quad l = 1,2,\cdots;$

(d) $\lim_{m+n\to\infty} \sup_{l\geq 1} |a_{m,n,k,l}| = 0, \quad k = 1,2,\cdots;$

and

(e) $\sup_{m,n,k,l} |a_{m,n,k,l}| < \infty$.

Proof. Proof of necessity.

Define the sequence $\{x_{k,l}\}$ as follows: For any fixed p, q, let

$$x_{k,l} = \begin{cases} 1, & \text{when } k = p,\ l = q; \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then

$$y_{m,n} = a_{m,n,p,q}.$$

Since $\{x_{k,l}\}$ has limit 0, it follows that (a) is necessary.

Define the sequence $\{x_{k,l}\}$ where $x_{k,l} = 1,\ k, l = 1, 2, \cdots$.
Now,

$$y_{m,n} = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}, \quad m, n = 1, 2, \cdots.$$

This shows that $\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}$ converges for $m, n = 1, 2, \cdots$. (4)

Since $\{x_{k,l}\}$ has limit 1, it follows that

$$\lim_{m+n\to\infty} \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} = 1,$$

so that (b) is necessary.

We now show that $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l}| = 0$ for all $l \in \mathbb{N}$. Suppose not. Then there exists $l_0 \in \mathbb{N}$ such that $\lim_{m+n\to\infty} \sup_{k\geq 1} |a_{m,n,k,l_0}| = 0$ does not hold. So, there exists an $\epsilon > 0$, such that

$$\left\{(m, n) : \sup_{k\geq 1} |a_{m,n,k,l_0}| > \epsilon\right\} \text{ is infinite.} \tag{5}$$

Let us choose $m_1 = n_1 = r_1 = 1$. Choose $m_2, n_2 \in \mathbb{N}$ such that $m_2+n_2 > m_1 + n_1$ and

$$\sup_{1\leq k\leq r_1} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using } (a);$$

and

$$\sup_{k\geq 1} |a_{m_2,n_2,k,l_0}| > \epsilon, \text{ using (5).}$$

Then choose $r_2 \in \mathbb{N}$ such that $r_2 > r_1$ and

$$\sup_{k>r_2} |a_{m_2,n_2,k,l_0}| < \frac{\epsilon}{8}, \text{ using (b).}$$

Inductively choose $m_p + n_p > m_{p-1} + n_{p-1}$ such that

$$\sup_{1\le k\le r_{p-1}} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}; \tag{6}$$

$$\sup_{k\ge 1} |a_{m_p,n_p,k,l_0}| > \epsilon; \tag{7}$$

and then choose $r_p > r_{p-1}$ such that

$$\sup_{k>r_p} |a_{m_p,n_p,k,l_0}| < \frac{\epsilon}{8}. \tag{8}$$

In view of (6), (7), (8), we have

$$\sup_{r_{p-1}<k\le r_p} |a_{m_p,n_p,k,l_0}| > \epsilon - \frac{\epsilon}{8} - \frac{\epsilon}{8} = \frac{3\epsilon}{4}.$$

We can now find $k_p \in \mathbb{N}$, $r_{p-1} < k_p \le r_p$ such that

$$|a_{m_p,n_p,k_p,l_0}| > \frac{3\epsilon}{4}. \tag{9}$$

Define the sequence $\{x_{k,l}\}$ as follows:

$$x_{k,l} = \begin{cases} 0, & l \ne l_0; \\ 1, & \text{if } l = l_0,\ k = k_p,\ p = 1, 2, \cdots. \end{cases}$$

We note that $\lim_{k+l\to\infty} x_{k,l} = 0$. Now, in view of (6),

$$\left| \sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| \le \sup_{1\le k\le r_{p-1}} \left| a_{m_p,n_p,k,l_0} \right| < \frac{\epsilon}{8}; \tag{10}$$

Using (8), we have,

$$\left| \sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0} \right| \le \sup_{k>r_p} \left| a_{m_p,n_p,k,l_0} \right| < \frac{\epsilon}{8}; \tag{11}$$

and using (9), we get,

$$\left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| = \left|a_{m_p,n_p,k_p,l_0}\right| > \frac{3\epsilon}{4}. \tag{12}$$

Thus

$$\begin{aligned}
|y_{m_p,n_p}| &= \left|\sum_{k=1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| \\
&\geq \left|\sum_{k=r_{p-1}+1}^{r_p} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| - \left|\sum_{k=1}^{r_{p-1}} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| - \left|\sum_{k=r_p+1}^{\infty} a_{m_p,n_p,k,l_0} x_{k,l_0}\right| \\
&\geq \left|a_{m_p,n_p,k_p,l_0}\right| - \sup_{1\leq k\leq r_{p-1}} \left|a_{m_p,n_p,k,l_0}\right| - \sup_{k>r_p} \left|a_{m_p,n_p,k,l_0}\right| \\
&> \frac{3\epsilon}{4} - \frac{\epsilon}{8} - \frac{\epsilon}{8}, \text{ using (10), (11) and (12)} \\
&= \frac{\epsilon}{2}, \quad p = 1, 2, \cdots.
\end{aligned}$$

Consequently $\lim_{m+n\to\infty} y_{m,n} = 0$ does not hold, which is a contradiction. Thus (c) is necessary. The necessity of (d) follows in a similar fashion.

To establish (e), we shall suppose that (e) does not hold and arrive at a contradiction. Since K is non-trivially valued, there exists $\pi \in K$ such that $0 < \rho = |\pi| < 1$. Choose $m_1 = n_1 = 1$. Using (a), (b), choose $m_2 + n_2 > m_1 + n_1$ such that

$$\sup_{1\leq k+l\leq m_1+n_1} |a_{m_2,n_2,k,l}| < 2, \text{ using (a);}$$

$$\sup_{k+l\geq 1} |a_{m_2,n_2,k,l}| > \left(\frac{2}{\rho}\right)^6;$$

and

$$\sup_{k+l>m_1+n_1} |a_{m_2,n_2,k,l}| < 2^2, \text{ using (b) and Lemma 1, Lemma 2.}$$

It now follows that

$$\sup_{k+l>m_2+n_2} |a_{m_2,n_2,k,l}| < 2^2.$$

Choose $m_3 + n_3 > m_2 + n_2$ such that

$$\sup_{1 \le k+l \le m_2+n_2} |a_{m_3,n_3,k,l}| < 2^2;$$

$$\sup_{k+l \ge 1} |a_{m_3,n_3,k,l}| > \left(\frac{2}{\rho}\right)^8;$$

and

$$\sup_{k+l > m_3+n_3} |a_{m_3,n_3,k,l}| < 2^4.$$

Inductively, choose $m_p + n_p > m_{p-1} + n_{p-1}$, such that

$$\sup_{1 \le k+l \le m_{p-1}+n_{p-1}} |a_{m_p,n_p,k,l}| < 2^{p-1}; \tag{13}$$

$$\sup_{k+l \ge 1} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2} \tag{14}$$

and

$$\sup_{k+l > m_p+n_p} |a_{m_p,n_p,k,l}| < 2^{2p-2}. \tag{15}$$

Using (13), (14), (15), we have,

$$\sup_{m_{p-1}+n_{p-1} < k+l \le m_p+n_p} |a_{m_p,n_p,k,l}| > \left(\frac{2}{\rho}\right)^{2p+2} - 2^{2p-2} - 2^{p-1}$$

$$\begin{aligned}
&\ge \left(\frac{2}{\rho}\right)^{2p+2} - \left(\frac{2}{\rho}\right)^{2p-2} - \left(\frac{2}{\rho}\right)^{p-1}, \text{ since } \frac{1}{\rho} > 1 \\
&= \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{p+3} - \left(\frac{2}{\rho}\right)^{p-1} - 1\right] \\
&\ge \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{p+3} - \left(\frac{2}{\rho}\right)^{p-1} - \left(\frac{2}{\rho}\right)^{p-1}\right], \text{ since } \left(\frac{2}{\rho}\right)^{p-1} \ge 1 \\
&= \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{4} \left(\frac{2}{\rho}\right)^{p-1} - 2\left(\frac{2}{\rho}\right)^{p-1}\right], \\
&> \left(\frac{2}{\rho}\right)^{p-1} \left[\left(\frac{2}{\rho}\right)^{4} \left(\frac{2}{\rho}\right)^{p-1} - \left(\frac{2}{\rho}\right)\left(\frac{2}{\rho}\right)^{p-1}\right], \text{ since } \frac{2}{\rho} > 2
\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{2}{\rho}\right)^{2p-1}\left[\left(\frac{2}{\rho}\right)^{3}-1\right] \\
&> \left(\frac{2}{\rho}\right)^{2p-1}[2^3-1], \text{ since } \frac{2}{\rho} > 2 \\
&= 7\left(\frac{2}{\rho}\right)^{2p-1} \\
&> 4\left(\frac{2}{\rho}\right)^{2p-1} \\
&= \frac{2^{2p+1}}{\rho^{2p-1}} \\
&> \frac{2^{2p+1}}{\rho^{p}}, \text{ since } \frac{1}{\rho} > 1.
\end{aligned} \tag{16}
$$

Thus there exist k_p and l_p, $m_{p-1}+n_{p-1} < k_p + l_p \leq m_p + n_p$ such that

$$
|a_{m_p,n_p,k_p,l_p}| > \frac{2^{2p+1}}{\rho^p}. \tag{17}
$$

Now, define the sequence $\{x_{k,l}\}$ as follows:

$$
x_{k,l} = \begin{cases} \pi^p, & \text{if } k = k_p, l = l_p, p = 1, 2, \cdots; \\ 0, & \text{otherwise.} \end{cases}
$$

We note that $\lim_{k+l\to\infty} x_{k,l} = 0$. Now,

$$
\begin{aligned}
|y_{m_p,n_p}| &= \left| \sum_{k=1,l=1}^{\infty,\infty} a_{m_p,n_p,k,l} x_{k,l} \right| \\
&\geq \left| \sum_{k+l=(m_{p-1}+n_{p-1})+1}^{m_p+n_p} a_{m_p,n_p,k,l} x_{k,l} \right| \\
&\quad - \left| \sum_{k+l=1}^{m_{p-1}+n_{p-1}} a_{m_p,n_p,k,l} x_{k,l} \right| \\
&\quad - \left| \sum_{k+l=(m_p+n_p)+1}^{\infty} a_{m_p,n_p,k,l} x_{k,l} \right|
\end{aligned}
$$

$$
\begin{aligned}
&\geq |a_{m_p,n_p,k_p,l_p}| \times |x_{k_p,l_p}| - \\
&\quad \sup_{1\leq k+l\leq m_{p-1}+n_{p-1}} |a_{m_p,n_p,k,l}| - \sup_{m_p+n_p<k+l<\infty} |a_{m_p,n_p,k,l}| \\
&> \frac{2^{2p+1}}{\rho^p}\rho^p - 2^{2p-2} - 2^{p-1}, \text{ using (13), (15) and (17)} \\
&= 2^{2p+1} - 2^{2p-2} - 2^{p-1} \\
&= 2^{2p-2}(2^3-1) - 2^{p-1} \\
&= 2^{2p-2}(7) - 2^{p-1} \\
&= 2^{p-1}[7\cdot 2^{p-1} - 1] \\
&\geq 2^{p-1}[7\cdot 2^{p-1} - 2^{p-2}] \\
&= 2^{p-1}[2^{p-2}(14-1)] \\
&= 2^{p-1}[13\cdot 2^{p-2}] \\
&= 13\cdot 2^{2p-3}
\end{aligned}
$$

i.e., $|y_{m_p,n_p}| > 13 \cdot 2^{2p-3}, \ p = 1, 2, \cdots,$
i.e., $\lim_{m+n\to\infty} y_{mn} = 0$ does not hold, which is a contradiction. Thus (e) is necessary.

Proof of Sufficiency.

Let $\lim_{m+n\to\infty} x_{m,n} = x$. Then

$$y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}x_{k,l} - x.$$

From (b) we have

$$\sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l} + r_{m,n} = 1,$$

where

$$\lim_{m+n\to\infty} r_{m,n} = 0. \tag{18}$$

Hence,

$$y_{m,n} - x = \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x.$$

Given $\epsilon > 0$, we can choose sufficiently large p and q such that

$$\sup_{k+l>p+q} |x_{k,l} - x| < \frac{\epsilon}{5H}, \tag{19}$$

where $H = \sup_{m,n,k,l\geq 1} |a_{m,n,k,l}|$. Observe that $H > 0$ (from (b)).
Let $L = \sup_{k+l\geq 1} |x_{k,l} - x|$. We now choose $N \in \mathbb{N}$ such that whenever $m+n \geq N$, the following are satisfied:

$$\text{(i)} \quad \sup_{1\leq k+l\leq p+q} |a_{m,n,k,l}| < \frac{\epsilon}{5pqL}, \text{ using (a);} \tag{20}$$

$$\text{(ii)} \quad \sup_{k\geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5qL}, \ l = 1, 2, \cdots, q, \text{ using (c);} \tag{21}$$

$$\text{(iii)} \quad \sup_{l\geq 1} |a_{m,n,k,l}| < \frac{\epsilon}{5pL}, \ k = 1, 2, \cdots, p, \text{ using (d);} \tag{22}$$

and

$$\text{(iv)} \quad |r_{m,n}| < \frac{\epsilon}{5|x|}, \text{ from the equation (18).} \tag{23}$$

Whenever $m + n \geq N$, we thus have,

$$\begin{aligned}
|y_{m,n} - x| &= \left| \sum_{k=1,l=1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) + r_{m,n}x \right| \\
&\leq \left| \sum_{k=1,l=1}^{p,q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=1,l=q+1}^{p,\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
&\quad + \left| \sum_{k=p+1,l=1}^{\infty,q} a_{m,n,k,l}(x_{k,l} - x) \right| + \left| \sum_{k=p+1,l=q+1}^{\infty,\infty} a_{m,n,k,l}(x_{k,l} - x) \right| \\
&\quad + |r_{m,n}|\ |x| \\
&< \frac{\epsilon}{5pqL} Lpq + \frac{\epsilon}{5pL} Lp + \frac{\epsilon}{5qL} Lq + \frac{\epsilon}{5H} H + \frac{\epsilon}{5|x|} |x| \\
&= \epsilon, \quad \text{using (19), (20), (21), (22) and (23).}
\end{aligned}$$

Thus

$$\lim_{m+n\to\infty} y_{m,n} = x,$$

which completes the proof of the theorem.

Nörlund means for simple sequences and series in complete, non-trivially valued, non-archimedean fields were introduced by Srinivasan [9] and studied later in detail by Natarajan (for instance, see [7]). Nörlund means for double sequences and series in classical analysis were introduced by Moore [5]. We now define Nörlund means for double sequences and series in complete, non-trivially valued, non-archimedean fields and apply Theorem 2 for these means.

Definition 4. Given a doubly infinite set of elements $p_{m,n} \in K$, $m,n = 0,1,2,\cdots$, where $p_{0,0} \neq 0$, $|p_{i,j}| < |p_{0,0}|$, $(i,j) \neq (0,0)$, $i,j = 0,1,2,\cdots$, let

$$P_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j}, \qquad m,n = 0,1,2,\cdots.$$

Given any double sequence $\{s_{m,n}\}$ we define

$$\sigma_{m,n} = (N, p_{m,n})(s_{m,n}) = \frac{S_{m,n}}{P_{m,n}} = \frac{\sum_{i,j=0}^{m,n} p_{m-i,n-j} s_{i,j}}{P_{m,n}}, \ m,n = 0,1,2,\cdots.$$

If $\lim_{m+n\to\infty} \sigma_{m,n} = \sigma$, we say that the double sequence $\{s_{m,n}\}$ is summable $(N, p_{m,n})$ to the value σ, written as

$$s_{m,n} \to \sigma(N, p_{m,n}).$$

Any double series $\sum_{m,n} u_{m,n}$ is said to be summable $(N, p_{m,n})$ to the value σ if the double sequence $\{s_{m,n}\}$, where

$$s_{m,n} = \sum_{i,j=0}^{m,n} u_{i,j}, \ m,n = 0,1,2,\cdots,$$

is summable $(N, p_{m,n})$ to the value σ.

Definition 5. Given the Nörlund means $(N, p_{m,n}), (N, q_{m,n})$, we say that they are consistent if

$$s_{m,n} \to \sigma(N, p_{m,n}) \text{ and } s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow \sigma = \sigma'.$$

We say that $(N, p_{m,n})$ is included in $(N, q_{m,n})$, written as

$$(N, p_{m,n}) \subseteq (N, q_{m,n}),$$

if

$$s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, q_{m,n}).$$

The two methods $(N, p_{m,n}), (N, q_{m,n})$ are said to be equivalent if

$$(N, p_{m,n}) \subseteq (N, q_{m,n}) \text{ and } (N, q_{m,n}) \subseteq (N, p_{m,n}).$$

In view of Theorem 2, it is easy to prove the following result.

Theorem 3. *The necessary and sufficient conditions for the regularity of the Nörlund means* $(N, p_{m,n})$ *are:*

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} |p_{m-i,n-j}| = 0, \quad 0 \le i \le m; \tag{24}$$

$$\lim_{m+n\to\infty} \sup_{0\le i\le m} |p_{m-i,n-j}| = 0, \quad 0 \le j \le n. \tag{25}$$

In the sequel let $(N, p_{m,n}), (N, q_{m,n})$ be two regular Nörlund methods such that each row and each column of the infinite matrices $(p_{m,n}), (q_{m,n})$ is a regular Nörlund mean for simple sequences.

Theorem 4. *Any two such regular Nörlund methods are consistent.*

Proof. Given two Nörlund methods $(N, p_{m,n})$ and $(N, q_{m,n})$, where each row and each column of the infinite matrices $(p_{m,n}), (q_{m,n})$ is a regular Nörlund mean for simple sequences, we define a third method $(N, r_{m,n})$ by the equation

$$r_{m,n} = \sum_{i,j=0}^{m,n} p_{i,j} q_{m-i,n-j}, \quad m, n = 0, 1, 2, \cdots.$$

We then readily obtain, for $s = \{s_{m,n}\}$,

$$(N, r_{m,n})(s) = \sum_{i,j=0}^{m,n} \gamma_{m,n,i,j} (N, q_{i,j})(s),$$

where

$$\gamma_{m,n,i,j} = p_{m-i,n-j} Q_{i,j} / \sum_{\mu,\nu=0}^{m,n} p_{m-\mu,n-\nu} Q_{\mu,\nu}.$$

Since $(N, p_{m,n})$ and $(N, q_{m,n})$ are regular, we have,

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} |p_{m-i,n-j}| = 0 = \lim_{m+n\to\infty} \sup_{0\le i\le m} |p_{m-i,n-j}|.$$

It now follows that

$$\lim_{m+n\to\infty} \sup_{0\le j\le n} \gamma_{m,n,i,j} = 0 = \lim_{m+n\to\infty} \sup_{0\le i\le m} \gamma_{m,n,i,j}.$$

Consequently $(N, r_{m,n})$ is regular. The regularity of this transformation enables us to infer that

$$s_{m,n} \to \sigma'(N, q_{m,n}) \Rightarrow s_{m,n} \to \sigma'(N, r_{m,n}).$$

Similarly we can show that

$$s_{m,n} \to \sigma(N, p_{m,n}) \Rightarrow s_{m,n} \to \sigma(N, r_{m,n}).$$

These imply that the two Nörlund methods $(N, p_{m,n})$ and $(N, q_{m,n})$ are consistent, completing the proof of the theorem.

If $(N, p_{m,n})$, $(N, q_{m,n})$ are regular, in view of conditions (24), (25), we have,

$$\begin{aligned} P(x,y) &= \sum P_{m,n} x^m y^n, \\ Q(x,y) &= \sum Q_{m,n} x^m y^n, \\ p(x,y) &= \sum p_{m,n} x^m y^n, \\ q(x,y) &= \sum q_{m,n} x^m y^n, \end{aligned}$$

all converge for $|x|$, $|y| < 1$. The series

$$\begin{aligned} k(x,y) &= \sum k_{m,n} x^m y^n = \frac{q(x,y)}{p(x,y)} = \frac{Q(x,y)}{P(x,y)}, \\ l(x,y) &= \sum l_{m,n} x^m y^n = \frac{p(x,y)}{q(x,y)} = \frac{P(x,y)}{Q(x,y)} \end{aligned}$$

are convergent for $|x|, |y| < 1$ and further

$$\sum_{i,j=0}^{m,n} k_{i,j} p_{m-i,n-j} = q_{m,n}; \quad \sum_{i,j=0}^{m,n} k_{i,j} P_{m-i,n-j} = Q_{m,n}, \tag{26}$$

$$\sum_{i,j=0}^{m,n} l_{i,j} q_{m-i,n-j} = p_{m,n}; \quad \sum_{i,j=0}^{m,n} l_{i,j} Q_{m-i,n-j} = P_{m,n}. \tag{27}$$

Theorem 5. *If* $(N,p_{m,n}),(N,q_{m,n})$ *are regular, then* $(N,p_{m,n}) \subseteq (N,q_{m,n})$ *if and only if* $\lim_{m+n\to\infty} k_{m,n} = 0$.

Proof. Let $s(x,y) = \sum s_{m,n}x^m y^n$. Then for $|x|,|y| < 1$, we have,

$$\begin{aligned}\sum Q_{m,n}(N,q_{m,n})(s)x^m y^n &= \sum \left(\sum_{i,j=0}^{m,n} q_{m-i,n-j}s_{i,j}\right) x^m y^n \\ &= q(x,y)s(x,y);\end{aligned}$$

similarly

$$\sum P_{m,n}(N,p_{m,n})(s)x^m y^n = p(x,y)s(x,y).$$

Thus

$$\sum Q_{m,n}(N,q_{m,n})(s)x^m y^n = \sum k_{m,n}x^m y^n \sum P_{m,n}(N,p_{m,n})(s)x^m y^n$$

which implies that

$$Q_{m,n}(N,q_{m,n})(s) = \sum_{i,j=0}^{m,n} k_{m-i,n-j}P_{i,j}(N,p_{i,j})(s).$$

Hence,

$$(N,q_{m,n})(s) = \sum_{i,j=0}^{m,n} c_{m,n,i,j}(N,p_{i,j})(s), \tag{28}$$

where

$$c_{m,n,i,j} = k_{m-i,n-j}P_{i,j}/Q_{m,n}.$$

If $(N,p_{m,n}) \subseteq (N,q_{m,n})$, $(c_{m,n,i,j})$ is regular and so, by Theorem 2 (a), $\lim_{m+n\to\infty} c_{m,n,0,0} = 0$,

i.e., $$\lim_{m+n\to\infty} \frac{|k_{m,n}|\ |p_{0,0}|}{|q_{0,0}|} = 0,$$

which implies that $\lim_{m+n\to\infty} k_{m,n} = 0$.

Conversely, if $\lim_{m+n\to\infty} k_{m,n} = 0$, we can easily verify that $(c_{m,n,i,j})$ is regular. Consequently, using (28), it follows that $(N,p_{m,n}) \subseteq (N,q_{m,n})$. This completes the proof of the theorem.

Theorem 6, stated below, is an immediate consequence of Theorem 5.

Theorem 6. *If* $(N, p_{m,n})$ *and* $(N, q_{m,n})$ *are regular Nörlund methods, then they are equivalent if and only if* $\lim_{m+n\to\infty} k_{m,n} = 0$ *and* $\lim_{m+n\to\infty} l_{m,n} = 0$.

Remark. For the analogue of Theorem 6 in the classical case, see [5], Theorem III. Theorem 5, Theorem 6, in the case of regular Nörlund means for simple sequences, were established earlier by Natarajan (see[7], Theorem 3, Theorem 4).

Acknowledgement. The authors profusely thank Prof. W.H. Schikhof for pointing out errors in the original version of the paper and for giving constructive and valuable suggestions to retain the main results of the paper.

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